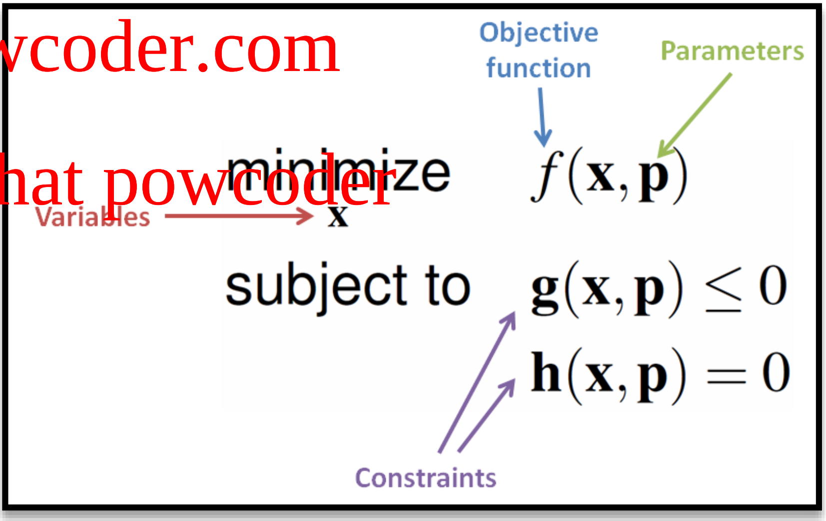
Formulating and analyzing a problem

ME 564/SYS 564
Wed Sep 5, 2018
Steven Hoffenson



Goal of Week 2: To learn some strategies to analyze and reduce an optimization formulation prior to solving

Recap: Class expectations

- Respect from everyone toward everyone
- Collaboration, learning, and helping each other
- Good discussions
- Ask questions when you have them
- Responsible phone/computer use
- Learn through practice
- Start on time (3:00pm) and take 1-2 breaks per session

Recap: How to optimize

1. **Formulate** the problem
   a) Define system boundaries
   b) Develop analytical models
   c) Explore/reduce the problem space ← TODAY
   d) Formalize optimization problem

(Weeks 1-2, 4, 9-12)

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & f(\mathbf{x}, \mathbf{p}) \\ \text{subject to} \quad & \mathbf{g}(\mathbf{x}, \mathbf{p}) \leq 0 \\ & \mathbf{h}(\mathbf{x}, \mathbf{p}) = 0 \end{aligned}$$

2. **Solve** the problem
   a) Choose the right approach/algorithm
   b) Solve (by hand, code, or software)
   c) Interpret the results
   d) Iterate if needed

(Weeks 3, 5-8, 12)

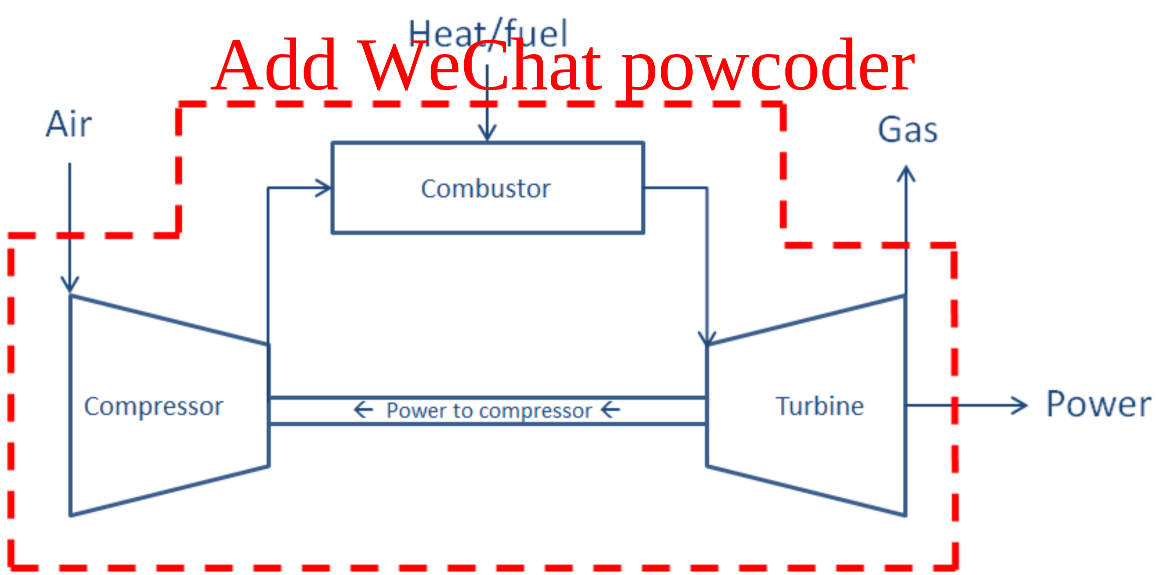
$$\mathbf{x}_{k+1} = \mathbf{x}_k - [\mathbf{H}(\mathbf{x}_k)]^{-1} \nabla f(\mathbf{x}_0)$$

Recap: Important problem attributes

Quantity	What it means	Battery examples
Objectives	What we want to maximize/minimize	Maximize capacity in kWh
Hard constraints	Must-haves, with specific thresholds	Must meet safety standards
Soft constraints	Wants, with specific thresholds	Weigh no more than 200 lb; Capacity of at least 30 kWh; Volume no more than 15 ft^3; Cost no more than $3,000
Variables	Things we can change	Dimensions, material choice, layout
Parameters	Quantities that we can't or won't change	Material properties, e.g., density of a particular lithium-ion battery; thresholds of soft constraints

Example: Gas turbine

Objective(s)	*maximize (power out)/(fuel in)*
Constraints	*power out must be at least 1 MW* *total cost must be no more than $1M*
Design variables	*air flow rate in, compressor ratio, fuel flow rate in*
Parameters	*Inlet air temperature & pressure, fuel specific heat*
Constants	*gas constant, gravity*



Recap: Optimization formulation

Objective function

Parameters

Variables

$$\underset{\mathbf{x}}{\text{minimize}} \quad f(\mathbf{x}, \mathbf{p})$$

$$\text{subject to} \quad \mathbf{g}(\mathbf{x}, \mathbf{p}) \leq 0$$

$$\mathbf{h}(\mathbf{x}, \mathbf{p}) = 0$$

Constraints

"negative null" form

<u>Negative null form</u>

$$\mathbf{h}(\mathbf{x}) = \mathbf{0}$$
$$\mathbf{g}(\mathbf{x}) \leq \mathbf{0}$$

<u>Positive null form</u>

$$\mathbf{h}(\mathbf{x}) = \mathbf{0}$$
$$\mathbf{g}(\mathbf{x}) \geq \mathbf{0}$$

<u>Negative unity form</u>

$$\mathbf{h}(\mathbf{x}) = \mathbf{1}$$
$$\mathbf{g}(\mathbf{x}) \leq \mathbf{1}$$

Example: Negative Null Form

Design an electric motor system with maximum efficiency $E(\mathbf{x})$ while power output $P_{out}(\mathbf{x})$ must be equal to P_0 and maximum speed $V_{max}(\mathbf{x})$ must be at least V_0.

Original		**Negative null form**	
maximize	$E(\mathbf{x})$	minimize	$-E(\mathbf{x})$ or $1/E(\mathbf{x})$
w.r.t.	$\mathbf{x}$	w.r.t.	$\mathbf{x}$
subject to	$P_{out}(\mathbf{x})=P_0$	subject to	$P_{out}(\mathbf{x})-P_0 = 0$
	$V_{max}(\mathbf{x})\geq V_0$		$V_0 - V_{max}(\mathbf{x}) \leq 0$

Explore the problem space

- Does a solution exist? (feasibility)
- Is the problem well-bounded?
- Are the constraints active?
- Are the functions monotonic?
- Can the formulation be simplified?

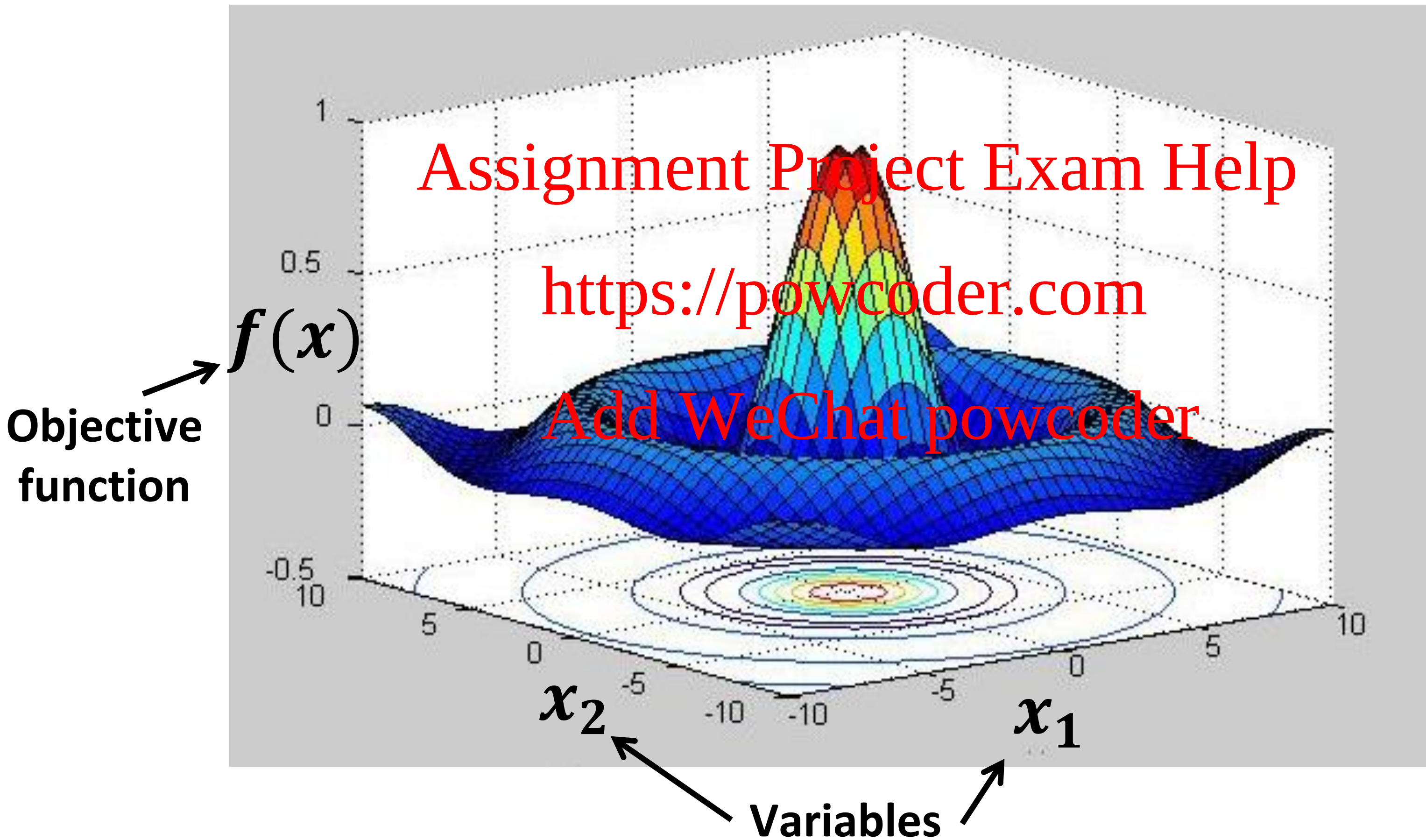
Answering these questions can help detect formulation errors and save time

Design space

The “design space” refers to the feasible region (satisfies the constraints) in the variable space

Design-objective space

$$
\begin{aligned}
\underset{\mathbf{x}}{\text{minimize}} \quad & f(\mathbf{x},\mathbf{p}) \\
\text{subject to} \quad & \mathbf{g}(\mathbf{x},\mathbf{p}) \le 0 \\
& \mathbf{h}(\mathbf{x},\mathbf{p}) = 0
\end{aligned}
$$



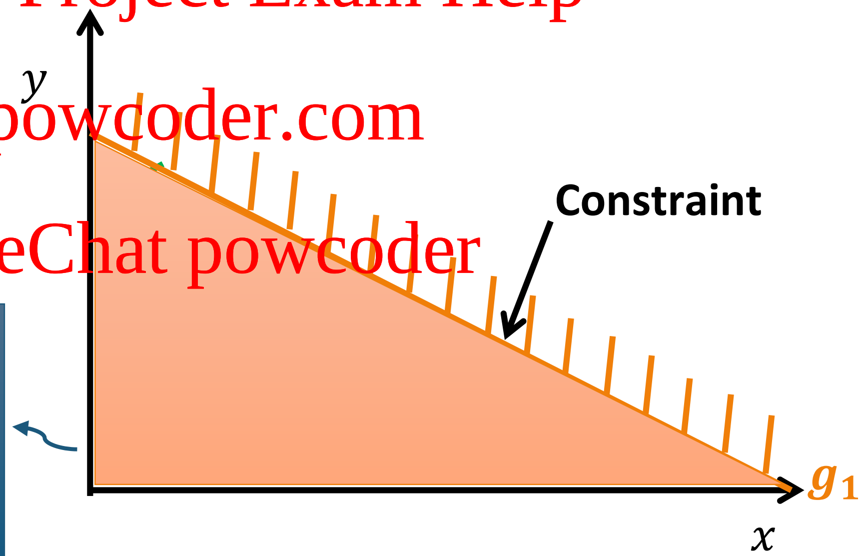
Design space - Example

$$\min_{\{x,y\}\in R^2} \quad f = -2x - y$$

$$\text{s.t.} \quad g_1 : x + 2y - 8 \leq 0$$

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & f(\mathbf{x},\mathbf{p}) \\ \text{subject to} \quad & \mathbf{g}(\mathbf{x},\mathbf{p}) \leq 0 \\ & \mathbf{h}(\mathbf{x},\mathbf{p}) = 0 \end{aligned}$$

Note: x and y are both in the design/variable space!



Choose a point from the orange triangular and plug it into g_1 to make sure the constraint is defined correctly

e.g.(0,0) 0+2×0-8≤ 0 ✓

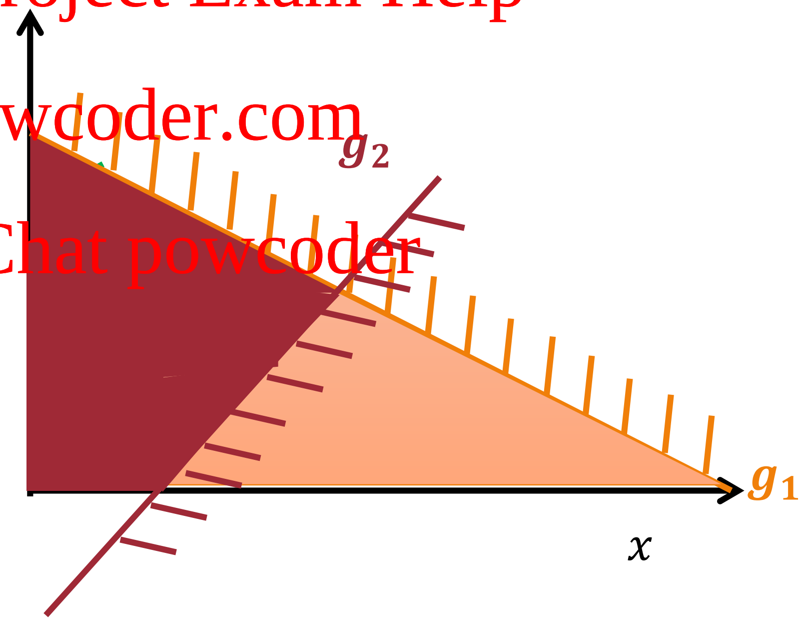
Design space - Example

$$\min_{\{x,y\}\in R^2} \quad f = -2x - y$$

s.t. $\quad g_1 : x + 2y - 8 \le 0$

$\quad g_2 : x - y - 1.5 \le 0$

minimize$_{\mathbf{x}}$	$f(\mathbf{x}, \mathbf{p})$
subject to	$\mathbf{g}(\mathbf{x}, \mathbf{p}) \le 0$
	$\mathbf{h}(\mathbf{x}, \mathbf{p}) = 0$



Design space - Example

$$\min_{\{x,y\}\in R^2} \quad f = -2x - y$$

s.t.

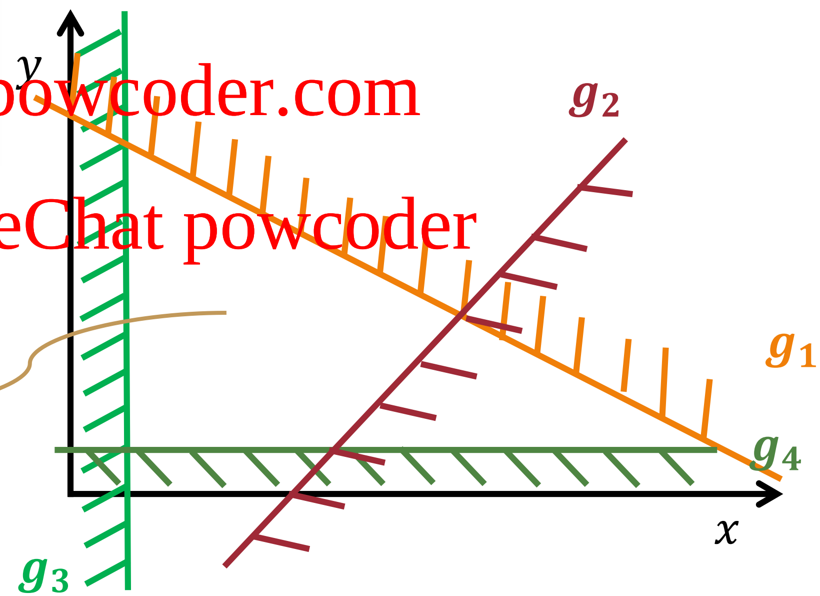
$$g_1 : x + 2y - 8 \leq 0$$

$$g_2 : x - y - 1.5 \leq 0$$

$$g_3 : -2x + 1 \leq 0$$

$$g_4 : -2y + 1 \leq 0$$

$$\begin{aligned} &\underset{\mathbf{x}}{\text{minimize}} && f(\mathbf{x}, \mathbf{p}) \\ &\text{subject to} && \mathbf{g}(\mathbf{x}, \mathbf{p}) \leq 0 \\ & && \mathbf{h}(\mathbf{x}, \mathbf{p}) = 0 \end{aligned}$$



The intersection of $g_1, g_2, g_3,$ & g_4 demonstrates the **feasible design space**

Equality constraints

Often, we can eliminate these algebraically.

Equality constraints

Objective function

Parameters

$$\text{minimize}_{\mathbf{x}} \quad f(\mathbf{x}, \mathbf{p})$$

Variables

$$\text{subject to} \quad \mathbf{g}(\mathbf{x}, \mathbf{p}) \le 0$$

$$\boxed{\mathbf{h}(\mathbf{x}, \mathbf{p}) = 0}$$

Constraints

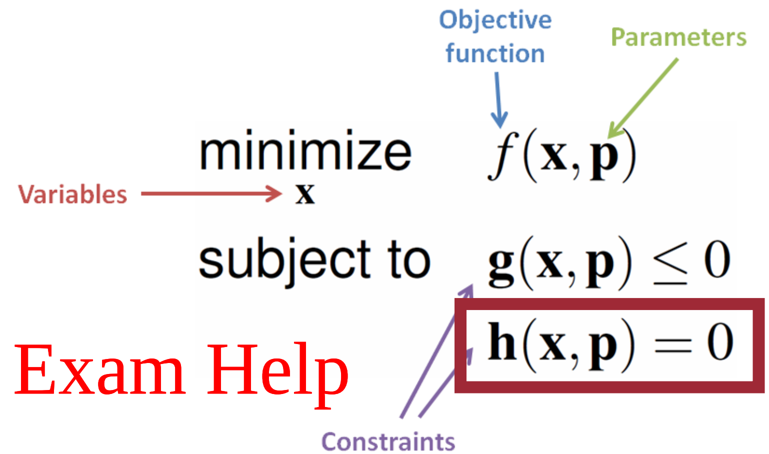
- Some algorithms cannot handle equality constraints (efficiently)
- When possible, substitute out the equality constraint

$$\boxed{\begin{array}{ll} \min\limits_{x \in R} & f(x,y) = x + 2x^2 y \\ \text{s.t.} & h(y) = y - 4 = 0 \end{array}} \Rightarrow \boxed{\min\limits_{x \in R} \quad f(x) = x + 8x^2}$$

- Otherwise, you may be able to direct it as an inequality (we'll discuss later)

Irrelevant variables

- Some variables or constraints are not relevant to the rest of the problem
- When possible, we can come up with a closed form equation for an irrelevant variable and remove it



$$\min_{x} \quad f(x) = x + 8x^2$$
$$\text{s.t.} \quad h(x,z) = x + z - 3 = 0$$

$$\min_{x} \quad f(x) = x + 8x^2$$
$$z = 3 - x$$

Variable z is irrelevant, since it does not affect the rest of the problem
Note: Variables that do not appear in the objective function are often relevant (we'll show examples later in this lecture).

Feasibility and Boundedness

Does a solution satisfying the constraints even exist?

Is there a finite optimal solution?

Feasibility

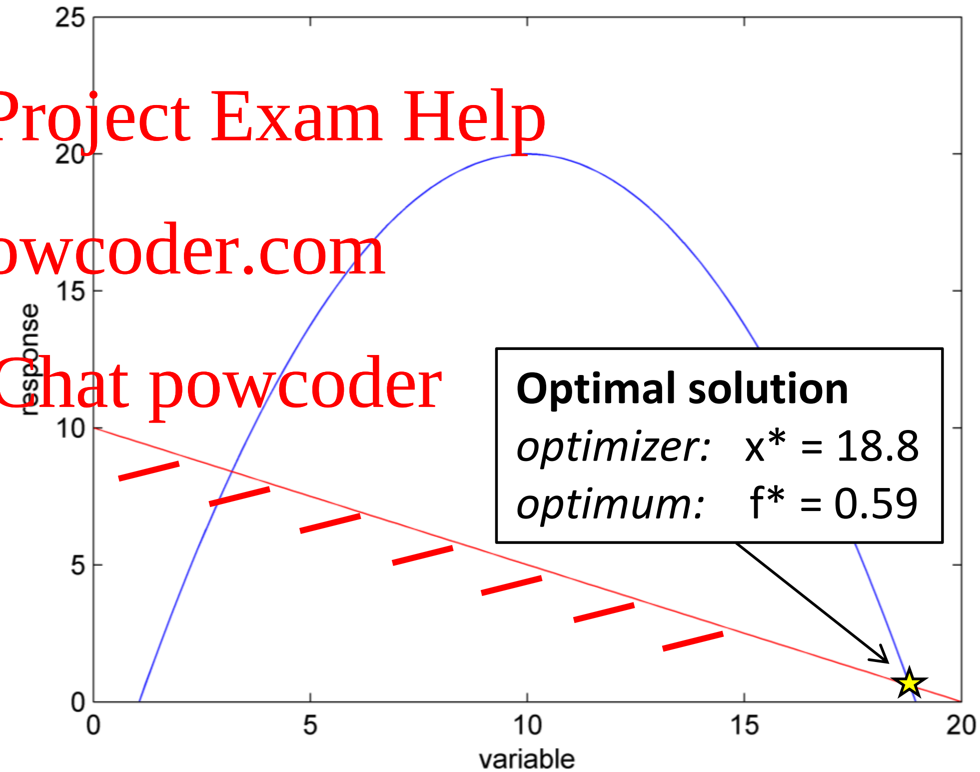
Consider the minimization problem:

$$\underset{x}{\text{minimize}} \quad \boxed{f(x) = 20 - \left(\frac{x-10}{2}\right)^2}$$
$$\text{subject to} \quad \boxed{f(x) \geq 10 - x/2}$$

Formally, we should get this into negative null form:

$$10 - x/2 \leq y$$
$$10 - x/2 - y \leq 0$$
$$10 - x/2 - \left(20 - \frac{x^2 - 20x + 100}{4}\right) \leq 0$$
$$10 - x/2 - \left(-\frac{x^2}{4} + 5x - 5\right) \leq 0$$
$$x^2 - 22x + 60 \leq 0$$

$$\boxed{\begin{array}{ll} \underset{x}{\text{minimize}} & f(x) = 20 - \left(\frac{x-10}{2}\right)^2 \\ \text{subject to} & g(x) = x^2 - 22x + 60 \leq 0 \end{array}}$$



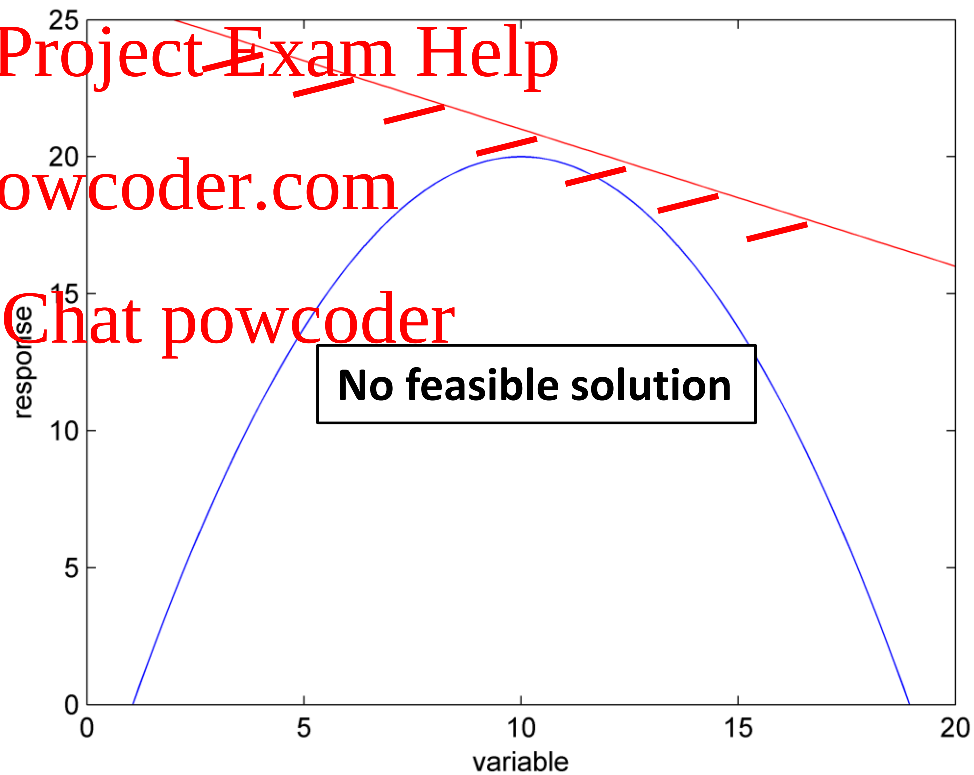
A problem is **well posed** when a solution exists
(This problem is well-posed)

Feasibility

Consider the minimization problem:

$$\underset{x}{\text{minimize}} \quad \boxed{f(x) = 20 - \left(\frac{x-10}{2}\right)^2}$$
$$\text{subject to} \quad \boxed{g(x) = x^2 - 22x + 124 \le 0}$$

*Regarding **feasibility**, we want to know whether a solution exists*



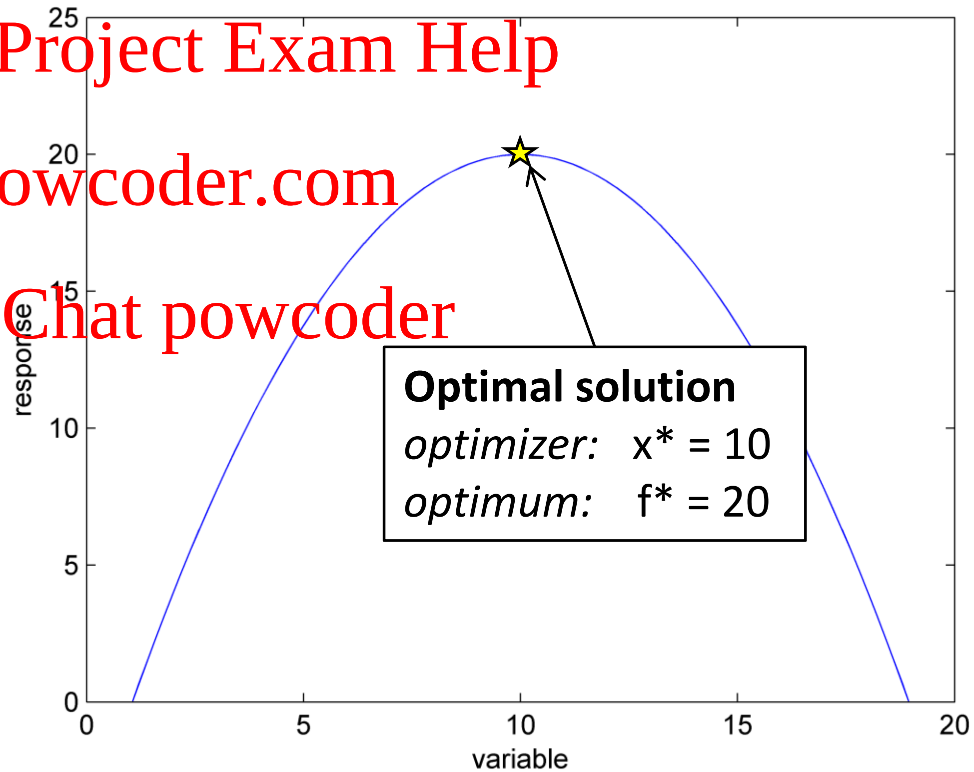
This problem is not well-posed

Feasibility

Consider the maximization problem:

$$\underset{x}{\text{maximize}} \quad f(x) = 20 - \left(\frac{x-10}{2}\right)^2$$

*Regarding **feasibility**, we want to know whether a solution exists*



This problem is well-posed

Feasibility

Is this problem well-posed?

$$\begin{aligned} &\min && f(\mathbf{x}) = x_1 + x_2 \\ &\text{w.r.t.} && \mathbf{x} \in \mathbb{R}^2 \\ &\text{s.t.} && g(\mathbf{x}) = x_1^2 + 1 \leq 0 \end{aligned}$$

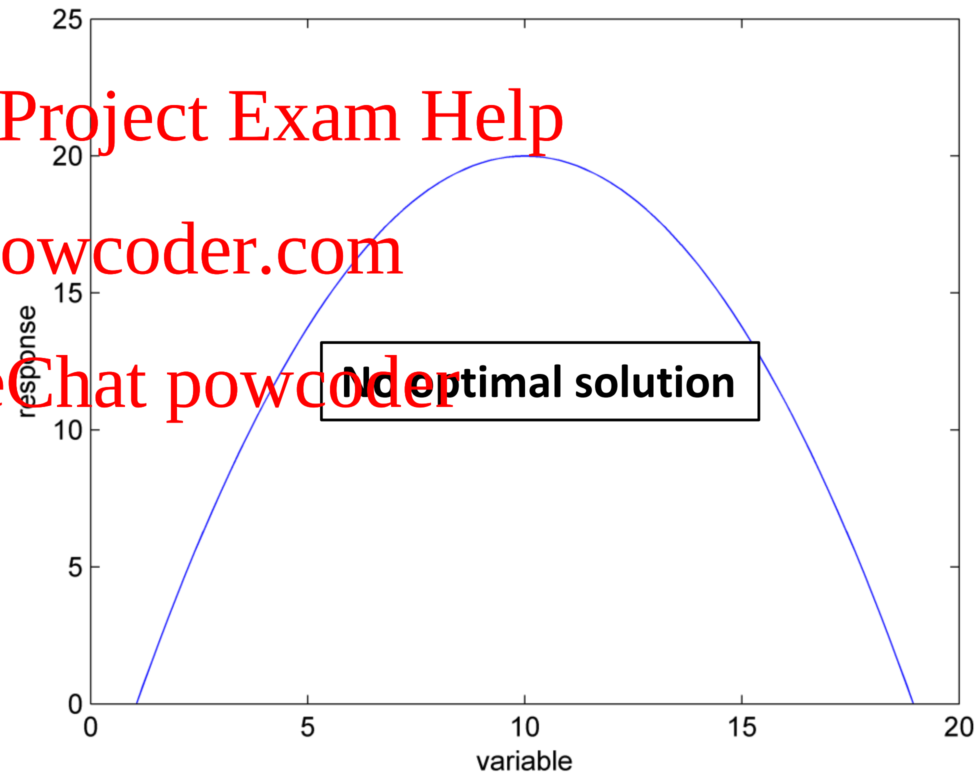
The feasible domain is empty.
This problem is **not** well-posed.

Boundedness

Consider the minimization problem:

$$\underset{x}{\text{minimize}} \quad f(x) = 20 - \left(\frac{x-10}{2}\right)^2$$

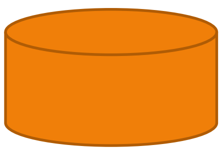
For any real "solution" you find (very high or very low number), you will always be able to find something better.



This problem is unbounded

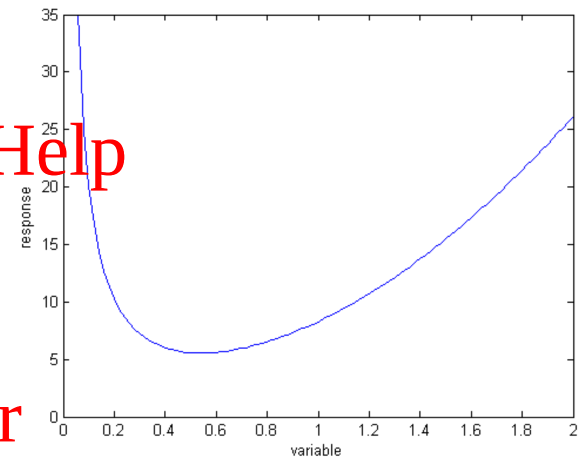
Boundedness

Consider the design of a cylindrical pill:



$$\underset{r,h}{\text{maximize}} \quad SA(r,h) = 2\pi r^2 + 2\pi r h$$

$$\text{subject to} \quad V(r,h) = \pi r^2 h = 1$$



Simplify:

$$h = \frac{1}{\pi r^2}$$

$$\underset{r}{\text{maximize}} \quad SA(r) = 2\pi r^2 + \frac{2}{r}$$

$$\text{subject to} \quad 0 < r < \infty$$

r is **unbounded** from above and **not well bounded** from below

Lower bound and infimum

Lower bound: A number l such that $f(\mathbf{x}) \geq l$ for all $\mathbf{x}$ in your domain

Infimum: the greatest lower bound; $g \geq l$ for all l

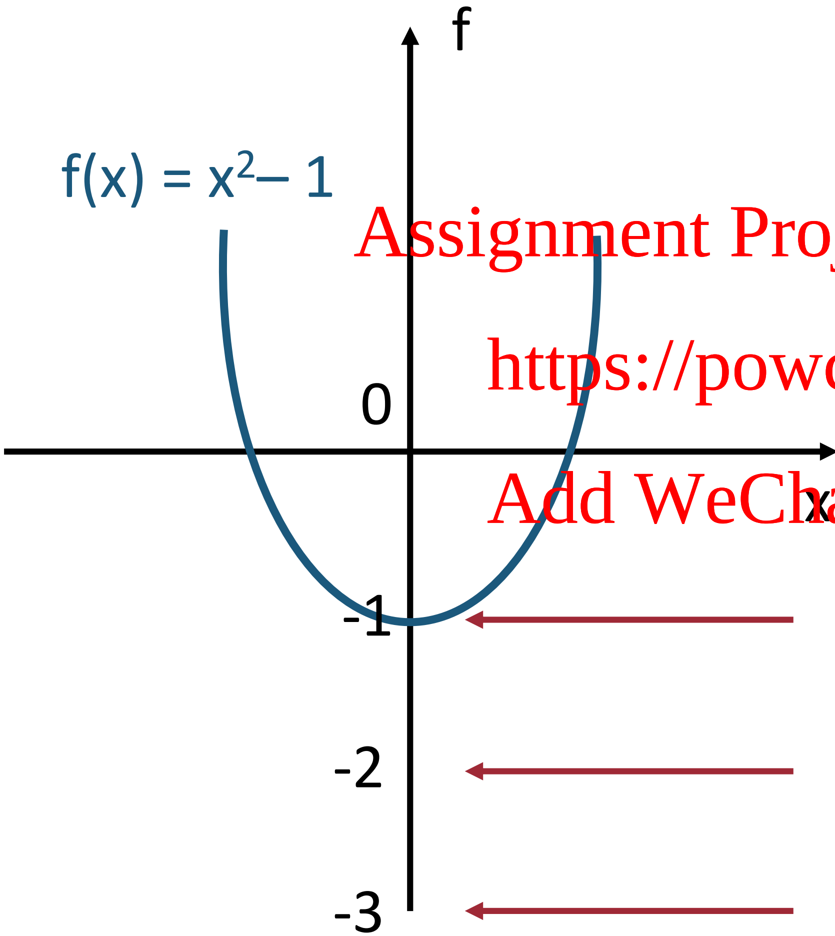
Argument of infimum: Value $\underline{\mathbf{x}}$ such that $f(\underline{\mathbf{x}}) = g$

If all arguments of the infimum are in your domain, then your minimization problem is **well-bounded**.

Note: We call the **least upper bound** of a model the **supremum**, with a similar definition, which is relevant for maximization problems

Lower bound and infimum

Argument of infimum $\underline{x} = 0$



Greatest Lower Bound (**infimum**)

Lower Bound

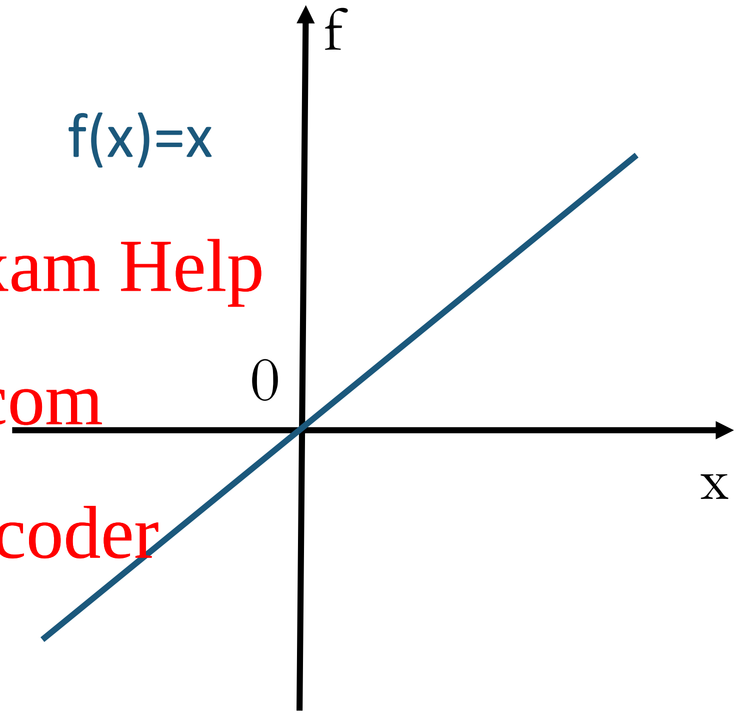
Lower Bound

Boundedness

Consider the following problem

$$\min \quad f(\mathbf{x}) = x$$
$$\text{w.r.t.} \quad \mathbf{x} \in \mathcal{P}$$

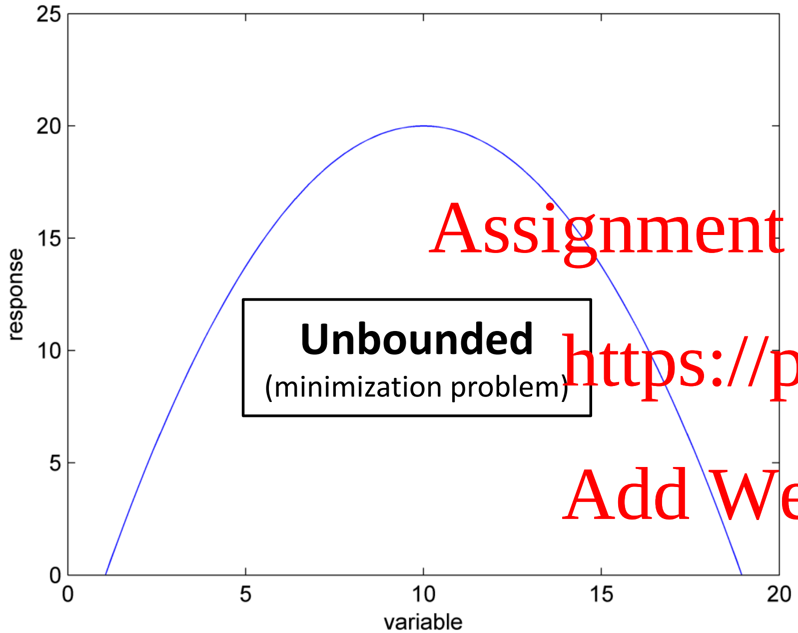
Domain $\mathcal{P}$: Positive real numbers, i.e. $0 < x < \infty$

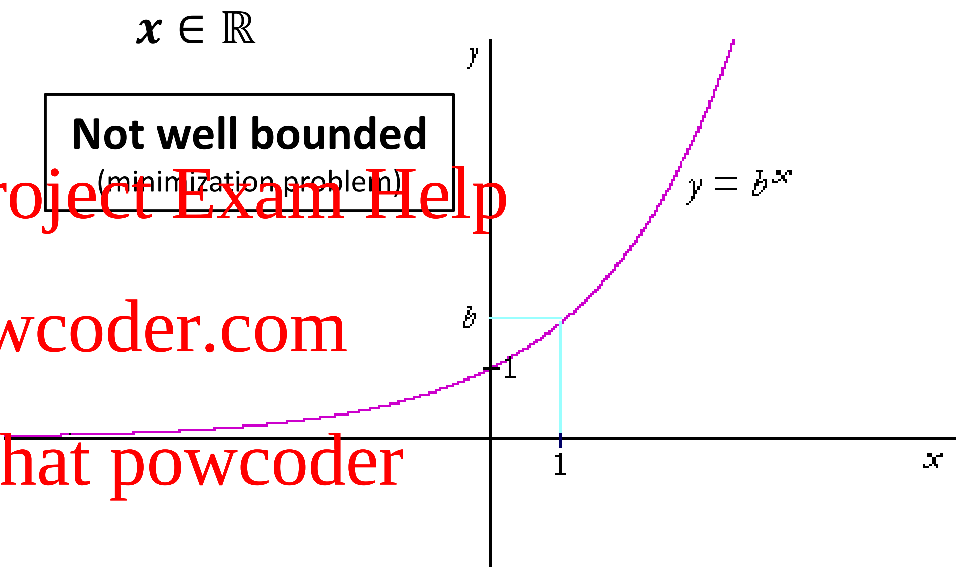


What is the solution?

Infimum in $\mathcal{P}$: $\quad f(\underline{\mathbf{x}}) = g = 0$
Arg. of infimum: $\underline{\mathbf{x}} = 0$
Since $0 \notin \boldsymbol{\mathcal{P}}$, this problem is **not well bounded**

Boundedness



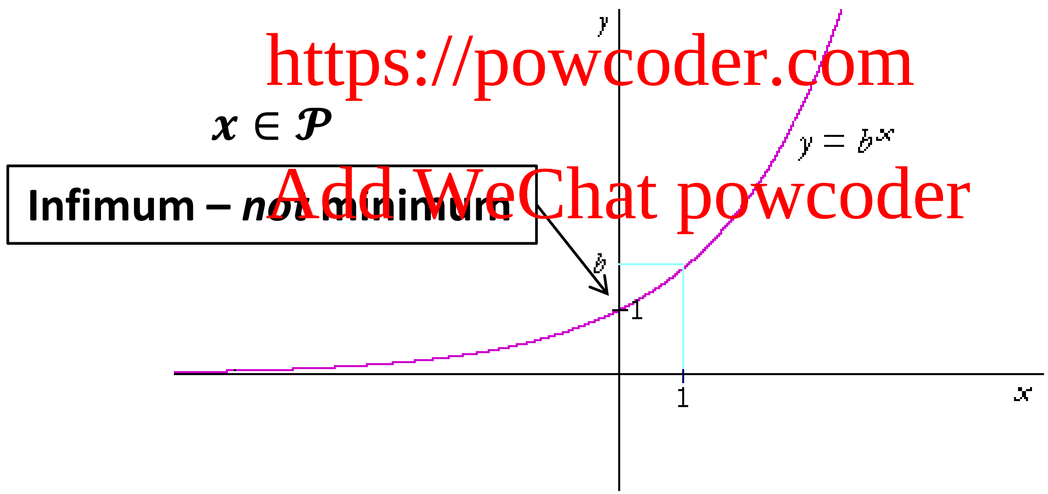


No infimum exists

No minimum exists

Boundedness

Since we normally have upper and lower bounds (constraints) on variables, this is often not a problem – the most common case is when we are minimizing in $\mathcal{P}$



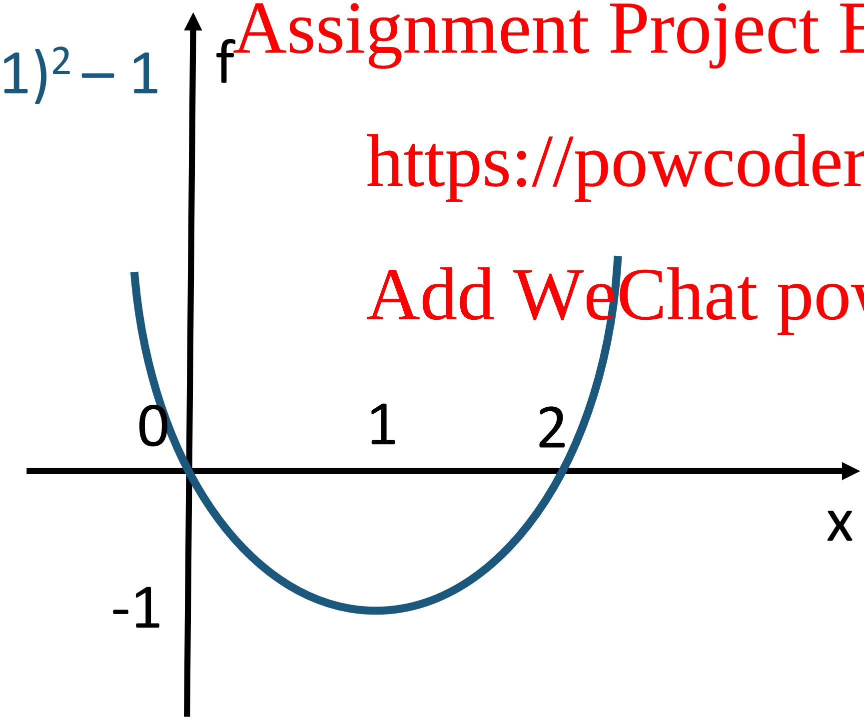
Here, since x > 0, we want x as close to 0 as possible… but there is no minimum

Exercise: Well bounded?

- Assume our domain is $\mathcal{P}$: Positive real numbers

  i.e. $0 < x < \infty$

$f(x)=(x-1)^2 - 1$



Infimum in $\mathcal{P}$: -1

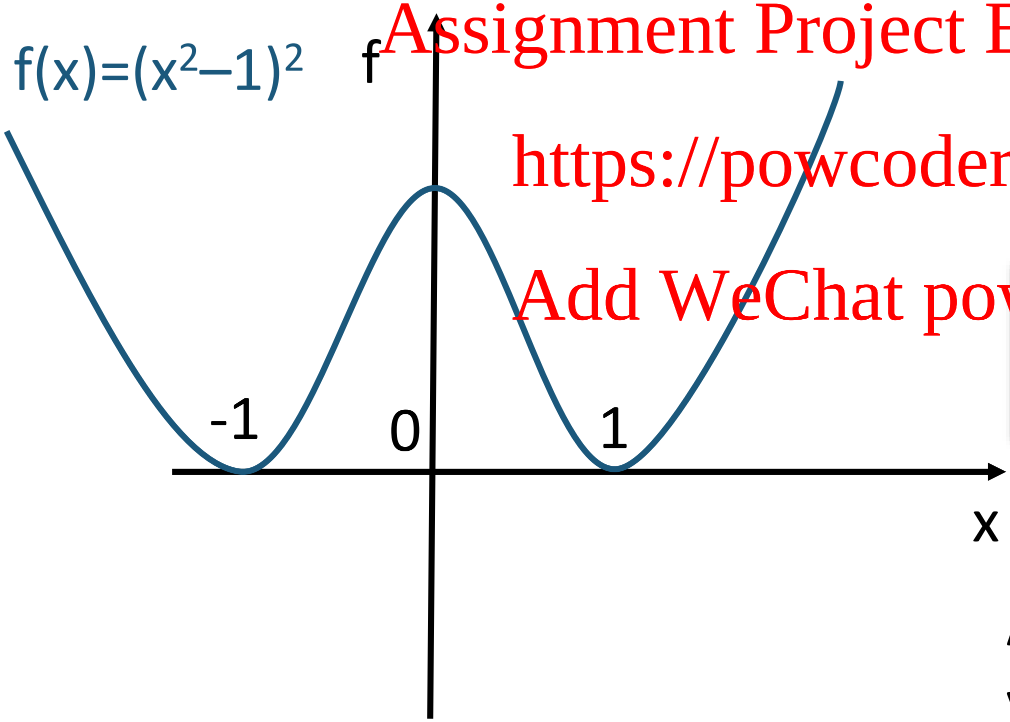
Arg. of infimum: 1

Since $1 \in \mathcal{P}$, well bounded

Exercises: Well Bounded?

- Assume our domain is $\mathcal{P}$: Positive real numbers

  i.e. $0 < x < \infty$



Infimum in $\mathcal{P}$: 0

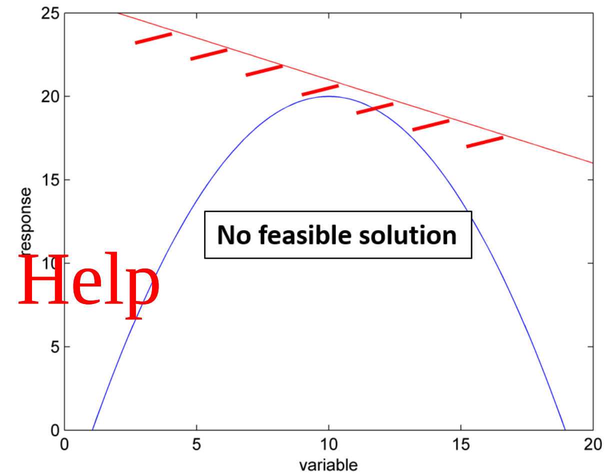
Arg. of infimum: ± 1

Since $1 \in \mathcal{P}$ but $-1 \notin \mathcal{P}$, not well bounded

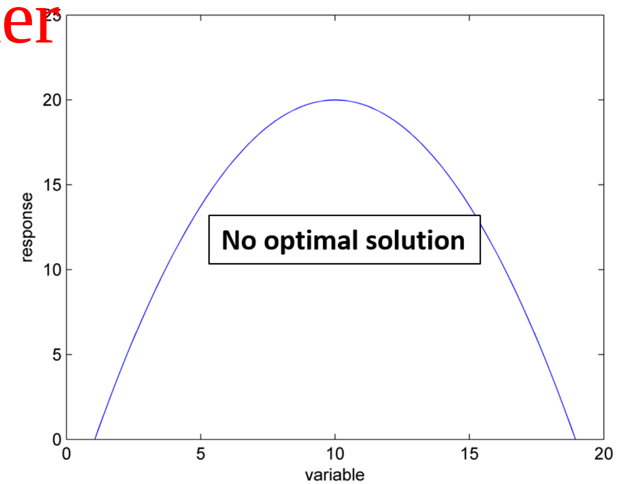
All arguments must be in your domain for boundedness by definition

Well-posed vs well-bounded

- A **well-posed** problem has feasible solutions in the domain
  - There are points that satisfy all constraints
  - We're just talking about existence of a feasible design



- A **well-bounded** problem has finite optimal solutions inside the domain
  - All arguments of the infimum must be in the domain
  - We're talking about feasibility of optima



Constraint activity

Can we simplify our problem formulation by identifying “active” inequality constraints?

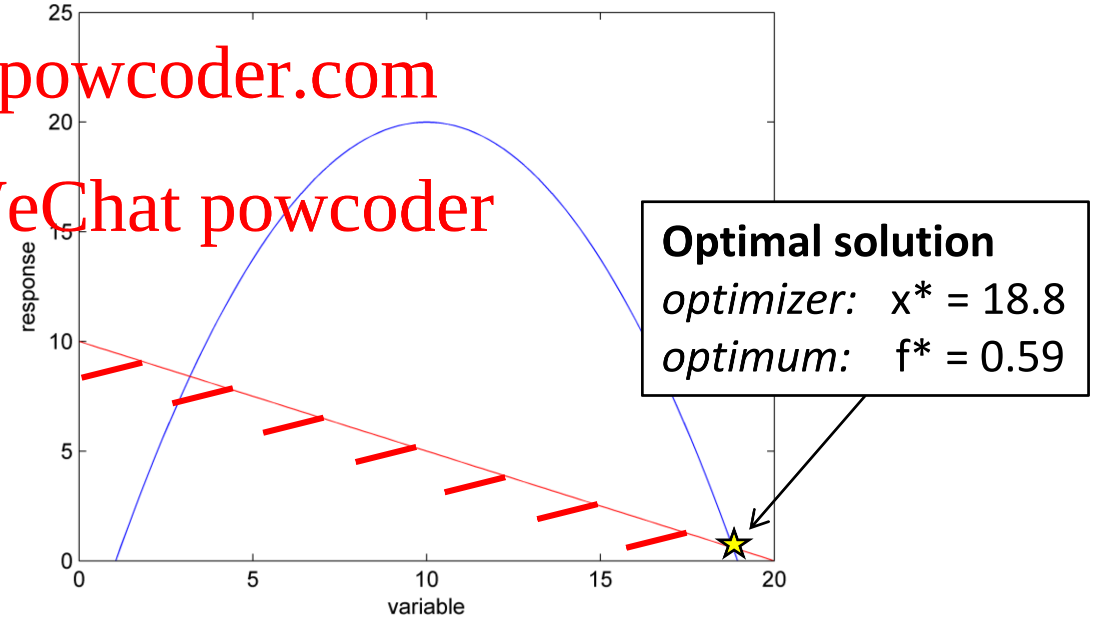
Constraint activity

*A constraint is **active** if removing it changes the optimization result*

$$\underset{x}{\text{minimize}} \quad f(x) = 20 - \left(\frac{x-10}{2}\right)^2$$

$$\text{subject to} \quad g(x) = x^2 - 22x + 60 \leq 0$$

*g(x) is **active***

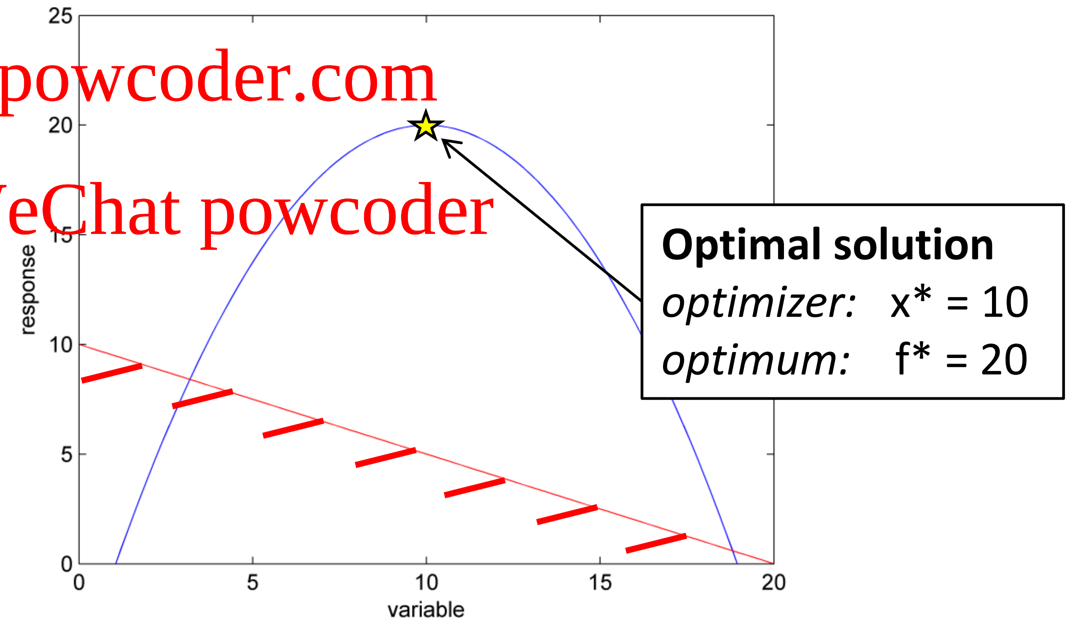


Constraint activity

*A constraint is **active** if removing it changes the optimization result*

$$\underset{x}{\text{maximize}} \quad f(x) = 20 - \left(\frac{x-10}{2}\right)^2$$

$$\text{subject to} \quad g(x) = x^2 - 22x + 60 \leq 0$$

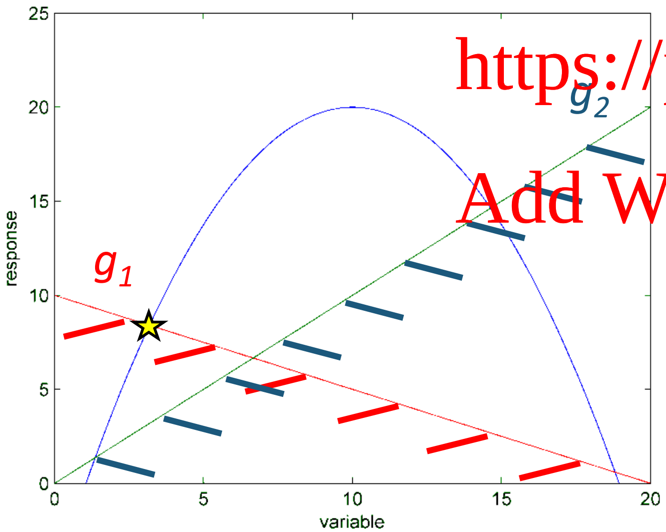


*g(x) is **inactive***

Constraint activity

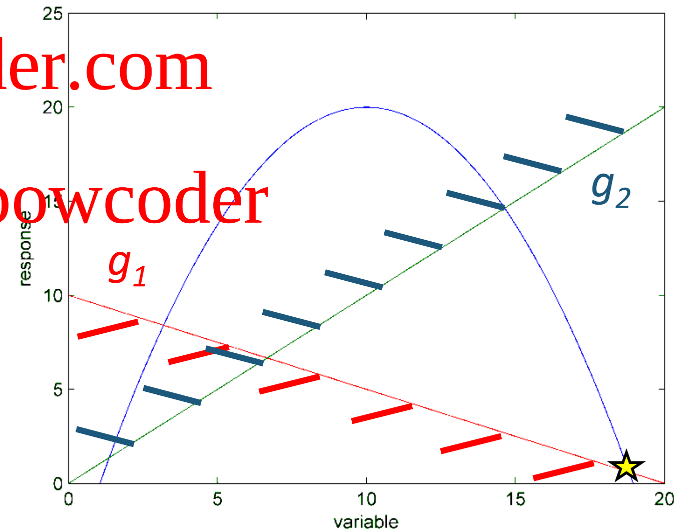
*A constraint is **active** if removing it changes the optimization result*

minimize



g_1 active, g_2 active

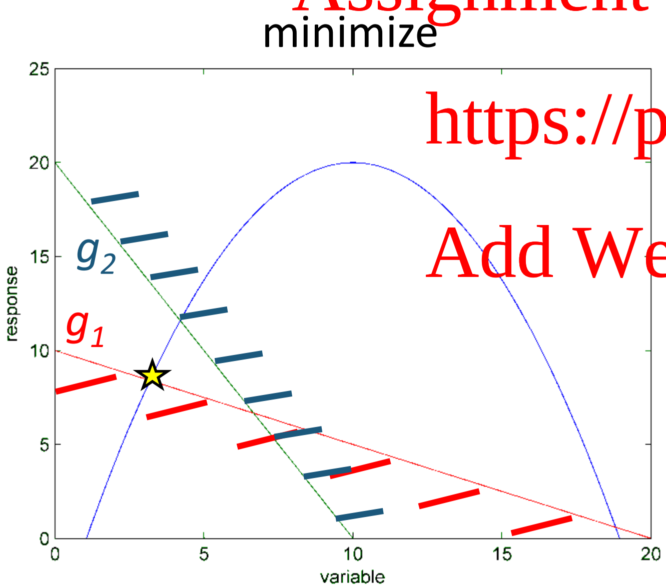
minimize



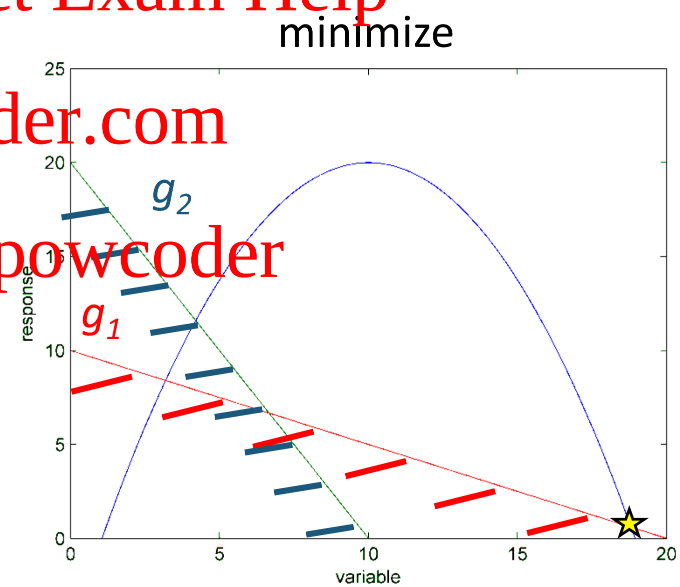
g_1 active, g_2 inactive

Constraint activity

*A constraint is **active** if removing it changes the optimization result*



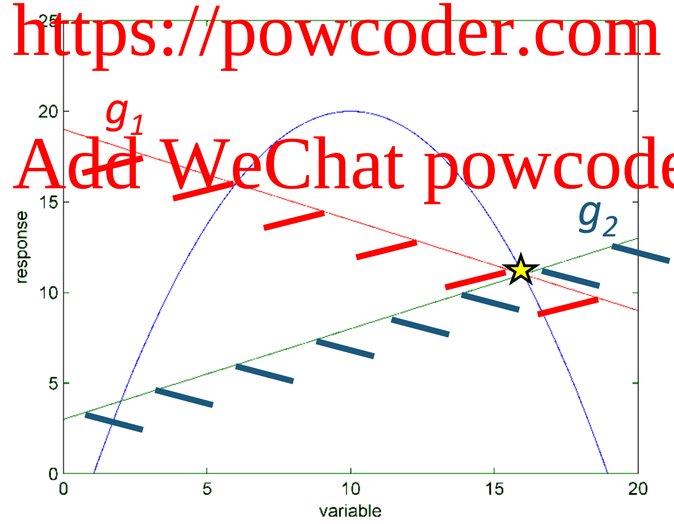
g_1 active, g_2 active



g_1 active, g_2 inactive

Constraint activity

*A constraint is **active** if removing it changes the optimization result*



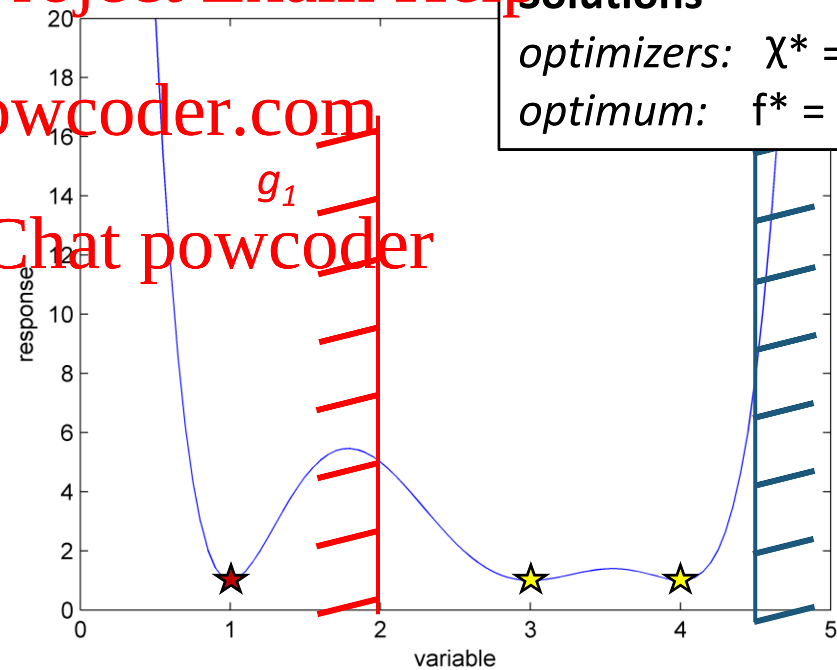
g_1 active, g_2 inactive

Constraint activity

A constraint is ***semiactive*** *if removing it adds to the existing set of optimizers*

$$
\begin{aligned}
&\underset{x}{\text{minimize}} && f(x) \\
&\text{subject to} && g_1 = -x + 2 \leq 0 \\
& && g_2 = x - 4.5 \leq 0
\end{aligned}
$$

Solutions to unconstrained problem
optimizers: $\chi^* = \{1, 3, 4\}$
optimum: $f^* = 1$



Solutions
optimizers: $\chi^* = \{3, 4\}$
optimum: $f^* = 1$

g_1 semiactive, g_2 inactive

Activity theorem

Relaxed problem solution when g_i is removed

Original solution

Constraint g_i is active if and only if $f(\mathbf{x}_i) < f(\mathbf{x}_*)$

I.e., the value of the objective at the minimizers of the *relaxed problem* is less than its value at the minimizers of the *original problem*

Activity check

Activity Definition

- Solve original problem
- Remove constraint
- Solve again
- Check if optimum changes

Relaxation

- Solve problem without constraint
- Add constraint
- Check if constraint violated

If changed

If violated

Constraint is active

Active constraints

- If active constraints are identified early you can reduce the problem size
  - **Active**: Equality
  - **Semiactive:** Do nothing
  - **Inactive:** Remove
- When a solution to the constrained problem found:
  - Active constraints limit you from improving your solution further
  - If the active constraint bound is a parameter, examine the impact of this parameter to the solution
- If an inequality constraint is satisfied equally is it active?
  - Not necessarily—think about min x^2, s.t. $x \geq 0$
  - This is very rare

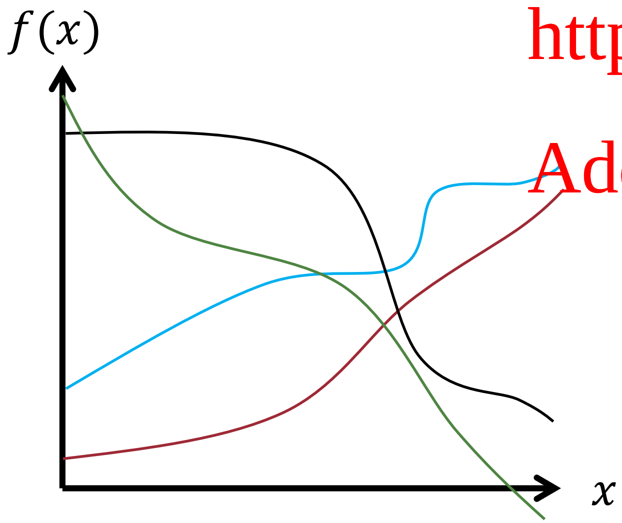
Monotonicity analysis

An easy way to identify active constraints

Monotonicity

A function $f(x)$ is ***monotonically increasing*** with respect to a variable x if:

$$\text{for every } x_2 > x_1,\ f(x_2) > f(x_1)$$

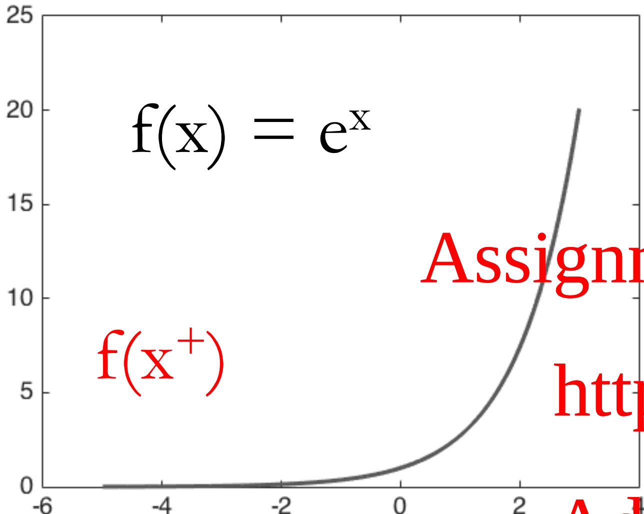


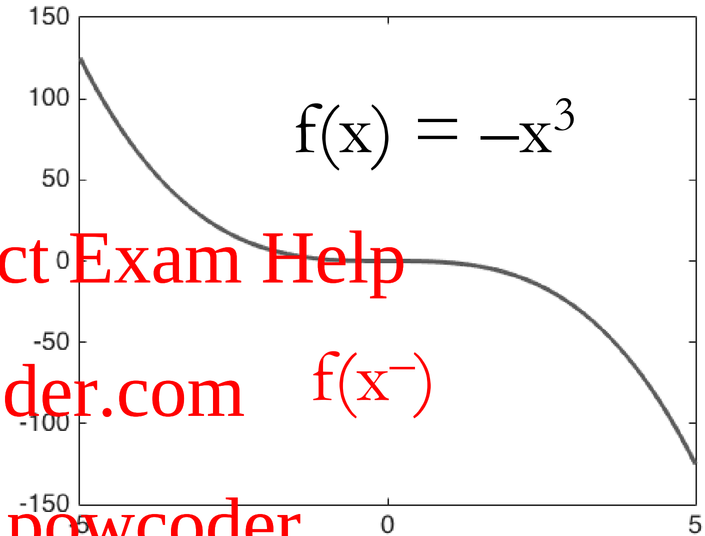
Other types of monotonicity

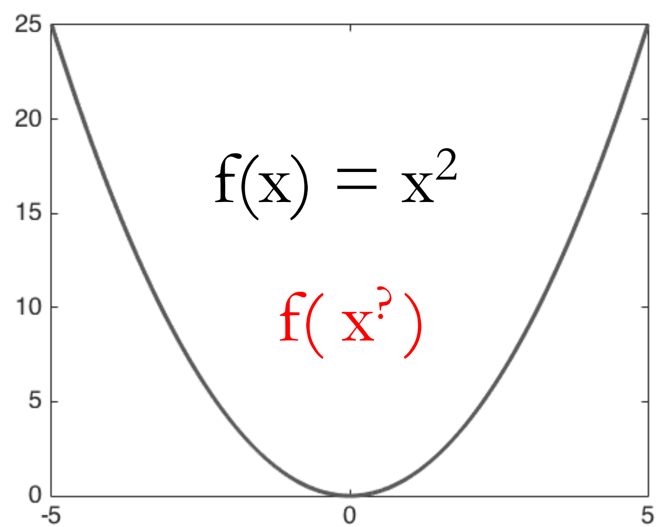
Replace monotonically increasing with:	Replace 2nd > with:
(strictly) monotonically increasing	>
weakly monotonically increasing/non-decr.	≥
(strictly) monotonically decreasing	<
weakly monotonically decreasing/non-incr.	≤

In all of these cases, we say the function is ***monotonic** with respect to x*

Monotonic functions







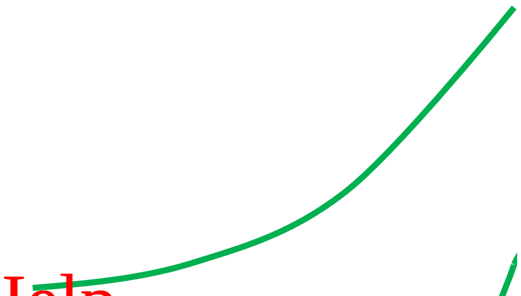
$x \geq 0 \quad f(x^+)$

$x < 0 \quad f(x^-)$

Checking for monotonicity

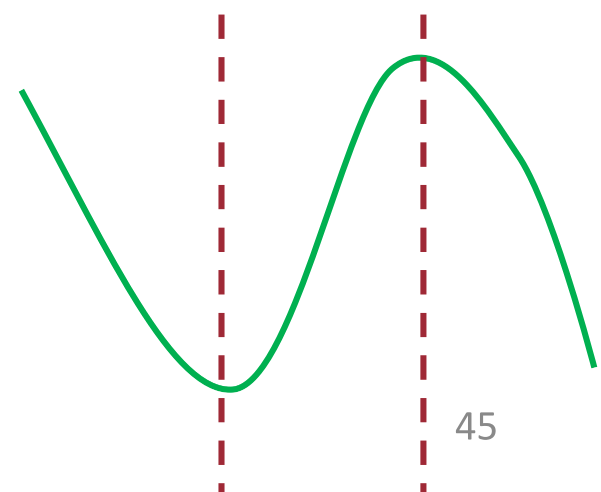
Check $\partial f / \partial x_i$:

- If $\partial f / \partial x_i > 0$ everywhere, then f is strictly increasing w.r.t. x_i



- If $\partial f / \partial x_i \geq 0$ everywhere, then f is weakly increasing w.r.t. x_i

- If the sign of $\partial f / \partial x_i$ flips, then divide it into regions and perform analysis on each region separately

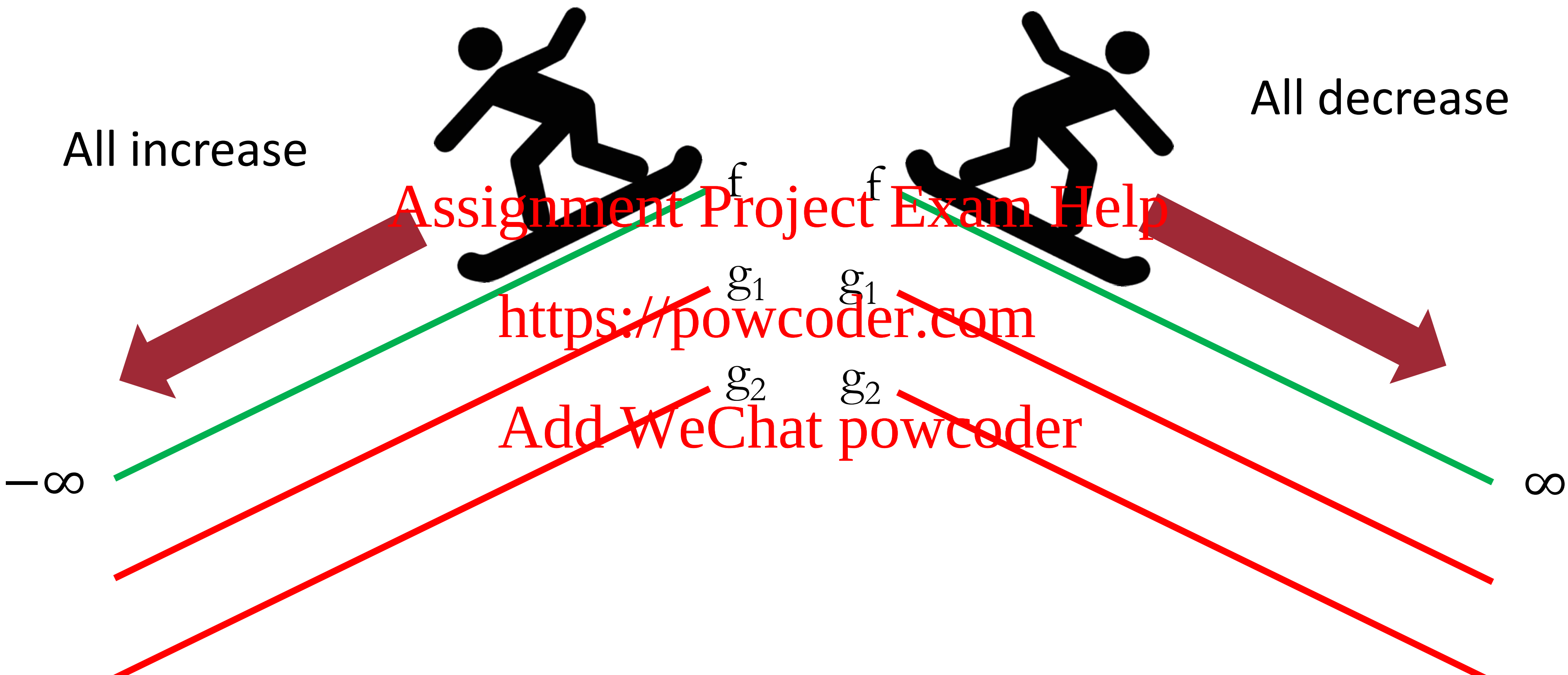


Monotonicity Theorem

If f(x) and the consistent constraint functions $g_i(x)$ all increase weakly or all decrease weakly with respect to x, the minimization problem domain is **not well constrained**.

$$
\begin{array}{lll}
\min & f(x^{+}) & \\
\text{s.t.} & g_1(x^{+}) \le 0 & \\
 & g_2(x^{+}) \le 0 & \\
 & g_3(x^{+}) \le 0 &
\end{array}
\qquad\qquad
\begin{array}{lll}
\min & f(x^{-}) & \\
\text{s.t.} & g_1(x^{-}) \le 0 & \\
 & g_2(x^{-}) \le 0 & \\
 & g_3(x^{-}) \le 0 &
\end{array}
$$

Monotonicity Theorem



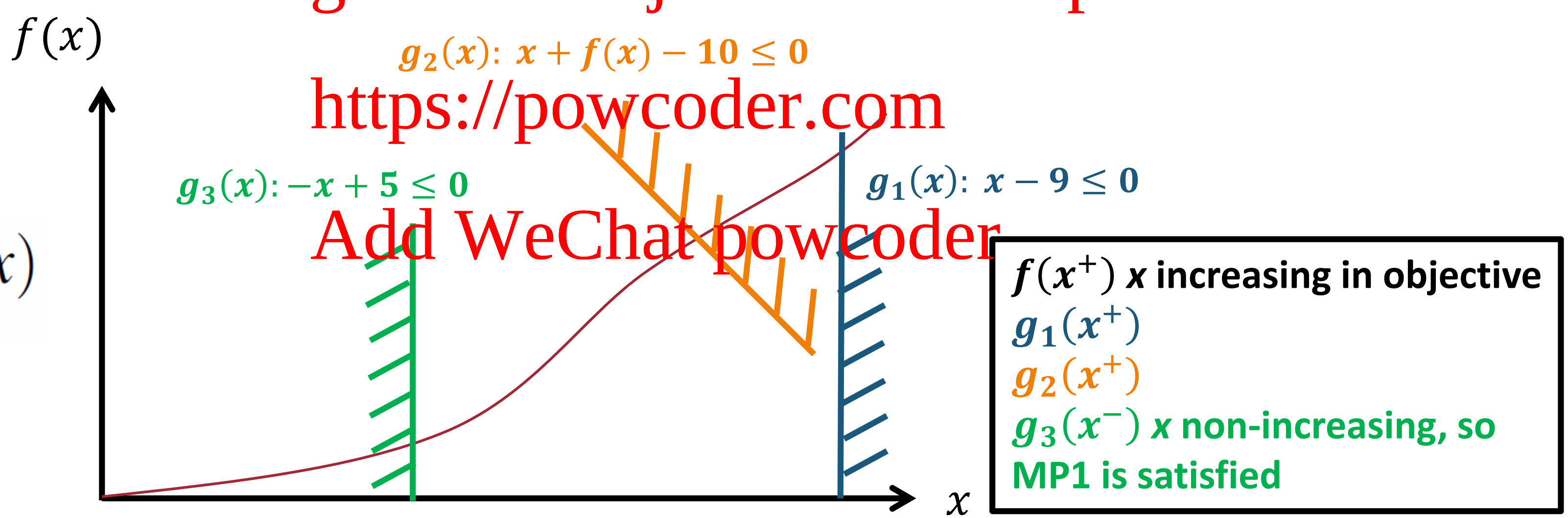
$$\begin{array}{ll} \min & f(x^{+}) = x \\ \text{s.t.} & g_1(x^{+}) = x-1 \leq 0 \\ & g_2(x^{+}) = x-2 \leq 0 \end{array}$$

$$\begin{array}{ll} \min & f(x^{-}) = -x \\ \text{s.t.} & g_1(x^{-}) = 1-x \leq 0 \\ & g_2(x^{-}) = 2-x \leq 0 \end{array}$$

Monotonicity Principle 1 (MP1)

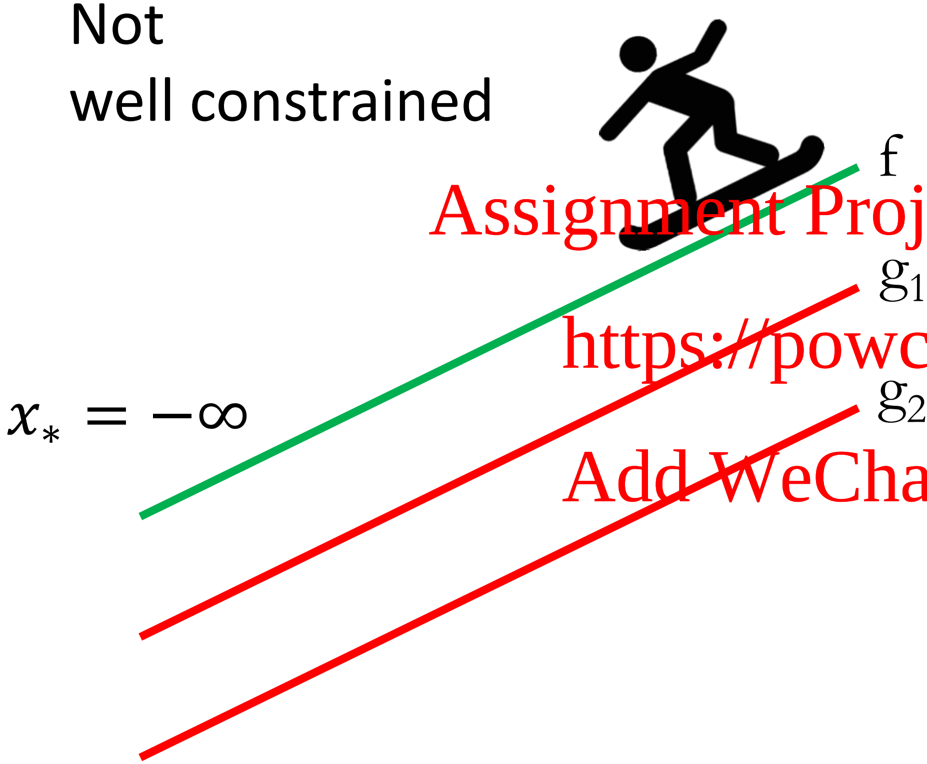
In a well-constrained minimization problem, every increasing variable (in the objective) is bounded below by at least one non-increasing active constraint

$$\min_{x \in P} \; f(x)$$

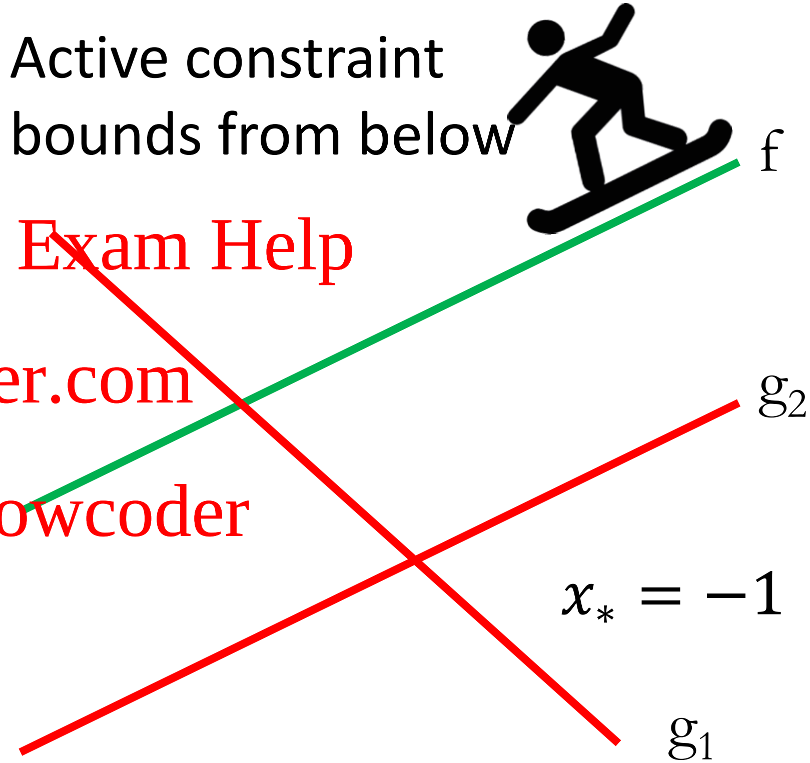


MP1

Not
well constrained



Active constraint
bounds from below



$$\begin{aligned} \min \quad & f(x^{+}) = x \\ \text{s.t.} \quad & g_1(x^{+}) = x - 1 \leq 0 \\ & g_2(x^{+}) = x - 2 \leq 0 \end{aligned}$$

$$\begin{aligned} \min \quad & f(x^{+}) = x \\ \text{s.t.} \quad & g_1(x^{-}) = -x - 1 \leq 0 \\ & g_2(x^{+}) = x - 2 \leq 0 \end{aligned}$$

MP1 Example 1

Apply MP1 to the problem

$$\begin{aligned} \min \quad & f = (1/3)(x_1+1)^3 + x_2 \\ \text{s.t.} \quad & g_1 = -x_1 + 1 \leq 0 \\ & g_2 = -x_2 \leq 0 \end{aligned}$$

Monotonicity Table

	x_1	x_2
f	$+$	$+$
g_1	$-$	
g_2		$-$

Active with respect to x_1

Active with respect to x_2

$$g_1 = 0,\ x_{1*} = 1$$
$$g_2 = 0,\ x_{2*} = 0$$

$$f_* = 8/3$$

MP1 Example 2

Apply MP1 to the problem

$$\max \quad f = x_1 - x_2$$

$$\text{s.t.} \quad g_1 = x_1 + 3x_2 - 10 \le 0$$

$$g_2 = -x_1 - 4x_2 + 2 \le 0$$

$$g_3 = -2x_1 + 7x_2 - 8 \le 0$$

Monotonicity Table

	x_1	x_2
$-f$	$-$	$+$
g_1	$+$	$+$
g_2	$-$	$-$
g_3	$-$	$+$

Negative null form ←

Active w.r.t. x_1 ←

MP1 Example 2

Eliminate x_1 using g_1

$$\begin{aligned} \min \quad & -f = -x_1 + x_2 \\ \text{s.t.} \quad & g_1 : x_1 = -3x_2 + 10 \\ & g_2 = -x_1 - 4x_2 + 2 \le 0 \\ & g_3 = -2x_1 + 7x_2 - 8 \le 0 \end{aligned}$$

$$\begin{aligned} \min \quad & -f = 4x_2 - 10 \\ \text{s.t.} \quad & g_2 = -x_2 - 8 \le 0 \\ & g_3 = 13x_2 - 28 \le 0 \end{aligned}$$

Monotonicity Table

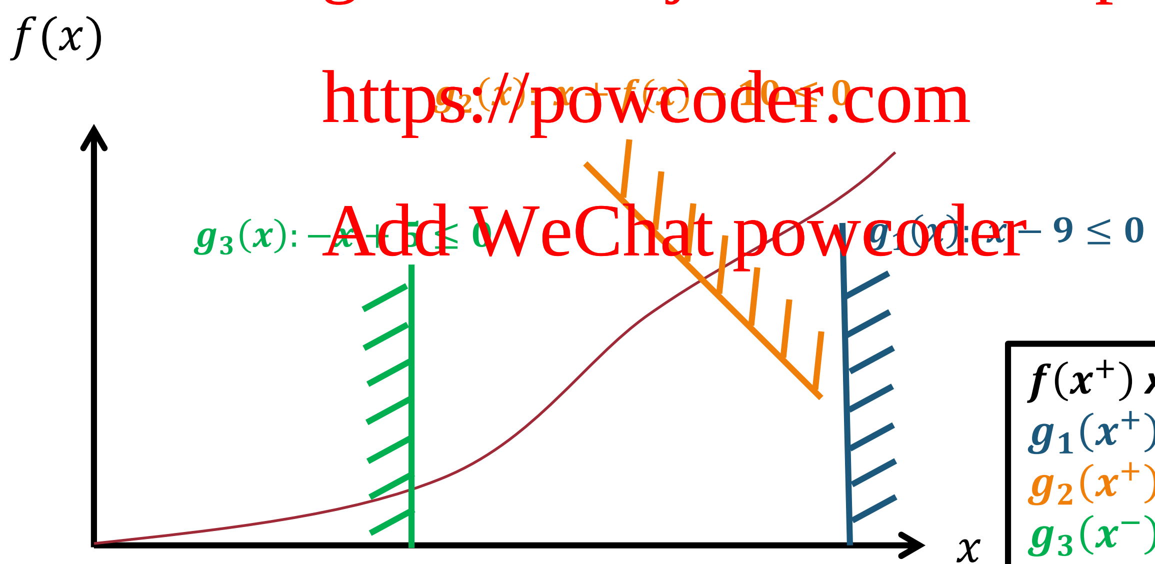
	x_2
$-f$	$+$
g_2	$-$
g_3	$+$

Negative null form

Active w.r.t. x_2

Critical constraints

*A constraint is **critical for x** in a well-constrained minimization problem if x is increasing in the objective and all other constraints*



$f(x)$

$f(x^+)$ ***x* increasing in objective**
$g_1(x^+)$
$g_2(x^+)$
$g_3(x^-)$ ***x* non-increasing, so this constraint is uniquely critical**

Monotonicity Principle 2

In a well-constrained minimization problem, every ***relevant nonobjective variable*** *is bounded both:*

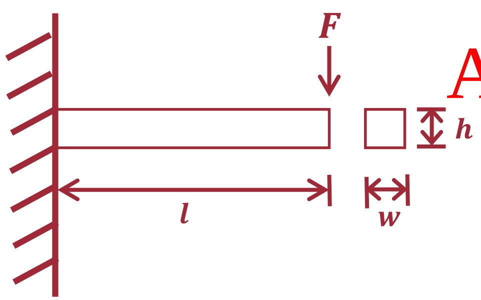
1. *below by at least one non-increasing semiactive constraint, &*
2. *above by at least one non-decreasing semiactive constraint.*

$$
\begin{aligned}
\min \quad & f(x_1^{+}) \\
\text{s.t.} \quad & g_1(x_1^{+}, x_2^{-}) \leq 0 \\
& g_2(x_1^{-}, x_2^{+}) \leq 0
\end{aligned}
$$

MP2 Example 1

*In a well-constrained minimization problem, every **relevant nonobjective variable** is bounded both:*

1. *below by at least one non-increasing semiactive constraint, &*
2. *above by at least one non-decreasing semiactive constraint.*



E = Young's modulus

$$\min_{l,w,h,E} \quad A$$

s.t. $h_1(w,h) = A - wh = 0$

$h_2 = F - 100N = 0$

multiply critical → $g_1(l^+, w^-, h^-, E^-) = Fl/AE - 5mm \le 0$ — **MP1: w, h**

uniquely critical → $g_2(l^-) = 1m - l \le 0$ — **MP2: l**

→ $g_3(E^+) = E - 2(10^9)Pa \le 0$ — **MP2: E**

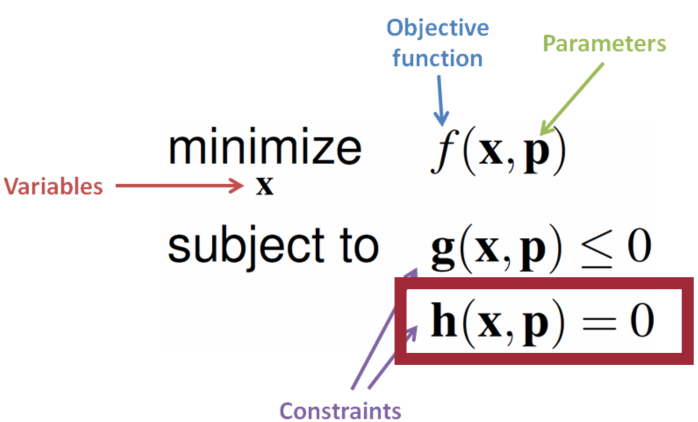
With monotonicity analysis, we can identify appropriate constraints to turn an unbounded problem into one that is solvable

Equality constraints

- When possible, substitute out the equality constraint

$$\boxed{\begin{array}{ll} \min\limits_{x\in R} & f(x,y) = x + 2x^2 y \\ \text{s.t.} & h(y) = y - 4 = 0 \end{array}} \Rightarrow \boxed{\min\limits_{x\in R} \quad f(x) = x + 8x^2}$$

- Otherwise, you can direct it as an inequality



$$\Rightarrow \boxed{\begin{array}{ll} \min\limits_{x\in R} & f(x,y^+) = x + 2x^2 y \\ \text{s.t.} & g(y^-) = -y + 4 \le 0 \end{array}}$$

Summary

- Identify if your problem is **well-posed** and **well-bounded** before attempting to solve
- Substitute out **equality constraints** when possible
- Identify inequality **constraint activity** (or inactivity) to reduce the problem
- **Monotonicity analysis** is an easy way to identify active constraints early on to:
  - Reduce (and potentially solve) the problem
  - Ensure the problem is well-bounded

Acknowledgements

- Much of this material came from Chapter 3 of the textbook, *Principles of Optimal Design*
- Some of the slides and examples came from Emrah Bayrak, Alex Burnap, and Namwoo Kang at the University of Michigan
- Some slides also contain material from: Bazaraa, Mokhtar S., John J. Jarvis, and Hanif D. Sherali. *Linear programming and network flows*. John Wiley & Sons, 2011.

Announcement

No office hours on Monday, September 10

Instead, I will be available:

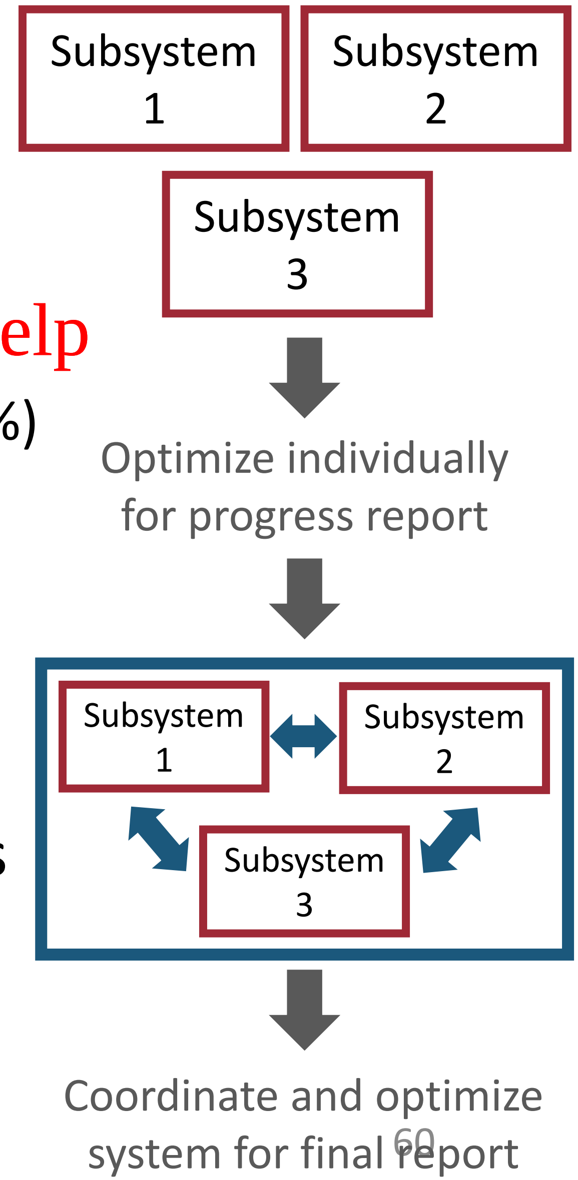
Wednesday, September 12, 9-11am

(or by appointment)

Team project (50% of grade)

- In teams of 3-4, you will define a system design problem with sub-systems, formulate optimization problems, solve them, and interpret/justify results
- There will be 4 graded deliverables:
  - ✓ Sep 26: Present project topic proposal (0%)
  - ✓ Oct 28: Progress report, written (10%)
  - ✓ Oct 31: Present progress (5%)
  - ✓ Nov 28: Present final project (10%)
  - ✓ Dec 2: Final report, written (25%)
- Let's discuss topics and teams during today's and next week's class, and teams will be formed before Week 3

Tomorrow, I will post to Canvas a survey on your skills and topic interests, due Tuesday at noon.



Examples

Active, Semiactive, Inactive?

Consider the following problem:

$$
\begin{aligned}
\text{min} \quad & f(\mathbf{x}) = x_1^2 + (x_2 - 1)^2 (x_2 - 3)^2 (x_2 - 4)^2 \\
\text{s.t.} \quad & g_1(\mathbf{x}) : x_1 \geq 1 \\
& g_2(\mathbf{x}) : x_2 \geq 2 \\
& g_3(\mathbf{x}) : x_2 \leq 5
\end{aligned}
$$

$x_1 = 1$ (**partial minimization**)

$$
\begin{aligned}
\text{min} \quad & f(\mathbf{x}) = 1 + (x_2 - 1)^2 (x_2 - 3)^2 (x_2 - 4)^2 \\
\text{s.t.} \quad & g_2(\mathbf{x}) : x_2 \geq 2 \\
& g_3(\mathbf{x}) : x_2 \leq 5
\end{aligned}
$$

$$\mathbf{X}_* = \{(1,3), (1,4)\}$$

Active, Semiactive, Inactive?

$$\begin{aligned} \min \quad & f(\mathbf{x}) = x_1^{\,2} + (x_2 - 1)^2 (x_2 - 3)^2 (x_2 - 4)^2 \\ \text{s.t.} \quad & g_1(\mathbf{x}) : x_1 \geq 1 \\ & g_2(\mathbf{x}) : x_2 \geq 2 \\ & g_3(\mathbf{x}) : x_2 \leq 5 \end{aligned}$$

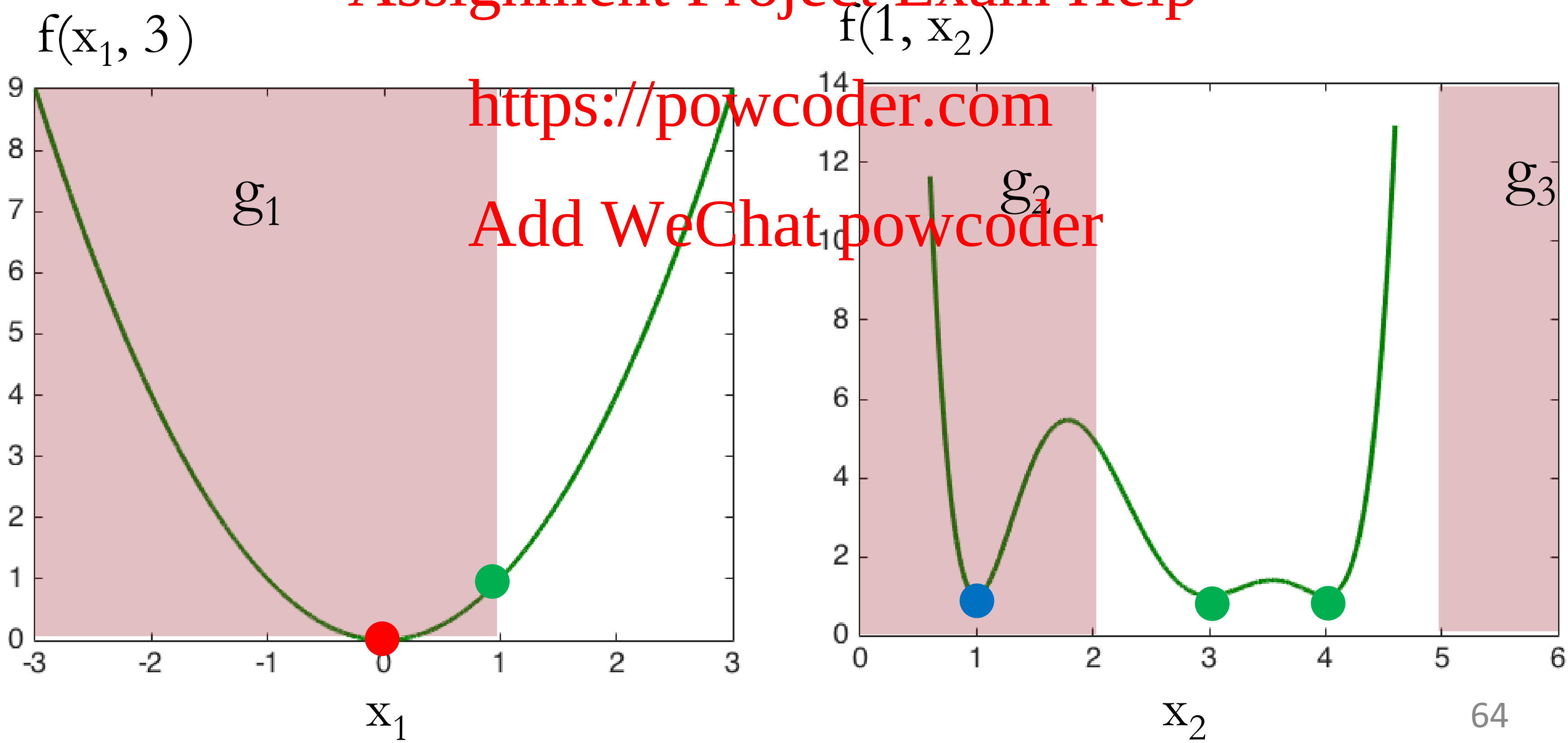
$$\boxed{\mathbf{X}_* = \{(1,3), (1,4)\}}$$

- Remove constraints one at a time (relaxation):
  - g_1 removed: $\longrightarrow$ $\boxed{\mathbf{X}_1 = \{(0,3), (0,4)\}}$ **Active**
  - g_2 removed: $\longrightarrow$ $\boxed{\mathbf{X}_2 = \{(1,1), (1,3), (1,4)\}}$ **Semiactive**
  - g_3 removed: $\longrightarrow$ $\boxed{\mathbf{X}_3 = \{(1,3), (1,4)\}}$ **Inactive**

Active, Semiactive, Inactive?

Relaxing:

- Active constraint: changes optimal set of variables and function value
- Semiactive constraint: does not change optimal function value, adds more variables to the optimal set of variables



Problem Reduction

min $\quad f(\mathbf{x}) = x_1{}^2 + (x_2 - 1)^2\,(x_2 - 3)^2\,(x_2 - 4)^2$

s.t.
$g_1(\mathbf{x}) : x_1 \geq 1$ (Active) $\longrightarrow$ **Equality**
$g_2(\mathbf{x}) : x_2 \geq 2$ (Semiactive) $\longrightarrow$ **Let it be**
$g_3(\mathbf{x}) : x_2 \leq 5$ (Inactive) $\longrightarrow$ **Remove**

min $\quad f(\mathbf{x}) = x_1{}^2 + (x_2 - 1)^2\,(x_2 - 3)^2\,(x_2 - 4)^2$

s.t.
$g_1(\mathbf{x}) : x_1 = 1$
$g_2(\mathbf{x}) : x_2 \geq 2$

Eliminate x_1

min $\quad f(\mathbf{x}) = 1 + (x_2 - 1)^2\,(x_2 - 3)^2\,(x_2 - 4)^2$

s.t. $\quad g_2(\mathbf{x}) : x_2 \geq 2$

MP1 Example 3

What if the functions are not monotonic?

Apply **regional monotonicity** to the problem

$$\min \quad f = x_1^2 - 3x_1 + x_2$$
$$\text{s.t.} \quad g_1 = x_1 - x_2 \leq 0$$

$$g_2 = 2x_1 + 3x_2 \leq 0$$

Monotonicity Table

	x_1	x_2
f	?	+
g_1	+	–
g_2	+	+

Active w.r.t. x_1

(g_1=0, x_2=x_1)

MP1 Example 3

Apply **regional monotonicity** to the problem

min $\quad f = x_1^{\,2} - 2x_1$

s.t. $\quad g_2 = 5x_1 \leq 0$

Monotonicity Table for $x_1 \geq 1$

	x_1
f	+
g_2	+
g_3	–

Check:

$$\frac{\partial f}{\partial x_1} = 2x_1 - 2$$

Case 1:

$$\frac{\partial f}{\partial x_1} \geq 0 \text{ for } x_1 \geq 1$$

$$\boxed{g_3 = 1 - x_1 \leq 0}$$

g_3 is active w.r.t. x_1

$x_{1*}=1$, but not feasible w.r.t g_2

No solution in this region!

MP1 Example 3

Apply **regional monotonicity** to the problem

$$\begin{aligned} \min \quad & f = x_1^2 - 2x_1 \\ \text{s.t.} \quad & g_2 = 5x_1 \leq 0 \end{aligned}$$

Check:

$$\frac{\partial f}{\partial x_1} = 2x_1 - 2$$

Case 2:

$$\frac{\partial f}{\partial x_1} \leq 0 \text{ for } x_1 \leq 1$$

$$\boxed{g_4 = x_1 - 1 \leq 0}$$

Monotonicity Table for $x_1 \geq 1$

	x_1
f	−
g_2	+
g_4	+

g_4 is dominated by g_2

g_2 is active w.r.t x_1

$x_{1*}=0$, $f(x_{1*})=0$

MP1 Example 4

Apply regional monotonicity to the problem:

$$\min \quad f = x$$
$$\text{s.t.} \quad g_1 = x^2 - 5x + 4 \leq 0$$

Check:

$$\frac{\partial g}{\partial x} = 2x - 5$$

Case 1:

$\frac{\partial g}{\partial x} \geq 0$ for x ≥ 2.5

$\boxed{g_2 = 2.5 - x \leq 0}$

$f(x^+)$
$g_1(x^+)$
$g_2(x^-)$

Case 1:

$f(x_*) = 2.5$

Case 2:

$\frac{\partial g}{\partial x} \leq 0$ for x ≤ 2.5

$\boxed{g_3 = x - 2.5 \leq 0}$

$f(x^+)$
$g_1(x^-)$
$g_3(x^+)$

Better solution

Case 2:

$f(x_*) = 1$

MP2 Example 2

Apply MP2 to the following problem

$$
\begin{array}{lll}
\min & f(x_1^{+}) & = x_1 \\
\text{s.t.} & g_1(x_1^{+}, x_2^{+}) = x_1^{3} + x_2^{3} \leq 0 & \\
 & g_2(x_1^{+}, x_2^{-}) = e^{x_1} - e^{x_2} \leq 0 &
\end{array}
$$

x_2 does not appear in the objective

x_2 is not relevant

The problem is unbounded