Section A (60 marks)

1. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables.

   (a) Suppose that the distribution is given by a probability density function/probability function $f(x, \theta)$, where $\theta \in \Theta$ is a real parameter. Define the maximum likelihood estimator of θ based on the sample $X_1, X_2, \ldots, X_n$.

   (b) Suppose that the common probability density function is

   $$f(x, \theta) = \frac{4\theta^4}{x^5} \quad \text{for} \quad x \geq \theta,$$

   where $\theta > 0$. Find the maximum likelihood estimator of the parameter θ.

(8 marks)

2. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables with a common distribution depending on an unknown parameter θ.

   (a) Explain the method of moments to estimate the parameter θ.

   (b) Suppose that the common probability density function is the same as in 1(b).

      (i) Find a moment estimator of the parameter θ.

      (ii) Is the estimator in (i) unbiased? Justify your answer.

(10 marks)

3. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables with a common probability density function/probability function $f(x, \theta)$.

(a) State, without proof, the Cramér-Rao theorem for an unbiased estimator $\hat{g}(\mathbf{X})$ of $g(\theta)$, where $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ and $g(\theta)$ is a differentiable function of the parameter θ.

(b) Suppose that the common distribution is Gamma $\Gamma(4, \theta)$, that is,

$$f(x, \theta) = \frac{x^3}{6\theta^4} \exp\left(-\frac{x}{\theta}\right) \quad \text{for } x > 0,$$

where $\theta > 0$. Find the Cramér-Rao lower bound for the variance of an unbiased estimator of θ.

(Assume that the regularity conditions are satisfied.)

(12 marks)

4. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables with a common probability density function/probability function $f(x, \theta)$.

(a) State, without proof, the theorem about the attainment of the Cramér-Rao lower bound.

(b) Suppose that the common distribution is Bernoulli $B(1, \theta)$. Using the theorem about the attainment of the Cramér-Rao lower bound, identify the minimum variance unbiased estimator for θ.

(Assume that the regularity conditions are satisfied.)

(10 marks)

5. Let $X_1, X_2, \ldots, X_n$ be random variables with a joint probability density function/probability function $f(\mathbf{x}, \theta) = f(x_1, x_2, \ldots, x_n, \theta)$, where $\theta \in \Theta$.

(a) Explain what is meant by saying that C is a test of size α.

(b) State, without proof, the Neyman-Pearson fundamental lemma.

(5 marks)

6. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables with the common probability density function

$$f(x, \theta) = \theta \exp(-\theta x) \quad \text{for} \quad x > 0,$$

where $\theta > 0$.

(a) Derive the Neyman-Pearson critical region for the test of the null hypothesis $H_0 : \theta = \theta_0$ against the alternative $H_1 : \theta = \theta_1$, with $\theta_0 < \theta_1$.
(You may assume that $2\theta \sum_{k=1}^{n} X_i$ has the $\chi^2(2n)$ distribution.)

(b) Specify the region when $\theta_0 = 2$, $\theta_1 = 3$, $n = 10$, and $\alpha = 0.05$, and find the power of the test.

(15 marks)

Section B (40 marks)

7. (a) Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables with a common probability density function/probability function $f(x, \theta)$ depending on an unknown parameter θ. Using the factorisation theorem (criterion), prove that any one-to-one function of a sufficient statistic is a sufficient statistic.

(b) Suppose that the common distribution is Binomial $B(N, \theta)$, that is,

$$f(x,\theta) = \binom{N}{x} \theta^x (1-\theta)^{N-x}, \; x = 0, 1, 2, \ldots, N,$$

where N is a known positive integer, $0 < \theta < 1$, and

$$g(\theta) = N\theta(1-\theta)^{N-1}.$$

(i) Let $\tilde{g}(\mathbf{X}) = 1$ if $X_1 = 1$ and $\tilde{g}(\mathbf{X}) = 0$ if $X_1 \neq 1$. Show that $\tilde{g}(\mathbf{X})$ is an unbiased estimator of $g(\theta)$.

(ii) Show that $T = \sum_{i=1}^{n} X_i$ is a complete sufficient statistic for θ.

(iii) Show that

$$\mathrm{E}(\tilde{g}(\mathbf{X}) | T = t) = \frac{N \binom{N(n-1)}{t-1}}{\binom{Nn}{t}}$$

(you may assume that if $X_1, X_2, \ldots, X_m$ are independent $B(N, \theta)$ random variables, then $\sum_{i=1}^{m} X_i \sim B(Nm, \theta)$).

(iv) Write down a minimum variance unbiased estimator for $g(\theta)$. Justify your answer.

(25 marks)

8. Let $X_1, X_2, \ldots, X_n$ be random variables with a joint probability density function/probability function $f(\mathbf{x}, \theta) = f(x_1, x_2, \ldots, x_n, \theta)$, where $\theta \in \Theta$. Suppose that $\Theta_0 \subset \Theta$ and consider testing a null hypothesis $\mathrm{H}_0 : \theta \in \Theta_0$ against the alternative $\mathrm{H}_1 : \theta \in \Theta_1$, where $\Theta_1 = \Theta \backslash \Theta_0$.

(a) Explain how the Neyman-Pearson fundamental lemma can in certain cases be extended to provide a uniformly most powerful test for testing a simple null hypothesis $H_0 : \theta = \theta_0$ against a composite alternative $H_1 : \theta \in \Theta_1$.

(b) Explain the application of the likelihood ratio test to composite hypotheses.

(15 marks)

Table of Standard Distributions

Binomial distribution $B(n,p)$

pf	$f(x,p) = \binom{n}{x} p^x(1-p)^{n-x};\ x = 0,1,\ldots,n;\ 0<p<1;\ n \geq 1$
mean	$\mathrm{E}(X) = np$
variance	$\mathrm{Var}(X) = np(1-p)$
mgf	$M(t) = ((1-p)+pe^t)^n$

Poisson distribution $P(\lambda)$

pf	$f(x,\lambda) = \lambda^x e^{-\lambda}/x!\ ;\ x = 0,1,2,\ldots;\ \lambda > 0$
mean	$\mathrm{E}(X) = \lambda$
variance	$\mathrm{Var}(X) = \lambda$
mgf	$M(t) = \exp\{\lambda(e^t-1)\}$

Geometric distribution $G(p)$

pf	$f(x,p) = (1-p)^{x-1}\,p;\ x = 1,2,\ldots;\ 0<p<1$
mean	$\mathrm{E}(X) = 1/p$
variance	$\mathrm{Var}(X) = (1-p)/p^2$
mgf	$M(t) = p(1-(1-p)e^t)^{-1},\ t < -\log(1-p)$

Normal distribution $N(\mu,\sigma^2)$

pdf	$f(x,\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}}\exp\{-\frac{1}{2\sigma^2}(x-\mu)^2\};\ -\infty<x<\infty,\ -\infty<\mu<\infty;\ \sigma>0$
mean	$\mathrm{E}(X) = \mu$
variance	$\mathrm{Var}(X) = \sigma^2$
mgf	$M(t) = \exp(\mu t + \sigma^2 t^2/2)$

Uniform distribution $U(a,b)$

pdf	$f(x,a,b) = \frac{1}{(b-a)};\ a \leq x \leq b;\ \ a<b$
mean	$\mathrm{E}(X) = (a+b)/2$
variance	$\mathrm{Var}(X) = (b-a)^2/12$
mgf	$M(t) = \frac{e^{bt}-e^{at}}{(b-a)t}$

Exponential distribution $\mathrm{M}(\theta)$

pdf	$f(x,\theta) = \theta\exp\{-\theta x\};\ x \geq 0;\ \theta > 0$
mean	$\mathrm{E}(X) = 1/\theta$
variance	$\mathrm{Var}(X) = 1/\theta^2$
mgf	$M(t) = (1-t/\theta)^{-1}$

Gamma distribution $\Gamma(\alpha,\beta)$

pdf	$f(x,\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha}x^{\alpha-1}\exp\left\{-\frac{x}{\beta}\right\};\ x>0;\ \alpha,\beta \geq 0$
mean	$\mathrm{E}(X) = \alpha\beta$
variance	$\mathrm{Var}(X) = \alpha\beta^2$
mgf	$M(t) = (1-\beta t)^{-\alpha}$

END

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