Due **Monday 19 October at 1pm** on blackboard.

Marks will be deducted for sloppy working. Clearly state your assumptions and conclusions, and justify all steps in your work.

The marked question 5 is required for MATH7861 students only. However, MATH1061 students are encouraged to try this also!

Q1 Let $f\colon \mathbb{Z}_{12} \to \mathbb{Z}_{12}$ be defined by $f(n) = 3n + 1 \pmod{12}$, and let $g\colon \mathbb{Q} \to \mathbb{Q}$ be defined by $g(q) = q^3 + 1$.

(a) Is f one-to-one?

(b) Is g one-to-one?

(c) Is f onto?

(d) Is g onto?

Prove all of your answers correct.

(12 marks)

Q2 Recall the Fibonacci sequence defined by $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.

(a) Prove that, for any integers a and b, $\gcd(a, b) = \gcd(a, a + b)$.

(b) Using induction, prove that $\gcd(F_n, F_{n-1}) = 1$ for all $n \geq 1$.

(10 marks)

Q3 Let $S = \{(a, b) \,|\, a, b \in \mathbb{Z} \text{ and } b \neq 0\}$. That is, S is the set of all pairs of integers where the second entry is non-zero.

Now define the relation ρ on S where $(a, b)\,\rho\,(x, y)$ if and only if $ay = bx$.
Prove that ρ is an equivalence relation.
Note: This is how you construct the rationals! The set $\mathbb{Q}$ is precisely the set of equivalence classes of ρ.

(10 marks)

Q4 Let X and Y be any non-empty sets, let $f\colon X \to Y$ be any function from X to Y, and let $A, B \subseteq X$ be any subsets of X.

(a) Is it always true that $f(A \cup B) = f(A) \cup f(B)$? If so, give a proof. If not, give a counterexample.

(b) Is it always true that $f(A \cap B) = f(A) \cap f(B)$? If so, give a proof. If not, give a counterexample.

(8 marks)

Q5 **[MATH7861 only]**
Again recall the Fibonacci sequence $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Now define a new sequence $G_0, G_1, \ldots$ by $G_0 = 0$, $G_1 = 1$ and $G_n = 3G_{n-1} - G_{n-2}$ for $n \geq 2$. Prove that $G_n = F_{2n}$ for all $n \geq 0$.

(10 marks)