MSBA 403:

Optimization

Lecture 3

November 20, 2020

UCLA Anderson School of Management

Example: Supply planning

- A manufacturer of hardware tools is deciding how many of two products -- wrenches and pliers – to produce per day, for the current quarter.

- The firm faces constraints on steel availability (in lbs), molding machine capacity (in hours), and assembly machine capacity (in hours) per day. Further, there are limits on the demand for both wrenches and pliers.

- Taking these constraints into account, the firm formulates the following linear program to determine how many wrenches (W) and pliers (P) to produce (in 1000s):

$$
\begin{aligned}
\max_{W,P}\quad & 130W + 100P \\
\text{s.t.}\quad & 1.5W + P \le 27 && \text{(Steel availability)} \\
& W + P \le 21 && \text{(Molding capacity)} \\
& 0.3W + 0.5P \le 9 && \text{(Assembly capacity)} \\
& W \le 15 && \text{(Wrench demand)} \\
& P \le 16 && \text{(Pliers demand)} \\
& W, P \ge 0.
\end{aligned}
$$

Modeling uncertainty

- We have so far assumed all model parameters are known

- What if there was uncertainty in some of the model parameters?
  - Uncertainty in demand
  - Uncertainty in profit due to fluctuating prices or production costs
  - Uncertainty in machine capacity

- One way to capture uncertainty in a linear or integer program is by using a *two-stage* modeling approach

Example: Supply planning

- In the current quarter, the firm had contracted with a steel supplier for 27,000 lbs of steel per day.

- Suppose the firm is now deciding how much steel to contract for with local suppliers for the next quarter. Steel contracts are arranged for daily delivery over the entire quarter, and the market price is $58/1,000 lbs.

- There are now two sources of uncertainty:

  1)The assembly machine capacity for next quarter is uncertain. The firm has ordered new assembly machines to replace failing equipment, but it is uncertain whether the new machines will be delivered on time. There is a 50% chance of successful delivery. If the new machines are successfully delivered, the assembly capacity will be 10,000 hours. If not, the assembly capacity will drop to 8,000 hours.

  2)Further, the profit margin on wrenches is uncertain due to fluctuations in market demand. For simplicity, assume that the profit per 1000 wrenches will be either $90 or $160 with equal probability.

- Assume that the uncertainty in assembly machine capacity and wrench profit are independent of each other.

Example: Supply planning

The sequence of events is now as follows:

1. The firm decides how much steel to purchase (Stage 1).
2. The assembly capacity and wrench profit becomes known.
3. After the uncertainty is resolved, the firm decides how many wrenches and pliers to produce (Stage 2).

There are now four *scenarios* to consider:

Scenario	Assembly Capacity (hours/day)	Wrench Profit ($/1000s)	Probability
1	8,000	160	0.25
2	10,000	160	0.25
3	8,000	90	0.25
4	10,000	90	0.25

How can we account for these uncertainties in a new linear programming model?

Example: Supply planning

- First we create a new decision variable, S, to represent the amount of steel to purchase
- We will also need new decision variables, W_s and P_s, to represent how many wrenches and pliers to produce under scenario $s = 1,2,3,4$
- Note: Because steel supply decision must be made before the uncertainties are realized, there is **only one** decision variable for steel
- What should the objective function be? A natural choice is the *expected* profit:

$$\underbrace{0.25(160W_1 + 100P_1)}_{\text{scenario 1}} + \underbrace{0.25(160W_2 + 100P_2)}_{\text{scenario 2}} + \underbrace{0.25(90W_3 + 100P_3)}_{\text{scenario 3}} + \underbrace{0.25(90W_4 + 100P_4)}_{\text{scenario 4}} - \underbrace{58S}_{\text{steel cost}}$$

Example: Supply planning

- Because assembly capacity depends on which scenario occurs, we require a separate assembly constraint for each scenario:

$$0.3W_1 + 0.5P_1 \leq 8 \quad \textbf{(scenario 1)}$$

$$0.3W_2 + 0.5P_2 \leq 10 \quad \textbf{(scenario 2)}$$

$$0.3W_3 + 0.5P_3 \leq 8 \quad \textbf{(scenario 3)}$$

$$0.3W_4 + 0.5P_4 \leq 10 \quad \textbf{(scenario 4)}$$

Example: Supply planning

- The full formulation is then given by

$$
\begin{aligned}
\max_{W,P,S}\ & 0.25(160W_1 + 100P_1) + 0.25(160W_2 + 100P_2) + 0.25(90W_3 + 100P_3) + 0.25(90W_4 + 100P_4) - 58S \\
\text{s.t. } & 0.3W_1 + 0.5P_1 \le 8 \\
& 0.3W_2 + 0.5P_2 \le 10 \\
& 0.3W_3 + 0.5P_3 \le 8 \\
& 0.3W_4 + 0.5P_4 \le 10 \\
& W_s + P_s \le 21, \quad s = 1,2,3,4 \\
& W_s \le 15, \quad s = 1,2,3,4 \\
& P_s \le 16, \quad s = 1,2,3,4 \\
& 1.5W_s + P_s \le S, \quad s = 1,2,3,4 \\
& W_s, P_s \ge 0, \quad s = 1,2,3,4 \\
& S \ge 0.
\end{aligned}
$$

Molding capacity and demand constraints are the same across all 4 scenarios

Steel supply is now a decision variable

Example: Supply planning

- The previous formulation maximizes the **expected value** of profit

- What if we wish to maximize profit under the **worst case** scenario?

Example: Supply planning

- The following LP maximizes a lower bound on the profit under all four scenarios

$$
\begin{aligned}
\max_{W,P,S,V} \quad & V \\
\text{s.t.} \quad & \left.\begin{aligned}
& V \le 160W_1 + 100P_1 - 58S \\
& V \le 160W_2 + 100P_2 - 58S \\
& V \le 90W_3 + 100P_3 - 58S \\
& V \le 90W_4 + 100P_4 - 58S
\end{aligned}\right\} \text{Profit under each of the 4 scenarios} \\
& 0.3W_1 + 0.5P_1 \le 8 \\
& 0.3W_2 + 0.5P_2 \le 10 \\
& 0.3W_3 + 0.5P_3 \le 8 \\
& 0.3W_4 + 0.5P_4 \le 10 \\
& W_s + P_s \le 21, \quad s = 1, 2, 3, 4 \\
& W_s \le 15, \quad s = 1, 2, 3, 4 \\
& P_s \le 16, \quad s = 1, 2, 3, 4 \\
& 1.5W_s + P_s \le S, \quad s = 1, 2, 3, 4 \\
& W_s, P_s \ge 0, \quad s = 1, 2, 3, 4 \\
& S \ge 0.
\end{aligned}
$$

Non-linear programming

- We have so far focused on problems where the objective and constraints are **linear** functions of the decision variables (which can be either continuous or integer)

- Certain optimization problems may require the use of a **non-linear** objective or constraint

- Not all non-linear optimization problems are easy to find optimal solutions to (more on this in Lecture 4)

Portfolio optimization

- One of the most well known applications of non-linear programming is *portfolio optimization*

- In finance, a **portfolio** is a collection of assets (e.g. stocks and bonds) defined by a set of **weights** over the possible assets

- The weight in asset i, w_i, is the fraction of your money invested in i

- The goal of portfolio optimization is to select a portfolio (i.e. a weight vector) that balances risk and reward

Example: 3-Asset Portfolio

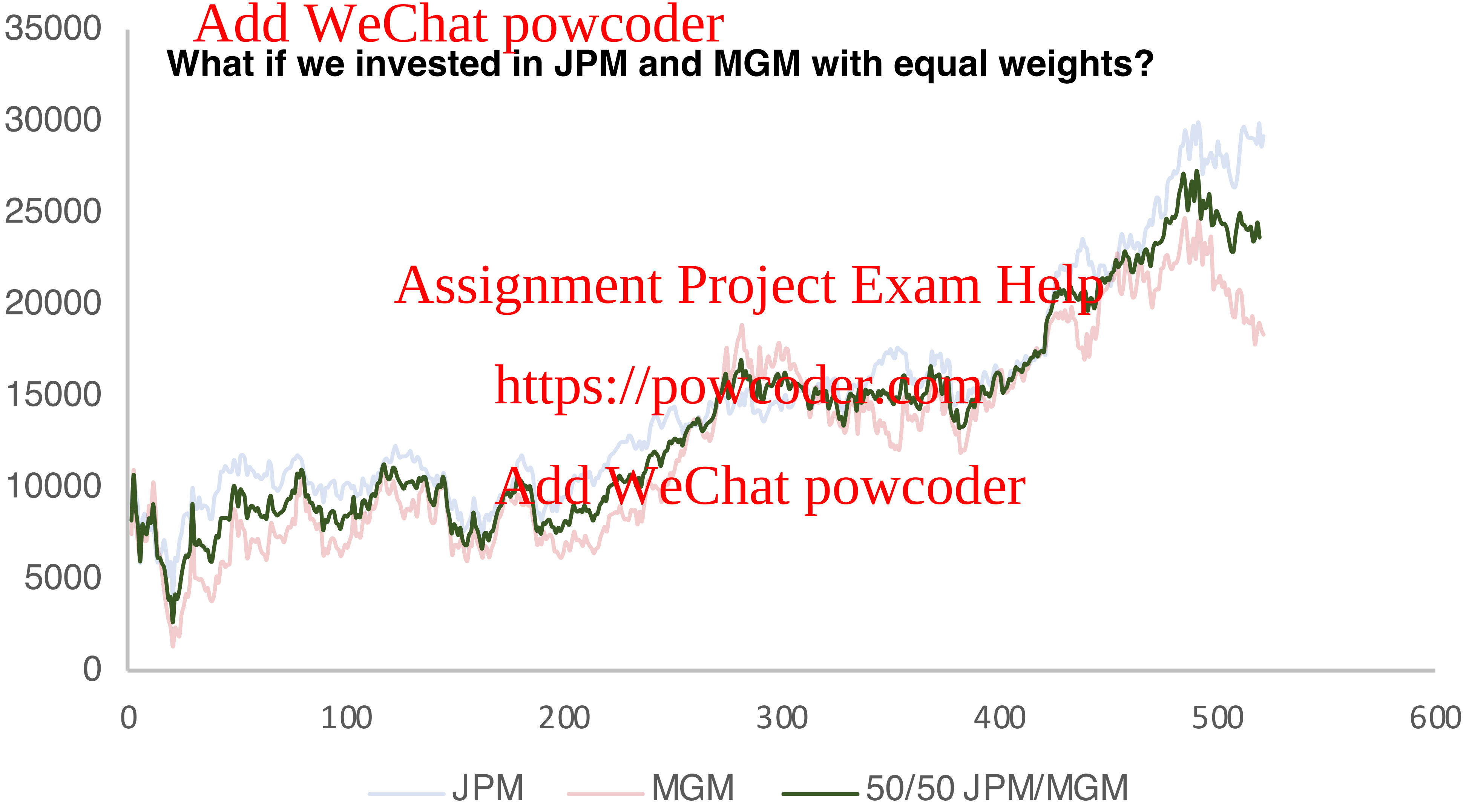
- Consider a portfolio constructed from three assets: MGM, JP Morgan, and Gold
- Each asset is a random variable, with an expected return and variance, based on historical data
- The expected monthly **returns** and **covariance** matrix of the three assets from Oct 2008 - Oct 2018 are:

	Return (%)
MGM	1.2
J.P.Morgan	0.5
	0.4

	MGM	J.P.Morgan	
MGM	28.9	7.1	-0.17
J.P.Morgan	7.1	6.4	-0.68
	-0.17	-0.68	2.5

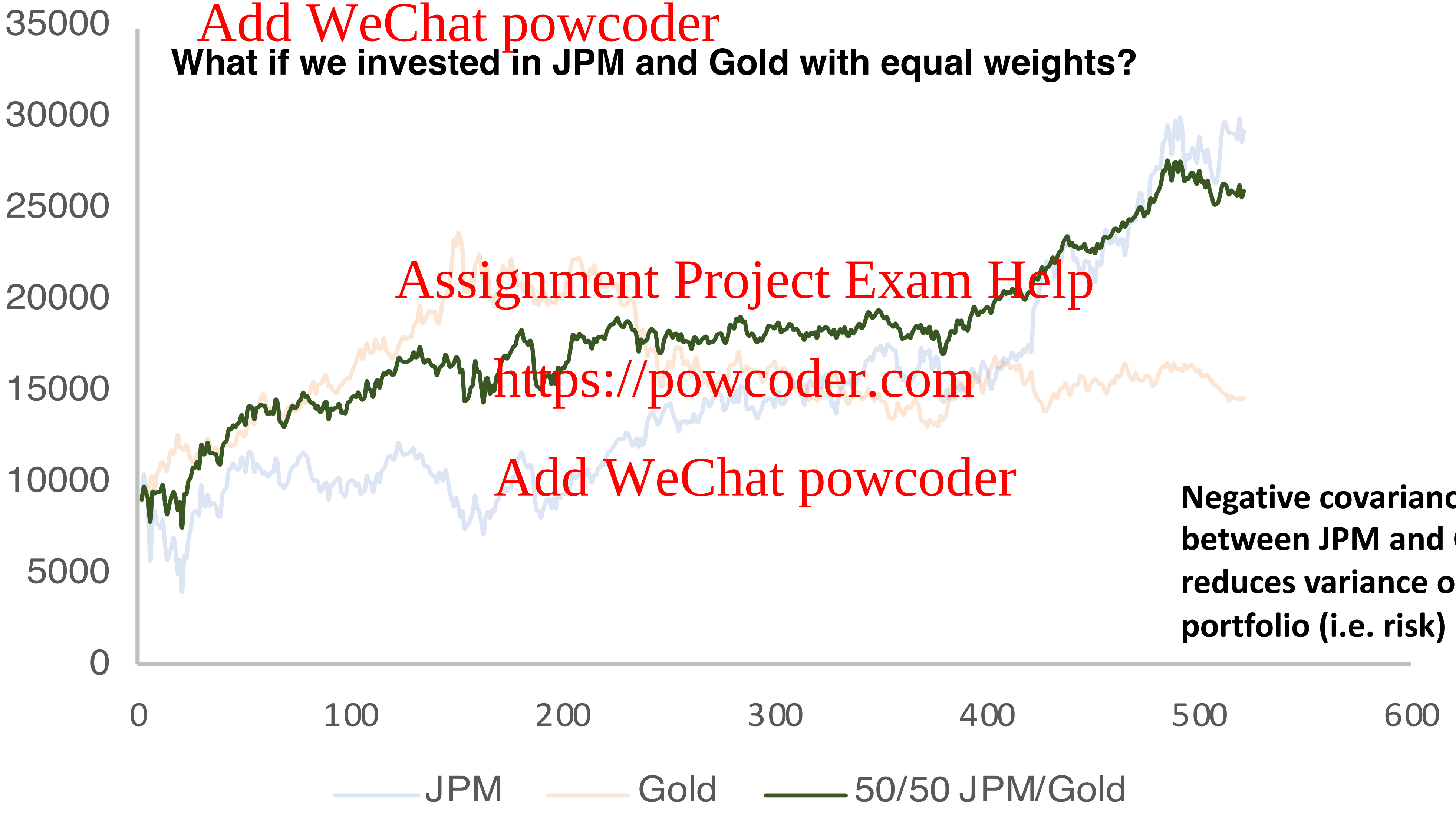
50/50 JPM and MGM

(October 2008 - October 2018)



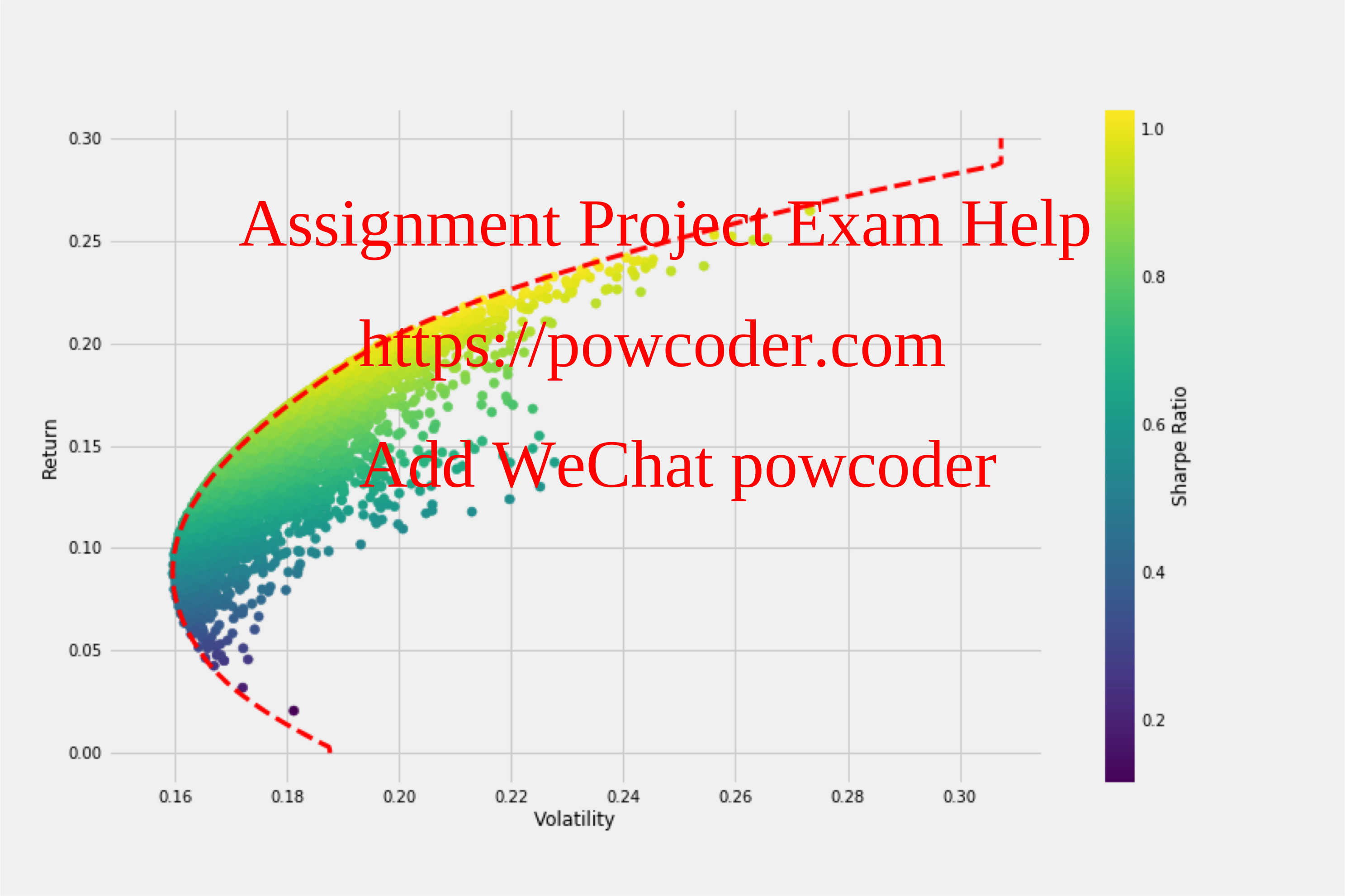
50/50 JPM and Gold

(October 2008 - October 2018)



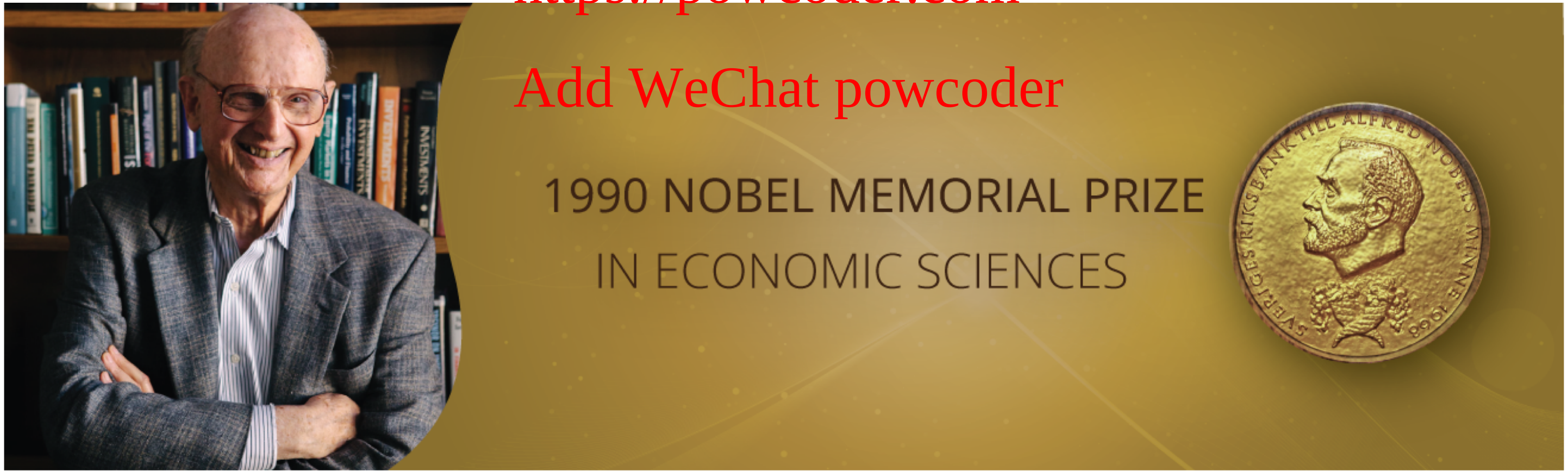
Negative covariance between JPM and Gold reduces variance of portfolio (i.e. risk)

How can we use optimization to construct an “optimal” portfolio?



The Markowitz Problem

- One approach: Minimize risk (i.e. portfolio variance) subject to the constraint that the expected return of the portfolio is at least ρ
  - This approach was pioneered by Harry Markowitz in 1952
  - Gave rise to modern finance theory and led to a Nobel Prize in Economics in 1990
  - Many mutual funds and portfolio managers use some variation of this model



Example: Portfolio optimization

- Suppose we want to build a portfolio by investing only in JPM and Gold

- Let w_J and w_G be the weight of our portfolio in JPM and Gold, respectively

- **How can we find values of w_J and w_G that will minimize risk (variance) while achieving a target expected monthly return of at least 0.45%?**

Example: Portfolio optimization

- Let J and G be random variables representing JP Morgan and Gold's monthly returns

- From our data, we can estimate the following:
- $E(J) = 0.5$
- $E(G) = 0.4$
- $Var(J) = 6.4$
- $Var(G) = 2.5$
- $Cov(J,G) = -0.68$

	Return (%)
MGM	1.2
J.P.Morgan	0.5
	0.4

	MGM	J.P.Morgan	
MGM	28.9	7.1	-0.17
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	-0.17	-0.68	2.5

- Using this data, we can now formulate a **quadratic** program to solve for the optimal w_J and w_G

- Reminder: we want to *minimize risk* subject to a constraint on expected return

Sum of 2 random variables

- Some useful things to know:
- Let X and Y be **any** random variables
- Then

$$E(aX + bY) = aE(X) + bE(Y)$$

$$\mathrm{Var}(aX + bY) = a^2\mathrm{Var}(X) + b^2\mathrm{Var}(Y) + 2ab\mathrm{Cov}(X, Y)$$

$$\mathrm{Cov}(X, X) = \mathrm{Var}(X)$$

Example: Portfolio optimization

- Expressing these formulas in terms of our variables, we get

$$\mathrm{Var}(w_J J + w_G G) = w_J^2 \mathrm{Var}(J) + w_G^2 \mathrm{Var}(G) + 2 w_J w_G \mathrm{Cov}(J, G)$$

$$\mathrm{E}(w_J J + w_G G) = w_J E(J) + w_G E(G)$$

- Using the estimates from our data, we can now formulate the portfolio optimization problem as the following quadratic program:

$$\begin{aligned} \min_{w_J, w_G} \quad & 6.4 w_J^2 + 2.5 w_G^2 - 1.36 w_J w_G \\ \text{s.t.} \quad & 0.5 w_J + 0.4 w_G \geq 0.45 \\ & w_J + w_G = 1 \\ & w_J, w_G \geq 0. \end{aligned}$$

What if we want to invest in *n* assets?

Sum of n random variables

- The portfolio is the sum of random variables, given by

$$\sum_{i=1}^{n} w_i X_i$$

where $\sum_{i=1}^{n} w_i = 1$.

- Note that

$$E\left(\sum_{i=1}^{n} w_i X_i\right) = \sum_{i=1}^{n} w_i E(X_i)$$

$$\mathrm{Var}\left(\sum_{i=1}^{n} w_i X_i\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \mathrm{Cov}(X_i, X_j)$$

The Markowitz Problem

- Let w_i be weight on asset i, r_i be expected return of asset i, C be covariance matrix, and ρ be the target return of the portfolio
- Then the portfolio optimization problem is

$$\min_{\mathbf{w}} \quad \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j C_{ij}$$

(Variance of returns / risk level)

$$\text{s.t.} \quad \sum_{i=1}^{n} w_i r_i \geq \rho$$

(Expected return should be at least ρ)

$$\sum_{i=1}^{n} w_i = 1$$

(Weights add up to 1)

$$w_i \geq 0, \quad i = 1, \ldots, n.$$

Reminder: Not all non-linear programs are computationally "easy" to solve, but this one generally is

In-class assignment 2:

Portfolio optimization