1. Let $X_1, X_2, \dots, X_n$ be independent random variables with a common distribution depending on an unknown parameter $\theta \in \Theta$.

(a) (i) Explain what is meant by saying that $\tilde{g}(\mathbf{X})$ is an unbiased estimator of a real-valued function $g(\theta)$ of the parameter θ.

(ii) Is the following true or false? You do NOT need to justify your answer.

(1) If $\hat{\theta}$ is a maximum likelihood estimator of the parameter θ, then $\hat{\theta}$ is unbiased.

(2) If $\hat{\theta}$ is a maximum likelihood estimator of the parameter θ, then $\hat{\theta}$ is efficient.

(3) If $\hat{\theta}$ is a maximum likelihood estimator of the parameter θ, then $\hat{\theta}$ is asymptotically efficient.

(4) If $\hat{\theta}$ is a moment estimator of the parameter θ, then $\hat{\theta}$ is unbiased.

(5) If $\hat{\theta}$ is a moment estimator of the parameter θ, then $\hat{\theta}$ is efficient.

(6) If $\hat{\theta}$ is a moment estimator of the parameter θ, then $\hat{\theta}$ is asymptotically efficient.

(7) Suppose that $\hat{\theta}$ is the maximum likelihood estimator of θ and $g(\theta)$ is a real-valued function of θ. Then $g(\hat{\theta})$ is the maximum likelihood estimator of $g(\theta)$.

(b) Suppose that the common distribution is the Gamma $\Gamma(2, \theta)$, that is, the common probability density function is

$$f(x, \theta) = \frac{1}{\theta^2} x e^{-\frac{x}{\theta}} \quad \text{for} \quad x > 0,$$

where $\theta > 0$.

(i) Find the maximum likelihood estimator $\hat{\theta}$ of the parameter θ.

(ii) Is the estimator in (i) unbiased? Justify your answer.

(c) Suppose that the common probability density function is

$$f(x, \theta) = \theta(\theta + 1) x^{\theta - 1}(1 - x) \quad \text{for} \quad 0 < x < 1,$$

where $\theta > 0$. Find a moment estimator of the parameter θ.

2. Let $X_1, X_2, \ldots, X_n$ be independent random variables with a common probability density function/probability function $f(x, \theta)$.

(a) (i) Give the definitions of a sufficient statistic and a minimal sufficient statistic.

(ii) State, without proof, the Rao-Blackwell theorem.

(iii) Given a complete sufficient statistic T for θ, explain how to obtain a minimum variance unbiased estimator $\hat{g}$ for a real-valued function $g(\theta)$.

(You need not define the notion of completeness.)

(b) Suppose that

$$f(x, \theta) = \theta^2 x e^{-\theta x} \quad \text{for} \quad x > 0,$$

where $\theta > 0$.

(i) By writing the probability density function in exponential family form, show that $T = \sum_{i=1}^{n} X_i$ is a complete sufficient statistic for θ.

(ii) Prove that $\tilde{\theta} = 1/X_1$ is an unbiased estimator of θ.

(iii) Assuming that the conditional probability density function of X_1 given $T = t$ is

$$f_{X_1|T}(x|t) = (2n-1)(2n-2)\frac{x}{t^{2n-1}}(t-x)^{2n-3} \quad \text{for} \quad 0 \leq x \leq t,$$

show that

$$\hat{\theta} = \frac{(2n-1)}{\sum_{i=1}^{n} X_i}$$

is the minimum variance unbiased estimator of θ.

3. Let $X_1, X_2, \ldots, X_n$ be independent random variables with a common probability density function/probability function $f(x, \theta)$.

(a) (i) Define the one-step Fisher information and the total Fisher information.

(ii) Assume that the regularity conditions of the Cramér-Rao Theorem hold. Give, without proof, an alternative formula for the one-step Fisher information.

(iii) State, without proof, the theorem about the attainment of the Cramér-Rao lower bound.

(*You need not list the regularity conditions.*)

(b) Suppose that the common distribution is the Gamma $\Gamma(4, \theta)$, that is

$$f(x, \theta) = \frac{x^3}{6\theta^4} \exp\left(-\frac{x}{\theta}\right) \quad \text{for} \quad x > 0 \text{ and } \theta > 0.$$

(i) Find the Cramér-Rao lower bound for the variance of an unbiased estimator of θ.

(Assume that the regularity conditions are satisfied.)

(ii) Using the theorem about the attainment of the Cramér-Rao lower bound, identify the minimum variance unbiased estimator for θ.

(Assume that the regularity conditions are satisfied.)

4. (a) Let $X_1, X_2, \ldots, X_n$ be random variables with a joint probability density function/probability function $f(\mathbf{x}, \theta) = f(x_1, x_2, \ldots, x_n, \theta)$, where $\theta \in \Theta$.

   (i) Define the power function of a test C.

   (ii) Suppose that C is a test for a null hypothesis $\mathrm{H}_0 : \theta \in \Theta_0$ against the alternative hypothesis $\mathrm{H}_1 : \theta \in \Theta_1$, where $\Theta_1 = \Theta \backslash \Theta_0$. Explain what is meant by saying that the test C is unbiased.

   (iii) State, without proof, the Neyman-Pearson fundamental lemma for the test of a simple hypothesis $H_0 : \theta = \theta_0$ against a simple alternative $H_1 : \theta = \theta_1$.

   (b) Let $X_1, X_2, \ldots, X_n$ be a random sample from the normal distribution $N(0, \sigma^2)$.

   (i) Obtain the Neyman-Pearson critical region of size α for the test of the null hypothesis $H_0 : \sigma = \sigma_0$ against the alternative $H_1 : \sigma = \sigma_1$ where $\sigma_1 > \sigma_0$. (*You may assume that* $\frac{1}{\sigma^2} \sum_{k=1}^{n} X_i^2$ *has the* $\chi^2(n)$ *distribution.*)

   (ii) Specify the critical region when $n = 21$, $\sigma_0 = 1.2$, $\sigma_1 = 1.8$ and $\alpha = 0.01$, and find the power.

5. (a) Explain the application of the likelihood ratio test to composite hypotheses.

   (b) Let $X_1, X_2, \ldots, X_n$ be a random sample from the normal distribution $N(\mu, \sigma^2)$, where both μ and σ are unknown. Suppose that C is the critical region for the likelihood ratio test for testing the null hypothesis $H_0 : \mu = \mu_0$ against the alternative $H_1 : \mu \neq \mu_0$.

   (i) Show that C can be written as

$$C = \left\{ \mathbf{x} : \left(\frac{\hat{\sigma}^2}{\hat{\sigma}_0^2} \right)^{\frac{n}{2}} \leq \lambda \right\},$$

   where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, $\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2$, $\hat{\sigma}_0^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \mu_0)^2$, $\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$.

   (ii) Show that the critical region of size α can be specified using tables of the t-distribution.
   (*You may assume that, if* $\lambda \leq 1$, $\left(\frac{a^2}{b^2} \right)^{n/2} \leq \lambda$ *is equivalent to* $\sqrt{\frac{b^2 - a^2}{a^2}} \geq \sqrt{\lambda^{-2/n} - 1}$.)

ROYAL HOLLOWAY UNIVERSITY OF LONDON

Table of Standard Distributions

Binomial distribution $B(n,p)$

pf	$f(x,p) = \binom{n}{x} p^x(1-p)^{n-x};\ x = 0,1,\ldots,n;\ 0 < p < 1;\ n \geq 1$
mean	$\mathrm{E}(X) = np$
variance	$\mathrm{Var}(X) = np(1-p)$
mgf	$M(t) = ((1-p)+pe^t)^n$

Poisson distribution $P(\lambda)$

pf	$f(x,\lambda) = \lambda^x e^{-\lambda}/x!\ ;\ x = 0,1,2,\ldots;\ \lambda > 0$
mean	$\mathrm{E}(X) = \lambda$
variance	$\mathrm{Var}(X) = \lambda$
mgf	$M(t) = \exp\{\lambda(e^t-1)\}$

Geometric distribution $G(p)$

pf	$f(x,p) = (1-p)^{x-1}\,p;\ x = 1,2,\ldots;\ 0 < p < 1$
mean	$\mathrm{E}(X) = 1/p$
variance	$\mathrm{Var}(X) = (1-p)/p^2$
mgf	$M(t) = p\,(1-(1-p)e^t)^{-1},\ t < -\log(1-p)$

Normal distribution $N(\mu,\sigma^2)$

pdf	$f(x,\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left\{-\frac{1}{2\sigma^2}(x-\mu)^2\right\};\ -\infty < x < \infty;\ -\infty < \mu < \infty;\ \sigma > 0$
mean	$\mathrm{E}(X) = \mu$
variance	$\mathrm{Var}(X) = \sigma^2$
mgf	$M(t) = \exp\{\mu t + \sigma^2 t^2/2\}$

Uniform distribution $U(a,b)$

pdf	$f(x,a,b) = \frac{1}{(b-a)};\ a \leq x \leq b;\ a < b$
mean	$\mathrm{E}(X) = (a+b)/2$
variance	$\mathrm{Var}(X) = (b-a)^2/12$
mgf	$M(t) = \frac{e^{bt}-e^{at}}{(b-a)t}$

Exponential distribution $\mathrm{M}(\theta)$

pdf	$f(x,\theta) = \theta\exp\{-\theta x\};\ x \geq 0;\ \theta > 0$
mean	$\mathrm{E}(X) = 1/\theta$
variance	$\mathrm{Var}(X) = 1/\theta^2$
mgf	$M(t) = (1-t/\theta)^{-1}$

Gamma distribution $\Gamma(\alpha,\beta)$

pdf	$f(x,\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha}x^{\alpha-1}\exp\left\{-\frac{x}{\beta}\right\};\ x > 0;\ \alpha,\beta > 0$
mean	$\mathrm{E}(X) = \alpha\beta$
variance	$\mathrm{Var}(X) = \alpha\beta^2$
mgf	$M(t) = (1-\beta t)^{-\alpha}$

END

T. Sharia