MTH6154: Financial Mathematics 1

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Contents

Preface

This course is an introduction to financial mathematics, and is the first in a series of three modules (along with **Financial Mathematics II** and **Mathematical Tools for Asset Management**[1]) that together give a thorough grounding in the theory and practice of modern financial mathematics. While this course primarily deals with financial mathematics in discrete time, **FMII** will cover financial mathematics in continuous time, and **MTAM** will focus on risk management and portfolio theory.

So what exactly is financial mathematics? A simple definition would be that it is **the use of mathematical techniques to model and price financial instruments**. These financial instruments include:

- Debt (also known as 'fixed-income'), such as cash, bank deposits, loans and bonds;
- Equity, such as shares/stock in a company; and
- Derivatives, including forwards, options and swaps.

As this definition suggests, there are two central questions in financial mathematics:

[1]Formally known as **Financial Mathematics III**.

1. What is a financial instrument worth? To answer this, we will develop principled ways in which to:
   - Compare the value of money at different moments in time; and
   - Take into account the uncertainty in the future value of an asset.
2. How can financial markets usefully be modelled? Note that modelling always involves a trade-off between accuracy and simplicity.

This course introduces you to the basics of financial mathematics, including:

- An overview of financial instruments, including bonds, forwards and options.
- An analysis of the fundamental ideas behind the rational pricing of financial instruments, including the central ideas of **the time-value of money** and **the no-arbitrage principle**. These concepts are **model-free**, meaning that they apply irrespective of the chosen method of modelling the financial market;
- An introduction to financial modelling, including the use of the **binomial model** to price European and American options.

The course culminates in a derivation, via an approximation procedure, of the celebrated **Black-Scholes formula** for pricing European options.

1 Interest rates and present value analysis

The value of money is not constant over time – a thousand pounds today is typically worth more than a contract guaranteeing $\pounds 1{,}000$ this time next year. To explain this, consider that

- If $\pounds 1{,}000$ was deposited into a bank account today, it would accumulate interest by this time next year;
- Inflation may reduce the purchasing power of $\pounds 1{,}000$ over the course of the year.

In this chapter we explore how to take into account interest rates and inflation when comparing assets that generate cash at different moments of time, using the concept of **present value**; this gives us a way to make a principled comparisons between investments. Later in the course we will use these ideas to assign a fair price to fixed-income securities,[2] such as bonds.

An important theme in this chapter is the notation of **standardisation**, i.e. adjusting interest rates and rates of return so that different rates can be compared fairly.

1.1 Interest rates

Suppose we borrow an amount P, called the **principal**, at **nominal interest rate** r. This means that if we repay the loan after 1 year, we will need to repay the principal plus an extra sum rP called the **interest**, so in total

$$P + rP = P(1+r). \tag{1}$$

Similarly, if you put an amount P in the bank at an annualised interest rate r, then in a year's time the account value will grow to $P(1+r)$. Note that here we have assumed that the nominal interest rate is **annualised**, meaning that interest is calculated once per year; **unless explicitly specified otherwise interest rates will always be assumed to be annualised in this course**.

Sometimes the interest is not calculated once per year, but instead is **compounded** every $\frac{1}{n}$-th of a year. This means that every $\frac{1}{n}$-th of a year you are charged (or, in the case of a bank account, gain) interest at rate r/n on the principal **as well as** on the interest that has already accumulated in previous periods. Continuing the example of the loan above, this would mean that after one year we would owe

$$P\Big(1+\frac{r}{n}\Big)^n. \tag{2}$$

If the interest rate is 'annualised' then it is compounded **annually** (i.e. once per year), which corresponds to $n=1$ and gives the same result as in (1).

Example 1.1. Suppose you borrow an amount P, to be repaid after one year at interest rate r, compounded **semi-annually**. Then the following will happen sequentially throughout the year. After half a year you will be charged interest at rate $r/2$, which is added on to the principal. Thus, after 6 months you owe

$$P\Big(1+\frac{r}{2}\Big).$$

At the end of the year you are again charged interest at rate $r/2$, with the interest accumulating on the entire sum owing, therefore at the end of the year you owe

$$P\Big(1+\frac{r}{2}\Big)\Big(1+\frac{r}{2}\Big) = P\Big(1+\frac{r}{2}\Big)^2.$$

[2] A security is a tradable financial instrument.

This corresponds to choosing $n = 2$ in (2).

If $n = 4$ in (2) we say that interest is compounded **quarterly**, and if $n = 12$ we say that interest is compounded **monthly**.

Example 1.2. Suppose that you borrow £100 at interest rate 18% compounded monthly. After one year you owe

$$100\left(1+\frac{0.18}{12}\right)^{12} = 100 \times 1.195 = £119.5.$$

In order to fairly compare interest rates we need to **standardise** to account for differences in the compounding frequency. In this context the interest rate r is called the **nominal rate**. To compare the effect of different compounding frequencies, we introduce the **effective rate**[3]

$$r_{\text{eff}} = \frac{(\text{amount after one year}) - P}{P}.$$

For an interest rate compounded every $\frac{1}{n}$-th of a year, we therefore have

$$r_{\text{eff}} = \frac{P(1+r/n)^n - P}{P} = \left(1+\frac{r}{n}\right)^n - 1 \qquad (3)$$

The effective rate quantifies the **net effect** of interest over the course of a year, and so is a fairer measure of the true effect of interest. Under UK consumer law, all credit providers must publish their effective rate.

Example 1.3. Suppose that you borrow £100 at nominal interest rate 18% compounded monthly. What is the effective rate?

Solution. We know from Example 1.2 that after one year you owe £119.5. Hence

$$r_{\text{eff}} = \frac{119.5 - 100}{100} = 0.195 = 19.5\%. \qquad \square$$

Example 1.4. Credit card company A charges a 24.5% nominal rate compounded daily whereas credit card company B charges a 24.7% nominal rate compounded monthly. Which company offers the better deal?

Solution. The nominal interest rates do not give a fair comparison, since they are compounded at different frequencies. A fairer comparison is given by the effective rates for both deals:

- For company A, the effective rate is

$$r_{\text{eff}} = \left(1+\frac{1}{365}24.5\%\right)^{365} - 1 = 0.2775.$$

- For company B, the effective rate is:

$$r_{\text{eff}} = \left(1+\frac{1}{12}24.7\%\right)^{12} - 1 = 0.2770.$$

[3] Effective rates are sometimes also called ‘annual percentage rates’ (APRs).

We conclude that company B offers the better deal even though the nominal rates advertised might suggest the opposite. $\square$

Imagine now that interest is compounded at very small intervals, i.e. interest is compounded every $\frac{1}{n}$-th of a year for n very large. In the limiting case, i.e. as $n \to \infty$, we say that interest is **continuously-compounded**, and in this case the amount owed after one year is

$$\lim_{n\to\infty} P\Big(1 + \frac{r}{n}\Big)^n = Pe^r$$

(we prove this rigorously in Coursework 1). The effective rate for continuously-compounded interest at nominal rate r is therefore

$$r_{\text{eff}} = \frac{Pe^r - P}{P} = e^r - 1.$$

Example 1.5. Suppose that a bank offers continuously-compounded interest at nominal rate 5%. The effective rate is

$$r_{\text{eff}} = e^{0.05} - 1 = 0.05127 = 5.127\%.$$

If a loan is taken out for T years, then the total amount owed after T years can be calculated using the effective rate r_{eff} as

$$P\underbrace{(1 + r_{\text{eff}}) \times \cdots \times (1 + r_{\text{eff}})}_{T \text{ times}} = P(1 + r_{\text{eff}})^T.$$

For example, if interest is compounded n times a year, this is

$$P\Big(1 + \frac{r}{n}\Big)^{nT},$$

and if it is compounded continuously

$$P(e^r)^T = Pe^{rT}.$$

Example 1.6 (The doubling rule)**.** If a bank offers continuously-compounded interest at nominal rate r, how long does it take for the amount of money in the bank to double?

Solution. Let T denote the time in years by the amount P doubles. Then

$$Pe^{rT} = 2P \quad \Rightarrow \quad e^{rT} = 2 \quad \Rightarrow \quad rT = \ln 2,$$

and therefore

$$T = \frac{\ln 2}{r} \approx \frac{0.693}{r}.$$

For example, if the nominal rate is $r = 12\%$, it takes 5.78 years for the amount to double. $\square$

Remark 1.7. In investment folklore, the doubling rule is sometimes referred to as the *'Rule of 72'*, since you can (roughly!) approximate $\ln 2 = 0.693$ by 0.72, which often makes the fraction easier to calculate, i.e. in the example above, $0.72/0.12 = 72/12 = 6$ years. This 'rule' first appeared in print in 1494, long before logarithms were invented.

1.2 Variable interest rates

Interest rates may not be constant over time. To describe variable interest rates we introduce the concepts of **instantaneous interest rate** and **yield curve**.

1.2.1 The instantaneous interest rate

Suppose that the interest rate at time t is equal to $r = r(t)$; we call $r(t)$ the **instantaneous interest rate**. We always assume that the instantaneous interest is **continuously-compounded**, which means that for small h (and as long as $r(t)$ is continuous at t), the amount of interest that accrues between times t and $t+h$ is equal to

$$e^{r(t)h} = 1 + r(t)h + \frac{1}{2}r(t)^2h^2 + \ldots \approx 1 + r(t)h.$$

We wish to calculate the amount $P(t)$ that accumulates by time t, given that the amount $P(0)$ is deposited (or loaned) at time 0. It turns out that $P(t)$ satisfies a simple differential equation:

Theorem 1.8. *Suppose that $r(t)$ is a piece-wise continuous function. Then*

$$P'(t) = P(t)r(t).$$

Proof. Let h denote a small time-period. Since interest is continuously compounded, the interest accruing between times t and $t+h$ is approximately equal to $r(t)h$, i.e.,

$$P(t+h) \approx P(t) + P(t)r(t)h,$$

or equivalently

$$\frac{P(t+h) - P(t)}{h} \approx P(t)r(t).$$

As h becomes smaller, the above approximation becomes more and more accurate, and so

$$\lim_{h\to 0} \frac{P(t+h) - P(t)}{h} = P(t)r(t).$$

Recognising the left-hand side as the definition of the derivative, we conclude that

$$P'(t) = P(t)r(t), \qquad \square$$

The above equation is a separable first order differential equation, easily solved by writing

$$\frac{P'(t)}{P(t)} = r(t)$$

and integrating both sides:

$$\int_0^t \frac{P'(u)}{P(u)}\, du = \int_0^t r(u)\, du.$$

Thus

$$[\ln P(u)]_{u=0}^{u=t} = \int_0^t r(u)\, du \implies \ln P(t) = \ln P(0) + \int_0^t r(u)\, du,$$

and so

$$P(t) = P(0) \exp\Big(\int_0^t r(u)\, du \Big), \tag{4}$$

which is the desired relation between the amount $P(t)$ in your bank at time t and the time-varying interest rate function $r(t)$.

Note that if $r(t) = r$ is constant, then $P(t) = P(0)e^{rt}$. Thus, the general formula (4) reduces to the familiar formula in the special case of constant interest rate.

1.2.2 The yield curve

Suppose that the instantaneous interest rate is $r(t)$. The **yield curve** $\bar{r}(t)$ is defined to be the average value of $r(t)$ on the interval $(0, t)$:

$$\bar{r}(t) = \frac{1}{t}\int_0^t r(u)\,du.$$

The yield curve allows us to write the formula for $P(t)$ in (4) as

$$P(t) = P(0)\exp\Big(\int_0^t r(u)\,du\Big) = P(0)e^{\bar{r}(t)t}.$$

The effective rate can also be expressed in terms of the yield curve. Since the interest rate is variable, this rate $r_{\text{eff}}(T)$ **depends on the period of time T the money is deposited/loaned**, and is equal to

$$r_{\text{eff}}(T) = e^{\bar{r}(T)} - 1.$$

Example 1.9. Suppose you deposit £100 in a bank offering the instantaneous interest rate

$$r(t) = \begin{cases} 3\% & t < 2, \\ 5\% & t \geq 2. \end{cases}$$

Find the yield curve $\bar{r}(t)$ and determine the value of the bank balance at time $t = 3$.

Proof. To calculate the yield curve we split the integral at $t = 2$. For $t < 2$,

$$\bar{r}(t) = \frac{1}{t}\int_0^t 3\%\,ds = 3\%$$

whereas for $t \geq 2$,

$$\bar{r}(t) = \frac{1}{t}\Big(\int_0^2 3\%\,ds + \int_2^t 5\%\,ds\Big) = \frac{0.06}{t} + \frac{0.05(t-2)}{t} = 5\% - \frac{4\%}{t}.$$

The value of the deposit is

$$100e^{\bar{r}(3)3} = 100e^{(0.05-0.04/3)3} = 100e^{0.11} = £112. \qquad \square$$

Example 1.10. Suppose a bank offers a loan with instantaneous interest rate increasing slowly from 3% to 4% via the function

$$r(t) = \frac{3\%}{1+t} + \frac{4\% \times t}{1+t}.$$

Find the effective rate for a loan until time $T = 5$.

Proof. We can rewrite the interest rate as

$$r(t) = \frac{3\%}{1+t} + \frac{4\% \times (1+t)}{1+t} - \frac{4\%}{1+t} = 4\% - \frac{4\% - 3\%}{1+t} = 4\% - \frac{1\%}{1+t}.$$

The yield curve is

$$\bar{r}(t) = \frac{1}{t}\int_0^t \Big(4\% - \frac{1\%}{1+u}\Big)\,du = \frac{1}{t}\big(4\% \times t - 1\% \times \ln(1+t)\big) = 4\% - \frac{1\%}{t}\ln(1+t).$$

The effective rate for the loan is

$$r_{\text{eff}}(5) = e^{\bar{r}(5)} - 1 = e^{0.04 - \frac{0.01}{5}\ln(6)} - 1 = 3.71\%;$$

as expected, this lies somewhere between 3% and 4%. $\square$

1.3 Inflation

We finish the chapter by taking a quick look at **inflation**, the phenomenon of prices increasing as a whole over time (or equivalently, the purchasing power of money eroding over time).

Inflation is most easily quantified by reference to an index, for example the Retail Price Index (RPI), which tracks the price of a basket of goods. Let r_{inf} be the (annualised) inflation rate, which means that £1 today will have the purchasing power of £$(1 + r_{\text{inf}})$ in one year. For example, if $r_{\text{inf}} = 3\%$, then a basket of goods costing £100 today, will cost £103 next year. Equivalently, the purchasing power of £103 in one year is equivalent to £100 today.

How should we take inflation into account when assessing interest rates? Suppose you deposit the amount P today. After one year the deposit will have grown to $P(1+r)$, where r is the interest rate. On the other hand, the purchasing power of the amount $P(1+r)$ in one year from now, taking into account inflation, is the same as the amount $P(1+r)\,(1+r_{\text{inf}})^{-1}$ today. So the 'real' interest rate (real in the sense that it reflects the decrease of purchasing power), also known as the **inflation adjusted interest rate**, is equal to

$$r_a = \frac{\frac{1+r}{1+r_{\text{inf}}}P - P}{P} = \frac{1+r}{1+r_{\text{inf}}} - 1 = \frac{r - r_{\text{inf}}}{1+r_{\text{inf}}}.$$

Example 1.11. Suppose that the interest rate is 5% and the rate of inflation is 3%. Then the inflation adjusted interest rate is

$$\frac{r - r_{\text{inf}}}{1+r_{\text{inf}}} = \frac{0.05 - 0.03}{1+0.03} = 0.0194 = 1.94\%\,.$$

In the case of compounding interest, we must use the effective interest rate in place of the nominal rate, i.e.,

$$r_a = \frac{r_{\text{eff}} - r_{\text{inf}}}{1+r_{\text{inf}}}.$$

Example 1.12. Suppose that the interest rate is 5% compounded monthly, and the rate of inflation is 3%. Then the inflation adjusted interest rate is

$$\frac{r_{\text{eff}} - r_{\text{inf}}}{1+r_{\text{inf}}} = \frac{((1+\frac{0.05}{12})^{12} - 1) - 0.03}{1+0.03} = 0.0205 = 2.05\%.$$

Since r_{inf} is usually much smaller than 100%, it is often sufficient to use the approximation

$$r_a = \frac{r_{\text{eff}} - r_{\text{inf}}}{1+r_{\text{inf}}} \approx r_{\text{eff}} - r_{\text{inf}}.$$

1.4 Present value analysis

A **cash-flow stream** is a sequence of payments that are specified by their **amount** and **transaction time**. Often, but not always, the cash-flow stream will consist of payments at regular intervals (for instance, one per year), in which case we can write the cash-flow stream simply as $\mathbf{a} = (a_1, a_2, \ldots, a_n)$, where a_i denotes the amount paid at the end of period i.

The question we consider in this section is:

How should we compare different cash-flow streams?

For this we introduce the concept of the **present value** of a cash-flow stream.

1.4.1 Defining the present value

Suppose that the interest rate is 10%. If somebody gives you $£100$ today, you could put it in the bank and after a year you would have $£110$. Thus $£100$ received today has a value of $£110$ in a year's time, or conversely, $£110$ received in a year's time has a present value of $£100$.

More generally, suppose that the interest rate is $r\%$, and suppose you put $(1+r)^{-i}V$ pounds in the bank today. At the end of year i you will have $(1+r)^{-i}V \cdot (1+r)^i = V$ pounds in the bank. Thus, $£V$ received in i years' time is worth $(1+r)^{-i}V$ in today's money. Given this, we define the **present value** of $£V$ at the end of year i to be

$$PV(V) = (1+r)^{-i}V.$$

Exchanging V for its present value $PV(V)$ is sometimes referred to as **discounting** the amount V, and the factor $(1+r)^{-i} = PV(1)$ is referred to as the **discount factor** (more on this later).

Similarly, if the interest is compounded every $\frac{1}{n}$-th of a year at nominal rate r, the present value of V pounds in i years is

$$PV(V) = \left(1 + \frac{r}{n}\right)^{-in} V.$$

We can generalise the notion of present value to a **cash-flow stream** $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ that pays a_i at the end of year i, for $i = 1, \ldots, n$. The present value[4] of $\mathbf{a}$ is

$$PV(\mathbf{a}) = \sum_{i=1}^{n} \frac{a_i}{(1+r)^i}.$$

To see this, observe that the cash-flow stream $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ can be replicated by first splitting the stream into the individual payments $a_1, a_2, \ldots$, and then depositing the corresponding amounts $PV(a_1), PV(a_2), \ldots$ needed to replicate these payments. Since

$$PV(\mathbf{a}) = PV(a_1) + PV(a_2) + \ldots + PV(a_n),$$

the total amount you need to deposit to replicate the cash-flow stream is $PV(\mathbf{a})$.

Remark 1.13. This is our first example of an argument that uses the idea of **replication** to assign a value to a financial instrument. Later in the course we will formalise this type of argument by using the **no-arbitrage assumption**.

Example 1.14. You are offered three different jobs. The salary paid at the end of each year (in thousands of pounds) is

Job\Year	1	2	3	4	5
A	32	34	36	38	40
B	36	36	35	35	35
C	40	36	34	32	30

Which job pays the best if the interest rate is (i) $r = 10\%$, (ii) $r = 20\%$, or (iii) $r = 30\%$?

[4] The term **net present value** (NPV) is sometimes used when discussing cash-flow streams as opposed to a single payment, but we will not use this term.

Solution. We shall compare the present values of the cash-flow streams. The present value for job A is

$$PV(A) = \frac{32}{1+r} + \frac{34}{(1+r)^2} + \frac{36}{(1+r)^3} + \frac{38}{(1+r)^4} + \frac{40}{(1+r)^5},$$

while the present value for job B is

$$PV(B) = \frac{36}{1+r} + \frac{36}{(1+r)^2} + \frac{35}{(1+r)^3} + \frac{35}{(1+r)^4} + \frac{35}{(1+r)^5}$$

and the present value for job C is

$$PV(C) = \frac{40}{1+r} + \frac{36}{(1+r)^2} + \frac{34}{(1+r)^3} + \frac{32}{(1+r)^4} + \frac{30}{(1+r)^5}.$$

Taking the different interest rates into account we get the following present values:

r	PV A	PV B	PV C
0.1	135.03	134.41	132.14
0.2	105.51	106.20	105.50
0.3	85.20	86.61	86.83

Thus,

- If $r = 10\%$, then A pays best, then B, then C;
- If $r = 20\%$, then B pays best, then A, then C;
- If $r = 30\%$, then C pays best, then B, then A.

and so which job pays best depends on the current interest rate. □

It is possible to consider cash-flow streams which go on forever ('in perpetuity'). Before doing so, recall some facts about geometric series:

Proposition 1.15. *Let $a \leq b$ be positive integers. Then if $r \neq 1$,*

$$\sum_{i=a}^{b} r^i = \frac{r^a - r^{b+1}}{1-r}.$$

Hence, if $|r| < 1$,

$$\sum_{i=a}^{\infty} r^i = \lim_{b\to\infty} \sum_{i=a}^{b} r^i = \lim_{b\to\infty} \frac{r^a - r^{b+1}}{1-r} = \frac{r^a}{1-r}.$$

These facts allow us to calculate the present value of perpetual cash-flow streams:

Example 1.16. A **perpetuity** entitles its owner to be paid an amount $a > 0$ at the end of each year indefinitely. What is its present value if the interest rate is r?

Solution. The cash-flow stream generated by the perpetuity is $\mathbf{a} = (a, a, a, \ldots)$, and so its present value is

$$PV(\mathbf{a}) = \sum_{i=1}^{\infty} \frac{a}{(1+r)^i} = a \times \frac{1/(1+r)}{1 - 1/(1+r)} = \frac{a}{(1+r)-1} = \frac{a}{r}.$$ □

Remark 1.17. Note that the present value $\frac{a}{r}$ of the perpetuity can also be justified by a simple **replication argument**. Suppose that instead of purchasing a perpetuity you place $\frac{a}{r}$ in a bank account. After one year you have $\frac{a}{r}(1+r)$ in the account. Suppose you withdraw a to replicate the payment of the perpetuity. Then you are left with

$$\frac{a}{r}(1+r) - a = \frac{a}{r} + a - a = \frac{a}{r}.$$

At the end of the second year, this will have grown to $\frac{a}{r}(1+r)$, so if you again withdraw a you are left with

$$\frac{a}{r}(1+r) - a = \frac{a}{r} + a - a = \frac{a}{r}.$$

If you continue like this you can withdraw a from the bank account each year forever, which replicates exactly the cash-flow stream generated by the perpetuity.

1.4.2 Discount Factors

It is often simpler to write the present value of a cash-flow stream using the concept of **discount factors**. This is especially the case if interest rates vary over time.

Definition 1.18. The discount factor to T years, $D(T)$, is the present value of a payment of 1 unit of cash in T years' time. Equivalently, it is the reciprocal of what a bank account holding 1 unit of cash now will be worth in T years' time.

If interest rates are positive (which is most often the case) the discount factor is smaller than 1 hence the phrase **'discount'**. This indicates that having money in the future is worth less than having the same amount of money today.

Examples 1.19.

1. If the interest is compounded annually at nominal rate r then the discount factor to T years is

$$D(T) = \frac{1}{(1+r)^T}.$$

2. If the interest is compounded at a different frequency, we can write the discount factor to T years in terms of the effective rate r_{eff} as

$$D(T) = \frac{1}{(1+r_{\text{eff}})^T}.$$

For example, if interest is compounded monthly then

$$D(T) = \frac{1}{\left(1+r\frac{1}{12}\right)^{12T}},$$

if interest is continuously-compounded then

$$D(T) = \mathrm{e}^{-rT}.$$

3. If $r(t)$ is an instantaneous interest rate then

$$D(T) = e^{-\bar{r}(T)\,T}.$$

Proposition 1.20. *If a cash-flow stream consists of payments $a_1, a_2, \ldots, a_n$ transacted at times $t_1, t_2, \ldots, t_n$, then the present value of this cash-flow stream is*

$$PV = \sum_{i=1}^{n} D(t_i) a_i.$$

1.4.3 Balancing present values

In many real-life situations it is essential that the present values of assets (i.e. incoming payments) and liabilities (i.e. outgoing payments) are equal. The following is a practical example of such a situation:

Example 1.21. Suppose that Alex is self-employed and wants to save for her retirement. She wants an income of $£1{,}000$ a month starting in 20 years' time and lasting for 30 years. What amount of money does she have to save every month (starting today) for the next 20 years to fund her retirement? Suppose that the nominal rate is 6% compounded monthly.

There exist various possible solutions to this problem. Here are two of them:

Solution 1. Let a denote the as yet unknown amount of pounds she must deposit every month, and let D denote the cash-flow stream generated by these deposits. Similarly, let $w = £1{,}000$, and let W denote the cash-flow stream generated by her monthly withdrawals for her retirement. In order to fund her retirement, the present value of her deposits must equal the present value of her withdrawals, i.e.

$$PV(D) = PV(W).$$

We first calculate $PV(D)$. It is convenient to work with the monthly discount factor

$$\beta = D(1/12) = \frac{1}{1+\frac{0.06}{12}} = \frac{1}{1.005}.$$

Since there are 240 months in 20 years,

$$PV(D) = a + a\beta + a\beta^2 + \cdots + a\beta^{239} = a(1+\beta+\beta^2+\cdots+\beta^{239}) = a\frac{1-\beta^{240}}{1-\beta}.$$

We next calculate $PV(W)$. Since her first withdrawal will take place in 240 months' time, and since there are 360 months in 30 years,

$$PV(W) = w\beta^{240} + w\beta^{241} + \cdots + w\beta^{240+359} = w\beta^{240}(1+\beta+\cdots+\beta^{359}) = w\beta^{240}\frac{1-\beta^{360}}{1-\beta}.$$

Since $PV(D) = PV(W)$, we conclude that

$$a\frac{1-\beta^{240}}{1-\beta} = w\beta^{240}\frac{1-\beta^{360}}{1-\beta} \quad \Rightarrow \quad a = w\beta^{240}\frac{1-\beta^{360}}{1-\beta^{240}} = £361$$

that is, she has to deposit $£361$ every month to fund her retirement. □

Solution 2. A alternative approach is to compare the value of the cash-flow streams at the end of year 20, instead of at the present time. Using the notation from the previous solution, the value of her deposits at the end of year 20 is

$$a\beta^{-240} + a\beta^{-239} + \ldots + a\beta^{-2} + a\beta^{-1} = a\beta^{-1}\frac{\beta^{-240}-1}{\beta^{-1}-1}$$

since β^{-1} is the amount that £1 accumulates in a deposit of one month. Similarly, the value at the end of year 20 of all her retirement withdrawals is

$$w + w\beta + w\beta^2 + \ldots + w\beta^{359} = w\,\frac{1-\beta^{360}}{1-\beta}.$$

Equating these values, we have

$$a\,\beta^{-1}\frac{\beta^{-240}-1}{\beta^{-1}-1} = w\,\frac{1-\beta^{360}}{1-\beta} \quad \Rightarrow \quad a = w\,\frac{1-\beta^{360}}{\beta^{-240}-1} = £360.9\,.$$

□

1.5 Rates of return

Suppose you invest an amount P (the 'principal') which at a later times results in a value of P_{final}. How can you assess the quality of this investment? Three possible quantifiers are:

- The **return** on the investment is the quantity P_{final}.
- The **profit** on the investment is the quantity $P_{\text{final}} - P$.
- The **rate of return** on the investment is the profit as a percentage of the principle

$$R = \frac{P_{\text{final}} - P}{P}. \tag{5}$$

Remark 1.22. Be careful when using the phrases 'profit' and 'return'; practitioners are not always consistent in their use.

Examples 1.23.

1. Suppose we invested $P = £150{,}000$ in a property which is now valued at $P_{\text{final}} = £200{,}000$. What is the rate of return? This is given by

$$R = \frac{200{,}000 - 150{,}000}{150{,}000} = 0.333 \approx 33.3\%.$$

2. What is the total return on an investment of $\$12{,}000$ that guarantees a 20% rate of return? What is the profit? Rearranging (5), the return is

$$P_{\text{final}} = P(1 + R) = \$12{,}000 \times (1 + 0.2) = \$14{,}400$$

and the profit is

$$P \times R = \$12{,}000 \times 0.2 = \$2{,}400 (= \$14{,}400 - \$12{,}000).$$

1.5.1 The annualised rate of return and the equivalent effective interest rate

The rate of return does not fairly evaluate the performance of an investment, since it does not take into account how much time it took for the investment to accrue its value.

For example, a rate of a return of 33.3% as in Example 1.23 would be spectacular if gained in two weeks, but would be less spectacular if the increase took 40 years. To fairly compare rates of return, we need to standardise them to account for differences in time-period.

A naive standardisation would simply divide the rate of return R by the time T it took for the investment to accrue its value, i.e.

$$r = \frac{R}{T}$$

called **annualised rate of return**. A better standardisation is to consider the **equivalent effective interest rate** r that a bank deposit would need to offer in order to match the return on the investment. This is defined to satisfy

$$P(1 + r)^T = P_{\text{final}} = P(1 + R).$$

Rearranging, this gives

$$r = (1 + R)^{1/T} - 1 = \left(\frac{P_{\text{final}}}{P}\right)^{1/T} - 1.$$

Examples 1.24.

1. Suppose we invested £$1{,}500$ in shares two year's ago, and these are now worth £$1{,}750$. The equivalent effective interest rate is

$$r = \left(\frac{P_{\text{final}}}{P}\right)^{1/T} - 1 = \left(\frac{1750}{1500}\right)^{1/2} - 1 = 0.0801 = 8.01\%, .$$

2. How much profit do we earn if we invest $\$2{,}200$ for two years in an investment that guarantees an equivalent effective interest rate of 3%? We have

$$P_{\text{final}} = P(1+r)^T = \$2{,}200(1+0.03)^2 = \$2{,}330$$

   and so the profit is

$$P_{\text{final}} - P = \$2{,}330 - \$2{,}200 = \$130.$$

1.5.2 Internal rate of return

When considered the performance of an investment that generates a cash-flow stream with payments at multiple instances of time, we need a measure of the annualised rate of return that takes into account all the payments. This is the **internal rate of return** (IRR).

Suppose that you invest amount P and after T years you get back amount P_{final}. Recall that the **equivalent effective interest rate** of this investment satisfies

$$P_{\text{final}} = P(1+r)^T \quad \Leftrightarrow \quad P = \frac{P_{\text{final}}}{(1+r)^T}.$$

More generally, suppose the initial investment P generates the cash-flow stream $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ with payments at times $t_1, t_2, \ldots, t_n$. Then we define the **internal rate of return** to be the number $r \in (-1, \infty)$ such that

$$P = \sum_{i=1}^{n} \frac{a_i}{(1+r)^{t_i}},$$

that is, the solution of the equation $f(r) = 0$, for the function

$$f(r) = -P + \sum_{i=1}^{n} a_i (1+r)^{-t_i}.$$

The internal rate of return is the 'hypothetical' effective interest rate that makes the present value of the cash-flow stream equal to the principal of the investment. In other words, it is the effective interest rate that a bank would need to offer in order to replicate the payment stream of the investment. An investment can be considered to be 'good' if the IRR exceeds the effective interest rate offered in the market, and 'bad' if the IRR is less than the effective interest rate offered in the market.

The definition of the IRR is only meaningful if the equation $f(r) = 0$ has a unique solution $r \in (-1, \infty)$. We next proves this to be true (at least, for positive cash-flow streams):

Theorem 1.25. *Suppose $P > 0$ and $a_i, t_i > 0$ for $i = 1, \ldots, n$. Then the equation*

$$f(r) = 0 \quad \Leftrightarrow \quad P = \sum_{i=1}^{n} a_i (1+r)^{-t_i} \tag{6}$$

has a unique solution in $(-1, \infty)$.

Proof. Observe the following properties of the function f on $(-1, \infty)$:

- f is strictly decreasing, since

$$f'(r) = \sum_{i=1}^{n} a_i \frac{-t_i}{(1+r)^{t_i+1}} < 0.$$

- $\lim_{r\to -1} f(r) = +\infty$ since

$$\lim_{r\to -1} f(r) = \lim_{r\to -1} \Big(-P + \sum_{i=1}^{n} a_i (1+r)^{-t_i} \Big) = -P + \sum_{i=1}^{n} a_i \lim_{r\to -1} (1+r)^{-t_i} = +\infty.$$

- $\lim_{r\to +\infty} f(r) = -P < 0$ since

$$\lim_{r\to +\infty} f(r) = \lim_{r\to +\infty} \Big(-P + \sum_{i=1}^{n} a_i (1+r)^{-t_i} \Big) = -P + \sum_{i=1}^{n} a_i \lim_{r\to +\infty} (1+r)^{-t_i} = -P.$$

Since f is continuous on $(-1, \infty)$ and strictly decreasing from $+\infty$ to $-P < 0$, f attains the value 0 at exactly one point in $r \in (-1, \infty)$. □

Remark 1.26. One by-product of the proof of Theorem 1.25 is that the present value of a cash-flow stream is a **decreasing** function of the interest rate r. We will study this fact in more detail in the next chapter.

In general calculating the IRR is difficult since it requires solving equation (6), which is a polynomial equation of degree n for which no closed solutions are available in general.[5] However the case $n = 2$ can be solved easily.

Example 1.27. Suppose that an investor bought shares in a company for £$1{,}000$ at the beginning of 2015, and sold them at the beginning of 2017 for £$1{,}050$. At the end of 2015 and 2016 the investor also received dividends (i.e. cash payments) of £10 and £20, respectively.

(a) What is the IRR of this investment?

(b) What is the IRR of this investment if the investor had to pay 10% tax on the dividends and 18% capital gains tax on the profit from the increase in share price?

(c) Suppose the nominal interest rate is 3.5%. By comparing the IRR to the nominal interest rate, judge whether the purchase of these shares was a good investment.

Solution.

(a) The IRR is the solution in $r \in (-1, \infty)$ of the equation

$$1{,}000 = \frac{10}{(1+r)} + \frac{20}{(1+r)^2} + \frac{1{,}050}{(1+r)^2} = \frac{10}{(1+r)} + \frac{1{,}070}{(1+r)^2}.$$

Rewriting this, we have

$$1{,}000(1+r)^2 - 10(1+r) - 1{,}070 = 0,$$

[5] Instead such equations must be solved numerically using trial and error, or sophisticated variations of this.

which is a quadratic equation in $(1+r)$ with solutions

$$1.0394 \quad \text{and} \quad -1.0294.$$

This implies that the solutions r are

$$0.0394 \quad \text{and} \quad -2.0294 \ \text{(which we reject, since } r > -1\text{)}.$$

Thus the IRR of this investment is $r = 3.94\%$.

(b) Since the two dividend payments were taxed at 10%, the investor received only $10-1 = £9$ at the end of 2015 and $20-2 = £18$ at the end of 2016. As the investor made a profit of $£1{,}050 - £1{,}000 = £50$ after selling the shares, they also had to pay $0.18 \cdot 50 = £9$ capital gains tax. This reduces the investor's return to $£1{,}050 - £9 = £1{,}041$.

The IRR in this case is the solution in $r \in (-1, \infty)$ of the equation

$$1{,}000 = \frac{9}{(1+r)} + \frac{18 + 1{,}041}{(1+r)^2}.$$

Again, this is a quadratic equation in r with solutions

$$0.0335 \quad \text{and} \quad -2.0245 \ \text{(which we reject, since } r > -1\text{)}.$$

Thus, in this case the IRR of this investment is $r = 3.35\%$. Note that this is smaller than the rate of return calculated in (a), which reflects the fact that the tax reduced the profitability of the investment.

(c) Ignoring taxes (as in part (a)), the IRR exceeded the nominal interest rate, so the purchase of the shares was a good investment. However, taking into account taxes, the IRR was lower than the nominal interest rate (as in part (b)), so overall the purchase of the shares was not a good investment. □

2 Immunisation of assets and liabilities

A company that expects to receive payments in the future (or needs to make payments in the future) is exposed to fluctuations in the interest rate. In this chapter we explore how the cash-flow of a company can be structured in a way that minimises this risk – this is referred to as the **immunisation** of assets and liabilities.

Manipulating a cash-flow stream so as to minimise exposure to fluctuations in financial markets more generally is called **risk management**, and is the focus of **Mathematical Tools for Asset Management**. The action of entering into transactions for the purpose of managing risk is referred to as **hedging**.

2.1 Measuring the effect of varying interest rates

Let us first consider how the present value of a cash-flow stream varies when the interest rate r varies. Let r denote the current interest rate (which may or may not be compounded), and let $D(T)$ be the discount factor to T years (sometimes we write $D(T;r)$ instead of $D(T)$ to stress the dependence on r). Suppose that the cash-flow stream consists of a sequence of payments C_i transacted at times t_i. The present value $V(r)$ of this cash-flow stream is then

$$V(r) = \sum_{i=1}^{m} C_i D(t_i; r).$$

We have already observed in Chapter 1.5.2 that $V(r)$ is a decreasing function of the interest rate r (at least, if all payments C_i are positive). We can analyse how rapidly $V(r)$ decreases with r by considering derivatives of $V(r)$.

First-order properties. A simple way to quantify how much $V(r)$ decreases with r is to compute $V'(r)$. For technical reasons explained below, practitioners prefer to use the following slightly different quantity:

Definition 2.1. If $V(r) > 0$, the **effective duration** of the cash-flow stream is defined as

$$\nu = -\frac{V'(r)}{V(r)}$$

Remarks 2.2. Let us explain why ν is preferred to $V'(r)$:

1. The derivative $V'(r)$ is measured in the same monetary units (e.g. pounds, dollars, pence etc.) as the present value $V(r)$. By dividing $V'(r)$ by $V(r)$, we standardise $V'(r)$ to remove the monetary units. In fact, the units in which ν is measured is **years** (why?).

2. Since $V(r)$ is decreasing in r for a positive cash-flow stream, the minus sign is included to ensure that ν is positive for such a stream (although it can be negative if the cash-flow stream has negative payments).

Why is the effective duration called a 'duration'? To answer this, let us first define a second quantity, called simply the 'duration':

Definition 2.3. If $V(r) > 0$, the **duration** of the cash-flow stream is defined as

$$\tau = \frac{\sum_{i=1}^{m} t_i C_i D(t_i)}{V(r)} = \frac{\sum_{i=1}^{m} t_i C_i D(t_i)}{\sum_{i=1}^{m} C_i D(t_i)}.$$

In the case of positive payments $C_i > 0$, τ can be considered a weighted average of the times t_i

$$\tau = \frac{\omega_1 t_1 + \cdots + \omega_m t_m}{\omega_1 + \cdots + \omega_m}$$

where the weights $\omega_i = C_i D(t_i) > 0$ are the present value of the payments.

What is the relationship between the effective duration ν and the duration τ? In the case of continuously-compounded interest, it turns out that they are **exactly the same thing**.

Proposition 2.4. *If interest is continuously-compounded, then the effective duration is equal to the duration, that is, $\nu = \tau$.*

Proof. The discount factors are $D(t_i) = e^{-rt_i}$. Calculating

$$V(r) = \sum_{i=1}^{m} C_i \mathrm{e}^{-rt_i} \qquad \text{and} \qquad V'(r) = \sum_{i=1}^{m} -t_i C_i \mathrm{e}^{-rt_i},$$

we have

$$\nu = -\frac{V'(r)}{V(r)} = -\frac{\sum_{i=1}^{m} -t_i C_i \mathrm{e}^{-rt_i}}{V(r)} = \frac{\sum_{i=1}^{m} t_i C_i D(t_i)}{V(r)} = \tau. \qquad \square$$

More generally, the duration is equal to the effective duration up to a constant.

Proposition 2.5. *There is a constant c_0, depending on the interest rate but not on the cash-flow stream, such that*

$$\nu = c_0 \tau.$$

Remark 2.6. As we have seen, in the case of continuously-compounded interest the constant $c_0 = 1$. The value of c_0 in the general case is more complex (and not very important), but if payments are made yearly and interest is compounded n times a year then (prove this!)

$$\nu = (1 + r/n)^{-1} \tau.$$

The equivalence between the effective duration and the duration leads to the following useful general principle:

> If cash-flow stream **A** has a larger duration than cash-flow stream **B**, then the present value of **A** decreases in value faster than the present value of **B** when the interest rate increases.

Remark 2.7. Whether it is 'better' to have a cash-flow stream with large duration depends on the context, in particular whether the cash-flow stream consists of assets or liabilities and whether the interest rate increases or decreases.

Example 2.8. You expect to be paid $£1{,}000$ at the end of this year and the end of the following year, and need to make a payment of $£2{,}215$ in three years' time. The interest rate is currently 7% and is compounded yearly.

(a) Verify that the present value of your assets (incoming payments) and your liability (outgoing payment) are equal.

(b) If the interest rate were to increase to 8%, would you be in a better or worse financial position (in terms of the net present value of assets and liabilities)?

Solution.

(a) The discount factor to one year is $\beta = 1/1.07$. The present value of the assets is

$$1{,}000\beta + 1{,}000\beta^2 = 1{,}808,$$

whereas the present value of the liability is

$$2{,}215\beta^3 = 1{,}808.$$

(b) The present value of both the assets and liability will decrease if the interest rate increases from 7% to 8%, so whether you are in a better financial position depends on which decreases more steeply. This can be answered by comparing durations.

Calculating the durations explicitly gives

$$\tau(A) = \frac{1 \times 1000\beta + 2 \times 1000\beta^2}{1808} = 1.483$$

and

$$\tau(L) = \frac{3 \times 2215\beta^3}{2215\beta^3} = 3$$

Hence the duration of the liability is larger than the duration of the assets, meaning that the present value of the liabilities will decrease more than the present value of the assets, so the increase in interest rates would put you in a **better** financial position.

Note that we actually didn't have to do the calculations explicitly: as a weighted average of payment times, the duration of the assets **must be between** 1 **and** 2, whereas the duration of the liability is equal to 3. □

Second-order properties. We can refine our analysis of how $V(r)$ varies with r by considering $V''(r)$. As for the effective duration, it is more natural to considered the normalised version $V''(r)/V(r)$, which is called the convexity.

Definition 2.9. If $V(r) > 0$, the **convexity** of the cash-flow stream is defined as

$$c = \frac{V''(r)}{V(r)}.$$

Proposition 2.10. *If interest is continuously-compounded, then the convexity is*

$$c = \frac{\sum_{i=1}^m t_i^2 C_i D(t_i)}{V(r)} = \frac{\sum_{i=1}^m t_i^2 C_i e^{-rt_i}}{\sum_{i=1}^m C_i e^{-rt_i}}. \tag{7}$$

Again, in the case of positive payments $C_i > 0$, the convexity c can be interpreted as a 'weighted average' of the squared payment times $t_1^2, \ldots, t_n^2$. However, this does not have a direct financial meaning.

Remark 2.11. In general, the relationship between the convexity c and the weighted average in (7) is more complex. For instance, if interest is compounded n times a year then

$$c = \frac{\sum_{i=1}^m t_i(t_i + 1/n) C_i D(t_i)}{V(r)} \times (1 + r/n)^{-2}.$$

2.2 Reddington immunisation

We now discuss how a company can minimise their exposure to fluctuations in the interest rate. Let the current interest rate be $r = r_0$. We assume that a company has a stream of positive future cash-flows (i.e. incoming payments), called assets, whose present value is $V_A(r_0)$, and a stream of negative future cash-flow (i.e. outgoing payments), called liabilities, with present value $-V_L(r_0)$. The present value of the combined cash-flow stream is therefore

$$V(r_0) = V_A(r_0) - V_L(r_0).$$

The 'immunisation' of these cash-flow streams refers to a sequence of three conditions that V_A and V_L need to satisfy in order to minimise expose to fluctuating interest rates; if all three conditions are satisfied the cash-flows are said to be **Reddington immune**.

Zero-order immunisation: Matching assets and liabilities Zero-order immunisation simply means that the present value of assets is matched by the present value of liabilities:

Definition 2.12. A cash-flow stream is said to be **zero-order immune** if the present value of assets and liabilities is the same, i.e. $V_A(r_0) = V_L(r_0)$.

Note that zero-order immunisation can in general only be achieved for the present value of the interest rate $r = r_0$, and will fail as soon as r differs from r_0.

First-order immunisation: Matching the effective durations of assets and liabilities Now assume that the cash-flow is zero-order immune, and consider the effect of a change in the interest rate to $r = r_0 + \varepsilon$, for ε a small number. As we previously observed, in general $V_A(r_0 + \varepsilon) \neq V_L(r + \varepsilon)$, and we can use Taylor expansions to approximate the difference:

$$V_A(r_0 + \varepsilon) \approx V_A(r_0) + V_A'(r_0)\varepsilon = V_A(r_0) - V_A(r_0)\nu_A\varepsilon = V_A(r_0)\left(1 - \nu_A\varepsilon\right) \quad (8)$$

$$V_L(r_0 + \varepsilon) \approx V_L(r_0) + V_L'(r_0)\varepsilon = V_L(r_0) - V_L(r_0)\nu_L\varepsilon = V_L(r_0)\left(1 - \nu_L\varepsilon\right) \quad (9)$$

where $\nu_A = -V_A'(r_0)/V_A(r_0)$ and $\nu_L = -V_L'(r_0)/V_L(r_0)$ are the effective durations of the cash-flow streams corresponding to the assets and liabilities. Hence

$$V(r_0 + \varepsilon) = V_A(r_0 + \varepsilon) - V_L(r + \varepsilon) \approx V_A(r_0)\left(1 - \nu_A\varepsilon\right) - V_L(r_0)\left(1 - \nu_L\varepsilon\right),$$

which in the case of zero-order immunisation, simplifies to

$$V(r_0 + \varepsilon) \approx V_A(r_0)\left(\nu_L - \nu_A\right)\varepsilon.$$

This quantity is zero if and only if the effective durations are equal, i.e. $\nu_A = \nu_L$. This is equivalent to the durations being equal, i.e. $\tau_A = \tau_B$.

Definition 2.13. A cash-flow stream is said to be **first-order immune** if (i) it is immune to zero-order, i.e. $V_A(r_0) = V_L(r_0)$, and (ii) if the effective durations (equiv. durations) of the cash-flow streams corresponding to the assets and liabilities are equal, i.e. if $\nu_A = \nu_B$ (equiv. $\tau_A = \tau_B$).

If a cash-flow stream is first-order immune, then the value of $V(r_0 + \varepsilon)$ is constant (up to small errors) for small ϵ. The relevant errors are very small since they correspond to terms of order ε^2 or higher, e.g., if $\varepsilon = 0.1\% = 0.001$ then $\varepsilon^2 = 0.000001$.

Reddington immunisation: Convexity We can use the notion of convexity to refine the approximations in (8) and (9):

$$
\begin{aligned}
V_A(r_0+\varepsilon) &\approx V_A(r_0) + V_A'(r_0)\varepsilon + \frac{1}{2}V_A''(r_0)\varepsilon^2 = V_A(r_0) - V_A(r_0)\nu_A\varepsilon + \frac{1}{2}V_A(r_0)c_A\varepsilon^2 \\
&= V_A(r_0)\Big(1 - \nu_A\varepsilon + \frac{1}{2}c_A\varepsilon^2\Big) \\
V_L(r_0+\varepsilon) &\approx V_L(r_0) + V_L'(r_0)\varepsilon + \frac{1}{2}V_L''(r_0)\varepsilon^2 = V_L(r_0) - V_L(r_0)\nu_L\varepsilon + \frac{1}{2}V_L(r_0)c_L\varepsilon^2 \\
&= V_L(r_0)\Big(1 - \nu_L\varepsilon + \frac{1}{2}c_L\varepsilon^2\Big)
\end{aligned}
$$

where $c_A = V_A''(r_0)/V_A(r_0)$ and $c_L = V_L''(r_0)/V_L(r_0)$ are the convexity of the cash-flow streams corresponding to the assets and liabilities respectively. Hence in the case of first-order immunisation,

$$
\begin{aligned}
V(r_0+\varepsilon) &\approx V_A(r_0)\Big(1 - \nu_A\varepsilon + \frac{1}{2}c_A\varepsilon^2\Big) - V_L(r_0)\Big(1 - \nu_L\varepsilon + \frac{1}{2}c_L\varepsilon^2\Big) \\
&= V_A(r_0)\frac{1}{2}(c_A - c_L)\varepsilon^2
\end{aligned}
$$

Clearly it would be advantageous to demand that $c_A = c_L$ so that $V(r_0+\varepsilon)$ was constant for small ε up to very small errors (of order ε^3). In practice, it is often sufficient to demand that $c_A \geq c_L$, which ensures that, up to very small errors, $V(r_0+\varepsilon) \geq V(r_0)$.

Definition 2.14. A cash-flow stream is Reddington immune if it is first-order immune, i.e.

$$V_A(r_0) = V_L(r_0) \qquad \text{and} \quad \nu_A(r_0) = \nu_L(r_0),$$

and if, in addition, the convexities of the assets and liabilities satisfy $c_A \geq c_L$.

If a cash-flow stream is Reddington immune, up to very small errors any change in the interest rates will increase $V(r)$ (or, at least, $V(r)$ will not decrease)

2.3 Immunisation in practice

Let us see how immunisation might be applied in practice. For this we will make use of **zero-coupon bonds**, which are simply assets that pay a fixed amount (say £1) at the end of a fixed number of years (bonds will be discussed in more detail in the next chapter).

Example 2.15. An investor has a liability of £20,000 to be paid in 4 years and another of £18,000 to be paid in 6 years. Suppose the interest rate is $r = 8\%$ and is continuously-compounded.

(a) Show that Reddington immunisation can be achieved by owning a combination of 2-year and 7-year zero-coupon bonds.

(b) Explain why first-order immunisation cannot be achieved if the fund owns a combination of 2-year and 3-year zero-coupon bonds.

Solution. (a) The yearly discount factor is $\beta = e^{-0.08} = 0.923$. Suppose we own 2-year bonds paying £P and 7-year bonds paying £Q. Zero-order immunisation requires that

$$P\beta^2 + Q\beta^7 = 20{,}000\beta^4 + 18{,}000\beta^6 = 25{,}661.$$

Moreover, first-order immunisation (i.e. $\tau_A = \tau_L$) requires that

$$2P\beta^2 + 7Q\beta^7 = 4 \times 20{,}000\beta^4 + 6 \times 18{,}000\beta^6 = 124{,}900.$$

Solving the two equations for P and Q gives the solution

$$P = 12{,}840, \quad Q = 25{,}770.$$

To determine whether Reddington immunisation is achieved for this solution, we calculate the convexities. Since interest is continuously-compounded, we have

$$c_A = \frac{\sum_{i=1}^n t_i^2 C_i \beta^{t_i}}{V_A(r)} = \frac{2^2 P\beta^2 + 7^2 Q\beta^7}{25{,}661} = 29.8$$

and

$$c_L = \frac{\sum_{i=1}^n t_i^2 C_i \beta^{t_i}}{V_L(r)} = \frac{4^2 \times 20{,}000\beta^4 + 6^2 \times 18{,}000\beta^6}{25{,}661} = 24.7.$$

Since $c_A > c_L$, Reddington immunisation is achieved.

(b) The duration of a combination of 2-year and 3-year bonds is at most 3 years, which cannot be equal to the duration of the liabilities (which is at least 4 years), so in this case first-order immunisation is impossible. □

2.4 Immunisation when interest is compounded yearly

If interest is compounded yearly at rate r, then the present value of a payment stream of payments C_i made at times t_i for $i = 1, 2, \ldots, n$ is

$$V(r) = \sum_{i=1}^n \frac{C_i}{(1+r)^{t_i}}.$$

Exactly the same definitions of ν and c can be made for this present value function as for continuously compounded interest and the same immunisation conditions apply. However, now

$$V'(r) = -\sum_{i=1}^n \frac{t_i C_i}{(1+r)^{t_i+1}}$$

and

$$V'(r) = \sum_{i=1}^n \frac{t_i(t_i+1)C_i}{(1+r)^{t_i+2}}.$$

With β defined as $\beta = 1/(1+r)$,

$$V(r) = \sum_{i=1}^n C_i \beta^{t_i},$$

$$V'(r) = -\sum_{i=1}^n t_i C_i \beta^{t_i+1},$$

and

$$V''(r) = \sum_{i=1}^n t_i(t_i+1) C_i \beta^{t_i+2},$$

The duration is

$$\tau = \frac{\sum_{i=1}^{n} t_i C_i \beta^{t_i}}{\sum_{i=1}^{n} C_i \beta^{t_i}}$$

while the effective duration is

$$\nu = -\frac{V'(r)}{V(r)} = \frac{\sum_{i=1}^{n} t_i C_i \beta^{t_i+1}}{\sum_{i=1}^{n} C_i \beta^{t_i}}.$$

As opposed to the case for continuously compounded interest, $\nu = \beta\tau$.

Example 2.16. A fund must make paments of $£50,000$ at the end of the sixth and eigth year from now. Show that immunisation can be acheived for 7% interest with a comination of 5-year zero-coupon bonds and 10-year zero-coupond bonds.

Solution. Suppose we purchase 5-year bonds paying P and 10-year bonds paying Q. Then, with $\beta = (1.07)^{-1}$,

$$V_A(0.07) = P\beta^5 + Q\beta^{10}.$$

We also have

$$V_L(0.07) = 50,000(\beta^6 + \beta^8) = 62,418$$

and so zero order immunisation implies that

$$P\beta^5 + Q\beta^{10} = 62,418. \tag{10}$$

The effective duration of the assets is

$$\nu_A = \frac{5P\beta^6 + 10Q\beta^{11}}{62418}$$

while the effective duration fo the liabilities is

$$\nu_L = \frac{50,000(6\beta^7 + 8\beta^9)}{62418} = \frac{404398}{62418}.$$

Setting $\nu_A = \nu_L$ gives

$$5P\beta^6 + 10Q\beta^{11} = 404398. \tag{11}$$

Solving (10) and (11) for P and Q gives

$$P = £53,710, \quad Q = £47,454. \tag{12}$$

We must still show that the third immunisation condition is satisfied. The convexity of the assets is

$$c_A = \frac{V_A''(0.07)}{V_A(0.07)} = \frac{(5)(6)P\beta^7 + (10)(11)Q\beta^{12}}{62418} = 53.21$$

while the convexity of the liabilities is

$$c_L = \frac{V_L''(0.07)}{V_L(0.07)} = \frac{50000((6)(7)\beta^8 + (8)(9)\beta^{10})}{62418} = 48.90.$$

Since $c_A \geq c_L$, the third condition is satisfied and (12) gives immunisation. □

3 Bonds and the term structure of interest rates

Bonds are an example of a **debt** or **fixed-income** security – these are tradable financial instruments whose cash-flow is fixed or predetermined. Other examples include bank deposits, loans, interest rate futures, interest rate swaps, swaptions, caps, floors, etc. (often credit derivatives are also included, e.g., CDSs (credit default swaps), CDOs (collateralised debt obligations), convertible bonds, etc., since their structure is similar). Note that, although the cash-flow is predetermined, the **present value** of a fixed-income security can fluctuate, since their value also depends on the prevailing interest rates.

Fixed-income securities can be contrasted with **equity**, such as shares and equity indices. These are securities whose cash-flow is **not** pre-determined. Other financial instruments of this type include FX (foreign exchange) and commodities, and collectively they are sometimes known as **variable-income** securities.

In this chapter we will use present value analysis and no-arbitrage arguments to assign a **fair price** to bonds. We also consider the **term structure** of bonds, which allow us to define a generalised notion of interest rates that depend, not just on the present time, but also on the period of time in which money is invested. Using this idea, we will see that all bond prices and all interest rates can be expressed in terms of the price of zero-coupon bonds.

3.1 Bonds

A **bond** is a contract issued by a government or a company in order to raise capital. Investors that purchase a bond are promised a stream of fixed payments known as **coupons**, plus an extra payment at the bond expiry, known as the **redemption**.

A bond is described by:

- An **issue price**, which is the amount the investor pays for the bond. It is sometimes determined by an auction.

- Its **face value**, also called the principal. This will determine the final redemption payment, and for some bonds will also determine the coupon values.

- The **expiry** (or 'maturity') of the bond. This is the date of the last coupon payment and/or when the investor receives the redemption payment.

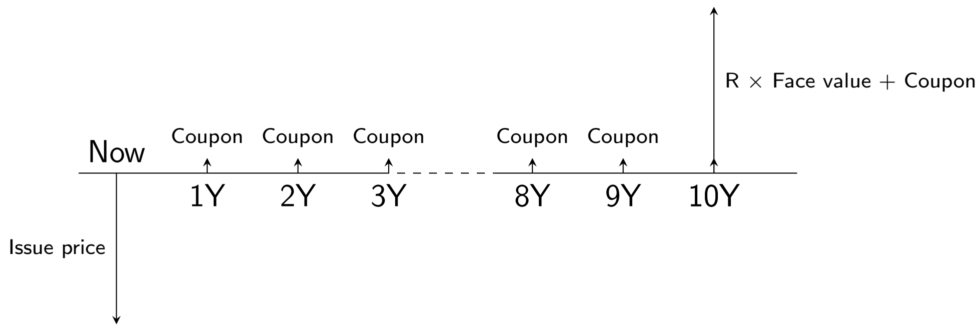
- The **coupon value and frequency**, which expresses the value and frequency of the coupon payments. The value is sometimes expressed as a fixed amount, and sometimes as a (annualised) **coupon rate**, which is a percentage of the face value.

  If there are m coupon payments in a year then each of them will be equal to the annual coupon divided by m. A **zero-coupon bond** is a bond with no coupon payments.

  Note that the coupon rate is in general different to the interest rate.

- The **redemption value**, which expresses the value of the redemption payment at expiry. This is again sometimes expressed as a fixed amount, and sometimes as a fraction R of the face value. If $R = 1$, then the bond is said to be 'redeemable at par', if $R > 1$, 'redeemable above par', and if $R < 1$, 'redeemable below par'. Most bonds are redeemable at par.

The cash-flow stream generated by a typical bond might be as follows:[6]



Examples 3.1.

1. A 5-year bond with a face value of $£100{,}000{,}000$ and an annual coupon rate of 3% will pay 5 annual coupon payments of $£3{,}000{,}000$. Additionally, if the the bond is redeemable at par ($R = 1$), at expiry the investor will be paid back the face value of the bond: \$100,000,000.

2. What coupons are paid by a 10-year bond with face value \$150,000,000 and semi-annual coupon paid at an annual rate of 2%? The bond will pay 20 coupon payments every half-year for 10 years, each worth

$$\$150{,}000{,}000 \times 2\% \times \frac{1}{2} = \$1{,}500{,}000.$$

3.1.1 The present value of a bond

The **present value of a bond** is simply equal to the present value of the cash-flow stream that the bond generates. By default we do **not** include the issue price in the cash-flow stream, unless specifically included (immediately after purchase, the initial payment is no longer relevant to the bond value).

Example 3.2. Assuming an interest rate of r, the present value of a zero-coupon bond with face value 1 that is redeemable at par in T years is equal to

$$\frac{1}{(1+r)^T}.$$

Example 3.3. Assuming an interest rate of r, the present value of a bond maturing in T years with yearly coupon payments of C, issue price P_0, face value P, and redemption price RP is

$$PV = \frac{1}{1+r}C + \frac{1}{(1+r)^2}C + \cdots + \frac{1}{(1+r)^T}C + \frac{1}{(1+r)^T}RP. \tag{13}$$

Example 3.4. Suppose the interest rate is 1.5%. The present value of a 3-year bond with face value £100,000, redeemable at par, with annual coupon payments of 1.7% is

$$\begin{aligned} PV &= \frac{0.017 \times 100{,}000}{1+0.015} + \frac{0.017 \times 100{,}000}{(1+0.015)^2} + \frac{0.017 \times 100{,}000}{(1+0.015)^3} + \frac{100{,}000}{(1+0.015)^3} \\ &= 1674.88 + 1650.12 + 1625.74 + 95631.70 \\ &= £100{,}582\,. \end{aligned}$$

Note that in this example the interest rate 1.5% is different to the coupon rate 1.7%.

[6]This picture reflects the cash-flow stream for the buyer: the buyer initially pays the issue price (negative, downwards arrow), and then receives (positive, upwards arrow) a stream of payments. The picture for the bond issuer would be the opposite: it starts with receiving the issue price, and continues with a stream of outgoing payments.

Recall the definition of **discount factors** from Section 1.4.2; these allow us to express the present value of a general bond:

Example 3.5. Let $D(T)$ be the discount factor for a payment after T years. Then the present value of a bond that pays coupons $C_1, \ldots, C_n$ at times $T_1, \ldots, T_n$ and is redeemable at value RP at the expiry time T_n, is equal to

$$PV = D(T_1) \cdot C_1 + D(T_2) \cdot C_2 + \cdots + D(T_n) \cdot C_n + D(T_n) \cdot RP. \tag{14}$$

3.1.2 The no-arbitrage principle and the fair price of a bond

How should we assign a fair price to a bond? To answer this, modern finance takes as its starting point the assumption that there are no arbitrage opportunities in the market. This is sometimes called the **no-arbitrage principle**, or the 'no-free-lunch principle'.

The definition of an arbitrage opportunity (or simply an 'arbitrage') is the following:

Definition 3.6. An **arbitrage** is an investment that guarantees a **risk-free profit** (relative to discounted values). More precisely, it is the creation of a portfolio of assets and liabilities worth $P(0) = 0$ at time 0 such that, at some time T in the future:

- The portfolio has a non-negative value $P(T)$ **under all possible outcomes**, i.e.

$$P(T) \geq 0 \quad \text{in all possible states of the world at time } T;$$

- There is **at least one outcome** under which the portfolio has a **strictly** positive value, i.e.

$$P(T) > 0 \quad \text{in at least one possible state of the world at time } T.$$

Remark 3.7. Equivalently, we could describe arbitrage in terms of a portfolio worth $P(0)$ at time 0, and then ask whether

$$P(T) \geq P(0)D(T)^{-1}$$

in all states of the world. Since $P(0)D(T)^{-1}$ is the amount that a bank account with $P(0)$ invested in it is worth at time T, arbitrage is essentially the condition that the **investment is guaranteed to outperform the risk-free rate**.

Definition 3.8. The **no-arbitrage principle** is the assumption, made about a financial market, that no arbitrage opportunities exist.

The justification for the no-arbitrage principle is that, in an idealised version of the market, the buying and selling activity of market participants will eliminate all arbitrage opportunities. This is since, the moment an arbitrage opportunity arises, there will be such demand for the relevant assets that it will increase their prices, eliminating the arbitrage opportunity.

Note that the no-arbitrage principle may not be reflected in real-life financial markets, especially those that are **illiquid** (meaning, in which trades are infrequent).

Along with the no-arbitrage principal, we will also make a series of related assumptions, namely that participants in the market may **freely create, buy, and sell assets without restriction or cost** (the so-called 'frictionless market' hypothesis).

It turns out that the no-arbitrage principle (combined with the 'frictionless market' hypothesis) is an extremely powerful tool with which to assign prices to financial instruments, such as bonds; this is known as 'arbitrage-free pricing' or 'rational pricing'. We will refer to the resulting price as the 'no-arbitrage' price, or sometimes simply the 'fair price'.

We begin with the **the law of one price**, a general expression of the no-arbitrage pricing principle:

Theorem 3.9 (Law of one price)**.** *Under the no-arbitrage assumption, if two financial instruments generate the same cash-flow streams then they have the same current price.*

In particular, the current price of a financial instrument that generates a fixed cash-flow stream is equal to the present value of this cash-flow stream.

Corollary 3.10. *The no-arbitrage price of a bond is equal to its present value.*

Proof. Suppose that financial instruments A and B generate the same cash-flow streams, and are currently priced at P_A and P_B respectively. Assume for the sake of contradiction that

$$P_A > P_B;$$

we will show that this creates an arbitrage opportunity. Consider selling instrument A and simultaneously buying instrument B, making a profit of

$$P_A - P_B > 0.$$

Since you own instrument B you will receive its associated cash-flow stream; since you sold instrument A, you need to pay the buyer the associated cash-flow stream. As these cash-flow streams are equal, the former can be used to pay off the latter, leaving you with a positive profit at the expiry of the cash-flow streams. This is an arbitrage opportunity, so $P_A > P_B$ must be false. Reversing the roles of A and B, we conclude that

$$P_A = P_B.$$

Now suppose we have an instrument that generates a fixed cash-flow stream. We already saw that if the present value of the cash-flow stream PV is placed in a bank account, it can be used to exactly replicate the cash-flow stream. Hence, by the law of one price, the price of the instrument equals the current value of the bank account, which is just the present value. $\square$

Two financial instruments that generate the same cash-flow streams are sometimes called **replicating portfolios**. Indeed, the law of one price is just a formal version of the 'replication' arguments we saw in previous chapters.

One consequence of the law of one price is that **the price of any bond can be expressed in terms of the price of unit zero-coupon bonds** (i.e. zero-coupon bonds that pay one unit of cash at expiry). This is since, by the law of one price, the price V_T of a unit zero-coupon bond with expiry T years is equal to the discount factor to T, that is,

$$V_T = D(T).$$

Then we can use formula (14), which expresses the present value, or equivalently the no-arbitrage price, of any bond in terms of the discount factors.

Example 3.11. Calculate the no-arbitrage price of a 4-year bond with face value $£200$, redeemable at par, with 4% annual coupons, if the price of unit zero-coupon bonds, with respective expiries $T = 1, 2, 3, 4$, are

$$(0.9, 0.72, 0.65, 0.53)\,.$$

Solution. By (14) the present value, or equivalently the no-arbitrage price, of the bond is

$$200 \times 4\% \times \sum_{i=1}^{4} D(i) + 200 \times D(4),$$

where $D(T)$ is the discount factor to time T, or equivalently the price V_T of a unit T-year ZCB. Inserting the values of V_T into the formula yields $£128$. $\square$

More generally, the no-arbitrage principle can also be used to fix relationships between different financial quantities that are not strictly speaking financial instruments, such as interest rates. We illustrate this with an example:

Example 3.12. Suppose that bank **A** offers us a monthly compounded interest rate r_{1M}, and bank **B** offers a yearly compounded interest rate r_{1Y}. Show that that the no-arbitrage principle requires that

$$1 + r_{1Y} = \left(1 + \frac{1}{12} r_{1M}\right)^{12}.$$

First Solution. Assume for the sake of contradiction that

$$1 + r_{1Y} < \left(1 + \frac{1}{12} r_{1M}\right)^{12} \qquad (15)$$

(the reverse inequality is treated similarly). Then suppose we took out a loan of $£1$ in bank **B**, which would require us to repay $(1 + r_{1Y})$ in one year's time. We could then invest this money in bank **A**. At the end of the year the account value would grow to $(1 + \frac{1}{12} r_{1M})^{12}$, which after paying back the loan gives us a profit of

$$\left(1 + \frac{1}{12} r_{1M}\right)^{12} - \left(1 + r_{1Y}\right).$$

Since by assumption this is strictly positive, we have made a risk-free profit which contradicts the no-arbitrage principle. Therefore inequality (15) cannot hold. $\square$

Second Solution. Put $\frac{1}{\left(1+\frac{1}{12}r_{1M}\right)^{12}}$ in bank $\mathbf{A}$ and $\frac{1}{1+r_{1Y}}$ in bank $\mathbf{B}$. Both invest investments result in a unit pay off after 1 year. By The Law of One Price,

$$\frac{1}{\left(1 + \frac{1}{12} r_{1M}\right)^{12}} = \frac{1}{1 + r_{1Y}},$$

or

$$1 + r_{1Y} = \left(1 + \frac{1}{12} r_{1M}\right)^{12}.$$

$\square$

Exercise 3.13. Explain why, in the example above, the reverse inequality

$$1 + r_{1Y} > \left(1 + \frac{1}{12} r_{1M}\right)^{12}$$

also leads to arbitrage.

3.1.3 The bond yield

If the interest rate is unknown, we cannot compute the present value. However, if we know the issue price of a bond, we can use no-arbitrage theory to deduce the 'hypothetical' interest rate that would make the bond fairly price. This is called the **bond yield**, and is nothing more than the 'internal rate of return' of an investment in the bond.

Definition 3.14. Suppose that a bond has issue price P_0, coupon payments of $C_1, \dots, C_n$ at times $t_1, \dots, t_n$ and redemption payment RP at expiry T. Then the **bond yield** is the unique solution $r \in (-1, \infty)$ to

$$P_0 = \sum_{i=1}^{n} \frac{C_i}{(1+r)^{t_i}} + \frac{1}{(1+r)^T} RP. \tag{16}$$

Equation (16) has a unique solution for the same reason that the IRR is guaranteed to exist (see Theorem 1.25). Just like the IRR, the bond yield can be used to determine whether the bond is a good investment.

Example 3.15. A broker proposes to sell a 2-year bond with face value £100,000, redeemable at par, with annual coupon payments of 2%. The broker quotes an issue price of $P_0 = £102{,}000$. Find the yield of the bond, and decide whether the bond is a good investment if the interest rate is 10%.

Solution. The yield is the (unique) solution $r \in (-1, \infty)$ to the equation

$$102{,}000 = \frac{2{,}000}{1+r} + \frac{2{,}000}{(1+r)^2} + \frac{100{,}000}{(1+r)^2}.$$

This is a quadratic in the variable $x = 1 + r$; solving this yields

$$x = -0.9902 \quad \text{or} \quad 1.0098.$$

Since $x > 0$ we can disregard the first solution, and we are left with

$$r = x - 1 = 0.98\%. \qquad \square$$

Since this is far less than the interest rate, the bond is not a good investment.

As shown by the example, the yield does **not** equal the coupon rate in general.

3.2 The term structure of interest rates

So far it has been assumed that there is a single interest rate $r = r(t)$ at each instant of time, regardless of the type or duration of the deposit/loan. In practice, interest rates usually depend on the term (i.e. the duration) of the deposit/loan. The main reason for this is supply and demand: if many people are asking for 20-year loans, then the the interest rate for 20-year deposits will increase. Likewise, if there is a good supply of one month deposits the 1-month interest rate will go down. This variation is referred to the **term structure** of interest rates.

For example, the sterling LIBOR rates[7] on 17th September 2019 were:[8]

[7] LIBOR stands for London Interbank Offered Rate, and is a set of benchmark interest rates fixed every day at 11am London time. LIBOR is used by many institutions to set short-term interest rates; the total value of instruments fixed by LIBOR is estimated to be in the hundreds of trillions of pounds.

[8] Source: The Intercontinental Exchange Benchmark Administration Ltd https://www.theice.com/marketdata/reports/170. In this page you can consult current or past LIBOR rates for a number of currencies, e.g. GBP, Euros, Yen and USD.

Term	LIBOR
Overnight	0.67150%
1 Week	0.68825%
1 Month	0.71225%
2 Month	0.75775%
3 Month	0.78463%
6 Month	0.84050%
1 Year	0.96213%

The fact that interest rates are term-dependent is an additional complication when pricing fixed-rate financial instruments. In this section we explore how these rates can be analysed. In particular, we use the price of **unit zero-coupon bonds** to relate various interest rates, via the no-arbitrage principle.

3.2.1 Spot rates

The first step in analysing term-dependent interest rates is to convert them to **effective rates**, analogous to the 'effective rates' we defined in Section 1.1 for different compounding frequencies. The name for these effective rates, in the context of term-dependent interest rates, is **spot rates**.

By default, interest rates are quoted as (annualised) **nominal rates**, this is the case for the LIBOR rates quoted above. This means that if you place P in a deposit for term T, at interest rate r_T, you will get back

$$P(1 + r_T T).$$

A fairer way to compare term-dependent rates is to compare their spot rates:

Definition 3.16. The **spot rate** s_T, for maturity T is defined as

$$s_T = (1 + r_T T)^{1/T} - 1 \tag{17}$$

or, equivalently,

$$(1 + s_T)^T = 1 + r_T T.$$

The function s_T is called the **spot rate curve**.

Remarks 3.17. Equation (17) can be compared to (3), which was the formula that we used to define 'effective rates' for various compounding frequencies. Just as for effective rates, spot rates are also 'artificial', in the sense that they are not directly quoted on the market.

Examples 3.18.

1. The 1-year spot rate s_1 is equal the nominal rate r_1.

2. The 2-year spot rate s_2 is equal to rate such that, compounding twice, has the same effect as the nominal 2-year rate r_2, i.e.

$$(1 + s_2)^2 = 1 + r_2 \cdot 2.$$

3. The 6-month spot rate $s_{0.5}$ has the same effect as the 6-month nominal rate compounded twice, i.e.

$$1 + s_{0.5} = \left(1 + \frac{1}{2} r_{0.5}\right)^2 \quad \Leftrightarrow \quad (1 + s_{0.5})^{1/2} = 1 + r_{0.5} \cdot \frac{1}{2}.$$

Example 3.19. Let $r_{1/12} = 0.712\%$ be the 1-month LIBOR rate quoted above. Then the associated spot rate satisfies

$$1 + s_{1/12} = \left(1 + 0.712\% \frac{1}{12}\right)^{12} = 1.00714$$

which yields a spot rate of $s_T = 0.714\%$.

In the case of term-dependent interest rates, the discount factors $D(T)$ that we use to do present value analysis must be computed in terms of the nominal/spot rates

$$D(T) = (1 + r_T T)^{-1} = (1 + s_T)^{-T}.$$

Example 3.20. Calculate the no-arbitrage price of a five-year bond with face value $£100{,}000$, redeemable at par, with 6% annual coupons, if the spot rate curve is

$$(s_1, s_2, s_3, s_4, s_5) = (7\%, 7.25\%, 7.5\%, 7.75\%, 8\%).$$

Solution. We use the discount factor $D(i) = (1 + s_i)^{-i}$ in the formula (14). This yields a no-arbitrage price of

$$6{,}000 \sum_{i=1}^{5} D(i) + 100{,}000 D(5) = £92{,}247. \qquad \square$$

Another point of view is that, under the no-arbitrage assumption, spot rates are determined by the price of unit zero-coupon bonds V_T with expiry T. This is since

$$s_T = D(T)^{-1/T} - 1 = V_T^{-1/T} - 1. \tag{18}$$

Examples 3.21. The following table gives some prices for unit zero-coupon bonds with varying maturities, alongside the implied spot rate calculated via the formula (18).

T =Year	Price	Spot rate s_n
1	0.94	6.4%
5	0.70	7.4%
10	0.47	7.8%
15	0.30	8.4%

Remark 3.22. Observe that the spot rates s_T we defined play the same role as the effective rate r_{eff}, in the sense that they define an equivalent annualised interest rate. Some practitioners instead convert r_T to an instantaneous spot rate s'_T that is the equivalent continuously-compounded rate. This satisfies

$$e^{s'_T T} = 1 + r_T T \quad \Leftrightarrow \quad s'_T = \frac{1}{T} \ln(1 + r_T T).$$

3.2.2 Forward rates

How much interest should we expect to earn on money deposited or loaned between two dates in the future? Surprisingly, the fair (i.e. no-arbitrage) value of such a 'forward rate' is completely determined by current spot rates.

Why is this? Suppose I expect to have £1m in one year's time and that I wish to deposit it for a further year. How much interest might I expect to receive? Well consider instead taking the following sequence of actions:

$t = 0$ • Take out a 1-year loan for $(1+s_1)^{-1}\times$ £1m. Place this amount in a 2-year deposit.

$t = 1$ • Pay £1m that you expect to have at this time to pay off the loan. Think of this money as an investment that will receive a return at time $t = 2$.

$t = 2$ • The deposit will now be worth $(1 + s_2)^2(1 + s_1)^{-1}\times$£1m.

Under this procedure, the £1m we expect to have in one year's time will grow to £1m to $(1 + s_2)^2/(1 + s_1)\times$£1m at the end of the second year. This implies that the money has earned an 'effective' interest rate $f_{1,2}$ satisfying

$$1 + f_{1,2} = \frac{(1 + s_2)^2}{1 + s_1}.$$

Let us now formalise this using no-arbitrage principles.

Definition 3.23. A **forward rate agreement** (FRA) from time n_1 to time n_2 is a contract whereby a party agrees to place a certain amount (the principal) in a deposit or a loan at time n_1 for an effective interest rate of f, and the deposit/loan expires at time n_2. In particular, a deposit of P at time n_1 will be worth, at the maturity time n_2,

$$P(1 + f)^{n_2-n_1}.$$

A signatory to a FRA is **obliged** to deposit (or loan) the money at the start time n_1 (i.e. there is no optionality).

Intuitively the fair value of f in a FRA might be thought of as both parties best guess as to what the interest rate will be at time n_1. However this is incorrect, since as we argued before f is completely determined by **today's spot rates**.

Proposition 3.24. *The fair (i.e. no-arbitrage) interest rate for a FRA from time n_1 years to n_2 years is*

$$f_{n_1,n_2} = \left(\frac{(1 + s_{n_2})^{n_2}}{(1 + s_{n_1})^{n_1}}\right)^{\frac{1}{n_2-n_1}} - 1,$$

where s_{n_1} and s_{n_2} are the spot rates to n_1 and n_2 years. Equivalently,

$$(1 + f_{n_1,n_2})^{n_2-n_1} = \frac{(1 + s_{n_2})^{n_2}}{(1 + s_{n_1})^{n_1}}.$$

Proof. The proof consists of formalising the previous argument using the no-arbitrage principle. Suppose for the purposes of contradiction that a bank proposes a FRA with forward rate f that is larger than the forward rate f_{n_1,n_2}. Then at time 0 we can (i) sign this FRA for a deposit between times n_1 and n_2, (ii) simultaneously take out a loan with term n_2 of amount

$$(1 + s_{n_1})^{-n_1},$$

and (iii) deposit this amount immediately into a bank account for term n_1. At time n_1, the deposit will have grown to amount

$$(1 + s_{n_1})^{-n_1} \times (1 + s_{n_1})^{n_1} = 1.$$

At this point we withdraw this money from the bank, and use it to fulfil the FRA. At time n_2, this will now be worth

$$(1 + f)^{n_2-n_1} > (1 + f_{n_1,n_2})^{n_2-n_1} = \frac{(1 + s_{n_2})^{n_2}}{(1 + s_{n_1})^{n_1}},$$

where the first inequality is by the assumption, and the second is the definition of the forward rate f_{n_1,n_2}. At the same time, our initial loan has grown to value

$$\frac{(1+s_{n_2})^{n_2}}{(1+s_{n_1})^{n_1}}$$

so we can use the deposit to pay off the loan, generating a risk-free profit. A similar argument (this time involving an FRA for a loan instead of a deposit) can be used to show that $f < f_{n_1,n_2}$ also gives rise to arbitrage (details are left as an exercise). □

Example 3.25. Suppose that the 2-year LIBOR rate is $r_2 = 1.43\%$ and the 5-year LIBOR rate is $r_5 = 1.98\%$. Find the fair forward rate $f_{2,5}$ for a FRA to deposit cash in two years time for a three year deposit.

Solution. The first step is to calculate spot rates using the formula

$$1 + s_T = (1 + r_T T)^{1/T}.$$

This yields

$$1 + s_2 = (1 + 1.43\% \cdot 2)^{1/2} \quad \text{and} \quad 1 + s_5 = (1 + 1.98\% \cdot 5)^{1/5},$$

which gives $s_2 = 1.42\%$ and $s_5 = 1.91\%$. Therefore we have

$$(1 + f_{2,5})^{5-2} = \frac{(1+s_5)^5}{(1+s_2)^2} = \frac{1.0990}{1.0286} = 1.0684,$$

which gives $f_{2,5} = 2.23\%$. For example, if a FRA is made to deposit £500 after 2 years at the forward interest rates, then in 5 years the deposit will be worth

$$500(1 + f_{2,5})^3 = 500 \times 1.0684 = £534.20$$

□

Observe that the ratio of unit zero-coupon bond prices with expiries n_1 and n_2 is given by

$$\frac{V_{n_1}}{V_{n_2}} = \frac{(1+s_{n_2})^{n_2}}{(1+s_{n_1})^{n_1}},$$

and therefore

$$\frac{V_{n_1}}{V_{n_2}} = (1 + f_{n_1,n_2})^{n_2 - n_1}.$$

It follows that, just as for discount rates and bond prices, forward rates are also completely determined by the prices of unit zero-coupon bonds.

Example 3.26. Suppose the price of 3-year and 5-year year unit zero-coupon bonds are 0.92 and 0.65 respectively. Calculate the 2-year forward rate $f_{3,5}$ from year 2 to year 5.

Proof. The ratio of the zero-coupon bond prices satisfies

$$\frac{V_3}{V_5} = (1 + f_{3,5})^{5-3}$$

and so

$$f_{3,5} = \left(\frac{0.92}{0.65}\right)^{1/2} - 1 = 18.97\%.$$

□

Notice that $f_{0,T} = s_T$, i.e. the forward rate starting from time 0 is equal to the spot rate. Moreovoer, for any sequence of years $n_1 < n_2 < \cdots n_j$, the following equality holds:

$$\begin{aligned}\prod_{i=1}^{j-1}(1+f_{n_i,n_{i+1}})^{n_{i+1}-n_i} &= \prod_{i=1}^{j-1}\frac{(1+s_{n_{i+1}})^{n_{i+1}}}{(1+s_{n_i})^{n_i}}\\ &= \frac{(1+s_{n_j})^{n_j}}{(1+s_{n_1})^{n_1}} = (1+f_{n_1,n_j})^{n_j-n_1}.\end{aligned}$$

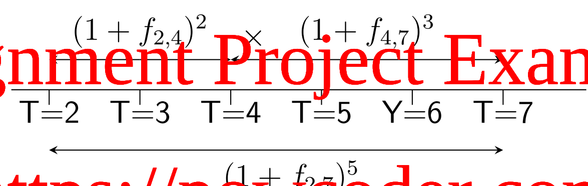
As an example,

$$(1+f_{2,4})^2(1+f_{4,7})^3 = (1+f_{2,7})^5,$$

which can be understood as saying that the following two ways of depositing money between times $T = 2$ and $T = 7$ must result in the same bank balance at $T = 7$:

2 First in a 2-year deposit between $T = 2$ and $T = 4$, then in a 3-year deposit between $T = 4$ and $T = 7$ (top of diagram below).

2 In a 5-year deposit between $T = 2$ and $T = 7$ (bottom of diagram below);



Used in combination, these relationships can help calculate unknown forward or spot rates.

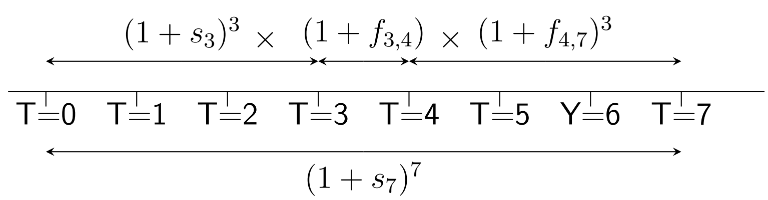
Example 3.27. Suppose the 3-year and 7-year year spot rates are 6% and 5% respectively, and that the 3-year forward rate $f_{4,7}$ is 5.2%. Calculate the forward rate $f_{3,4}$.

Proof. We can calculate $f_{3,4}$ via the relationship (see diagram below)

$$(1+s_3)^3(1+f_{3,4})(1+f_{4,7})^3 = (1+s_7)^7.$$

This yields

$$1+f_{3,4} = \frac{1.05^7}{1.06^3 \times 1.052^3} = 1.0148 \implies f_{3,4} = 1.48\%. \qquad \square$$



4 Stochastic interest rates

Up until now interest rates have been considered to be deterministic (i.e. known in advance); in this chapter we study a model of interest rates that allows for **uncertainty**.

Why is it useful to be able to model uncertain (or 'stochastic') interest rates? Often financial contracts are of a long term nature (for example, a 20-year mortgage), and usually the initial interest rate is only valid for a short time; after that, the new prevailing interest rate is applied. Since these future rates are uncertain, we need a model that allows for uncertainty.

4.1 A fixed interest rate model

The most elementary model of stochastic interest rates consists of a single interest rate R, fixed throughout the period of the investment, that is a random variable (which could be discrete or continuous). Suppose an investor places the amount P in the bank. Then the value of the deposit after time n is

$$P(1+R)^n,$$

which is itself a random variable. Without further information, we cannot say anything more precise about this random variable. For instance, all we can say about its mean is that it is

$$\mathbb{E}(P(1+R)^n) = P\mathbb{E}((1+R)^n)$$

which is *not* equal to $P(1+\mathbb{E}(R))^n$. In fact, using Jensen's inequality (which is not a topic of this module), one may show that

$$P\mathbb{E}((1+R)^n) \geq P(1+\mathbb{E}(R))^n.$$

Example 4.1. Suppose the interest rate R has probability distribution

$$R = \begin{cases} 0.06 & \text{with probability } 0.2, \\ 0.08 & \text{with probability } 0.7, \\ 0.10 & \text{with probability } 0.1. \end{cases}$$

Find the mean and standard deviation of the accumulated value of a deposit of $£5{,}000$ for five years.

Solution. Let $S_5 = (1+R)^5$, so that the unknown accumulated value is

$$P(5) = 5{,}000 \times S_5.$$

Then

$$\mathbb{E}(S_5) = 0.2 \times 1.06^5 + 0.7 \times 1.08^5 + 0.1 \times 1.10^5 = 1.457,$$

and we can use the formula $\mathrm{Var}(S_5) = \mathbb{E}(S_5^2) - \mathbb{E}(S_5)^2$ to compute

$$\begin{aligned} \mathrm{Var}(S_5) &= \mathbb{E}[(1+R)^{10}] - 1.457^2 \\ &= 0.2 \times 1.06^{10} + 0.7 \times 1.08^{10} + 0.1 \times 1.10^{10} - 1.457^2 \\ &= 0.00528, \end{aligned}$$

and so the standard deviation of S_5 is

$$\mathsf{sd}(S_5) = \sqrt{0.00528} = 0.0726.$$

Using linearity,

$$\mathbb{E}[P(S_5)] = 5{,}000 \times \mathbb{E}(S_5) = £7{,}285 \qquad \text{and} \qquad \text{sd}(P(5)) = 5{,}000 \times \text{sd}(S_5) = £363.3.$$

On the other hand, notice that the mean rate of interest is

$$\mathbb{E}(R) = 0.2 \times 0.06 + 0.7 \times 0.08 + 0.1 \times 0.10 = 7.8\%,$$

and so the accumulated value at the mean rate of interest is

$$5{,}000(1 + \mathbb{E}R)^5 = 5{,}000 \times 1.078^5 = £7{,}279,$$

which is indeed less than the mean accumulated value $\mathbb{E}[P(5)]$. $\square$

Example 4.2. Suppose you deposit P into a bank with random interest rate R as in Example 4.1. Calculate the expected time until the deposit value doubles to $2P$.

Solution. Let T be the time until the deposit value doubles, which satisfies

$$P(1 + R)^T = 2P \quad \Leftrightarrow \quad T = \frac{\ln 2}{\ln(1 + R)}.$$

We can then calculate $\mathbb{E}(T)$ as

$$\begin{aligned}
\mathbb{E}(T) &= \ln 2 \times \mathbb{E}[1/\ln(1 + R)] \\
&= \ln 2 \times \left(0.2 \times 1/\ln(1 + 0.06) + 0.7 \times 1/\ln(1 + 0.08) + 0.1 \times 1/\ln(1 + 0.10)\right) \\
&= 9.41 \text{ years.}
\end{aligned}$$

Note that this is **not** equal to

$$\frac{\ln 2}{\ln(1 + \mathbb{E}(R))} = \frac{\ln 2}{\ln(1 + 0.078)} = 9.23 \text{ years.}$$ $\square$

4.2 A varying interest rate model

We next consider a model that allows for varying interest rates. Let R_i be the interest rate applicable between year i and year $i + 1$, with each R_i a random variable. In order to make calculations tractable, we will assume that R_i are mutually independent. Later on we will, in addition, assume that the interest rates R_i are also identically distributed.

In reality the assumption that interest rates are i.i.d. **may not be realistic**:

1. Interest rates in different periods will not generally be independent. For instance, if interest rates are low now then we expect interest rates in the immediate future to be low as well, i.e. in general we expect that R_i and R_{i+1} to be **positively correlated**.

2. The interest rates R_i for large i are more uncertain than the interest rate, say, next year, and so we would expect $\mathrm{Var}(R_i)$ to be larger than $\mathrm{Var}(R_1)$.

Nevertheless, for simplicity in this course we will persist with these assumptions.

4.2.1 The growth of a single deposit

Suppose that we make a single deposit of £1 at time 0. In n years this will grow in value to

$$S_n = (1+R_0)(1+R_1)\cdots(1+R_{n-1}).$$

More generally, if we deposit £P then this will grow in value to PS_n. In order to analyse the performance of this investment, we calculate the mean and variance of the accumulated value S_n.

The mean of the accumulated value PS_n is

$$\mathbb{E}\left(PS_n\right) = P\mathbb{E}\Big(\prod_{i=0}^{n-1}(1+R_i)\Big) = P\prod_{i=0}^{n-1}\mathbb{E}\left(1+R_i\right) = P\prod_{i=0}^{n-1}(1+\mathbb{E}(R_i)),$$

with the second equality holding since we assume R_i are independent.

In order to calculate $\mathrm{Var}(PS_n)$, we use the formula

$$\mathrm{Var}(PS_n) = P^2\left(\mathbb{E}(S_n^2) - \left(\mathbb{E}\left(S_n\right)\right)^2\right).$$

We calculate $\mathbb{E}(S_n^2)$ as follows:

$$\begin{aligned}\mathbb{E}\left(S_n^2\right) &= \mathbb{E}\Big(\prod_{i=0}^{n-1}(1+R_i)^2\Big) = \prod_{i=0}^{n-1}\mathbb{E}\left((1+R_i)^2\right) = \prod_{i=0}^{n-1}\mathbb{E}\left(1+2R_i+R_i^2\right)\\ &= \prod_{i=0}^{n-1}\left(1+2\mathbb{E}(R_i)+\mathbb{E}(R_i^2)\right) = \prod_{i=0}^{n-1}\left(1+2\mathbb{E}(R_i)+(\mathbb{E}(R_i))^2+\mathrm{Var}(R_i)\right)\\ &= \prod_{i=0}^{n-1}\left((1+\mathbb{E}(R_i))^2+\mathrm{Var}(R_i)\right)\end{aligned}$$

where we again used the independence of the R_i, as well as the formula

$$\mathbb{E}(R_i^2) = (\mathbb{E}(R_i))^2 + \mathrm{Var}(R_i).$$

Combining with our previous calculation of $\mathbb{E}(S_n)$,

$$\begin{aligned}\mathrm{Var}(PS_n) &= P^2\left(\mathbb{E}(S_n^2) - \left(\mathbb{E}\left(S_n\right)\right)^2\right)\\ &= P^2\left(\prod_{i=0}^{n-1}\left((1+\mathbb{E}(R_i))^2+\mathrm{Var}(R_i)\right) - \prod_{i=0}^{n-1}(1+\mathbb{E}(R_i))^2\right).\end{aligned}$$

If the interest rates R_i are also identically distributed with common mean μ and variance σ^2 then

$$\mathbb{E}\left(PS_n\right) = P\prod_{i=0}^{n-1}\left(1+\mu\right) = P\left(1+\mu\right)^n$$

and

$$\mathrm{Var}(PS_n) = P^2\left(\left((1+\mu)^2+\sigma^2\right)^n - (1+\mu)^{2n}\right). \tag{19}$$

Remark 4.3. In the case that the interest rates are deterministic, we would expect that the deposit value PS_n is also deterministic. This is indeed the case, since setting $\sigma^2 = 0$ in the formula (19) yields

$$\mathrm{Var}(PS_n) = P^2\left(\left((1+\mu)^2+\sigma^2\right)^n - (1+\mu)^{2n}\right) = P^2\left(\left((1+\mu)^2\right)^n - (1+\mu)^{2n}\right) = 0.$$

Remark 4.4. If the variance of the interest rates goes up, then so does the variance of the deposit value S_n. This can be seen by inspecting the formula (19) and checking that the derivative of the right-hand side with respect to σ is positive (exercise!).

Example 4.5. Suppose that R_i are all distributed as in Example 4.1, and suppose that we again invest £5,000 for $n = 5$ years. Find the mean and variance of the accumulated value of the investment.

Solution. In Example 4.1 we already calculated the mean accumulated value

$$P(1+\mu)^5 = 5{,}000(1+\mathbb{E}[R_i])^5 = \pounds 7{,}279.$$

We can calculate $\sigma^2 = \mathrm{Var}(R_i)$ via

$$\begin{aligned}
\sigma^2 &= \mathbb{E}(R_i^2) - \mathbb{E}(R_i)^2 \\
&= (0.2 \times 0.06^2 + 0.7 \times 0.08^2 + 0.1 \times 0.10^2) - \mu^2 = 0.0062 - 0.078^2 \\
&= 0.000116\,.
\end{aligned}$$

Hence, applying formula (19), the variance of the accumulated value is

$$25{,}000{,}000\big[(1.078^2 + 0.000116)^5 - (1.078)^{10}\big] = \pounds 26{,}500$$

□

4.2.2 The growth of regular deposits

We next consider an investment in which additional deposits are made every year. Assume that at time $i \in \{0, 1, \ldots, n-1\}$ we deposit $\pounds P_i$.

Then the accumulated value of the deposit A_n at time n (by convention, just before the new deposit is made) satisfies a recursion:

$$A_0 = 0 \quad \text{and} \quad A_n = (A_{n-1} + P_{n-1})(1 + R_{n-1}), \quad n \geq 1. \tag{20}$$

This recursion can be interpreted as follows: the value of the deposit at time n is equal to the value at time $n-1$, plus the $\pounds P_{n-1}$ contribution from the previous year, all increased by the interest rate applicable from time $n-1$ to time n.

As before, in order to analyse the accumulated value A_n we can compute its expectation and variance. The recursion formula (20) can be manipulated to give a recursion formula for the expectation. Specifically, we compute the expectation of both sides of (20):

$$\begin{aligned}
\mathbb{E}(A_n) &= \mathbb{E}\left((A_{n-1} + P_{n-1})(1 + R_{n-1})\right) \\
&= \mathbb{E}(A_{n-1} + P_{n-1}) \times \mathbb{E}(1 + R_{n-1}) \\
&= \left(\mathbb{E}(A_{n-1}) + P_{n-1}\right)\left(1 + \mathbb{E}(R_{n-1})\right)
\end{aligned}$$

where we used the fact that $A_{n-1} + P_{n-1}$ and $(1 + R_{n-1})$ are independent (since A_{n-1} and R_{n-1} are independent – explain this).

In general, solving the resulting recursive formula for $\mathbb{E}(A_n)$ is quite complicated. In the special case that $P_i = P$ (i.e. all deposit amounts are the same), and R_i are i.i.d. with common mean μ we have, for example,

$$\begin{aligned}
&\mathbb{E}(A_1) = P(1+\mu) \\
&\mathbb{E}(A_2) = \left(\mathbb{E}(A_1 + P)\right)(1+\mu) = P(2+\mu)(1+\mu) \\
&\mathbb{E}(A_3) = \left(\mathbb{E}(A_2) + P\right)(1+\mu) = P\left(1 + (2+\mu)(1+\mu)\right)(1+\mu).
\end{aligned}$$

Example 4.6. If $P_i = P$ and the R_i are i.i.d. with common mean $\mu = 0$, prove that

$$\mathbb{E}(A_n) = Pn$$

In other words, if the expected interest rate is zero, the expected value of a fund where we deposit £P every year is £Pn.

Solution. In the case $\mu = 0$ the recursive formula reduces to

$$\mathbb{E}(A_n) = \mathbb{E}(A_{n-1}) + P,$$

which shows that $\mathbb{E}(A_n) = Pn$ by induction. □

Remark 4.7. Note that in the case that the interest rate **was** zero (i.e. $\mu = 0$ and $\sigma^2 = 0$), then $A_n = Pn$, so certainly $\mathbb{E}(A_n) = Pn$ as well.

A recursive formula for the variance can also be derived via a similar method. The formulae are too complicated to manipulate by hand but could be easily programmed into Excel, for example. We explore this in Coursework 3.

4.3 Log-normally distributed interest rates

In practice, the easiest stochastic interest rate model to work with is usually the **log-normal model**, in which the interest rates are assumed to follow a **log-normal process**. This model will appear again in the course when we discuss share prices.

A continuous random variable Y is said to be **log-normally distributed** with parameters μ and σ^2, written $Y \sim \mathrm{LogNormal}(\mu, \sigma^2)$, if

$$\ln Y \sim \mathcal{N}(\mu, \sigma^2)$$

Equivalently, $Y = e^X$, where $X \sim \mathcal{N}(\mu, \sigma^2)$. The p.d.f. of Y is

$$f_Y(y) = \begin{cases} \frac{1}{\sqrt{2\pi}\sigma y} \exp\left(-\frac{(\ln y - \mu)^2}{2\sigma^2}\right), & \text{if } y > 0, \\ 0, & \text{if } y \leq 0. \end{cases}$$

Let us consider the growth of a single deposit in the case that $1 + R_i$ are assumed to be an i.i.d. sequence of log-normally distributed random variables. In that case the distribution of

$$S_n = (1 + R_0)(1 + R_1) \cdots (1 + R_{n-1})$$

can be computed exactly (and not merely the mean and variance as we have previously done). The underlying reason is that log-normal distributions behave well under multiplication. Suppose that $Y_1 \sim \mathrm{LogNormal}(\mu_1, \sigma_1^2)$ and $Y_2 \sim \mathrm{LogNormal}(\mu_2, \sigma_2^2)$ are independent. Then we can write $Y_1 = e^{X_1}$ and $Y_2 = e^{X_2}$, where $X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2)$ and $X_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$ are independent. Therefore,

$$Y_1 Y_2 = e^{X_1} e^{X_2} = e^{X_1 + X_2},$$

and since

$$X_1 + X_2 \sim \mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2),$$

it follows that

$$Y_1 Y_2 \sim \mathrm{LogNormal}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2).$$

By extending the previous results to n variables we can show the following:

Theorem 4.8. *Suppose that* $1 + R_i$ *are a sequence of i.i.d. random variables with* $1 + R_i \sim \mathrm{LogNormal}(\mu, \sigma^2)$. *Then*

$$S_n = \prod_{i=0}^{n-1}(1 + R_i) \sim \mathrm{LogNormal}(n\mu, n\sigma^2).$$

Using the above theorem, the c.d.f. of S_n can be computed as

$$\mathbb{P}(S_n \le x) = \mathbb{P}(e^X \le x) = \mathbb{P}(X \le \ln x) = \mathbb{P}\Big(\frac{X - n\mu}{\sqrt{n}\sigma} \le \frac{\ln x - n\mu}{\sqrt{n}\sigma}\Big)$$

where $X \sim \mathcal{N}(n\mu, n\sigma^2)$. Therefore

$$\mathbb{P}(S_n \le x) = \Phi\Big(\frac{\ln x - n\mu}{\sqrt{n}\sigma}\Big),$$

where Φ denotes the c.d.f. of the standard normal distribution.

Example 4.9. Suppose £1000 is deposited in a bank account for five years, and suppose the interest rates R_i are a sequence of i.i.d. random variable such that $1 + R_i$ is log-normally distributed with parameters $\mu = 0.075$ and $\sigma^2 = 0.025^2$. What is the probability that the accumulated value of the deposit exceeds £1,500? Find the upper and lower quartiles for the accumulated value of the deposit.

Solution. The accumulated value $A_n = 1000 S_5$ has distribution

$$A_n \sim 1000\,\mathrm{LogNormal}(5\mu, 5\sigma^2).$$

The probability that $A_n > 1500$ is

$$\begin{aligned}\mathbb{P}(A_n > 1500) &= \mathbb{P}(\mathrm{LogNormal}(5\mu, 5\sigma^2) > 1.5) = 1 - \Phi\Big(\frac{\ln 1.5 - 5\mu}{\sqrt{5}\sigma}\Big)\\ &= 1 - \Phi(0.545) = 1 - 0.7017 = 0.2929,\end{aligned}$$

where the value of the normal c.d.f. has to be looked up (in a table for instance).

By definition of the quartile, the accumulated value A_n will exceed the upper quartile u with probability 0.25, i.e.

$$0.75 = \mathbb{P}(A_n \le u) = \mathbb{P}(1000 S_5 \le u) = \mathbb{P}(S_5 \le u/1000).$$

Hence u satisfies

$$0.75 = \Phi\left(\frac{\ln(u/1000) - 5\mu}{\sqrt{5}\sigma}\right).$$

Looking up the normal c.d.f. we find that $\Phi(0.6745) = 0.75$, so therefore

$$\frac{\ln(u/1000) - 5\mu}{\sqrt{5}\sigma} = 0.6745,$$

that is,

$$u = 1000 \exp\left(5\mu + 0.6745\sigma\sqrt{5}\right) = £1{,}510\,.$$

Similarly, the lower quartile is

$$\ell = 1000 \exp\left(5\mu - 0.6745\sigma\sqrt{5}\right) = £1{,}400\,. \qquad \square$$

5 Equities and their derivatives

In previous chapters we considered fixed-income securities. Equities and derivatives differ from fixed-income securities in that their cash-flow stream is not pre-determined, and instead depends on market fluctuations. This complicates their analysis.

A **derivative** is a financial instrument (i.e. a contract between two or more parties) whose value is 'derived' from one or more **underlying** assets (e.g. stocks, bonds, foreign exchange, interest rates, commodities etc). There exist many types of derivatives; in this course we study only the most common examples being **forwards** and **options**, but there are many other 'exotic' derivatives. Derivatives can be bought and traded either through an exchange such as the *Chicago Mercantile Exchange* (CME) or over-the-counter (off-exchange). The most common use of derivatives is to reduce expose to risk ('hedge'), or conversely to increase exposure to risk ('speculate').

5.1 Shares

The basic instrument of equity is the **share** (or 'stock'),[9] which is a unit of ownership of a company. Owning shares translates to partly owning the company which issues the shares.

Example 5.1. Apple currently has 4.52 billion shares. If you were to buy one share in Apple (current price $\$240.24$) you would own

$$\frac{1}{4{,}520{,}000{,}000} = 0.0000000221\%$$

of the company. The current market capitalisation of Apple (i.e. the total value its shares) is

$$4.52 \text{ billion} \times \$240.24 = \$1.086 \text{ trillion}$$

Sometimes shares entitle the holder to additional benefits, for instance to **dividends** (cash payments made by the company to its shareholders). For simplicity, we will generally only consider non-dividend paying shares.

Shares are usually traded on a **stock exchange**, such as the *London Stock Exchange* or the *New York Stock Exchange*. We will always assume that:

1. Shares can be bought and sold by an investor in any amount (even fractional amounts), at any time, without transaction costs.

2. An investor can borrow a share and then subsequently sell it. This operation is called **short-selling**, and can be viewed as owning a negative number of shares.

These assumptions are another instance of the 'frictionless market hypothesis'.

5.2 Forwards

A **forward** (or a **forward contract**) is an agreement between two parties in which the seller promises to sell a specified asset (the 'underlying') for a specified price F (the 'forward price') at a specified time T (the 'settlement date') in the future, while the buyer promises to buy it. Usually the underlying is either a commodity (e.g. oil, gas, gold etc.) or foreign exchange.

[9] These terms are basically synonymous; *shares* is mainly used in the UK, while *stock* or *common stock* is mainly used in the US.

Forward contract are not bought or sold, but are 'entered into'. The investor can either enter a **long** position in the forward, if they are buying the underlying asset at time T, or a **short** position in the forward, if they are selling the underlying asset at time T.

As with all derivatives, forward contracts can be used to:

- *Reduce exposure to risk, i.e. to hedge*. Suppose that a company needs 1,000 barrels of oil in 6 months' time, and wants to eliminate the risk that the oil price will move significantly higher by then. Or suppose they trade in the UK but need to buy oil in US dollars. By entering into a forward contract they can fix the price of the future purchase, and so eliminate their exposure to price/currency fluctuations.

- *Increase exposure to risk, i.e. to speculate*. Suppose that an investment bank expects the price of oil to increase, but has no need for oil. By entering into a forward contract, they can agree to buy oil at a fixed (low) price, and then at the settlement date, when the oil is received, they can immediately sell for a profit (in practice, they would never even take possession of the oil).

Example 5.2. Suppose a company enters into a forward contract with an oil supplier to purchase 1,000 barrels of oil on 1st January 2020 for $42,000. Suppose that the price of oil per barrel on 1 January 2020 is $50. Then the company, who agreed to buy 1,000 barrels on 1 January for $42,000 could immediately sell the oil for $50,000, yielding an instant profit of

$$1{,}000 \cdot \$50 - \$42{,}000 = \$8{,}000.$$

Conversely, if the price on 1 January 2020 is $35, then on 1 January the company takes an instant loss of

$$1{,}000 \cdot \$35 - \$42{,}000 = -\$7{,}000.$$

Even though two companies are free to enter into a forward contract at any forward price F they agree on, in reality the principles of no-arbitrage pricing determine a single 'fair' forward price for every asset:

Proposition 5.3. *Suppose that an asset has current price S at time $t = 0$. By the no-arbitrage principle, the forward price F agreed at $t = 0$ for the delivery of the underlying asset at $t = T$ must be equal to*

$$F = S/D(T)$$

where $D(T)$ is the discount factor to time T.

Proof. We prove this result using 'replicating portfolios' (i.e. we apply the law of one price). Consider the following two portfolios formed at t = 0:

- Portfolio A. A portfolio consisting of the asset.

- Portfolio B. A portfolio consisting of:

  - a long position in the forward contract; and
  - T-year zero-coupon bonds with face-value F (or, equivalently, $D(T)F$ deposited in a bank account).

We claim that both portfolios will have the same value at time T. To see this, observe that the value of Portfolio A is equal to the value of the asset at time T. On the other hand, Portfolio B also consists of the asset (guaranteed through the forward contract), less the forward price F that needs to be paid, plus the F that the bond pays upon maturity. It total, we see that Portfolio B has the same value as the asset at time T.

Since portfolios A and B are replicating, by the law of one price they have the same price at t = 0. Let us now examine these prices. The price of Portfolio A at time 0 is equal to the price of the asset, which is S. The price of Portfolio B is equal to the price of the zero-coupon bond (remember that we do not pay anything to enter into the forward contract), which is $D(T)F$. Equating these, we conclude that $S = D(T)\,F \Leftrightarrow F = S/D(T)$. □

The current price of an asset of often referred to as the **spot price**. It is equal to the forward price of a contract with immediate delivery.

Example 5.4. Interest is continuously-compounded at nominal rate $r = 5\%$. What is the fair forward price F for a contract to buy $1{,}000$ barrels of oil in one year's time if the spot price is $\$37$ per barrel.

Solution. Since $D(1) = e^{-0.05} = 0.9512$, the fair forward price is

$$F = 37{,}000/0.9512 = \$38{,}897.$$

□

Example 5.5 (Not covered in lectures)**.** Suppose that the effective rate of interest in the UK is $r_{£} = 0.5\%$, and in the Eurozone is $r_{€} = 1\%$. Suppose also that the current exchange rate is €1.13 to £1. What is the fair forward price £F to buy €$1{,}000$ in three months' time?

Solution. By the above proposition,

$$F = S/D(1/4),$$

where S is the current price (in pounds) of € $1{,}000$

$$S = \frac{1000}{1.13} = 885,$$

and $D(1/4)$ is the discount factor to 3 months. The complicating thing about this example is determining the relevant discount factor $D(T)$, since there are two interest rates: $r_{£}$ and $r_{€}$.

To resolve this, let us take a more direct approach using 'replicating portfolios'. Consider two portfolios, both of which will be worth €$1{,}000$ at time $T = 1/4$:

- Portfolio (A). A long position in a forward contract to buy € $1{,}000$ at time T, and £ $F/(1 + r_{£})^T$ deposited in a UK bank until time T.
- Portfolio (B). € $1{,}000/(1 + r_{€})^T$ deposited in a bank in Germany until time T.

By the law of one price, since the two investments are worth the same amount at time T, they are also worth the same amount at time 0. Thus, in pounds,

$$0 + F/(1 + r_{£})^T = \frac{1{,}000/(1 + r_{€})^T}{1.13}$$

which implies

$$F = \frac{1{,}000}{1.13}\left(\frac{1 + r_{£}}{1 + r_{€}}\right)^T = \frac{1{,}000}{1.13}\left(\frac{1.005}{1.01}\right)^{1/4} = £883.9.$$

Here we see that the relevant discount factor was

$$D(T) = \left(\frac{1 + r_{£}}{1 + r_{€}}\right)^{-T},$$

which implies that value is 'discounted' with respect to the 'currency adjusted' rate

$$\frac{1 + r_{£}}{1 + r_{€}} - 1 = \frac{r_{£} - r_{€}}{1 + r_{€}} = -0.495\%. \tag{21}$$

Note that the expression (21) is mathematically the same as the 'inflation adjusted' interest rate we saw in Section 1.3, with $r_{€}$ playing the role of the inflation rate. □

5.3 Options

An **option** is a derivative contract that gives the buyer the right, but not the obligation, to buy or sell an underlying asset for a specific **strike price** K (also known as the 'exercise price') on, and sometimes also prior to, the **expiry time** T (also known as the 'maturity'). The seller of the option is then obliged to fulfil the transaction. The buyer pays a certain fee to purchase an option.

Options are traded at the *Chicago Board Options Exchange* (CBOE), or over-the-counter as customised contracts between a single buyer and seller.

Remark 5.6. Be sure you understand the difference between a forward contract and an option. A forward contract gives you the **obligation** to buy or sell an underlying asset for a specified price (the forward price F), while an option gives you the **right, but not the obligation**, to buy or sell an underlying asset for a specified price (the strike price K). On the other hand, forward contracts are free to enter, whereas there is a cost to buy an option.

There are two main categories of options:

- **Call options**, which give the owner the right to **buy** the asset at the strike price;
- **Put options**, which give the owner the right to **sell** the asset at the strike price.

These are further divided into two main categories:

- **European options** which give the owner the right to buy or sell the asset at the strike price **at** the expiry date.
- **American options**, which give the owner the right to buy or sell the underlying asset at the strike price **at any time up to and including the expiry**.

Example 5.7. A European call option on a share, with strike K and expiry T, is a contract that gives the right, but not the obligation, to purchase the share at time T for price K. A European put option is similar, but the right to purchase is replaced by a right to sell.

Example 5.8. An American call option on a share, with strike K and expiry T, is a contract that gives the right, but not the obligation, to purchase the share at any time up to and including time T for price K. An American put option is similar, but the right to purchase is replaced by a right to sell.

American options are more expensive (or at least as expensive) than the equivalent European options (i.e., with the same expiry T and strike K), since they offer their holder more optionality.

Exercise 5.9. Use a no-arbitrage argument to verify that an American option must cost at least as much as the equivalent European option.

Remarks 5.10.

1. The options described above are actually only a small sample of the enormous variety of options that are issued and traded. For this reason they are sometimes referred to as 'vanilla' options, with other variety of options known as 'exotic'.

2. Although options can be written on any underlying asset, for simplicity we will work only with options written on an underlying share.

3. In practice an option will be bought for multiple shares simultaneously. For simplicity, we will normally assume that the option refers to a single share.

Let $S(t)$ denote the price of a share at time t. The owner of a European call option has only one choice to make: at expiry T they must decide whether to exercise the option. A moment's reflection shows that:

- If $S(T) > K$ (known as the stock being 'in-the-money'), then it is beneficial for the owner to exercise the right, buy the stock, and make a profit by selling it immediately on the market. This will yield a profit of $S(T) - K$.

- If $S(T) < K$ (known as the stock being 'out-of-the-money'), then the owner will not exercise the option because the stock can be bought cheaper on the market. This means that the the investor earns zero from the option in this case.

Summarising this discussion, the **payoff** (another word for 'return') of a European call option is equal to

$$\text{Payoff}(S(T)) = \begin{cases} 0, & \text{if } S(T) < K, \\ (S(T) - K), & \text{if } S(T) \geq K. \end{cases}$$

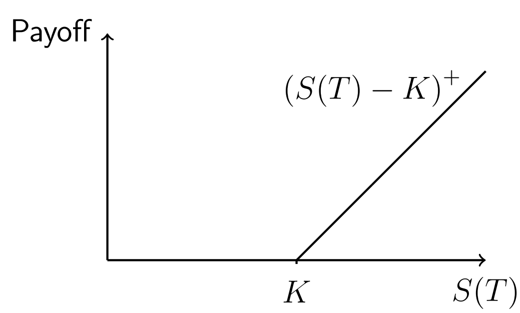
Using the function x^+ defined by

$$x^+ = \begin{cases} 0, & \text{if } x < 0, \\ x, & \text{if } x \geq 0, \end{cases}$$

the payoff can also be written as

$$\text{Payoff}(S(T)) = (S(T) - K)^+.$$

Here is a plot of the payoff as a function of $S(T)$:



Exercise 5.11. Show that the payoff of a European put option is equal to $(K - S(T))^+$, and plot its graph as a function of $S(T)$.

5.4 Pricing options: model-independent pricing principles

Despite the fact that options contracts have been traded for many centuries, assigning a fair (i.e. no-arbitrage) price to an option is a very challenging task, and is the subject of much ongoing research.

The simple reason for this complication is that, unlike for bonds or forwards, the no-arbitrage price of an option depends on the model that is used to model the underlying asset.

A major milestone in the field of option pricing was the approach introduced by F. Black and M.S. Scholes in 1973,[10] for which M.S. Scholes was ultimately awarded the 1997 Nobel Prize in Economics (along with R. Merton; F. Black had passed away by that point). We will study the Black-Scholes approach later in this course, and in much more detail in **Financial Mathematics II**. In the aftermath of Black and Scholes' work, trading and academic interest in options increased significantly, and nowadays trillions of pounds' worth of options are traded worldwide each year.[11]

Nevertheless, there are certain general **model-independent** pricing principles that are simple to deduce from no-arbitrage arguments; these principles are the focus of this section.

Our first example is an important relationship between the prices of a European call and put options with the same strike and expiry, known as **put-call parity**:

Theorem 5.12 (Put-call parity)**.** *Let C be the price of a European call option on a share with strike price K and expiry T. Let P be the price of a European put option on the same stock with the same strike and expiry. Suppose the underlying share has current price $S(0)$. Then*

$$P + S(0) = C + K \cdot D(T)$$

where $D(T)$ is the discount factor at time T.

Proof. We argue using replicating portfolios. Consider the following two portfolios, both formed at time 0, consisting of:

- Portfolio A. One share and one put option with strike K and expiry T.
- Portfolio B. One call option with strike K and expiry T, and the amount $KD(T)$ deposited in the bank.

Let $S(T)$ denote the price of the stock at time T. Then at time T, Portfolio A is worth

$$S(T) + (K - S(T))^+,$$

whereas Portfolio B is worth

$$(S(T) - K)^+ + KD(T)/D(T) = (S(T) - K)^+ + K.$$

Notice that

$$S(T) + (K - S(T))^+ = \max\{S(T), K\} = (S(T) - K)^+ + K$$

[10] F. Black and M.S. Scholes (1973). The Pricing of Options and Corporate Liabilities. *Journal of Political Economy*, **81 (3)**, pp. 637–654.

[11] The overall derivative market is estimated to be worth in the quadrillions of pounds, more than the entire worldwide GDP.

so the two portfolios have the same value at time T. Hence, by the law of one price, they also have the same value at time 0. Since at time 0 the value of Portfolio A and Portfolio B are respectively

$$S(0) + P \quad \text{and} \quad C + KD(T),$$

we deduce that

$$S(0) + P = C + KD(T). \qquad \square$$

We next show that it is never worth exercising an American call option early:

Theorem 5.13. *An American call option should never be exercised before the expiry time.*

Remarks 5.14. The above theorem has some important caveats:

- It only applies to American **call** options, and does not apply to American **put** options, which in some circumstances should be exercised early.
- It only applies to non-dividend paying shares. If a share pays a dividend, then again in some circumstances it is worth exercising an American call option early.
- It only applies when interest rates are positive.

Proof. Suppose you exercised the call option early, i.e. at some time $t_0 < T$ such that $S(t_0) > K$ (if $S(t_0) \leq K$ there would be no point exercising early). Then the payoff you receive at t_0 is $S(t_0) - K$. If you then deposit this money in the bank, your payoff at time T would be

$$(S(t_0) - K)/D,$$

where $D = D(T)/D(t_0)$ is the discount factor that applies between t_0 and T.

However, it turns out that there exists an alternative strategy that is more profitable than exercising the option, namely, (short-)selling the share to receive $S(t_0)$, and depositing this money in the bank. At time T the payoff of this alternative strategy is the sum of:

- The money in the bank, which has grown to $S(t_0)/D$;

- The share you owe from the short-selling, which is worth $S(T)$;
- The payoff of the option, which is $(S(T) - K)^+$.

In other words, the total payoff of this strategy is

$$S(t_0)/D - S(T) + (S(T) - K)^+.$$

It remains to argue that

$$\frac{S(t_0)}{D} - S(T) + (S(T) - K)^+ \geq \frac{S(t_0) - K}{D}.$$

To show this, we first observe that

$$-S(T) + (S(T) - K)^+ = \left.\begin{cases} -S(T) + (S(T) - K) = -K & \text{if } S(T) \geq K \\ -S(T) + 0 \geq -K & \text{if } S(T) < K \end{cases}\right\} \geq -K,$$

so

$$\frac{S(t_0)}{D} - S(T) + (S(T) - K)^+ \geq \frac{S(t_0)}{D} - K.$$

Then since $D \leq 1$ (at least, if interest rates are positive),

$$\frac{S(t_0)}{D} - K \geq \frac{S(t_0)}{D} - \frac{K}{D} = \frac{S(t_0) - K}{D},$$

which proves the claim. $\square$

Exercise 5.15. What goes wrong if the argument is attempted for American put options?

5.5 Pricing options via the risk-neutral distribution

To develop model-independent pricing principles further we need to introduce a new kind of no-arbitrage argument based on the **risk-neutral distribution**.

5.5.1 Betting strategies, the risk-neutral distribution, and the arbitrage theorem

Buying an option can be viewed as a kind of **bet** on the outcome of the financial markets, and the general setting of **betting strategies** is a useful way to introduce the theory of risk-neutral distributions.

Suppose there is an event (e.g. an experiment, sports event, financial outcome etc.) with m different mutually exclusive outcomes $\{1, 2, \dots, m\}$. A **wager** is a type of investment such that, if you pay an initial amount P (called a **bet**), there is a function $r(j)$, called the **return function**, such that if the outcome of the event is j you are returned an amount $P_{\text{final}}(j) = P \times r(j)$. The **profit** from such a transaction is

$$P_{\text{final}}(j) - P = P(r(j) - 1).$$

Example 5.16. Suppose that you buy 2 barrels of oil at the current price $\$50$, and tomorrow the oil price will either rise to $\$60$ a barrel or fall to $\$20$ a barrel. This transaction can be viewed as a bet of $2 \times \$50 = \$100$ on a wager with return function

$$r(\text{price rises}) = \frac{60}{50} = 1.2\ , \quad r(\text{price falls}) = \frac{20}{50} = 0.4.$$

Example 5.17. For example, consider the upcoming England vs Montenegro football match, for which a bookmaker is quoting the following odds:

Outcome	Odds
England win	$1/16$
Draw	$14/1$
Montenegro win	$40/1$

This this means that the bookmaker is offering three wagers r_E, r_D and r_M with respective return functions

$$\begin{array}{lll} r_E(\text{E wins}) = £1.06 & r_D(\text{E wins}) = £0 & r_M(\text{E wins}) = £0 \\ r_E(\text{Draw}) = £0 & r_D(\text{Draw}) = £15 & r_M(\text{Draw}) = £0 \\ r_E(\text{M wins}) = £0 & r_D(\text{M wins}) = £0 & r_M(\text{M wins}) = £41 \end{array}$$

For instance, if you bet $£2$ on England winning and the match was a draw you would be returned $£0$ so your profit would be $-£2$.

In general, if a bookmaker offers **odds** o_i on outcome i this means they are offering the wagers with return functions

$$r_i(j) = \begin{cases} 1 + o_i & \text{if } j = i, \\ 0 & \text{if } j \neq i. \end{cases}$$

Let's now suppose that there exists n different wagers available. A **betting strategy** is a vector $\mathbf{x} = (x_1, \dots, x_n)$ in which x_i is bet on wager i; these numbers can be negative, which we interpret as **offering ('selling') a bet** to another person. Notice that the **profit** from the betting strategy $\mathbf{x}$ when the outcome of the event is j is given by

$$P_{\mathbf{x}}(j) = \sum_{i=1}^{n} x_i \left(r_i(j) - 1\right).$$

From the point of view of the investor, it is clearly of interest whether there exists a betting strategy that **guarantees a profit** no matter the outcome of the event (i.e. there is an arbitrage opportunity). Let's consider this question in some simple examples:

Example 5.18. Consider an NBA match[12] between the **Atlanta Hawks** (A) and the **Boston Celtics** (B), and a bookmaker quotes you odds of

Outcome	Odds
Atlanta win	$5/2$
Boston Celtics win	$2/3$

Is there a betting strategy that guarantees a profit?

Solution. We seek x and y such that betting x on outcome A and y on outcome B leads to a guaranteed profit no matter who wins. The profit from such a bet is

$$P_{x,y} = \begin{cases} \frac{5}{2}x - y & \text{if } A \text{ wins,} \\ \frac{2}{3}y - x & \text{if } B \text{ wins.} \end{cases}$$

In order to make a guaranteed profit, it is enough that

$$5x/2 - y > 0 \quad \text{and} \quad 2y/3 - x > 0,$$

or equivalently

$$1.5x < y < 2.5x$$

Hence for any $x > 0$ there is an interval $(1.5x, 2.5x)$ from which we can choose y in order to guarantee a profit. For example, we could choose $x = £1$ and $y = £2$, in which case the profit is

$$P_{1,2} = \begin{cases} \frac{5}{2} - 2 = \frac{1}{2} & \text{if } A \text{ wins,} \\ \frac{2}{3} \cdot 2 - 1 = \frac{1}{3} & \text{if } B \text{ wins.} \end{cases}$$ □

Example 5.19. Consider the same set-up as in the previous example, except for the odds that B wins have been reduced to $2/5$. Is there still a betting strategy that guarantees a profit?

Solution. Now the profit is

$$P_{x,y} = \begin{cases} \frac{5}{2}x - y & \text{if } A \text{ wins,} \\ \frac{2}{5}y - x & \text{if } B \text{ wins,} \end{cases}$$

and so a guaranteed profit is only possible if we can find x and y such that

$$2.5x - y \geq 0 \quad \text{and} \quad 0.4y - x \geq 0$$

and at least one of these inequalities is strict. Since this is equivalent to

$$2.5x \geq y \quad \text{and} \quad y \geq 2.5x$$

finding such an x and y is impossible. □

We next state an important theorem in financial mathematics, often called simply 'the arbitrage theorem', states that **exactly one** of the following is true:

[12] We consider basketball for the simple reason that there are no draws!

- Either there exists a betting strategy that generates a profit irrespective of the outcome of the experiment (i.e. there is an arbitrage opportunity);
- Or, we can prescribe specific probabilities (called the 'risk-neutral distribution') to the possible outcomes in such a way that the expected profit on any betting strategy will be zero (in which case we say that all bets are fair).

Let us state this theorem more formally (we will not discuss the proof):

Theorem 5.20 (The arbitrage theorem)**.** *Suppose there is an event with m outcomes and there are n possible wagers on these outcomes with respective return functions r_i. Then exactly one of the following two statements is true:*

(i) There exists a betting strategy $\mathbf{x} = (x_1, \ldots, x_n)$ such that the profit satisfies

$$P_{\mathbf{x}}(j) \geq 0 \quad \textit{for all } j = 1, \ldots, m$$

and

$$P_{\mathbf{x}}(j) > 0 \quad \textit{for at least one } j = 1, \ldots, m;$$

(ii) There exists a probability vector $\tilde{\mathbf{p}} = (\tilde{p}_1, \ldots, \tilde{p}_m)$ over the comes, called the 'risk-neutral distribution', such that $\tilde{p}_j > 0$ for all $j = 1, \ldots, m$, and

$$\sum_{j=1}^{m} \tilde{p}_j \left(r_i(j) - 1\right) = 0 \quad \textit{for all } i = 1, \ldots, n.$$

The up-shot of the arbitrage theorem is that, under the no-arbitrage principle, we are **guaranteed** the existence of a 'risk-neutral distribution'. This will be crucial in developing the pricing principles in the following subsection.

Remark 5.21. One important distinction to make is that the 'risk-neutral distribution' does not reflect anything about 'real-world' probabilities. Instead the 'risk-neutral distribution' should be thought of as artificial construct.

Remark 5.22. In general the risk-neutral distribution guaranteed by the arbitrage theorem **may not be unique**, which poses problems for no-arbitrage pricing theory. Nevertheless, in all the models that we consider in this course the risk-neutral distribution **will** be unique.

Example 5.23. Suppose there is an event with two outcomes, and there are two wagers with return functions $r_1(1) = 2$, $r_1(2) = -1$, $r_2(1) = -1$, and $r_2(2) = 5$. Does there exist a betting strategy that guarantees a profit?

Solution. We shall try to find a risk-neutral distribution, i.e. numbers $\tilde{p}_1, \tilde{p}_2 > 0$ with $\tilde{p}_1 + \tilde{p}_2 = 1$ such that the following system of linear equations

$$\begin{array}{rcrcl} \tilde{p}_1(r_1(1) - 1) & + & \tilde{p}_2(r_1(2) - 1) & = & 0 \\ \tilde{p}_1(r_2(1) - 1) & + & \tilde{p}_2(r_2(2) - 1) & = & 0 \end{array}$$

holds, that is,

$$\begin{array}{rcrcl} \tilde{p}_1 & - & 2\tilde{p}_2 & = & 0 \\ -2\tilde{p}_1 & + & 4\tilde{p}_2 & = & 0. \end{array}$$

Note that, the second equation is a multiple of the first, thus $\tilde{p}_1 = 2\tilde{p}_2$. Using $\tilde{p}_1 + \tilde{p}_2 = 1$ now yields

$$2\tilde{p}_2 + \tilde{p}_2 = 1.$$

Thus

$$\tilde{p}_1 = \frac{2}{3} \quad \text{and} \quad \tilde{p}_2 = \frac{1}{3}$$

is a risk-neutral distribution. Therefore, the arbitrage theorem implies that there is no betting strategy that guarantees a profit. □

Assume that, for a event with n outcomes, a bookmaker quotes odds of $o_1, \ldots, o_n$. What can we say about when these odds permit a betting strategy that guarantees a profit?

Proposition 5.24. *Define the numbers* $\tilde{p}_i = 1/(1 + o_i)$. *Then the odds* o_i *do not allow for an arbitrage opportunity if and only if the numbers* $\tilde{p}_i$ *are a valid probability distribution on the set of outcomes.*

Proof. Recall that a bet on outcome i has return function

$$r_i(j) = \begin{cases} 1 + o_i & \text{if } j = i, \\ 0 & \text{if } j \neq i. \end{cases}$$

Let $(p_1, \ldots, p_n)$ be a probability distribution over the outcomes. If this is a risk-neutral distribution, then the expected profit on a bet on outcome i is equal to zero, i.e.,

$$0 = \sum_j p_j(r_i(j) - 1) = p_i \times o_i + \sum_{j \neq i} p_j \times (-1) = p_i o_i - (1 - p_i) = p_i(o_i + 1) - 1,$$

which implies that

$$p_i = 1/(1 + o_i) = \tilde{p}_i,$$

i.e. this distribution is 'risk-neutral' **if and only if** $p_i = \tilde{p}_i$ for all i. Hence by Theorem 5.20 either (i) $\tilde{p}_i$ is a valid probability distribution and is also the (unique) risk-neutral distribution, or else (ii) $\tilde{p}_i$ is not a valid probability distribution, in which case no 'risk-neutral' distribution can exist. □

Exercise 5.25. Let us recall the odds from the England vs Montenegro football match:

Outcome	Odds
England win	$1/16$
Draw	$14/1$
Montenegro win	$40/1$

Does there exist an arbitrage opportunity?

Solution. We calculate $\tilde{p}_i = 1/(1 + o_i)$:

$$\tilde{p}_E = \frac{1}{1 + 1/16} = 0.941, \ \tilde{p}_D = \frac{1}{1 + 14} = 0.0667, \ \tilde{p}_M = \frac{1}{1 + 40} = 0.0244.$$

Since these numbers sum to 1.03 they do not define a valid probability distribution, there is no risk-neutral distribution and so there does exist an arbitrage opportunity. □

Remark 5.26. In the previous example we argued that there must exist a betting strategy that guarantees a risk-free profit. Before you get excited, remember than in general such a betting strategy will involve offering ('selling') bets as well as placing ('buying') them, and in reality the only person likely to be able to exploit this arbitrage opportunity is the bookmaker. Note that gambling does not satisfy the 'frictionless market hypothesis'!

5.5.2 Options pricing using the risk-neutral distribution

Recall that the no-arbitrage principle guarantees the existence of a probability distribution on the set of outcomes, called the **risk-neutral distribution**, such that the expected profit of any betting strategy is zero.

The purchase of a share can be interpreted as a betting strategy, since we are in effect betting on the future value of the share. Therefore we deduce the following important principle:

Proposition 5.27. *Under the no-arbitrage principle, there exists a probability distribution on the future price of share, called the risk-neutral distribution, under which* **the expected profit** *on the purchase of a share is zero.*

The risk-neutral distribution will be denoted $\widetilde{\mathbb{P}}$, to distinguish it from the real-world probabilities $\mathbb{P}$. The corresponding expectation will be denoted $\widetilde{\mathbb{E}}$.

Remark 5.28. Our argument for the existence of the risk-neutral distribution required there be only a **finite** number of outcomes of the experiment, whereas in reality a share price may have an infinite number of future possible values. We will not dwell on this subtlety.

Let us analyse the implications of this principle by considering the expected profit on the purchase of a share. To buy a share at price $S(0)$ we need to borrow this amount to fund the purchase. At a later time T we will posses the share, but the amount that we owe has grown to $S(0)/D(T)$, where $D(T)$ is the discount factor to time T. Hence this transaction yields a profit of

$$S(T) - S(0)/D(T).$$

By the definition of the risk-neutral measure, the expected profit must be 0, so we conclude that

$$0 = \widetilde{\mathbb{E}}\big(S(T) - S(0)/D(T)\big) = \widetilde{\mathbb{E}}\big(S(T)\big) - S(0)/D(T).$$

This leads us the following important result:

Theorem 5.29. *Under the no-arbitrage principle, the current price of a share is equal to its (discounted) expected future price under the risk-neutral distribution, i.e.*

$$S(0) = D(T)\widetilde{\mathbb{E}}(S(T)).$$

The above argument generalises to the purchase of any asset, and in the case of **European options** leads to a general model-independent pricing formula for these options (the case of American options is more complicated):

Theorem 5.30. *Under the no-arbitrage principle, the current price of a European option is equal to the (discounted) expected value of its payoff under the risk-neutral distribution.*

Remark 5.31. We stress that the expected value of the payoff is being calculated under the risk-neutral distribution, which does **not** necessarily equal the real-world probabilities. In fact, real-world probabilities are irrelevant for the no-arbitrage pricing of financial instruments.

Example 5.32. Assume interest is continuously-compounded at nominal rate r. Then the no-arbitrage price of a European call option with strike K and expiry T is

$$\mathrm{e}^{-rT}\,\widetilde{\mathbb{E}}\left[(S(T)-K)^{+}\right],$$

where $\widetilde{\mathbb{E}}$ indicates that the expectation is calculated under the risk-neutral measure. Similarly, the no-arbitrage price of a European put option is

$$\mathrm{e}^{-rT}\,\widetilde{\mathbb{E}}\left[(K-S(T))^{+}\right].$$

We reiterate that the above formulas are **model-independent**. However, in the absence of a model, these formulas cannot be used to compute an actual numerical 'price' of an option. In order to compute a numerical price, we need to specify a model (or a risk-neutral distribution) for the share price $S(T)$. Let's see some simple examples:

Example 5.33. Assume interest is compounded annually at rate 2%. Suppose that a stock has a risk-neutral distribution at $T = 3$ given by

$$S(3) = \begin{cases} 20 & \text{with probability } 0.3, \\ 45 & \text{with probability } 0.5, \\ 60 & \text{with probability } 0.2. \end{cases}$$

Calculate the no-arbitrage price of a European put option with strike 50 and expiry $T = 3$.

Solution. The price of the put is given by

$$P = \left(\frac{1}{1.02}\right)^3 \widetilde{\mathbb{E}}[(50 - S(3))^+]$$

where $\widetilde{\mathbb{E}}$ denotes expectation under the risk-neutral distribution. Hence

$$P = \left(\frac{1}{1.02}\right)^3 \left((50-20)^+ \times 0.3 + (50-45)^+ \times 0.5 + (50-60)^+ \times 0.2\right)$$

$$= \left(\frac{1}{1.02}\right)^3 (30 \times 0.3 + 5 \times 0.5) = 10.8.$$ □

Example 5.34. Assume interest is continuously-compounded at rate 3%. Suppose that a stock has a uniform risk-neutral distribution at $T = 5$

$$S(5) \sim \mathcal{U}[20, 40],$$

Calculate the no-arbitrage price of a European call option on this stock with strike $K = 35$ and expiry $T = 5$.

Solution. The price of the call is $C = e^{-0.03\times 5}\widetilde{\mathbb{E}}[(S(5) - 35)^+]$ where $\widetilde{\mathbb{E}}$ denotes expectation under the risk-neutral distribution. Hence

$$Ce^{-0.03\times 5} \int_{20}^{40} \frac{1}{20}(x-35)^+ dx.$$

To calculate the integral it is simplest to split the domains into the regions $(20, 35)$ and $(35, 40)$. On the domain $(20, 35)$ the integrand is 0 (since the option is 'out-of-the-money'), so the integral contributes nothing. Hence we are left with

$$C = e^{-0.03\times 5} \int_{35}^{40} \frac{1}{20}(x-35)dx = \frac{e^{-0.03\times 5}}{20}\left(40^2/2 - 35^2/2 - 35 \times 5\right) = 0.538.$$ □

One useful feature of the general options pricing formula is that it can be used to infer how the price depends on the option parameters, such as the strike price K.

Proposition 5.35. *The price of a European call option is a decreasing function of the strike.*

Proof. Since the payoff $(S(T) - K)^+$ is a decreasing function of the strike K, so is the price of the call option $C = D(T)\,\widetilde{\mathbb{E}}\left[(S(T) - K)^+\right]$. □

Exercise 5.36. Prove that the price of a European put option is a increasing function of K.

Exercise 5.37. Reprove the put-call parity relation using risk-neutral pricing formulae.

The pricing of American options is significantly more challenging than European options, because we need to take into account all possible **exercise strategies** (i.e. whether to exercise the option or not). We will briefly analyse American options in the next section.

6 The binomial model

In the previous chapter we showed that the no-arbitrage price of a European option is equal to the discounted expected value of its payoff under the risk-neutral distribution. Unfortunately, this does not tell us how to actually **calculate** the no-arbitrage price, since in order to determine the risk-neutral distribution we will usually need to specify a model for the underlying share.

In this chapter we study the simplest model of a share in which the share price is assumed to move exactly once per discrete time-period to one of **two possible future states** (called the 'up' state and the 'down' state). This is known as the **binomial model** ('bi'='two').

Even though this is a crude model of a share, it is **widely used** by practitioners. The reason is that the length of the discrete time-periods can be chosen to be very small (e.g. a millisecond), which makes the model more powerful that it first appears. The downside is that doing computations over many tiny time-periods is complex (necessitating computer aide).

6.1 The single-period binomial model

Let $S(0) = S$ denote the current price of a share, and assume that price of the share at the end of a single time-period is either $S(1) = S\,u$ or $S(1) = S\,d$, where u and d are positive real numbers satisfying $0 < d < u$; we call the outcome $S(1) = S\,u$ the 'up' state, and the outcome $S(1) = S\,d$ the 'down' state. This model is called the **single-period binomial model**, and is illustrated in Figure 1.

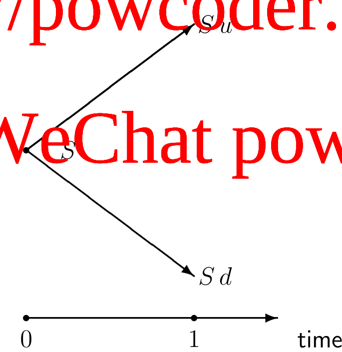


Figure 1: A share price in the single-period binomial model

In order to fully specify the binomial model, we also introduce the nominal interest rate r per time-period; this means that the discount factor to the end of the single time-period is $D(1) = 1/(1+r)$ in the case of interest compounded once per time-period ('simple interest'), or $D(1) = e^{-r}$ in the case of continuously-compounded interest.

Remark 6.1. Sometimes the binomial model is specified by giving the values of $S = S(0)$ and the values of $S(1)$ in the 'up' and 'down' state, rather than giving the multiplication factors u and d. It is easy to convert between these specifications.

6.1.1 The no-arbitrage condition and the risk-neutral distribution

An important question that must be asked of any financial model is whether the no-arbitrage principle applies (which could depend on the model parameters).

Proposition 6.2. *The no-arbitrage principle is valid in the single-period binomial model if and only if*

$$d < 1/D(1) < u,$$

where $D(1) = 1/(1+r)$ *in the case of simple interest and* $D(1) = e^{-r}$ *in the case of continuously-compounded interest. If it exists, the risk-neutral distribution for the share price at the end of the time-period is*

$$S(1) = \begin{cases} uS & \text{with probability } \frac{1/D(1)-d}{u-d}, \\ dS & \text{with probability } \frac{u-1/D(1)}{u-d}. \end{cases}$$

Proof. We analyse the model in terms of betting strategies. There is only one possible wager in the model: at time 0 you buy the share (with borrowed money), and then sell the share at $t = 1$ (you could also short-sell the share, but really this is just the 'negation' of the first wager). As we saw in the previous chapter, in order for this bet to be 'fair' the risk-neutral distribution $\widetilde{P}$ (if it exists) over $S(1)$ must satisfy

$$S = D(1)\widetilde{E}[S(1)] = D(1)(\tilde{p}_u \times uS + \tilde{p}_d \times dS),$$

where $(\tilde{p}_u, \tilde{p}_d)$ are the risk-neutral probabilities of the 'up' and 'down' outcomes. Dividing through by S, and since $\tilde{p}_u + \tilde{p}_d = 1$ (these define a probability distribution!), the above is equivalent to

$$1 = D(1)(\tilde{p}_u u + (1 - \tilde{p}_u)d) \implies \tilde{p}_u = \frac{1/D(1) - d}{u - d},$$

and so

$$\tilde{p}_d = 1 - \tilde{p}_u = 1 - \frac{1/D(1) - d}{u - d} = \frac{u - 1/D(1)}{u - d}.$$

We conclude that the risk-neutral distribution, if it exists, must equal

$$\tilde{p}_u = \frac{1/D(1) - d}{u - d} \quad \text{and} \quad \tilde{p}_d = \frac{u - 1/D(1)}{u - d}.$$

To be a valid risk-neutral distribution, we also require that $\tilde{p}_u > 0$ and $\tilde{p}_d > 0$. The condition $\tilde{p}_u > 0$ implies that

$$1/D(1) - d > 0 \implies d < 1/D(1),$$

whereas the condition $\tilde{p}_d > 0$ implies that

$$u - 1/D(1) > 0 \implies 1/D(1) < u.$$

Hence we conclude that the condition for the existence of a valid risk-neutral distribution, and also the condition for no-arbitrage, is $d < 1/D(1) < u$. □

Exercise 6.3. Prove Proposition 6.2 directly by showing that, if either $1/D(1) < d$ or $1/D(1) > u$, then an arbitrage opportunity arises.

Exercise 6.4. Consider a single-period binomial model such that the current price of the stock is $S = S(0) = 40$, and the price of the stock is either $S(1) = 42$ in the 'up' state or $S(1) = 30$ in the 'down' state. The interest rate is $r = 10\%$ per time-period. Argue that there **is** arbitrage in this model.

Solution. We have

$$u = \frac{S(1) \text{ in 'up' state}}{S(0)} = \frac{42}{40} = 1.05 \qquad \text{and} \qquad d = \frac{S(1) \text{ in 'down' state}}{S(0)} = \frac{30}{40} = 3/4.$$

Hence the condition for no-arbitrage is

$$d < 1 + r < u \Leftrightarrow 0.75 < 1.1 < 1.05,$$

which is false, and so there is arbitrage in the model. □

6.1.2 Option pricing in the single-period binomial model

Assume that a share price follows the single-period binomial model. What is the 'fair' price of a European call option written on this share with expiry at the end of the single time-period?

To assign a fair price we must assume that the no-arbitrage principle holds. For simplicity let us work with simple interest, in which case the condition for no-arbitrage is

$$d < 1 + r < u.$$

By the general no-arbitrage pricing principle in Theorem 5.30, the no-arbitrage price of the call option C is

$$C = \frac{1}{1+r}\widetilde{\mathbb{E}}((S(1) - K)^+).$$

Evaluating the expectation with respect to the risk-neutral measure

$$\tilde{p}_u = \frac{1+r-d}{u-d} \qquad \text{and} \qquad \tilde{p}_u = \frac{u-(1+r)}{u-d},$$

yields the pricing formula

$$C = \frac{1}{1+r}\Big((Su - K)^+ \frac{1+r-d}{u-d} + (Sd - K)^+ \frac{u-(1+r)}{u-d}\Big).$$

Exercise 6.5. Find the corresponding no-arbitrage price P for a European put option with strike price K, and corresponding formulae in the case of continuously-compounded interest.

Remark 6.6. It will usually be the case that the strike price K satisfies

$$Sd < K < Su,$$

since otherwise either (i) $K \geq Su$, in which case the option is always out-of-the money and so is worthless, or (ii) $K \leq Sd$, in which case the option is always in-the-money, and so is equivalent to a portfolio consisting of a single share S minus K units of cash, and so is not an interesting financial product. In this case the formula simplifies to

$$C = \frac{(1+r-d)(Su-K)}{(1+r)(u-d)}.$$

Example 6.7. Consider a single-period binomial model with parameters $S = S(0) = 40$, $u = 3/2$, $d = 3/4$ and continuously-compounded interest rate $r = 5\%$. Verify the no-arbitrage principle holds and compute the price C of a European call option with strike $K = 45$.

Solution. The no-arbitrage principle holds since

$$d < e^r < u \Leftrightarrow 0.75 < 1.051 < 1.5.$$

and the risk-neutral distribution is

$$\tilde{p}_u = \frac{e^{0.05} - 0.75}{1.5 - 0.75} = 0.402 \qquad \text{and} \qquad \tilde{p}_d = 1 - \tilde{p}_u = 0.598.$$

Since the payoff of the option is

$$\text{Payoff} = \begin{cases} (40 \times 3/2 - 45)^+ = 15 & \text{if } S(1) = Su \\ (40 \times 3/4 - 45)^+ = 0 & \text{if } S(1) = Sd \end{cases},$$

we calculate the no-arbitrage price of the option to be

$$C = e^{-0.05}(0.40169 \times 15 + 0.59831 \times 0) = 5.73. \qquad \square$$

Example 6.8. With the model as in the previous example, find the no-arbitrage price of a European put option with strike price $K = 45$.

Solution. We could price the put option directly, but a simpler way is to use 'put-call parity' (since we have already computed the price of the corresponding call option).

By put-call parity, the no-arbitrage price P of the put option is equal to

$$P = C - S + KD(T),$$

where $S = 40$, $K = 45$, $D(T) = e^{-0.05}$, and the price of the call option was previously computed to be $C = 5.73$. Evaluating P, we see that

$$P = 5.73 - 40 + 45 \times e^{-0.05} = 8.54 \qquad \square$$

The same argument that we used to price European call options can be extended to price **any** European derivative, i.e. any instrument that generates a pay-off at $t = 1$ depending on the value of the stock. E.g. consider a European derivative with payoff

$$V(1) = \begin{cases} V_u(1) & \text{if } S(1) = Su, \\ V_d(1) & \text{if } S(1) = Sd. \end{cases}$$

Then by Theorem 5.30 the no-arbitrage price of this option is

$$V(0) = D(1)\,\widetilde{\mathbb{E}}(V(1)) = D(1)\big(\tilde{p}_u V_u(1) + \tilde{p}_d V_d(1)\big).$$

Example 6.9. Consider a single-period binomial model with parameters $S = S(0) = 60$, $u = 2$, $d = 2/3$ and nominal interest rate $r = 4\%$. Verify that the no-arbitrage principle holds and compute the no-arbitrage price of a European derivative with payoff

$$V(1) = \begin{cases} 40 & \text{if } S(1) = Su, \\ 20 & \text{if } S(1) = Sd. \end{cases}$$

Solution. The no-arbitrage principle holds since

$$d < 1 + r < u \Leftrightarrow 2/3 < 1.04 < 2,$$

and the risk-neutral distribution is

$$\tilde{p}_u = \frac{1.04 - 2/3}{2 - 2/3} = 0.28 \qquad \text{and} \qquad \tilde{p}_d = 1 - \tilde{p}_u = 0.72.$$

Hence the no-arbitrage price of the option is

$$V(0) = \frac{1}{1.04}\,(0.28 \times 40 + 0.72 \times 20) = 24.6. \qquad \square$$

So far we have only considered European options. For American options the pricing formulae have to be adapted slightly to take in account the ability to exercise the option early.

The basic idea is to interpret the European pricing formula in Theorem 5.30 as the **value** of the American option **were it forbidden to exercise the option early**. Since this might be **less** than the payoff of the option if it were exercised **immediately**, the value of the American option at $t = 0$ is the **maximum** of the value computed via Theorem 5.30 and the value were the option to be exercised immediately.

This analysis also allows us to determine the **optimal exercise** strategy – it is only optimal to exercise early (at $t = 0$) if the value of exercising early exceeds the European value.

Example 6.10. Consider a single-period binomial model with parameters $S = S(0) = 40$, $u = 6/5$, $d = 4/5$ and continuously-compounded interest rate $r = 5\%$. Find the no-arbitrage price P of an American put option with strike price $K = 52$. Is it optimal to exercise early?

Proof. The no-arbitrage condition holds since

$$4/5 < e^{0.05} < 6/5,$$

and the risk-neutral distribution is given by

$$\tilde{p}_u = \frac{e^r - d}{u - d} = \frac{e^{0.05} - 4/5}{6/5 - 4/5} = 0.628 \qquad \text{and} \qquad \tilde{p}_d = 1 - \tilde{p}_u = 0.372.$$

If the option were European (so must be exercised at $t = 1$) its payoff would be

$$\text{Payoff} = \begin{cases} (K - S(1))^+ = (52 - 40 \times 6/5)^+ = 4 & \text{if } S(1) = S(0)u \\ (K - S(1))^+ = (52 - 40 \times 4/5)^+ = 20 & \text{if } S(1) = S(0)d \end{cases}$$

Hence, according to the pricing formula in Theorem 5.30, the value of the option **were it forbidden to exercised early** would be

$$e^{-r}\left(\tilde{p}_u \times 4 + \tilde{p}_d \times 20\right) = e^{-0.05}\left(0.628 \times 4 + 0.372 \times 20\right) = 9.46\,.$$

On the other hand, if the option were exercised at $t = 0$ it would generate a payoff of

$$(K - S(0))^+ = (52 - 40)^+ = 12.$$

Since the option is worth more if exercised immediately ($= 12$) than its European value ($= 9.464$), its no-arbitrage price equals the current payoff, i.e. $P = 12$, and it **is** optimal to exercise at $t = 0$. ☐

Example 6.11. Consider a single-period binomial model with parameters $S = S(0) = 60$, $u = 3/2$, $d = 2/3$ and nominal interest rate $r = 5\%$. Find the no-arbitrage price C of an American call option with strike price $K = 55$. Is it optimal to exercise early?

Solution. The no-arbitrage condition holds since

$$2/3 < 1.05 < 3/2,$$

and the risk-neutral distribution is given by

$$\tilde{p}_u = \frac{1 + r - d}{u - d} = \frac{1.05 - 2/3}{3/2 - 2/3} = 0.46 \qquad \text{and} \qquad \tilde{p}_d = 1 - \tilde{p}_u = 0.54.$$

If the option were European, its payoff at $t = 1$ would be

$$\text{Payoff} = \begin{cases} (S(1) - K)^+ = (60 \times 3/2 - 55)^+ = 35 & \text{if } S(1) = S(0)u \\ (S(1) - K)^+ = (60 \times 2/3 - 55)^+ = 0 & \text{if } S(1) = S(0)d \end{cases}$$

Hence, according to the pricing formula in Theorem 5.30, the value of the option **were it forbidden to be exercised at** $t = 0$ would be

$$\frac{1}{1.05}\left(\tilde{p}_u \times 35 + \tilde{p}_d \times 0\right) = \frac{1}{1.05}\left(0.46 \times 35\right) = 15.33\,.$$

On the other hand, if the option were exercised immediately (at $t = 0$) it would generate a payoff of

$$(S(0) - K)^+ = (60 - 55)^+ = 5.$$

Since the option is worth less if exercised immediately ($= 5$) than its current value if it were a European option ($= 15.33$), its no-arbitrage price equals its value were it to be exercised at $t = 1$, i.e. $C = 15.33$, and it is **not** optimal to exercise early. ☐

Remark 6.12. These examples are in agreement with the general phenomena that we proved in the previous chapter: that American call options should never be exercised early (unlike American put options, which should be exercised early in some situations).

6.1.3 Replicating portfolios and the delta-hedging formula

In the previous subsection we priced European options (and more general European derivatives) in the single-period binomial model using the risk-neutral distribution. An alternate method to price European derivatives in this model is via a 'replicating portfolio' argument; this is just another version of the no-arbitrage principle.

Recall that the aim of 'replicating portfolios' arguments is to set up two portfolios that have the same value at some future time. Suppose that at $t = 0$ we form a portfolio consisting of Ψ units of share and Φ amount of money deposited in the bank; this portfolio is worth

$$P(0) = \Psi S + \Phi.$$

Now, at $t = 1$ this portfolio is worth

$$P(1) = \begin{cases} \Psi\, S\, u + (1+r)\Phi, & \text{if } S(1) = Su, \\ \Psi\, S\, d + (1+r)\Phi, & \text{if } S(1) = Sd. \end{cases}$$

Note that we are assuming the case of simple interest here.

Now suppose there is a certain European derivative with payoff at $t = 1$ equal to

$$V(1) = \begin{cases} V_u(1), & \text{if } S(1) = Su, \\ V_d(1), & \text{if } S(1) = Sd. \end{cases} \tag{22}$$

Is there a way to choose (Ψ, Φ) such that $P(1)$ replicates the payoff $V(1)$ at $t = 1$? For this to occur we would need the following two equations in two unknowns to be satisfied:

$$\Psi\, S\, u + (1+r)\Phi = V_u(1) \qquad \text{and} \qquad \Psi\, S\, d + (1+r)\Phi = V_d(1).$$

Solving these equations for Ψ and Φ yields

$$\Psi = \frac{V_u(1) - V_d(1)}{S\, u - S\, d} \qquad \text{and} \qquad \Phi = \frac{1}{1+r}\frac{(-V_u(1)\, S\, d + V_d(1)\, S\, u)}{S\, u - S\, d} \tag{23}$$

as the replicating portfolio for the derivative; this is known as the **delta-hedging formula**. If Ψ is negative, the portfolio consists of a **negative** number of shares, and we interpret this as shares being **(short-)sold**. If Φ is negative, the portfolio consists of a **negative** amount of money deposited in the bank, and we interpret this as **money borrowed from the bank**.

There are two ways in which this formula might be applied in practice:

1. Since the initial value $P(0)$ of the replicating portfolio is equal to the initial price $V(0)$ of the derivative (a consequence of 'the law of one price'), we deduce that

   $$V(0) = P(0) = \Psi S + \Phi,$$

   which is an alternative method of pricing derivatives in the single-period binomial model.

2. The risk of selling options (i.e. for an investment bank) can be mitigated by constructing the replicating portfolio as a counterbalance; this procedure is known as 'hedging'. The 'delta' in the formula's name refers to the fact that the formula for Ψ can be considered the 'discrete derivative' of $V(1)$ with respect to the share price $S(1)$, and derivatives are traditionally denoted by the symbol Δ.

Exercise 6.13. By plugging in the expressions for Ψ, Φ, $\tilde{p}_u$ and $\tilde{p}_d$, verify that the single-periodic binomial model pricing formula via replicating portfolios

$$V(0) = \Psi S + \Phi$$

coincides with the pricing formula via the risk-neutral distribution

$$V(0) = \frac{1}{1+r}\widetilde{\mathbb{E}}[V(1)].$$

Exercise 6.14. Find the equivalent expression for Ψ and Φ in (23) in the case of continuously-compounded interest.

Taking the point of view of an investment bank, for which the option price $V(0)$ is known, the delta-hedging formula takes the following simpler form:

Proposition 6.15 (Delta-hedging formula)**.** *A European derivative with current price $V(0)$ and payoff given by (22) can be replicated with a portfolio consisting of*

$$\Psi = \frac{V_u(1) - V_d(1)}{S\,u - S\,d}$$

shares (bought or short-sold, depending on if this is positive or negative), and the amount

$$\Phi = V(0) - S\Psi$$

deposited in the bank (or borrowed from the bank, if this is negative).

Example 6.16. Find the replicating portfolio for the European call option in Example 6.7.

Solution. Recall that $S = S(0) = 40$, $K = 45$, $u = 3/2$, $d = 3/4$, and interest is continuously-compounded at nominal rate $r = 5\%$. The pay-off of the option is $(Su - K)^+ = 15$ if $S(1) = uS$, and $(Sd - K)^+ = 0$ if $S(1) = dS$. Applying the delta-hedging formula, the replicating portfolio consists of

$$\Psi = \frac{15-0}{60-30} = \frac{15}{30} = 1/2$$

units of shares, and the remaining

$$\Phi = C - \Phi\,S = C - (1/2)\cdot 40 = C - 20$$

deposited in, or loaned from, the bank. Since in Example 6.7 we calculated that $C = 5.73$,

$$\Phi = 5.73 - 20 = -14.27.$$

i.e., to construct the portfolio we need to loan 14.27 from the bank. □

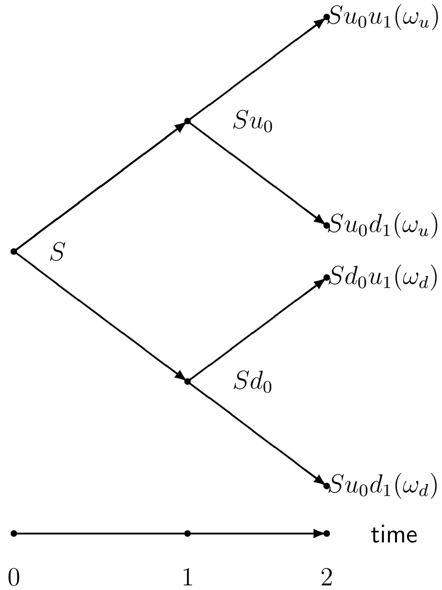


Figure 2: A share price in the two-period binomial model

6.2 The two-period binomial model

We now extend the one-period binomial model to model the share over two successive time-periods. This is known as the 'two-period binomial model', and is illustrated in Figure 2.

To help discuss the various nodes of the binomial tree, we introduce some notation. We will denote by ω_u and ω_d the nodes at $t = 1$ corresponding to the event that the share moved 'up' and 'down' respectively, and by ω_{uu}, ω_{ud}, ω_{du} and ω_{dd} the nodes at $t = 2$ corresponding to the events that the share moved 'up' and then 'up' again, 'up' and then 'down', etc.

6.2.1 Options pricing in the two-period binomial model

The no-arbitrage principle holds if and only if it holds for each of the single-period binomial models that make up the two-period model, in other words, using the notation of Figure 2, if and only if

$$d_0 < 1/D(1) < d_1 \quad \text{and} \quad \max\{d_1(\omega_u), d_1(\omega_d)\} < 1/D(1;2) < \min\{u_1(\omega_u), u_1(\omega_d)\},$$

where $D(1;2) = D(2)/D(1)$. Observe that $1/D(1) = 1/D(1;2) = 1 + r$ in the case of interest compounded once per time-period, and $1/D(1) = 1/D(1;2) = e^r$ in the case of continuously-compounded interest.

Let us suppose that the no-arbitrage principle holds. Consider a European option in the two-period binomial model whose payoff at time 2 given by

$$V(2) = \begin{cases} V_{uu}(2) & \text{if } \omega = \omega_{uu} \\ V_{ud}(2) & \text{if } \omega = \omega_{ud} \\ V_{du}(2) & \text{if } \omega = \omega_{du} \\ V_{dd}(2) & \text{if } \omega = \omega_{dd}. \end{cases}$$

The no-arbitrage price of this option at time 0 can be found by applying a **backwards recursion** algorithm. To be precise, since the value of the option at time 2 is known (it is equal to the payoff), we begin by calculating the no-arbitrage price of the option at time 1, at both the

ω_u and ω_d states, by solving the corresponding single-period binomial models between times 1 and 2. Then we calculate the no-arbitrage price at time 0 by solving the single-period binomial tree between times 0 and 1. This algorithm is illustrated in the following example:

Example 6.17. Consider the two-period binomial model for a share price depicted in Figure 3. Assume that interest is continuously-compounded at nominal rate (per time-period) of 5%. Verify the no-arbitrage condition, and find the no-arbitrage price of a European call option with strike price 45 and expiry $T = 2$.

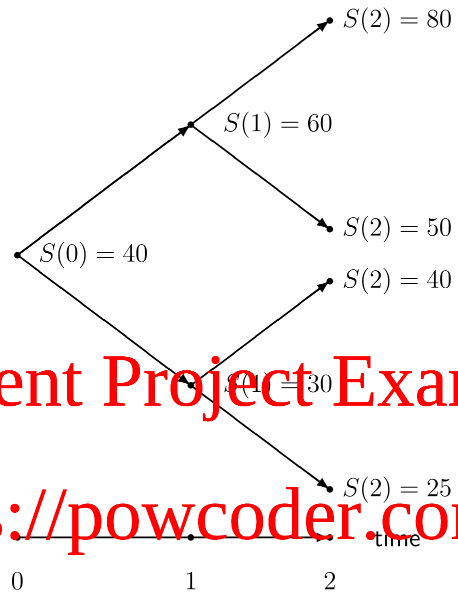


Figure 3: The share price tree in Example 6.17

Solution. In terms of the parameters in Figure 2, the model has parameters values $S = 40$, $(u_0, d_0) = (3/2, 3/4)$ and $(u_1(\omega_u), d_1(\omega_u)) = (u_1(\omega_d), d_1(\omega_d)) = (4/3, 5/6)$, since, for example, considering states ω_{du} and ω_{dd} we see that

$$u_1(\omega_d) = \frac{40}{30} = \frac{4}{3} \qquad \text{and} \qquad d_1(\omega_d) = \frac{25}{30} = \frac{5}{6}.$$

The no-arbitrage condition holds since

$$5/6 = \max\{3/4, 5/6\} < e^{0.05} < \min\{3/2, 4/3\} = 4/3.$$

Let $V(n)$ denote the value of the option at time n. As described above, we work backwards through the tree. The value of the option at time 2 is equal to its payoff, i.e.,

$$V(2) = (S(2) - K)^+ = \begin{cases} (80-45)^+ = 35 & \text{if } \omega = \omega_{uu}, \\ (50-45)^+ = 5 & \text{if } \omega = \omega_{ud}, \\ (40-45)^+ = 0 & \text{if } \omega = \omega_{du}, \\ (25-45)^+ = 0 & \text{if } \omega = \omega_{dd}. \end{cases}$$

Figure 4 shows this information.

We next compute $V_u(1)$ and $V_d(1)$ by solving the respective single-period binomial model between times 1 and 2. Here we view the option as a asset, existing between times 1 and

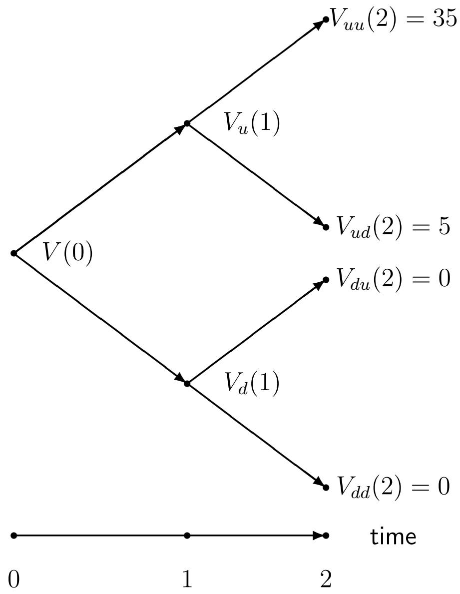


Figure 4: The price tree after computing the terminal values

2, with payoff at time 2 given by the already computed values of $V(2)$; this fits into our framework for computing no-arbitrage prices of any European derivative in the single-period binomial model.

Let us first consider $V_u(1)$. We define $(\tilde{p}_u(\omega_u), \tilde{p}_d(\omega_u))$ to be the risk-neutral distribution associated to the single-period binomial model from $t = 1$ to $t = 2$ in the state $\omega = \omega_u$. We can calculate this risk-neutral distribution to be

$$\tilde{p}_u(\omega_u) = \frac{e^r - d_1(\omega_u)}{u_1(\omega_u) - d_1(\omega_u)} = \frac{e^{0.05} - 5/6}{4/3 - 5/6} = 0.4359 \quad \text{and} \quad \tilde{p}_d(\omega_u) = 1 - \tilde{p}_u(\omega_u) = 0.5641.$$

Hence,

$$\begin{aligned} V_u(1) &= e^{-r}\left(\tilde{p}_u(\omega_u)V_{uu}(2) + \tilde{p}_d(\omega_u)V_{ud}(2)\right) \\ &= e^{-0.05}\left(0.43588 \times 35 + 0.5641 \times 5\right) \\ &= 17.195. \end{aligned}$$

Similarly, for $V_d(1)$ the risk-neutral distribution for the corresponding single-period model is also

$$\tilde{p}_u(\omega_d) = \frac{e^r - d_1(\omega_d)}{u_1(\omega_d) - d_1(\omega_d)} = \frac{e^{0.05} - 5/6}{4/3 - 5/6} = 0.4359 \quad \text{and} \quad \tilde{p}_d(\omega_d) = 1 - \tilde{p}_u(\omega_d) = 0.5641,$$

and hence

$$\begin{aligned} V_d(1) &= e^{-r}\left(\tilde{p}_u(\omega_d)V_{du}(2) + \tilde{p}_d(\omega_d)V_{dd}(2)\right) \\ &= e^{-0.05}\left(0.4359 \times 0 + 0.5641 \times 0\right) \\ &= 0. \end{aligned}$$

Figure 5 shows this information.

Finally we consider the single-period binomial model between times 0 and 1. The risk-neutral measure corresponding to this model is

$$\tilde{p}_u = \frac{e^r - d_0}{u_0 - d_0} = \frac{e^{0.05} - 3/4}{3/2 - 3/4} = 0.4017 \qquad \text{and} \qquad \tilde{p}_d = 1 - \tilde{p}_u = 0.5983$$

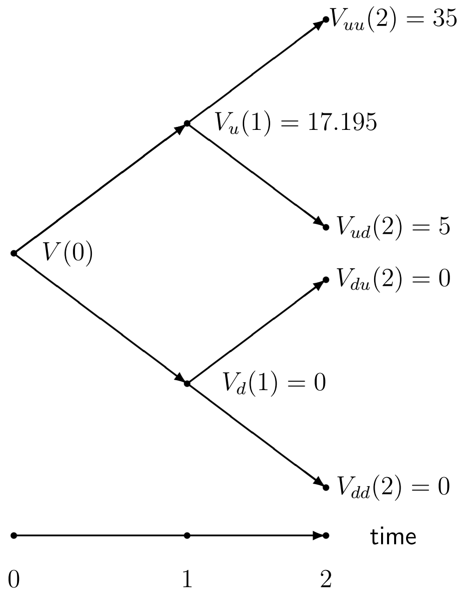


Figure 5: The price tree after computing the values at times 1 and 2

and so

$$
\begin{aligned}
V(0) &= e^{-r}\left(\tilde{p}_u V_u(1) + \tilde{p}_d V_d(1)\right) \\
&= e^{-0.05}\left(0.4017 \times 17.195 + 0.5983 \times 0\right) \\
&= 6.570
\end{aligned}
$$

In conclusion, we have shown that the fair price of the option is 6.570. □

Remark 6.18. The same backwards recursion algorithm can be used to find the value of any European derivative in the two-period model.

To price American options we need to modify slightly the procedure explained above, since we need to account for the possibility that it is optimal to exercise these options early. As such, at each stage of the procedure (say at time $t = i$), we must consider whether the option is in fact worth more if it is exercised immediately, i.e., whether its 'exercise value' exceeds its 'European value'. If it does, then **the 'exercise value' must be used when continuing the backwards recursion algorithm** to find the value of the option at time $i - 1$, since the 'exercise value', and not the 'European value', reflects the true value of the option.

As a by-product of this procedure, we also obtain **the optimal exercise strategy** for the American option, i.e., we determine under what circumstances is it optimal to exercise the American option early. These are simply the states at which the 'exercise value' was found to exceed the 'European value'.

Example 6.19. Consider the two-period binomial model for a share price depicted in Figure 6. Assume that interest is continuously-compounded at nominal rate 5%. Verify the no-arbitrage condition, find the no-arbitrage price of an American put option with strike 52 and expiry $T = 2$, and determine the optimal exercise strategy.

Proof. In terms of the parameters in Figure 2, the model has parameters values $S = 50$ and $(u_0, d_0) = (u_1(\omega_u), d_1(\omega_u)) = (u_1(\omega_d), d_1(\omega_d)) = (6/5, 4/5)$. The no-arbitrage condition holds since

$$4/5 < e^{0.05} < 6/5.$$

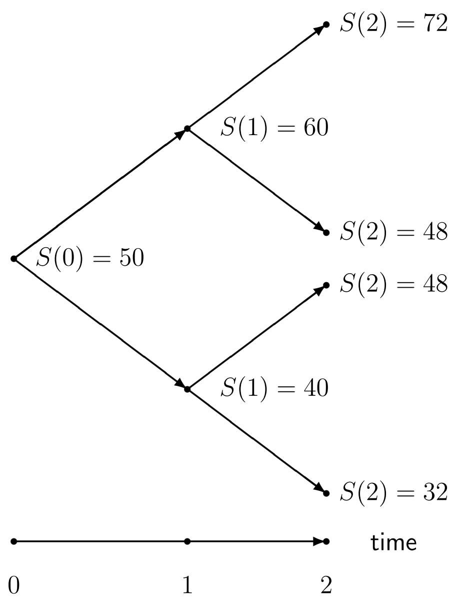


Figure 6: The share price tree in Example 6.19

Hence the risk-neutral distribution **is the same at each node of the tree**, and is given by

$$\tilde{p}_u = \frac{e^r - d}{u - d} = \frac{e^{0.05} - 4/5}{6/5 - 4/5} = 0.6282 \qquad \text{and} \qquad \tilde{p}_d = 1 - \tilde{p}_u = 0.3718\,.$$

The value of the option at time 2 is equal to its payoff, i.e.,

$$V(2) = (K - S(2))^+ = \begin{cases} (52 - 72)^+ = 0 & \text{if } \omega = \omega_{uu}, \\ (52 - 48)^+ = 4 & \text{if } \omega = \omega_{ud}, \\ (52 - 48)^+ = 4 & \text{if } \omega = \omega_{du}, \\ (52 - 32)^+ = 20 & \text{if } \omega = \omega_{dd}. \end{cases}.$$

Let us consider the value of the option at time 1, beginning with state ω_u. Solving the corresponding single-period binomial model, we see that the **European value** of the option at this state is

$$e^{-r}\left(\tilde{p}_u V_{uu}(2) + \tilde{p}_d V_{ud}(2)\right) = e^{-0.05}\left(0.6282 \times 0 + 0.3718 \times 4\right) = 1.4147\,.$$

On the other hand, the **exercise value** of the option at this state is

$$(K - S_u(1))^+ = (52 - 60)^+ = 0$$

which is less than the European value. Hence the value of the option $V_d(1)$ at this state is equal to its European value of 1.4147.

Turning to state ω_d, the **European value** of the option at this state is

$$e^{-r}\left(\tilde{p}_u V_{du}(2) + \tilde{p}_d V_{dd}(2)\right) = e^{-0.05}\left(0.6282 \times 4 + 0.3718 \times 20\right) = 9.4639\,.$$

On the other hand, the **exercise value** of the option at this state is

$$(K - S_d(1))^+ = (52 - 40)^+ = 12,$$

and so in this case the exercise value **exceeds** the European value. Therefore it is optimal to exercise at this state, and the value of the option $V_d(1)$ is equal to its **exercise value** of 12.

Finally, we consider the value of the option at time 0. Solving the corresponding single-period binomial model, we see that the **European value** of the option at this state is

$$e^{-r}\left(\tilde{p}_u V_u(1) + \tilde{p}_d V_d(1)\right) = e^{-0.05}\left(0.6282 \times 1.4147 + 0.3718 \times 12\right) = 5.0894\,.$$

(Note that we used the **exercise value** ($= 12$) in the second term!) On the other hand, the **exercise value** of the option at this state is

$$(K - S(0))^+ = (52 - 50)^+ = 2$$

which is less than the European value. Hence the value of the option $V(0)$ at this state is equal to its European value of 5.0894.

In conclusion, the no-arbitrage price of the option is 5.0894. We deduce the optimal exercise strategy by considering at which nodes in the backwards recursion algorithm the exercise value of the option exceeded the European value. In this case, this occurred only at the state ω_d, and so the optimal exercise strategy is to only exercise early at the state ω_d, i.e. if and only if the first movement of the share price is down. □

Exercise 6.20. Assuming a two-period binomial model as in Exercise 6.19, find the no-arbitrage price and optimal exercise strategy for an American call option with strike 40. Does this agree with the general fact that American call options should never be exercised early?

Solution. We have already verified that no-arbitrage condition holds, and calculated that the risk-neutral distribution is the same at each node of the tree and is given by

$$\tilde{p}_u = \frac{e^r - d}{u - d} = \frac{e^{0.05} - 4/5}{6/5 - 4/5} = 0.6282 \quad \text{and} \quad \tilde{p}_d = 1 - \tilde{p}_u = 0.3718\,.$$

The value of the option at time 2 is equal to its payoff, i.e.,

$$V(2) = (S(2) - 40)^+ = \begin{cases} (72 - 40)^+ = 32 & \text{if } \omega = \omega_{uu}, \\ (48 - 40)^+ = 8 & \text{if } \omega = \omega_{ud}, \\ (48 - 40)^+ = 8 & \text{if } \omega = \omega_{du}, \\ (32 - 40)^+ = 0 & \text{if } \omega = \omega_{dd}. \end{cases}$$

Let us consider the value of the option at time 1, beginning with state ω_u. Solving the corresponding single-period binomial model, we see that a **European value** of the option at this state is

$$e^{-r}\left(\tilde{p}_u V_{uu}(2) + \tilde{p}_d V_{ud}(2)\right) = e^{-0.05}\left(0.6282 \times 32 + 0.3718 \times 8\right) = 21.95\,.$$

On the other hand, the **exercise value** of the option at this state is

$$(S_u(1) - K)^+ = (60 - 40)^+ = 20$$

which is less than its European value. Hence the value of the option $V_u(1)$ at this state is equal to its European value of 21.95.

Turning to state ω_d, the **European value** of the option at this state is

$$e^{-r}\left(\tilde{p}_u V_{du}(2) + \tilde{p}_d V_{dd}(2)\right) = e^{-0.05}\left(0.6282 \times 8 + 0.3718 \times 0\right) = 4.78\,.$$

On the other hand, the **exercise value** of the option at this state is

$$(S_d(1) - K)^+ = (40 - 40)^+ = 0,$$

and so again in this case the exercise value is less than its European value. Hence the value of the option at this state is equal to its European value of 4.78.

Finally, we consider the value of the option at time 0. Solving the corresponding single-period binomial model, we see that the **European value** of the option at this state is

$$e^{-r}\left(\tilde{p}_u V_u(1) + \tilde{p}_d V_d(1)\right) = e^{-0.05}\left(0.6282 \times 21.95 + 0.3718 \times 4.78\right) = 14.81\,.$$

On the other hand, the **exercise value** of the option at this state is

$$(S(0) - 40)^+ = (50 - 40)^+ = 10$$

which is less than the European value. Hence the value of the option $V(0)$ at this state is equal to its European value of 14.81.

In conclusion, the no-arbitrage price of the option is 14.81, and the optimal exercise strategy is to **never exercise the option early**. □

6.2.2 Replicating portfolios in the two-period binomial model

Just like in the one-period model, we can also analyse European options in the two-period binomial model by considering **replicating portfolios**. Once again we reduce the two-period binomial model to two successive single-period binomial models, in each of which we apply the 'delta hedging' formula.

Note that the replicating portfolio will be **different at each time-step**, and in the case $t = 1$ it may also depend on the node ω_u and ω_d of the binomial tree. In practice this means that the replicating portfolio needs to be **adjusted dynamically** in order to ensure that it replicates the payoff of the option for all possible outcomes of the share price in the model.

6.2.3 Option pricing via the two-period risk-neutral distribution

Up until now we have been working with risk-neutral distributions that are defined for single-period binomial models. In fact, it is possible to define a risk-neutral distribution directly for the two-period binomial model. This leads to an **alternative method** to price European options in the two-period binomial model that **avoids** the backward recursion algorithm.

Recall that $(\tilde{p}_u, \tilde{p}_d)$ denotes the risk-neutral distribution that corresponds to the single-period binomial model between times 0 and 1, and $(\tilde{p}_u(\omega_u), \tilde{p}_d(\omega_u))$ and $(\tilde{p}_u(\omega_d), \tilde{p}_d(\omega_d))$ denote the risk-neutral distributions that correspond to the single-period model between times 1 and 2 in the states ω_u and ω_d respectively. Then the risk-neutral distribution at $t = 2$ of the two-period binomial model, written as $\widetilde{\mathbb{P}}$, is equal to

$$\widetilde{\mathbb{P}}(\omega) = \begin{cases} \tilde{p}_u \times \tilde{p}_u(\omega_u) & \text{if } \omega = \omega_{uu}, \\ \tilde{p}_u \times \tilde{p}_d(\omega_u) & \text{if } \omega = \omega_{ud}, \\ \tilde{p}_d \times \tilde{p}_u(\omega_d) & \text{if } \omega = \omega_{du}, \\ \tilde{p}_d \times \tilde{p}_d(\omega_d) & \text{if } \omega = \omega_{dd}. \end{cases}$$

Recall that by the general no-arbitrage pricing principle we can express the no-arbitrage price at $t = 0$ of any European option $V(0)$ as

$$V_0 = D(2)\,\widetilde{\mathbb{E}}(V(2)),$$

where $V(2)$ is the payoff of the option at time 2.

Example 6.21. Consider the two-period binomial model as in Exercise 6.17. Find the risk-neutral distribution for the share price $S(2)$ at time 2, and use it to price a European put option with strike 45 and expiry $T = 2$.

Solution. Recall that in Exercise 6.17 we calculated the risk-neutral distribution for the one-period binomial model at time $t = 0$,

$$\tilde{p}_u = 0.4017 \qquad \text{and} \qquad \tilde{p}_d = 0.5983,$$

and the risk-neutral distributions for the one-period binomials models at time $t = 1$ to be

$$\tilde{p}_u(\omega) = 0.4359 \quad \text{and} \quad \tilde{p}_d(\omega) = 0.5641$$

for both $\omega = \omega_u$ or $\omega = \omega_d$ (i.e. the risk-neutral probabilities are the same at $t = 1$ for both the 'up' and 'down' states). Hence we can calculate the overall risk-neutral distribution $\widetilde{\mathbb{P}}$ at $t = 2$ to be

$$\widetilde{\mathbb{P}}(\omega) = \begin{cases} \tilde{p}_u \times \tilde{p}_u(\omega_u) = 0.175 & \text{if } \omega = \omega_{uu}, \\ \tilde{p}_u \times \tilde{p}_d(\omega_u) = 0.227 & \text{if } \omega = \omega_{ud}, \\ \tilde{p}_d \times \tilde{p}_u(\omega_d) = 0.261 & \text{if } \omega = \omega_{du}, \\ \tilde{p}_d \times \tilde{p}_d(\omega_d) = 0.337 & \text{if } \omega = \omega_{dd}. \end{cases}$$

Notice that these probabilities sum to one, as they should. Via the general pricing formula, the price of the put option is

$$P = e^{-2r}\widetilde{\mathbb{E}}\big((K - S(2))^+\big).$$

Considering the possible values of $S(2)$ from Exercise 6.17, only the outcomes $S(2) = 40$ if $\omega = \omega_{du}$ and $S(2) = 25$ if $\omega = \omega_{dd}$, are 'in-the-money'. Evaluating the expectation with respect to the risk-neutral measure, we have

$$P = e^{-0.05\times 2}\big(0.261 \times (45 - 40) + 0.337 \times (45 - 25)\big) = 7.28. \qquad \square$$

Remark 6.22. We can verify that our answers to Exercises 6.17 and 6.21 satisfy put-call parity. We found the prices of European call and put options with strike $K = 45$ and expiry $T = 2$ to be $C = 6.57$ and $P = 7.28$. Since the initial stock price in the model was $S = 40$, we indeed have put-call parity:

$$P + S = 7.28 + 40 = 47.3 \quad \text{and} \quad C + D(2)K = 6.57 + e^{-2\times 0.05}45 = 47.3.$$

Remark 6.23. Although this method is perhaps simpler than the backwards recursion algorithm, it is less general since it only works for European options. In particular, it cannot be used to price American options.

6.3 The multi-period binomial model

The two-period binomial model can be extended to any number of time-periods; this is called the **multi-period binomial model**. A major potential disadvantage of this model is that the number of possible states grows exponentially; at $t = n$, the share price can be in 2^n possible states. For this reason we consider a simplified model in which the 'up' and 'down' factors u and d are equal for every state. This can be interpreted as an independence assumption: the potential range of future price movements does not depend on the current state.

Suppose a share has initial price $S(0) = S$. We model the share price $S(i)$ for $i = 1, \dots, n$ by specifying that $S(i)$ is either equal to $u\,S(i-1)$ or to $d\,S(i-1)$, where $0 < d < u$ are parameters.

Let us consider the possible share prices $S(n)$ at time n. Since the share price either moves 'up' or 'down' at each step, the possible values of $S(n)$ are

$$S(n) = S\,u^k d^{n-k}$$

where $k = \{0, 1, \dots, n\}$ represents the number of 'up' steps out of the n total steps (the number of 'down' steps is therefore $n-k$). Notice that $S(n)$ can take only $n+1$ possible values; for large n this is an enormous simplification over the general multi-period binomial model, in which the share price has 2^n possible states.

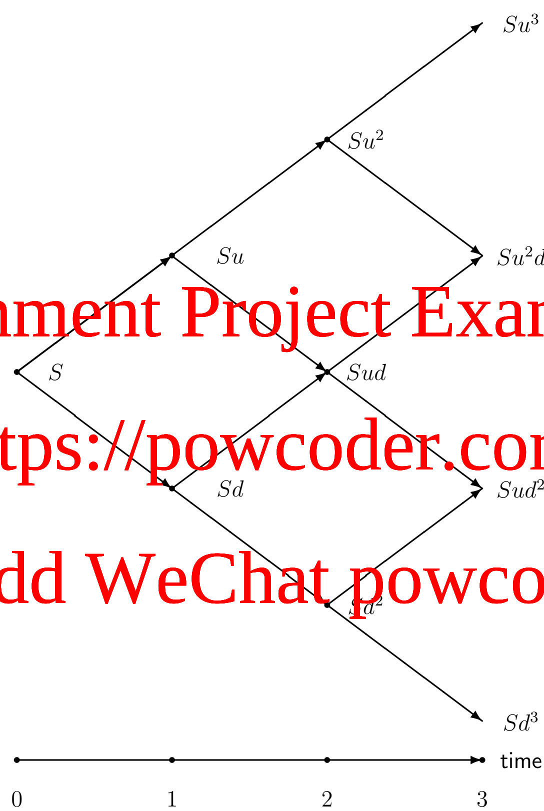


Figure 7: A share in the 3-period multi-period binomial model

6.3.1 Options pricing in the multi-period binomial model

Let $D(1)$ be the discount factor per time-period. Just as in the single-period case, the no-arbitrage principle holds if and only if

$$d < 1/D(1) < u,$$

in which case the risk-neutral distribution corresponding to each single-period model is

$$\tilde{p}_u = \frac{1/D(1) - d}{u - d} \qquad \text{and} \qquad \tilde{p}_d = \frac{u - 1/D(1)}{u - d}.$$

As we did in Section 6.2.3, we will analyse the multi-period model by considering the risk-neutral distribution for $S(n)$. Recall that

$$S(n) = S\,u^k d^{n-k}$$

where $k = \{0, 1, \ldots, n\}$ represents the number of 'up' steps. An equivalent way to write this is to introduce the random variables

$$X_i = \begin{cases} 1\,, & \text{if } S(i) = S(i-1)\,u \\ 0\,, & \text{if } S(i) = S(i-1)\,d \end{cases}, \qquad K = \sum_{i=1}^{n} X_i,$$

and to write

$$S(n) = S\,u^K d^{n-K}.$$

Notice that the X_i are Bernoulli trials that record a 'success' if the share moved upwards, and a 'failure' if the share moved downwards. Since each X_i is an independent Bernoulli($\tilde{p}_u$) random variable under the risk-neutral distribution, $K = \sum_{i=1}^{n} X_i$ is a *Binomial* random variable with distribution

$$\widetilde{\mathbb{P}}(K = k) = \binom{n}{k} \tilde{p}_u^k (1 - \tilde{p}_u)^{n-k}.$$

Hence we have:

Proposition 6.24. *The risk-neutral distribution for $S(n)$ is given by*

$$S(n) = S u^k d^{n-k} \quad \textit{with probability} \binom{n}{k} \tilde{p}_u^k (1 - \tilde{p}_u)^{n-k}, \quad k = 0, 1, \ldots, n\,.$$

We can now use the general no-arbitrage pricing theory to price European options in the model:

Theorem 6.25. *Assume that a share follows a multi-period binomial model with parameters u and d and interest rate r. Consider a European option with payoff at expiry time $t = n$ equal to*

$$V(n) = g(S(n))$$

for g a given payoff function. Then the no-arbitrage price at time 0 of the option is

$$V(0) = D(n) \sum_{k=0}^{n} g\left(S u^k d^{n-k}\right) \binom{n}{k} \tilde{p}_u^k (1 - \tilde{p}_u)^{n-k},$$

where $\tilde{p}_u = (1/D(1) - d)/(u - d)$.

Proof. By the general no-arbitrage pricing principle,

$$V(0) = D(n)\,\widetilde{\mathbb{E}}\,[V(n)] = D(n)\,\widetilde{\mathbb{E}}[g(S(n))].$$

The result follows by evaluating the expectation using the risk-neutral distribution for $S(n)$. □

Example 6.26. Assume interest is compounded once per time-period. The no-arbitrage price of a European call option with strike price K and expiry time n is

$$C = \frac{1}{(1+r)^n} \sum_{k=0}^{n} (S u^k d^{n-k} - K)^+ \binom{n}{k} \tilde{p}_u^k (1 - \tilde{p}_u)^{n-k}.$$

Example 6.27. Consider a share following a multi-period binomial model with parameters $S = 200$, $u = 1.1$ and $d = 0.9$, and interest rate per time-period $r = 1\%$. Find the no-arbitrage price of a European call option with strike $K = 425$ and expiry time 10.

Solution. The no-arbitrage condition holds since

$$d < 1 + r < u \Leftrightarrow 0.9 < 1.01 < 1.1$$

and so the the risk-neutral probabilities are

$$\tilde{p}_u = \frac{1 + r - d}{u - d} = 0.55 \qquad \text{and} \qquad \tilde{p}_d = 1 - \tilde{p}_u = 0.45\,.$$

The payoff of the call option is

$$(S(10) - K)^+ = (Su^k d^{n-k} - K)^+ = (200 \times 1.1^k \times 0.9^{\,10-k} - 425)^+.$$

This is 'in-the-money' (i.e. has positive payoff) if and only if

$$200 \times 1.1^k \times 0.9^{\,10-k} > 425.$$

Examining the possible values of k, if $k = 10$ then the left-hand side is $519 > 425$, whereas if $k \leq 9$ the left-hand side is at most $424 < 425$. Hence the only state that is 'in-the-money' is $k = 10$, which occurs with probability 0.55^{10}. Therefore the price of the option is

$$\begin{aligned} C &= \frac{1}{1.01^{10}} \widetilde{\mathbb{E}}[(S(10) - 425)^+] \\ &= \frac{1}{1.01^{10}} (519 - 425) \times 0.55^{10} = 0.215\,. \end{aligned}$$

□

Example 6.28. Suppose a share is modelled by assuming each month it either increases by 2% or decreases by 1%. The current price of the share is 130. Assume also that interest is continuously-compounded at rate 12% per year. Find the no-arbitrage price of 1 million European put options with strike 120 and expiry in one year.

Proof. The share follows a multi-period binomial model with parameters $S = 130$, $u = 1.02$ and $d = 0.99$, where each time-step is a month. The interest rate per time-step is

$$r = 12\%/12 = 1\%.$$

The no-arbitrage condition is satisfied since

$$d < e^r < u \Leftrightarrow 0.99 < 1.01 < 1.02$$

and the risk-neutral probabilities are

$$\tilde{p}_u = \frac{e^r - d}{u - d} = \frac{e^{0.01} - 0.99}{1.02 - 0.99} = 0.669 \qquad \tilde{p}_d = 1 - \tilde{p}_d = 0.332\,.$$

The payoff of each put option is

$$V(12) = (K - S(12))^+ = (120 - 130 \times 1.02^k \times 0.99^{\,12-k})^+,$$

which we calculate to be

$$V(12) = \begin{cases} 4.77 & \text{if } k = 0 \\ 1.28 & \text{if } k = 1 \\ 0 & \text{if } k \geq 2. \end{cases}$$

Since the outcome $k = 0$ occurs with probability 0.3317^{12} and the outcome $k = 1$ occurs with probability $\binom{12}{1} \times 0.6683 \times 0.3317^{11}$, the no-arbitrage price of the 1 million put options is

$$\begin{aligned} P &= 1{,}000{,}000 \times e^{-0.12} \widetilde{\mathbb{E}}[V(12)] \\ &= 1{,}000{,}000 \times e^{-0.12} \big(4.77 \times 0.332^{12} + 1.28 \times 12 \times 0.668 \times 0.332^{11}\big) \\ &= 56.1\,. \end{aligned}$$

□

7 From the multi-period binomial model to the Black-Scholes formula

In this final chapter of the course we derive the **Black-Scholes formula**, which is a celebrated pricing formula for European options first published in 1973. This formula arises from the Black-Scholes model, in which the stock price is assumed to follow a continuous-time stochastic process known as **geometric Brownian motion**. While this is outside the scope of the course (it will be studied in detail in **FM II**), we can nevertheless derive the Black-Scholes formula via an approximation argument using the multi-period binomial model.

7.1 General discrete-time models

The most general discrete-time model for the stock price would be to define $(S(i), i = 0, 1, \ldots)$ as a general sequence of random variables, which could be discrete or continuous; this is known as a **discrete time stochastic process**. It is convention to take $S(0)$ as deterministic (non-random), since the stock price is assumed known at time 0. Note that the time-step in such a model could be anything: a year, a week, a second etc.

One possible example of such a model is to take $S(i)$ as an i.i.d. sequence. This will **not** be a good model for the stock price for the same reasons that i.i.d. sequences are a poor model for interest rates (see Section 4.2), namely since (i) the price of stock at one point in time will generally be positively correlated with, and not independent to, the price at another point in time, (ii) the variance in the stock price should grow over time rather than stay constant.

A more realistic model is to define

$$S(n) = S(0) \times \prod_{i=1}^{n} Y_i,$$

where Y_i is a sequence of i.i.d. random variables, i.e. we assume that the **multiplicative increments** $S(n)/S(n-1)$ of the stock prices, rather than the prices themselves, are an i.i.d. sequence. The multi-period binomial model is an example of such a model; in this case Y_i are discrete random variables taking values u and d with some (real-world) probabilities p_u and p_d.

7.2 The log-normal process

As we have seen, in order to price options we need to be able to analyse the (risk-neutral) distribution of $S(n)$. One choice of model that facilitates this is to let Y_i be an i.i.d. sequence of **log-normal** random variables, i.e., defining

$$Y_i \sim \mathrm{LogNormal}(\mu, \sigma^2).$$

The resulting model for $S(n)$ is called the **log-normal process**, and a remarkable property of this model is that we can compute the distribution of $S(n)$ explicitly:

Theorem 7.1. *If $S(n)$ is modelled by the log-normal process, then*

$$S(n) \sim S(0) \times \mathrm{LogNormal}(n\mu, n\sigma^2).$$

Proof. Since

$$\frac{S(n)}{S(0)} = Y_1 Y_2 Y_3 \cdots Y_n,$$

and since $Y_1, Y_2, \ldots, Y_n$ are independent and distributed as $\mathrm{LogNormal}(\mu, \sigma^2)$ random variables, by Theorem 4.8 we have the result. $\square$

Example 7.2. Assume that the stock price follows a log-normal process with parameters $\mu = 0.0165$ and $\sigma = 0.0730$ and time-step equal to a week. Calculate the probability that

(a) The price increases over the first week,

(b) The price increases in each of the first two weeks;

(c) The price is higher at the end of the second week than at the starting time 0.

You may assume that $\Phi(0.23) = 0.591$ and $\Phi(0.32) = 0.626$.

Proof. Let $S(n)$ denote the stock price at the end of the n^{th} week.

(a) The desired probability is

$$P(S(1) > S(0)) = P\Big(\frac{S(1)}{S(0)} > 1\Big) = P\Big(\ln\frac{S(1)}{S(0)} > 0\Big).$$

Since

$$\ln\frac{S(1)}{S(0)} \sim \mathcal{N}(\mu, \sigma^2),$$

we deduce that

$$\begin{aligned} P(S(1) > S(0)) &= P\Big(\frac{\mathcal{N}(\mu,\sigma^2) - \mu}{\sigma} > -\frac{\mu}{\sigma}\Big) \\ &= 1 - \Phi\Big(-\frac{\mu}{\sigma}\Big) = \Phi\Big(\frac{\mu}{\sigma}\Big) = \Phi(0.23) = 0.591\,. \end{aligned}$$

(b) Let $p = 0.591$ be the probability that the price increases over the first week. Since the random variables $S(1)/S(0)$ and $S(2)/S(1)$ are i.i.d. the probability that the price increases in each of the first two weeks is $p^2 = 0.349$.

(c) Similarly to in (a), since

$$\ln\frac{S(2)}{S(0)} \sim \mathcal{N}(2\mu, 2\sigma^2),$$

the desired probability is

$$\begin{aligned} P(S(2) > S(0)) &= P\Big(\ln\frac{S(2)}{S(0)} > 0\Big) \\ &= P\Big(\frac{\ln\frac{S(2)}{S(0)} - 2\mu}{\sqrt{2}\sigma} > -\frac{2\mu}{\sqrt{2}\sigma}\Big) \\ &= 1 - \Phi\Big(-\frac{\sqrt{2}\mu}{\sigma}\Big) = \Phi\Big(\frac{\sqrt{2}\mu}{\sigma}\Big) = \Phi(0.32) = 0.626\,. \end{aligned}$$ □

7.3 The Black-Scholes formula

Suppose that a share is modelled using a log-normal process. The celebrated **Black-Scholes formula** for the price of European options is the result of applying the general no-arbitrage pricing theory that we developed in Chapter 5 to this model.

Theorem 7.3 (Black-Scholes formula)**.** *Suppose a share price is modelled by a log-normal process with initial value S and log-normal parameters (μ, σ^2). Assume interest is continuously-compounded with nominal rate r. Then the no-arbitrage price of a European call option with strike price K and maturity time T is*

$$C = C(S, T, K, \sigma, r) = S\Phi(\eta) - Ke^{-rT}\Phi(\eta - \sigma\sqrt{T}), \tag{24}$$

where

$$\eta = \frac{\ln \frac{S}{K} + rT}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T}.$$

Remark 7.4. By put-call parity, one can deduce a similar formula for European put options.

Remark 7.5. The parameters μ and σ are called the 'drift' and 'volatility' of the stock price respectively. Remarkably, the price C does not depend on the drift μ. One explanation is that μ affects only the 'real-world' probabilities, which are irrelevant in no-arbitrage pricing. A more intuitive explanation is that the price of a call option reflects its usefulness as a way for investors to manage risk, and since μ results in a **deterministic** average increase in the stock price, it is irrelevant to the ability of the option to manage risk.

Example 7.6. Suppose a share is currently traded at a price of £30 and interest is continuously-compounded at rate 8%. Assume that the volatility is $\sigma = 0.2$. Find the Black-Scholes price C of a European call option with expiry in 3 months and strike £34, and the price P of the European put option with the same strike and expiry (you may use $\Phi(1.00) = 0.841$ and $\Phi(1.10) = 0.864$).

Solution. If time is measured in years, we have the parameter settings $T = 3/12 = 0.25$, $r = 0.08$, $\sigma = 0.2$, $K = 34$, and $S = 30$. Hence

$$\eta = -1.00 \quad \text{and} \quad \eta - \sigma\sqrt{T} = -1.10,$$

and the Black-Scholes formula gives

$$C = 30\Phi(-1.00) - 34e^{-0.02}\Phi(-1.10) = 30(1 - 0.841) - 34e^{-0.02}(1 - 0.864) = 0.24.$$

By put-call parity, the price of put options is

$$P = C - S + Ke^{-rT} = 0.24 - 30 + 34e^{-0.02} = 3.57. \qquad \square$$

Remark 7.7. Note that the interest rate r is known to investors but the volatility σ is not, which raises the question of how to implement the formula in practice. One view is that past data can be used to approximate the volatility. Another view is that price data from options that are traded on the open market can be plugged into the formula to derive an 'implied volatility' (this requires a computer). In reality, practitioners use a mixture of these views.

To prove the Black-Scholes formula we will require an intermediate result about the **risk-neutral distribution** in the model. We will prove this in the following section.

Proposition 7.8. *Suppose a share price follows a log-normal process with initial value S and parameters (μ, σ^2), and assume that interest is continuously-compounded with nominal rate r. Then there exists a risk-neutral measure $\widetilde{\mathbb{P}}$ under which the share price follows a log-normal process with initial value S and parameters*

$$\tilde{\mu} = r - \frac{1}{2}\sigma^2 \qquad \textit{and} \qquad \tilde{\sigma}^2 = \sigma^2.$$

Proof of Theorem 7.3. By general principles of no-arbitrage pricing,

$$C = e^{-rT}\,\widetilde{\mathbb{E}}\big[(S(T)-K)^+\big],$$

where $\widetilde{\mathbb{E}}$ denotes expectation under the risk-neutral distribution. Making use of the indicator random variable $\mathbb{1}_{S(T)>K}$ (that equals 1 if $S(T)>K$ occurs and 0 otherwise), this can be written as

$$\begin{aligned} C &= e^{-rT}\,\widetilde{\mathbb{E}}\big[(S(T)-K)\mathbb{1}_{S(T)>K}\big] \\ &= e^{-rT}\,\widetilde{\mathbb{E}}\big[S(T)\mathbb{1}_{S(T)>K}\big] - e^{-rT}\,\widetilde{\mathbb{E}}\big[K\mathbb{1}_{S(T)>K}\big] \\ &= e^{-rT}\,\widetilde{\mathbb{E}}\big[S(T)\mathbb{1}_{S(T)>K}\big] - e^{-rT}K\,\widetilde{\mathbb{P}}(S(T)>K), \end{aligned} \tag{25}$$

since $\mathbb{E}[\mathbb{1}_A] = \mathbb{P}(A)$. By Proposition 7.8, under the risk-neutral distribution

$$S(T) \overset{d}{\sim} S \times \mathsf{LogNormal}(\tilde{\mu}T, \tilde{\sigma}^2 T) \overset{d}{\sim} S e^{(r-\sigma^2/2)T+\sigma\sqrt{T}Z}$$

where $Z \overset{d}{\sim} \mathcal{N}(0,1)$, and hence

$$\begin{aligned} S(T) > K &\Leftrightarrow S e^{(r-\sigma^2/2)\,T+\sigma\sqrt{T}\,Z} > K \\ &\Leftrightarrow (r-\sigma^2/2)\,T + \sigma\sqrt{T}\,Z > \ln K/S \\ &\Leftrightarrow Z > \frac{\ln K/S - (r-\sigma^2/2)\,T}{\sigma\sqrt{T}} \\ &\Leftrightarrow Z > \sigma\sqrt{T} - \frac{\ln S/K + r\,T}{\sigma\sqrt{T}} - \frac{1}{2}\sigma\sqrt{T} \\ &\Leftrightarrow Z > \sigma\sqrt{T} - \eta. \end{aligned}$$

Thus we simplify the two terms in (25) as

$$\begin{aligned} e^{-rT}\widetilde{\mathbb{E}}\big[S(T)\mathbb{1}_{S(T)>K}\big] &= e^{-rT}\mathbb{E}\big[S e^{(r-\sigma^2/2)\,T+\sigma\sqrt{T}\,Z}\mathbb{1}_{Z>\sigma\sqrt{T}-\eta}\big] \\ &= S\,\mathbb{E}\big[e^{(-\sigma^2/2)\,T+\sigma\sqrt{T}\,Z}\mathbb{1}_{Z>\sigma\sqrt{T}-\eta}\big] \end{aligned} \tag{26}$$

and

$$e^{-rT}K\,\widetilde{\mathbb{P}}\big[S(T)>K\big] = e^{-rT}K\,\mathbb{P}\big[Z>\sigma\sqrt{T}-\eta\big]. \tag{27}$$

To compute (26) we evaluate the expectation (using the p.d.f. of the standard normal):

$$\begin{aligned} S\,\mathbb{E}\big[e^{(-\sigma^2/2)T+\sigma\sqrt{T}\,Z}\mathbb{1}_{Z>\sigma\sqrt{T}-\eta}\big] &= S\int_{\sigma\sqrt{T}-\eta}^{\infty} e^{(-\sigma^2/2)T+\sigma\sqrt{T}\,z}\frac{1}{\sqrt{2\pi}}e^{-z^2/2}\,dz \\ &= S\int_{\sigma\sqrt{T}-\eta}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-((\sigma^2/2)T-\sigma\sqrt{T}\,z+z^2/2)}\,dz \\ &= S\int_{\sigma\sqrt{T}-\eta}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-(\sigma\sqrt{T}-z)^2/2}\,dz. \end{aligned}$$

Changing variables $y \mapsto \sigma\sqrt{T} - z$, the above equals

$$-S\int_{\eta}^{-\infty}\frac{1}{\sqrt{2\pi}}e^{-y^2/2}\,dy = S\int_{-\infty}^{\eta}\frac{1}{\sqrt{2\pi}}e^{-y^2/2}\,dy$$

which we can write in terms of the standard normal c.d.f. as $S\,\Phi(\eta)$. The term (27) can also be expressed in term of the standard normal c.d.f.:

$$e^{-rT}K\mathbb{P}\big[Z>\sigma\sqrt{T}-\eta\big] = e^{-rT}K\big(1-\Phi(\sigma\sqrt{T}-\eta)\big) = e^{-rT}K\Phi(\eta-\sigma\sqrt{T}).$$

Plugging the expressions for (26) and (27) into (25), we get the desired formula:

$$C = S\Phi(\eta) - Ke^{-rT}\Phi(\eta-\sigma\sqrt{T})\,. \qquad \square$$

7.3.1 Properties of the Black-Scholes formula

We briefly study the dependency of the Black-Scholes formula on the parameters S, T, K, σ, and r.

Proposition 7.9. *The partial derivatives of $C = C(S, T, K, \sigma, r)$ are:*

$$\frac{\partial C}{\partial K} = -e^{-rT}\Phi(\eta - \sigma\sqrt{T}) \ , \quad \frac{\partial C}{\partial S} = \Phi(\eta) \ , \quad \frac{\partial C}{\partial r} = KTe^{-rT}\Phi(\eta - \sigma\sqrt{T})$$

$$\frac{\partial C}{\partial \sigma} = S\sqrt{T}\Phi'(\eta) \quad \text{and} \quad \frac{\partial C}{\partial T} = \frac{\sigma}{2\sqrt{T}}S\Phi'(\eta) + Kre^{-rT}\Phi(\eta - \sigma\sqrt{T}).$$

Remark 7.10. In investment jargon these partial derivatives are known as the 'greeks' (since they are traditionally denoted by Greek letters) and are an important investment tool with which to analyse options. As an example, the continuous time analogue of the 'delta-hedging' formula states that the replicating portfolio should contain $\partial C/\partial S$ units of stock.

Remark 7.11. The above proposition shows that C is a decreasing function of K (when the other parameters remain constant), and is an increasing function of S, T, σ, r. This can be explained intuitively:

- We have already proven in Propositions 5.35 that the price of a European call option is a decreasing function of the strike K. This is a model-free result, so applies equally to the Black-Scholes formula.
- Similarly, the fact that C is an increasing function of S is a model-free result, as can be seen from the fact that $C = e^{-rT}\widetilde{\mathbb{E}}[(S(T) - K)^+]$ and since $(S(T) - K)^+$ is an increasing function of S.
- The fact that C increases with σ (and also T) can be explained by considering that the call option is a way to manage risk, and the more volatile the underlying asset (and the longer time to which the option applies) the more it should cost to manage risk.

Proof. We only prove the formula for $\partial C/\partial K$, since the others are similar. By the chain rule,

$$\begin{aligned}\frac{\partial C}{\partial K} &= S\Phi'(\eta)\frac{\partial \eta}{\partial K} - e^{-rT}\Phi(\eta - \sigma\sqrt{T}) - Ke^{-rT}\Phi'(\eta - \sigma\sqrt{T})\frac{\partial \eta}{\partial K} \\ &= \frac{\partial \eta}{\partial K}\left[S\Phi'(\eta) - Ke^{-rT}\Phi'(\eta - \sigma\sqrt{T})\right] - e^{-rT}\Phi(\eta - \sigma\sqrt{T}).\end{aligned}$$

It remains to show that

$$Ke^{-rT}\Phi'(\eta - \sigma\sqrt{T}) = S\Phi'(\eta).$$

First, rearranging the definition of η, we have that

$$\eta = \frac{\ln\frac{S}{K} + rT}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T} \quad \Leftrightarrow \quad \sigma\sqrt{T}\eta - rT - \frac{1}{2}\sigma^2 T = \ln\frac{S}{K}. \tag{28}$$

Recalling also that $\Phi'(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$, we have

$$\begin{aligned}K\,e^{-rT}\,\Phi'(\eta - \sigma\sqrt{T}) &= K\,e^{-rT}\,\frac{1}{\sqrt{2\pi}}e^{-(\eta-\sigma\sqrt{T})^2/2} \\ &= K\,\frac{1}{\sqrt{2\pi}}e^{-\eta^2/2}\,e^{\sigma\sqrt{T}\eta - rT - \frac{1}{2}\sigma^2 T} \\ &\overset{(28)}{=} K\,\Phi'(\eta)\,e^{\ln(S/K)} \\ &= S\Phi'(\eta)\,.\end{aligned}$$

□

7.4 The log-normal process as an approximation of the bin. model

We finish the course by justifying, from first principles, our use of the log-normal process in deriving the Black-Scholes formula. We also prove Proposition 7.8 on the risk-neutral distribution for the log-normal process.

Recall that in the multi-period binomial model we assume that the stock price moves up or down each time-period, and we can choose the time-period as small as we like. Even though taking smaller time-periods leads to a more accurate model, a major problem arises in that the pricing formulae become more and more complex.

The way out is to **approximate** a multi-period binomial model by a log-normal process; as we will show, this approximation gets better as the time-period gets smaller.

Let us suppose that the time-period of the multi-period binomial model is $1/n$, for n very large. This means that the up/down multiplication factors should be **very close to one**, since share prices should not move much in any single time-period. It turns out that, in order to get convergence to a log-normal process, we need to choose

$$u = e^{\sigma/\sqrt{n}} \qquad \text{and} \qquad d = e^{-\sigma/\sqrt{n}}, \tag{29}$$

where $\sigma > 0$ is the **volatility** parameter of the log-normal process.

Theorem 7.12. *Fix parameters $\mu, r \in \mathbb{R}$ and $\sigma > 0$. Consider a multi-period binomial model for a share price $S(t)$ with time-period $1/n$, up/down multiplication factors (per time-period) as in (29). Then the following hold for any $t > 0$:*

- *Suppose the 'real world' probability of up/down steps are*

$$p_u = \frac{1}{2}\Big(1 + \frac{\mu}{\sigma\sqrt{n}}\Big) \quad \textit{and} \quad p_d = \frac{1}{2}\Big(1 - \frac{\mu}{\sigma\sqrt{n}}\Big).$$

  Then, as $n \to \infty$,

$$S(t) \overset{d}{\Rightarrow} S(0)\textit{LogNormal}(\mu t, \sigma^2 t).$$

- *Suppose interest is continuously-compounded at rate r. Then, as $n \to \infty$,*

$$S(t) \overset{d}{\Rightarrow} S(0)\textit{LogNormal}(\tilde{\mu} t, \tilde{\sigma}^2 t)$$

  under the risk-neutral distribution $\widetilde{\mathbb{P}}$, where

$$\tilde{\mu} = r - \frac{1}{2}\sigma^2 \qquad \textit{and} \qquad \tilde{\sigma}^2 = \sigma^2.$$

Before proving this theorem let us remark that Proposition 7.8 is an immediate consequence:

Proof of Proposition 7.8. Theorem 7.12 shows that a log-normal process with parameters (μ, σ) can be approximated by a multi-period binomial model. Moreover, the risk-neutral distribution of this multi-period binomial model also approximates a log-normal process with parameter $(\tilde{\mu}, \tilde{\sigma})$. Passing to the limit, the latter log-normal process must be a risk-neutral distribution for the former log-normal process. □

Proof of Theorem 7.12. Let us begin with the first claim of the theorem (the convergence under the real-world probability distribution). There are two steps to the proof:

1) Express $\ln(S(t)/S(0))$ as the sum of a sequence of i.i.d. random variables;

2) Apply a central limit theorem to deduce the result.

Step 1. The stock price at t in the multi-period binomial model with time-period $1/n$ can be written

$$S(t) = S(0) \prod_{i \leq nt} Y_{n,i}$$

where, for each n, $Y_{n,i}$ are an i.i.d. sequence of random variables with distribution

$$Y_{n,i} \sim \begin{cases} u = \exp(\sigma/\sqrt{n}) & \text{with probability } p_u = \frac{1}{2}\big(1 + \frac{\mu}{\sigma\sqrt{n}}\big) \\ d = \exp(-\sigma/\sqrt{n}) & \text{with probability } p_d = \frac{1}{2}\big(1 - \frac{\mu}{\sigma\sqrt{n}}\big). \end{cases}$$

Taking logarithms,

$$\ln(S(t)/S(0)) = \sum_{i \leq nt} \ln Y_{n,i},$$

where

$$\ln Y_{n,i} \sim \begin{cases} \sigma/\sqrt{n} & \text{with probability } p_u = \frac{1}{2}\big(1 + \frac{\mu}{\sigma\sqrt{n}}\big), \\ -\sigma/\sqrt{n} & \text{with probability } p_d = \frac{1}{2}\big(1 - \frac{\mu}{\sigma\sqrt{n}}\big). \end{cases}$$

Step 2. Let us now apply a central limit theorem to this sum of i.i.d. random variables. Since we are dealing with a triangular array of random variables, we are in the setting of Theorem 6.4 from the *Review of Probability Theory* (in fact, the setting is almost identical to Example 6.6 of that document), which states that if as $n \to \infty$,

$$n\mathbb{E}[\ln Y_{n,1}] \to \mu \quad \text{and} \quad n\mathrm{Var}[\ln Y_{n,1}] \to \sigma^2,$$

(and a 'technical condition' $n\mathbb{E}[|\ln Y_{n,1}|^p] \to 0$ holds for some $p > 2$), then

$$\sum_{i \leq n} Y_{n,i} \overset{d}{\Rightarrow} \mathcal{N}(\mu, \sigma^2). \tag{30}$$

Calculating

$$n\mathbb{E}[\ln Y_{n,1}] = n\big((\sigma/\sqrt{n})p_u - (\sigma/\sqrt{n})(1 - p_u)\big) = \sigma\sqrt{n}(2p_u - 1) = \sigma\sqrt{n}\frac{\mu}{\sigma\sqrt{n}} = \mu$$

and

$$\begin{aligned} n\mathrm{Var}[\ln Y_{n,1}] &= n\Big(\mathbb{E}[(\ln Y_{n,1})^2] - \mathbb{E}[\ln Y_{n,1}]^2\Big) \\ &= n\big((\sigma^2/n)p_u + (\sigma^2/n)(1 - p_u) - (\mu/n)^2\big) \\ &= \sigma^2 - \mu^2/n \to \sigma^2, \end{aligned}$$

and applying the 'triangular CLT' (the 'technical condition'

$$n\mathbb{E}[|\ln Y_{n,1}|^4] \sim \frac{n\sigma^4}{n^2} \to 0$$

clearly holds), we see that we have proven (30). Now, replacing n by nt in the above argument shows that $\ln(S(t)/S(0)) = \sum_{i \leq nt} \ln Y_{n,i}$ converges, for large n, to the normal distribution $\mathcal{N}(\mu t, \sigma^2 t)$, i.e. $S(t)/S(0)$ converges to a log-normal random variable with the same parameters. This completes the first claim of the theorem.

The proof of the second claim is similar, except we work under the risk-neutral probabilities. Again we express

$$\ln(S(t)/S(0)) = \sum_{i \le n} \ln Y_{n,i},$$

but now, under the risk-neutral distribution,

$$\ln Y_{n,i} \sim \begin{cases} \sigma/\sqrt{n} & \text{with probability } \tilde{p}_u, \\ -\sigma/\sqrt{n} & \text{with probability } \tilde{p}_d. \end{cases}$$

Let us calculate the risk-neutral probabilities. The discount factor for each period is $e^{-r/n}$, and so the risk-neutral probability for the up-state is

$$\tilde{p}_u = \frac{e^{r/n} - d}{u - d} = \frac{e^{r/n} - e^{-\sigma/\sqrt{n}}}{e^{\sigma/\sqrt{n}} - e^{-\sigma/\sqrt{n}}}.$$

Using the Taylor expansion of the exponential function

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots,$$

we can approximate $\tilde{p}_u$ for large n, as

$$\begin{aligned} \tilde{p}_u &= \frac{(1 + r/n + \cdots) - \left(1 - \sigma/\sqrt{n} + \frac{1}{2}\sigma^2/n + \cdots\right)}{\left(1 + \sigma/\sqrt{n} + \frac{1}{2}\sigma^2/n + \cdots\right) - \left(1 - \sigma/\sqrt{n} + \frac{1}{2}\sigma^2/n + \cdots\right)} \\ &\approx \frac{\sigma/\sqrt{n} + (r - \frac{1}{2}\sigma^2)/n}{2\sigma/\sqrt{n}} \\ &= \frac{1}{2}\Big(1 + \frac{r - \frac{1}{2}\sigma^2}{\sigma\sqrt{n}}\Big) \\ &= \frac{1}{2}\Big(1 + \frac{\tilde{\mu}}{\sigma\sqrt{n}}\Big). \end{aligned}$$

The CLT now applies in the same way, since all that has changed under the risk-neutral distribution is that the risk-neutral parameters $(\tilde{\mu}, \tilde{\sigma}^2) = (r - \sigma^2/2, \sigma^2)$ play the role of the original parameters (μ, σ^2). We conclude that for large n, $\ln(S(t)/S(0)) = \sum_{i \le nt} \ln Y_{n,i}$ converges to a normal distribution with parameters

$$\tilde{\mu}t = (r - \sigma^2/2)t \qquad \text{and} \qquad \tilde{\sigma}^2 t = \sigma^2 t,$$

i.e. $S(t)/S(0)$ converges to a log-normal random variable with the same parameters. $\square$

Remark 7.13. There is a different proof of the Black-Scholes formula that uses a continuous time stochastic process known as **Brownian motion**; this proof bypasses the multi-period binomial model altogether.