Lecture 7:
MIP Models for Scheduling

Outline

- Time-indexed Formulation for JSP
- Disjunctive Formulation for JSP

Readings

- Presentations by Stefan Heinz & Timo Berthold, Zuse Institute Berlin
- Applegate & Cook, A Computational Study of the Job-Shop Scheduling Problem, *ORSA Journal on Computing*, 3(2), 149-156, 1991
- Ku & Beck, Revisiting Off-the-Shelf Mixed Integer Programming and Constraint Programming Models for Job Shop Scheduling, *Computers & Operations Research*, 73, 165-173, 2016

MILP Basics

$$\min \sum_{i \in V} c_i x_i$$

Objective function

Could be ≤, =, or ≥

$$\text{s.t.} \sum_{i \in V} \sum_{j \in C} a_{ij} x_i \le b_j$$

Constraints

$$V = V_I \cup V_R$$

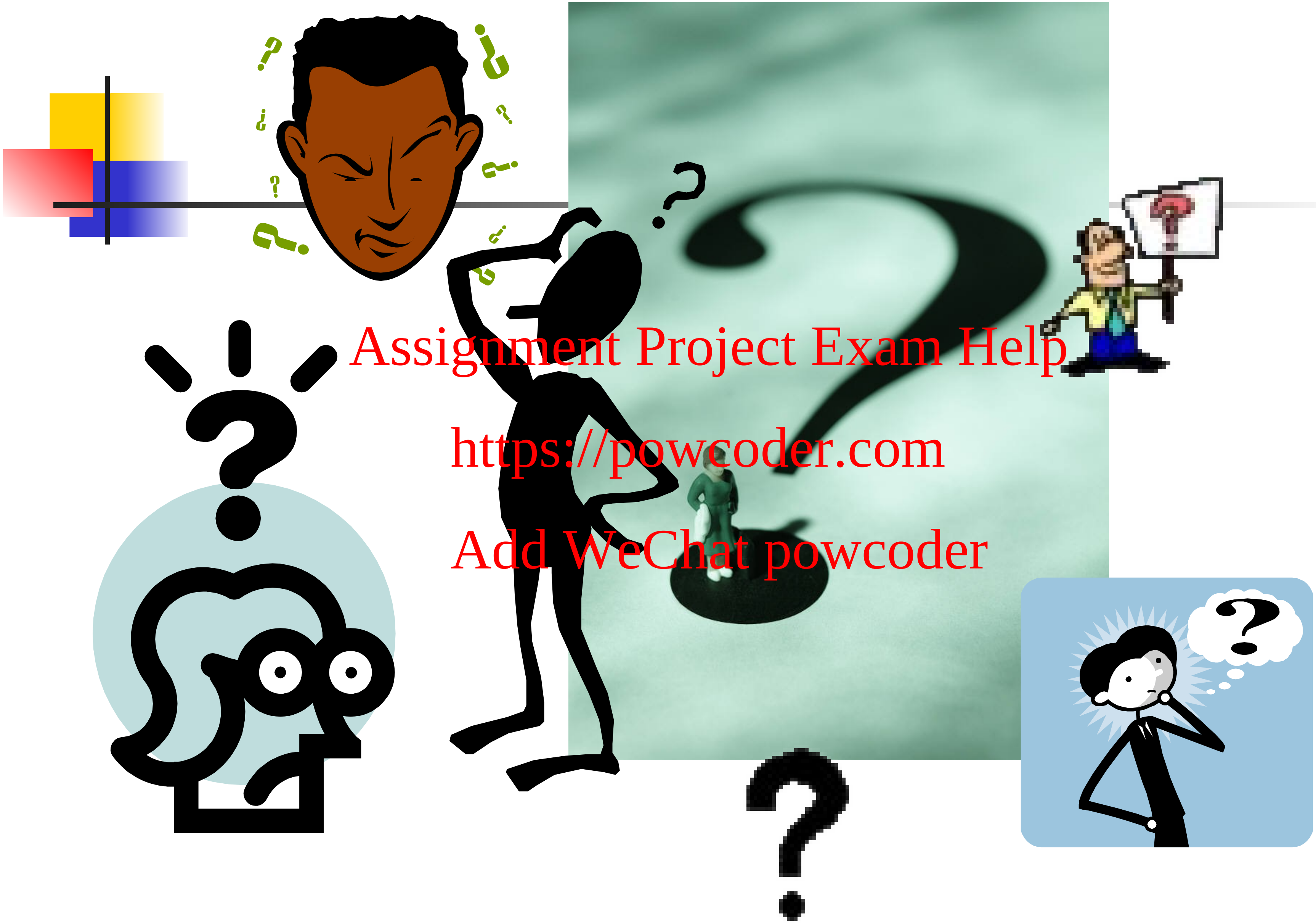
$$x_i \cancel{\in} \mathbb{Z}, \forall i \in V_I$$

Integer variables

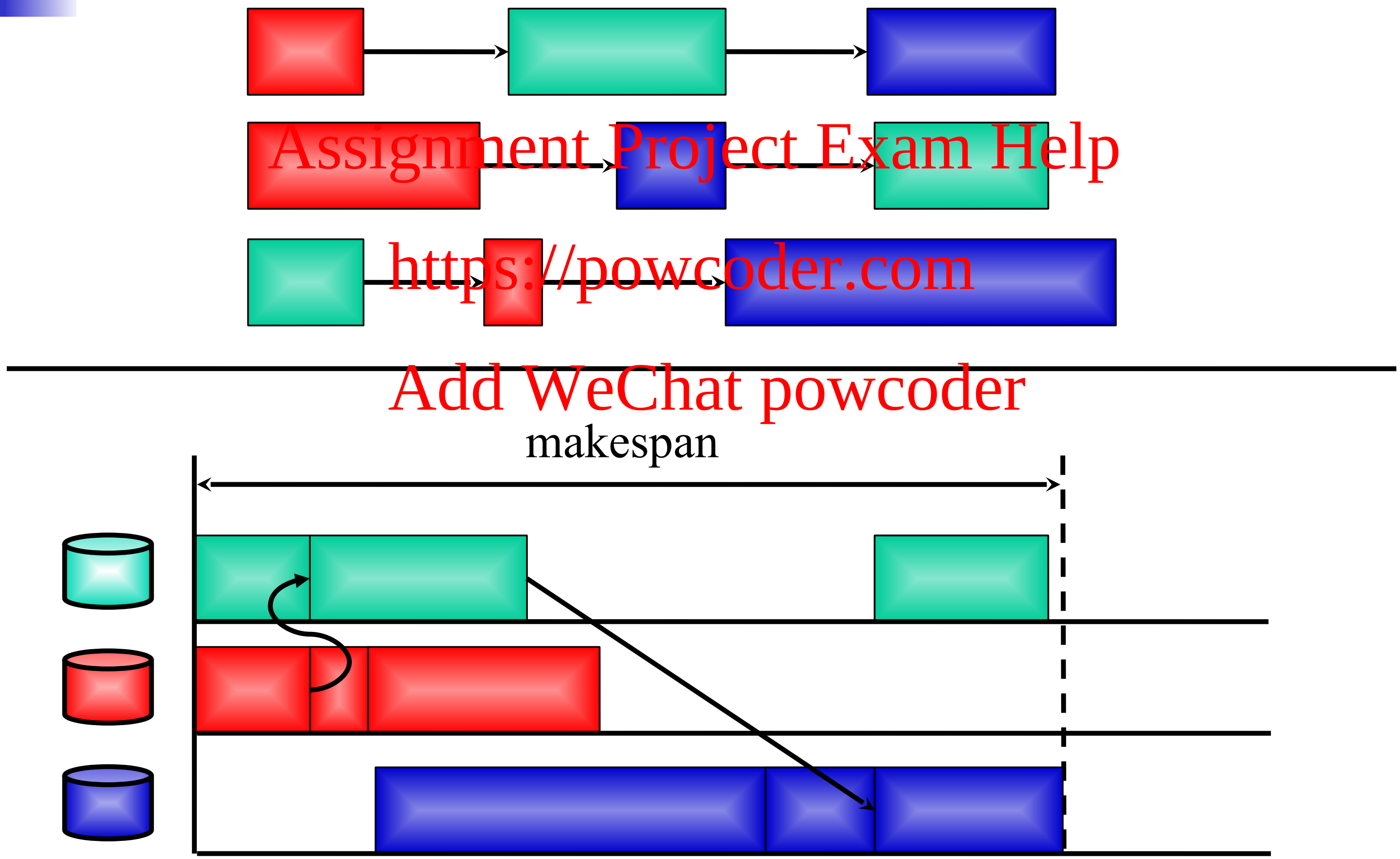
$$x_i \in \mathbb{R}, \forall i \in V_R$$

Continuous variables

Continuous (linear) relaxation: poly-time soluble!



Job Shop Scheduling



Where Do We Start?

Time-Indexed MIP

- The main decision variable represents whether a job starts at time t or not
  - Variables are "indexed" by time

Notation

- $\mathcal{H}$ is the set of all time-points
- $T_{jt} = \{t - p_j + 1, \ldots, t\}$ Set of time points defined by t and j
- $\mathcal{J}_k$ is the set of operations on resource $k \in \mathcal{K}$
- $\mathcal{E}$ is the set of all operation pairs (j, i) s.t. i is constrained to be after j
- $x_{jt} = 1$ iff operation j starts at time t

What Constraints Do We Need?

Where:

- $\mathcal{H}$ is the set of all time-points
- $T_{jt} = \{t - p_j + 1, \ldots, t\}$
- $\mathcal{J}_k$ is the set of operations on resource $k \in \mathcal{K}$
- $\mathcal{E}$ is the set of all operation pairs (j, i) s.t. i is constrained to be after j
- $x_{jt} = 1$ iff operation j starts at time t

Time-Indexed JSP MIP

$$\min \quad C_{max}$$

$$\text{s.t.} \quad \sum_{t \in \mathcal{H}} x_{jt} = 1$$

$$\sum_{j \in \mathcal{J}_k} \sum_{t' \in T_{jt}} x_{jt'} \leq 1$$

$$\sum_{t \in \mathcal{H}} (t + p_j) x_{jt} \leq C_{max}$$

$$\sum_{t \in \mathcal{H}} (t + p_j) x_{jt} \leq \sum_{t \in \mathcal{H}} (t) x_{it}$$

$$x_{jt} \in \{0, 1\} \qquad \forall j \in \mathcal{J}, \forall t \in \mathcal{H}$$

All operations start only once

Resource constraints

C_{max} is the largest end-time

Precedence constraints

Where:

- $\mathcal{H}$ is the set of all time-points
- $T_{jt} = \{t - p_j + 1, \ldots, t\}$
- $\mathcal{J}_k$ is the set of operations on resource $k \in \mathcal{K}$
- $\mathcal{E}$ is the set of all operation pairs (j, i) s.t. i is constrained to be after j
- $x_{jt} = 1$ iff operation j starts at time t

$$\sum_{j \in \mathcal{J}_k} \sum_{t' \in T_{jt}} x_{jt'} \leq 1 \qquad \forall k \in \mathcal{K}, \forall t \in \mathcal{H}$$

$$T_{jt} = \{t - p_j + 1, \ldots, t\}$$

Goal: Make sure we are not over-capacity on the blue resource at t = 10

Time-Indexed JSP MIP

$$\min \quad C_{max}$$

$$\text{s.t.} \quad \sum_{t \in \mathcal{H}} x_{jt} = 1$$

All activities start only once

$$\sum_{j \in \mathcal{J}_k} \sum_{t' \in T_{jt}} x_{jt'} \leq 1$$

Resource constraints

$$\sum_{t \in \mathcal{H}} (t + p_j) x_{jt} \leq C_{max}$$

C_{max} is the largest end-time

$$\sum_{t \in \mathcal{H}} (t + p_j) x_{jt} \leq \sum_{t \in \mathcal{H}} (t) x_{it}$$

Precedence constraints

$$x_{jt} \in \{0, 1\} \qquad \forall j \in \mathcal{J}, \forall t \in \mathcal{H}$$

Where:

- $\mathcal{H}$ is the set of all time-points
- $T_{jt} = \{t - p_j + 1, \ldots, t\}$
- $\mathcal{J}_k$ is the set of operations on resource $k \in \mathcal{K}$
- $\mathcal{E}$ is the set of all operation pairs (j, i) s.t. i is constrained to be after j
- $x_{jt} = 1$ iff operation j starts at time t

Questions?

Disjunctive MIP

- The main decision variable represents the sequence between each pair of operations
  - A before B or B before A

Notation

- $\mathcal{J}_k$ is the set of operations on resource $k \in \mathcal{K}$
- $\mathcal{E}$ is the set of all operation pairs (j, i) s.t. i is constrained to be after j
- M a big constant
- S_j is the start-time of operation j
- $z_{ji} = 1$ iff operation i is scheduled after operation j

What Constraints Do We Need?

Where:

- $\mathcal{J}_k$ is the set of operations on resource $k \in \mathcal{K}$
- $\mathcal{E}$ is the set of all operation pairs (j, i) s.t. i is constrained to be after j
- M a big constant
- S_j is the start-time of operation j
- $z_{ji} = 1$ iff operation i is scheduled after operation j

Disjunctive JSP MIP

$$
\begin{aligned}
\min \quad & C_{max} \\
\text{s.t.} \quad & S_j + p_j \le C_{max} \\
& S_j + p_j \le S_i \\
& S_j \ge S_i + p_i - M \cdot z_{ji} \\
& S_i \ge S_j + p_j - M \cdot (1 - z_{ji}) \\
& 0 \le S_j \le |\mathcal{H}| & \forall j \in \mathcal{J} \\
& z_{ji} \in \{0, 1\} & \forall j, i \in \mathcal{J}_k, \forall k \in \mathcal{K}
\end{aligned}
$$

C_{max} is the largest end-time

Precedence constraints

Resource constraints

Where:

- $\mathcal{J}_k$ is the set of operations on resource $k \in \mathcal{K}$
- $\mathcal{E}$ is the set of all operation pairs (j, i) s.t. i is constrained to be after j
- M a big constant
- S_j is the start-time of operation j
- $z_{ji} = 1$ iff operation i is scheduled after operation j

$$S_j \geq S_i + p_i - M \cdot z_{ji} \qquad j, i \in \mathcal{J}_k, j < i, \forall k \in \mathcal{K}$$

$$S_i \geq S_j + p_j - M \cdot (1 - z_{ji}) \qquad j, i \in \mathcal{J}_k, j < i, \forall k \in \mathcal{K}$$

- $z_{ji} = 1$ ➔ operation i is after operation j
  - $S_j \geq S_i + p_i - M$: redundant since $S_j \geq 0$
  - $S_i \geq S_j + p_j - M \times 0$: precedence constraint
- $z_{ji} = 0$ ➔ operation j is after operation i
  - $S_j \geq S_i + p_i - M \times 0$: precedence constraint
  - $S_i \geq S_j + p_j - M$: redundant since $S_i \geq 0$

Disjunctive JSP MIP

$$
\begin{array}{lll}
\min & C_{max} & \\
\text{s.t.} & S_j + p_j \leq C_{max} & \\
& S_j + p_j \leq S_i & \\
& S_j \geq S_i + p_i - M \cdot z_{ji} & \\
& S_i \geq S_j + p_j - M \cdot (1 - z_{ji}) & \\
& 0 \leq S_j \leq |\mathcal{H}| & \forall j \in \mathcal{J} \\
& z_{ji} \in \{0, 1\} & \forall j, i \in \mathcal{J}_k, \forall k \in \mathcal{K}
\end{array}
$$

C_{max} is the largest end-time

Precedence constraints

Resource constraints

Where:

- $\mathcal{J}_k$ is the set of operations on resource $k \in \mathcal{K}$
- $\mathcal{E}$ is the set of all operation pairs (j, i) s.t. i is constrained to be after j
- M a big constant
- S_j is the start-time of operation j
- $z_{ji} = 1$ iff operation i is scheduled after operation j

Questions?

Which is Best?

So is time-indexed useless?

Problem	Disjunctive		Time-Indexed	
	Time (geo/arith)	Opt	Time (geo/arith)	Opt
3×3	**0.00 / 0.00**	10	0.02 / 0.01	10
4×3	0.01 / 0.01	10	0.04 / 0.03	10
5×3	**0.01 / 0.01**	10	0.17 / 0.17	10
3×6	0.01 / 0.01	10	0.18 / 0.18	10
3×8	**0.01** / 0.01	10	0.44 / 0.42	10
3×10	**0.01 / 0.01**	10	0.94 / 0.85	10
8×8	**0.59 / 0.58**	10	3001.69 / 2478.13	2
10×10	**5.95 / 5.30**	10	-[10]	-
15×15	-	-	#	#
20×15	-	-	#	#

Results with CPLEX (default parameters)

What Happened Today?

- MIP models for JSP
  - Time-indexed model
  - Disjunctive model