THE UNIVERSITY OF QUEENSLAND
AUSTRALIA

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Exam information	
Course code and name	MATH7861 Discrete Mathematics
Semester	Semester 2, 2020
Exam type	Online, non-invigilated
Exam date and time	Please refer to your personalised timetable
Exam duration	Working time: 120 minutes + additional online allowance: 30 minutes. TOTAL exam duration: 2 hrs 30 minutes from the advertised exam commencement time.
Additional time	30 minutes additional time has been incorporated in recognition of the online environment and the different circumstances that students face in their home environments. This includes time for download and upload and allowances for network or connection issues.
Reading time	Reading time has not been formally allocated for online exams, however students are encouraged to review and plan their approach for the exam before they start. The total exam time should be sufficient to do this.
Exam window	You must commence your exam at the time listed in your personalised timetable. The exam will remain open only for the duration of the exam.
Weighting	This exam is weighted at either 50% or 70% of your total mark for this course, whichever gives you the highest overall grade. There are 80 marks available in total on this exam.
Permitted materials	This is an open book exam – all materials permitted. Any calculator is permitted.
Instructions	Answer all questions. You can print the exam and write in the exam paper, or write your answers on blank paper (clearly label your solutions so that it is clear which problems they are solutions to), or annotate an electronic file on a suitable device. You must submit your answers as a single electronic file **in PDF format** through Blackboard before the end of the allowed time. You should include your **name and student number** on the first page of the file that you submit.
Who to contact	If you have any concerns or queries about a particular question, or need to make any assumptions to answer the question, state these at the start of your solution to that question. You may also include queries you may have made with respect to a particular question, should you have been able to 'raise your hand' in an examination room. If you experience any **technical difficulties** during the exam, contact the Library AskUs service for advice (open 7am–10pm, 7 days a week, Brisbane time): Chat: https://support.my.uq.edu.au/app/chat/chat_launch_lib/p/45 Phone: +61 7 3506 2615 Email: examsupport@library.uq.edu.au You should also ask for an email documenting the advice provided so you can provide this on request. In the event of a **late submission**, you will be required to submit evidence that you completed the exam in the time allowed. We recommend you use a phone camera to take photos (or a video) of every page of your exam. Ensure that the photos are time-stamped.

	If you submit your exam after the due time then you should send details (including any evidence) to SMP Exams (exams.smp@uq.edu.au) as soon as possible after the end of the exam.
Important exam condition information	The normal academic integrity rules apply. You cannot cut-and-paste material other than your own work as answers. You are not permitted to consult any other person – whether directly, online, or through any other means – about any aspect of this assessment during the period that this assessment is available. If it is found that you have given or sought outside assistance with this assessment then that will be deemed to be cheating and will result in disciplinary action. By undertaking this online assessment you will be deemed to have acknowledged UQ's academic integrity pledge to have made the following declaration: *"I certify that my submitted answers are entirely my own work and that I have neither given nor received any unauthorised assistance on this assessment item".*

1. (a) For statement variables p and q, prove that

$$p \vee (p \to q)$$

is a tautology.

You may use either truth tables or logical equivalences. If you use logical equivalences, you should name the laws you use at each step.

(3 marks)

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(b) Prove that the following argument is valid. You should use only the laws of inference (i.e., the list of valid argument forms from lectures) and/or logical equivalences.

$$\begin{array}{ll} 1. & (a \vee b) \to c \\ 2. & d \to a \\ 3. & \sim d \to b \\ & \\ \therefore & c \end{array}$$

(5 marks)

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2. (a) Using the Euclidean algorithm, compute $\gcd(588, 90)$. Show all your working.

(3 marks)

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(b) Using your answer to part (a), compute $\mathrm{lcm}(588, 90)$.

(2 marks)

3. (a) Prove the following statement:
For all $n \in \mathbb{Z}$, if n is odd then $(n+2)^2 \equiv n^2 \pmod 8$.

(4 marks)

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(b) Find a counterexample to disprove the following statement:
For all $n \in \mathbb{Z}$, $(n+2)^2 \equiv n^2 \pmod 8$.

(2 marks)

4. Let $a_0, a_1, a_2, \ldots$ be the sequence defined recursively by $a_0 = 0$, $a_1 = 1$, and

$$a_n = \frac{n}{2} \cdot (3 + a_{n-1} - a_{n-2}) \text{ for all integers } n \geq 2.$$

(a) Compute the terms a_2, a_3 and a_4.

(3 marks)

(b) Guess an explicit formula for this sequence, and prove that your formula is correct.
(Hint: If you have trouble guessing, you may find that computing more terms can help.)

(5 marks)

5. Let $b_0, b_1, b_2, \ldots$ be the sequence defined recursively by $b_0 = 0$, $b_1 = 1$, and

$$b_n = 2(b_{n-1} + b_{n-2}) \text{ for all integers } n \geq 2.$$

Using strong mathematical induction, prove that for all integers $n \geq 0$, $b_n \equiv n \pmod 3$.

(5 marks)

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6. (a) Write out the elements of $\mathcal{P}(\{1\})$ and $\mathcal{P}(\mathcal{P}(\{1\}))$.

(2 marks)

(b) Suppose sets A, B and C have cardinalities $|A| = 3$, $|B| = 4$ and $|C| = 5$. What is the cardinality of $A \times B \times C$?

(2 marks)

6. (c) Recall that $[a, b]$ denotes the interval from a to b *including* its endpoints, that (a, b) denotes the interval from a to b *excluding* its endpoints, and that $|S|$ denotes the cardinality of the set S.
Prove that $|[0, 1] \cup [2, 3]| = |(4, 5)|$.
(Hint: You might wish to use the Schröder-Bernstein theorem.)

(5 marks)

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7. Consider the function $f : \mathbb{Z} \to \{0, 1, 2\}$ defined by $f(n) = n^2 \bmod 3$.

(a) What is the image of the set $\{0, 1, 2\}$? You do not need to show your working.

(2 marks)

(b) What is the preimage of 0? You do not need to show your working.

(2 marks)

(c) What is the image of $\mathbb{Z}$? You do not need to show your working.

(2 marks)

(d) Is f one-to-one? Is f onto? You do not need to explain your answers.

(2 marks)

8. Define a relation ρ on $\mathbb{N}$ by $x \rho y$ if and only if $x \times y$ is a perfect square.

Determine whether ρ is reflexive, whether ρ is symmetric, and whether ρ is transitive. For each property, explain why or why not.

(Recall that, in this course, $\mathbb{N} = \{1, 2, 3, \ldots\}$*, and a perfect square is a number of the form* n^2 *where* $n \in \mathbb{Z}$*.)*

(8 marks)

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9. It is time to elect the University President! There are two candidates: Donald and Joe.

The university contains six faculties, and each faculty gets one vote. To win the election, you need a majority of votes (i.e., four or more faculties must vote for you).

(a) How many different ways could the faculties cast their votes to ensure a perfect tie (i.e., Donald and Joe each receive exactly three votes)?
Give your answer as a single integer.

(2 marks)

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(b) If each faculty votes for either Donald or Joe at random (e.g., by flipping a coin), what is the probability that Donald wins the election?
Give your answer in the form of a fraction.

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(3 marks)

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9. (c) Now suppose that the Science faculty publicly declares that they will not choose their own vote, but instead they will vote in the same way as the Arts faculty.

If each of the *other* five faculties (including Arts but not Science) votes for either Donald or Joe at random, and then Science votes exactly the same way as Arts, what is the probability that Donald wins the election?

Give your answer in the form of a fraction.

(4 marks)

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10. Let G denote the group $(\mathbb{Z}_2 \times \mathbb{Z}_2, +)$, and let H denote the group $(\{1, 3, 5, 7\}, \circ)$ where $x \circ y$ is defined to be $x \times y \pmod 8$.

(a) Write out the full Cayley tables for G and H.

(4 marks)

(b) Are groups G and H isomorphic? Explain why or why not.

(4 marks)

11. (a) A graph has 12 edges, and every vertex of the graph has degree 3. How many vertices does the graph have?

(3 marks)

(b) In a tree, every vertex has degree 1 or degree 3. If there are exactly four vertices of degree 3, how many vertices have degree 1?

(3 marks)

END OF EXAMINATION

MATH1061/MATH7861 Examination Formula Pages

Logical Equivalences

Given any statement variables p, q and r, the following logical equivalences hold, where $\mathbf{t}$ denotes a tautology and $\mathbf{c}$ denotes a contradiction:

Commutative laws	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
Associative laws	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
Distributive laws	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
Identity laws	$p \wedge \mathbf{t} \equiv p$	$p \vee \mathbf{c} \equiv p$
Negation laws	$p \vee \sim p \equiv \mathbf{t}$	$p \wedge \sim p \equiv \mathbf{c}$
Double negative laws	$\sim (\sim p) \equiv p$	
Idempotent laws	$p \wedge p \equiv p$	$p \vee p \equiv p$
Universal bound laws	$p \vee \mathbf{t} \equiv \mathbf{t}$	$p \wedge \mathbf{c} \equiv \mathbf{c}$
De Morgan's laws	$\sim (p \wedge q) \equiv \sim p \vee \sim q$	$\sim (p \vee q) \equiv \sim p \wedge \sim q$
Absorption laws	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
Negations of $\mathbf{t}$ *and* $\mathbf{c}$	$\sim \mathbf{t} \equiv \mathbf{c}$	$\sim \mathbf{c} \equiv \mathbf{t}$

Valid Argument Forms

Modus Ponens	$p \to q$ p $\therefore q$	Elimination	(a) $p \vee q$ $\sim q$ $\therefore p$ (b) $p \vee q$ $\sim p$ $\therefore q$
Modus Tollens	$p \to q$ $\sim q$ $\therefore \sim p$	Transitivity	$p \to q$ $q \to r$ $\therefore p \to r$
Generalization	(a) p $\therefore p \vee q$ (b) q $\therefore p \vee q$	Proof by division into cases	$p \vee q$ $p \to r$ $q \to r$ $\therefore r$
Specialization	(a) $p \wedge q$ $\therefore p$ (b) $p \wedge q$ $\therefore q$		
Conjunction	p q $\therefore p \wedge q$	Contradiction Rule	$\sim p \to \mathbf{c}$ $\therefore p$

Set Identities

Given any sets A, B and C that are subsets of a universal set U, the following laws hold:

1. Commutative Laws — (a) $A \cup B = B \cup A$ (b) $A \cap B = B \cap A$
2. Associative Laws — (a) $(A \cup B) \cup C = A \cup (B \cup C)$ (b) $(A \cap B) \cap C = A \cap (B \cap C)$
3. Distributive Laws — (a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
4. Identity Laws — (a) $A \cup \emptyset = A$ (b) $A \cap U = A$
5. Complement Laws — (a) $A \cup A^c = U$ (b) $A \cap A^c = \emptyset$
6. Double Complement Law — $(A^c)^c = A$
7. Idempotent Laws — (a) $A \cup A = A$ (b) $A \cap A = A$
8. Universal Bound Laws — (a) $A \cup U = U$ (b) $A \cap \emptyset = \emptyset$
9. De Morgan's Laws — (a) $(A \cup B)^c = A^c \cap B^c$ (b) $(A \cap B)^c = A^c \cup B^c$
10. Absorption Laws — (a) $A \cup (A \cap B) = A$ (b) $A \cap (A \cup B) = A$
11. Complement of U and $\emptyset$ — (a) $U^c = \emptyset$ (b) $\emptyset^c = U$
12. Set Difference Law — $A - B = A \cap B^c$

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The Quotient-Remainder Theorem Given any integer n and any positive integer d, there exist unique integers q and r such that $n = dq + r$ and $0 \le r < d$.

Unique Factorisation Theorem Given any integer $n > 1$, there exists a positive integer k, distinct prime numbers $p_1, p_2, \ldots, p_k$, and positive integers $e_1, e_2, \ldots, e_k$ such that $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$, and any other expression for n as a product of prime numbers is identical to this except perhaps for the order in which the factors are written.

Schröder-Bernstein Theorem For all sets A and B, if $|A| \le |B|$ and $|A| \ge |B|$ then $|A| = |B|$.

Binomial Theorem Given any real numbers a and b, and any nonnegative integer n,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.$$

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