1. Let $X_1, X_2, \ldots, X_n$ be independent random variables with a common distribution.

(a) Suppose that the random variables are continuous with the common distribution function $F(x)$ and the common probability density function $f(x)$. Denote $Y = \min(X_1, X_2, \ldots, X_n)$. Show that

$$F_Y(x) = 1 - [1 - F(x)]^n$$

and hence deduce that

$$f_Y(x) = nf(x)(1 - F(x))^{n-1},$$

where $F_Y(x)$ and $f_Y(x)$ are the distribution function and the probability density function of Y respectively.

(b) Suppose that the distribution is given by a probability density function/probability function $f(x, \theta)$, where $\theta \in \Theta$ is a real parameter. Explain the principle of the maximum likelihood estimation of θ based on the sample $X_1, X_2, \ldots, X_n$.

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(c) Suppose that the common probability density function is

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$$f(x, \theta) = \frac{3\theta^3}{x^4}, \text{ where } x \geq \theta > 0.$$

(i) Find a moment estimator of the parameter θ.

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(ii) Find the maximum likelihood estimator of the parameter θ.

(iii) Are the estimators in (i) and (ii) unbiased? Justify your answer.

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2. Let $X_1, X_2, \ldots, X_n$ be independent random variables with a common probability density function/probability function $f(x, \theta)$, depending on an unknown parameter θ.

(a) (i) State the factorisation theorem (criterion) for finding a sufficient statistic for θ.

(ii) Prove that any one-to-one function of a sufficient statistic is a sufficient statistic.

(b) Suppose that the common distribution is Poisson $P(\theta)$ and let $g(\theta) = e^{-\theta}$.

(i) Denote $\mathbf{X} = (X_1, \ldots, X_n)$, let $\tilde{g}(\mathbf{X}) = 1$ if $X_1 = 0$ and let $\tilde{g}(\mathbf{X}) = 0$ if $X_1 \neq 0$. Show that $\tilde{g}(\mathbf{X})$ is an unbiased estimator of $g(\theta)$.

(ii) Show that $T = \sum_{i=1}^{n} X_i$ is a complete sufficient statistic for θ.

(iii) Assuming that the conditional distribution of X_1 given $T = t$ is Binomial $B\left(t, \frac{1}{n}\right)$, find $\mathrm{E}(\tilde{g}(\mathbf{X})|T = t)$.

(iv) Write down a minimum variance unbiased estimator for $g(\theta)$. Justify your answer.

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3. Let $X_1, X_2, \ldots, X_n$ be independent random variables with a common probability density function/probability function $f(x, \theta)$.

(a) State, without proof,

(i) the Cramér-Rao theorem for an unbiased estimator $\hat{g}(\mathbf{X})$ of $g(\theta)$, where $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ and $g(\theta)$ is a differentiable function of the parameter θ; and

(ii) the theorem about a necessary and sufficient condition for attaining the Cramér-Rao lower bound.

(You need not list the regularity conditions.)

(b) Suppose that the common distribution is Bernoulli $\mathrm{B}(1, \theta)$.

(i) Find the Cramér-Rao lower bound for the variance of an unbiased estimator of θ.

(Assume that the regularity conditions are satisfied.)

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(ii) Using the theorem about the attainment of the Cramér-Rao lower bound, identify the minimum variance unbiased estimator for θ.

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(Assume that the regularity conditions are satisfied.)

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4. (a) Let $X_1, X_2, \ldots, X_n$ be random variables with a joint probability density function/probability function $f(\mathbf{x}, \theta) = f(x_1, x_2, \ldots, x_n, \theta)$, where $\theta \in \Theta$. Suppose that $\Theta_0 \subset \Theta$ and consider testing a null hypothesis $\mathrm{H}_0 : \theta \in \Theta_0$ against the alternative $\mathrm{H}_1 : \theta \in \Theta_1$, where $\Theta_1 = \Theta \backslash \Theta_0$.

(i) Explain what is meant by saying that C is a test of size α.

(ii) Explain how the Neyman-Pearson fundamental lemma can in certain cases be extended to provide a uniformly most powerful test for testing a simple null hypothesis $H_0 : \theta = \theta_0$ against a composite alternative $H_1 : \theta \in \Theta_1$.

(b) Let $X_1, X_2, \ldots, X_n$ be a random sample from the normal distribution $N(\theta, 1)$.

(i) Obtain the Neyman-Pearson critical region of size α for the test of the null hypothesis $H_0 : \theta = 2$ against the alternative $H_1 : \theta = 3$.

(ii) Specify the region when $n = 16$ and the size of the test is $\alpha = 0.05$, and find the power of the test.

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5. (a) Let $X_1, X_2, \ldots, X_n$ be random variables with a joint probability density function/probability function $f(\mathbf{x}, \theta) = f(x_1, x_2, \ldots, x_n, \theta)$, where $\theta \in \Theta$. Suppose that $\Theta_0 \subset \Theta$ and consider testing a null hypothesis $\mathrm{H}_0 : \theta \in \Theta_0$ against the alternative $\mathrm{H}_1 : \theta \in \Theta_1$, where $\Theta_1 = \Theta \backslash \Theta_0$.

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(i) Define the power function of a test C.

(ii) Explain what is meant by saying that the test C is *unbiased*.

(iii) Describe briefly the informal argument underlying the likelihood ratio test.

(b) Let $X_1, X_2, \ldots, X_n$ be a random sample from the normal distribution $N(\mu, 1)$. Find the critical region of size α for the likelihood ratio test of the null hypothesis $H_0 : \mu = \mu_0$ against the alternative $H_1 : \mu \neq \mu_0$.

(You may assume that the maximum likelihood estimator of μ is $\bar{X}$.)

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Table of Standard Distributions

Binomial distribution $B(n, p)$

pf	$f(x,p) = \binom{n}{x} p^x (1-p)^{n-x}$; $x = 0, 1, 2, \ldots, n$; $0 < p < 1$; $n \geq 1$
mean	$\mathrm{E}(X) = np$
variance	$\mathrm{Var}(X) = np(1-p)$
mgf	$M(t) = ((1-p) + pe^t)^n$

Poisson distribution $P(\lambda)$

pf	$f(x, \lambda) = \lambda^x e^{-\lambda}/x!$; $x = 0, 1, 2, \ldots$; $\lambda > 0$
mean	$\mathrm{E}(X) = \lambda$
variance	$\mathrm{Var}(X) = \lambda$
mgf	$M(t) = \exp\{\lambda(e^t - 1)\}$

Geometric distribution $G(p)$

pf	$f(x, p) = (1-p)^{x-1}\, p$; $x = 1, 2, \ldots$; $0 < p < 1$
mean	$\mathrm{E}(X) = 1/p$
variance	$\mathrm{Var}(X) = (1-p)/p^2$
mgf	$M(t) = p(1-(1-p)e^t)^{-1}$, $t < -\log(1-p)$

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Normal distribution $N(\mu, \sigma^2)$

pdf	$f(x, \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2\sigma^2}(x-\mu)^2\right\}$; $-\infty < x < \infty$; $-\infty < \mu < \infty$; $\sigma > 0$
mean	$\mathrm{E}(X) = \mu$
variance	$\mathrm{Var}(X) = \sigma^2$
mgf	$M(t) = \exp\{\mu t + \sigma^2 t^2/2\}$

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Uniform distribution $U(a, b)$

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pdf	$f(x, a, b) = \frac{1}{(b-a)}$; $a \leq x \leq b$; $a < b$
mean	$\mathrm{E}(X) = (a+b)/2$
variance	$\mathrm{Var}(X) = (b-a)^2/12$
mgf	$M(t) = \frac{e^{bt} - e^{at}}{(b-a)t}$

Exponential distribution $\mathrm{M}(\theta)$

pdf	$f(x, \theta) = \theta \exp\{-\theta x\}$; $x \geq 0$; $\theta > 0$
mean	$\mathrm{E}(X) = 1/\theta$
variance	$\mathrm{Var}(X) = 1/\theta^2$
mgf	$M(t) = (1 - t/\theta)^{-1}$

Gamma distribution $\Gamma(\alpha, \beta)$

pdf	$f(x, \alpha, \beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} \exp\left\{-\frac{x}{\beta}\right\}$; $x > 0$; $\alpha, \beta > 0$
mean	$\mathrm{E}(X) = \alpha\beta$
variance	$\mathrm{Var}(X) = \alpha\beta^2$
mgf	$M(t) = (1 - \beta t)^{-\alpha}$

END

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