THE UNIVERSITY OF SYDNEY
MATH3888

Semester 2 | Interdisciplinary Project (Stream 1) | 2022

## WEEK 6 HOMEWORK GUIDELINES

**Submission**:
As outlined in the information sheet of this interdisciplinary project course, you will create reports using the (maths) editing software LaTeX:

`https://en.wikibooks.org/wiki/LaTeX`

You are encouraged to use *Overleaf* to create your LaTeX report which you can access via your browser through your University of Sydney account:

`https://www.overleaf.com`

Use the following basic setup for your LaTex file:

```
\documentclass[11pt]{article}
\usepackage{fullpage,amsmath,graphicx}
...
\begin{document}
...
\end{document}
```



Submission of the corresponding pdf file is via Canvas/turnitin (where it will be checked for plagariasm). As outlined in the course info sheet, this report is worth 5% of your final mark.

Deadline is **Thursday, week 7 (September 15th), 23:59**. No late submission will be accepted!

**Constraints**:
The 'fontsize' is strictly 11 points and the margins of the document are automatically set by the 'fullpage' package (as instructed above).
The package 'amsmath' might be needed for the mathematical editing, and I let you figure out what the 'graphicx' package is needed for. Add any other packages, if needed.
Additional LaTeX instructions are given within the text. Please follow them to avoid losing marks!

## Bifurcation Theory

This week you will check analytic criteria regarding basic bifurcations discussed in previous lectures. Based on that experience you will hopefully appreciate the work MatCont is doing for you.

1. Given the differential equation

$$\frac{dx}{dt} = e^{-3(x+\mu)} + 3x - 5 =: f(x,\mu)\,, \qquad x \in \mathbb{R}, \mu \in \mathbb{R} \tag{1}$$

   (a) Show that (1) undergoes a saddle-node bifurcation, i.e. (1) possesses equilibrium states $(\bar{x},\bar{\mu})$ where $D_x f(\bar{x},\bar{\mu}) = 0$ (i.e. has a zero eigenvalue), and the map

$$(x,\mu) \mapsto (f(x,\mu), D_x f(x,\mu))$$

   is regular at $(\bar{x},\bar{\mu})$. (Recall: a map is regular at $(\bar{x},\bar{\mu})$ if its linearisation has full rank.)

   (b) Based on the regularity of the map, which theorem can you appeal to?
   **Hint:** it allows you to conclude explicitly the location of saddle-node points in $(x,\mu)$-space.)

   (c) Verify by plotting a corresponding bifurcation diagram in $(\mu,x)$-space using MatCont. Compare the *non-degeneracy parameter* `a` calculated by MatCont with the value of $D_{xx} f(\bar{x},\bar{\mu})$? What information does this parameter encode?
   Provide your plot (not a screenshot) in an appropriate LaTeX-figure environment (width = 8cm). Include a figure caption with relevant information that explains type of bifurcation observed, its location, and stability properties of each branch.

2. Given the system of differential equation $dx/dt = f(x,\mu)$, $x \in \mathbb{R}^2$ and $\mu \in \mathbb{R}$, by

$$\begin{aligned} \frac{dx_1}{dt} &= (-x_2 + \mu)(1 + \frac{3}{16}x_2^2) \\ \frac{dx_2}{dt} &= x_1 - \frac{1}{2}\mu x_2 - x_1(2x_1 - x_2^2)(x_2 - \mu) \end{aligned} \tag{2}$$

   (a) Show that (2) undergoes an Andronov-Hopf bifurcation at a bifurcation point $(\bar{x},\bar{\mu})$, i.e. there exists a unique local branch of equilibria parametrised by $\mu \in B_\delta(\bar{\mu})$ with eigenvalues $\lambda_{1/2}(\mu)$ such that $\operatorname{Re}\lambda_{1/2}(\bar{\mu}) = 0$ and $\operatorname{Im}\lambda_{1/2}(\bar{\mu}) = \pm\omega \neq 0$, as well as

$$\frac{d}{d\mu}\operatorname{Re}\lambda_{1/2}(\bar{\mu}) \neq 0 \qquad \text{and} \qquad l_1(\bar{\mu}) \neq 0,$$

   where $l_1$ denotes the *first Lyapunov coefficient*.
   **Note:** google the *Scholarpedia* website on *'Andronov-Hopf bifurcation'* for how to calculate this coefficient. The planar case is discussed at the very end. Check that your system is in the correct form as stated there. Define the right hand side explicitly that you are using in your calculation.

   (b) The sign of $l_1$ determines the criticality of the AH-bifurcation. Do you observe a sub- or supercritical AH bifurcation?

   (c) Verify by plotting a corresponding bifurcation diagram in $(\mu, x_1, x_2)$-space using MatCont; $\mu \in [\bar{\mu} - 1/2, \bar{\mu} + 1/2]$ and appropriate scaling of $(x_1, x_2)$. Compare your first Lyapunov coefficient with the one calculated by MatCont.
   Provide your 3D plot (not a screenshot) in an appropriate LaTeX-figure environment (width = 8cm). Include a figure caption with relevant information that explains the type of bifurcation observed, stability properties of equilibrium and limit cycle branches.