Final Exam

Question 1 (30 points)

Consider the following neoclassical growth model. The equilibrium conditions of the model are given by

$$K_{t+1} = I_t + K_t(1-\delta)$$

$$Y_t = A_t K_t^{\alpha} L^{1-\alpha}$$

$$(1+\tau_{c,t})^{-1} C_t^{-\sigma} = (1+\tau_{c,t+1})^{-1} \beta C_{t+1}^{-\sigma} [A_{t+1} \alpha K_{t+1}^{\alpha-1} L^{1-\alpha} + (1-\delta)]$$

$$C_t + I_t + G_t = Y_t$$

- Log-linearize the second, third, and fourth equations around the steady state. Note that "L" is a constant, not a variable.

Answer to Question 1

Here, we use the formulas of log-linearization introduced in the practice session because this is an easier way. You can also derive the same results by following procedures in "Slides_NK_LL."

Recall the following formulas:

$$z_t = x_t y_t \Rightarrow \hat{z}_t = \hat{x}_t + \hat{y}_t \quad (1)$$

$$z_t = x_t^{\alpha} \Rightarrow \hat{z}_t = \alpha \hat{x}_t \quad (2)$$

$$z_t = x_t + y_t \Rightarrow \hat{z}_t = \frac{x_{ss}}{z_{ss}} \hat{x}_t + \frac{y_{ss}}{z_{ss}} \hat{y}_t . \quad (3)$$

For the production function, it follows from (1) and (2) that

$$\hat{Y}_t = \hat{A}_t + \alpha \hat{K}_t.$$

For the LHS and RHS of the Euler equation,

$$(1+\tau_{c,t})^{-1} C_t^{-\sigma} = (1+\tau_{c,t+1})^{-1} \beta C_{t+1}^{-\sigma} [A_{t+1} \alpha K_{t+1}^{\alpha-1} L^{1-\alpha} + (1-\delta)],$$

we first use (1) and (2):

$$-\widehat{(1+\tau_{c,t})} - \sigma \hat{C}_t = -\widehat{(1+\tau_{c,t+1})} - \sigma \hat{C}_{t+1} + \hat{H}_{t+1}, \quad (4)$$

where $H_t = A_{t+1} \alpha K_{t+1}^{\alpha-1} L^{1-\alpha} + (1-\delta)$. Then, by using (3), we obtain

$$\widehat{(1+\tau_{c,t})} = \frac{\tau_{c,ss}}{1+\tau_{c,ss}} \hat{\tau}_{c,t}$$

$$\widehat{(1+\tau_{c,t+1})} = \frac{\tau_{c,ss}}{1+\tau_{c,ss}} \hat{\tau}_{c,t+1}$$

$$\hat{H}_{t+1} = \frac{A_{ss} \alpha K_{ss}^{\alpha-1} L^{1-\alpha}}{A_{ss} \alpha K_{ss}^{\alpha-1} L^{1-\alpha} + (1-\delta)} \{\hat{A}_{t+1} + (\alpha-1) \hat{K}_{t+1}\}.$$

Substituting these results into (4) yields

$$\hat{C}_t = \hat{C}_{t+1} + \frac{\tau_{c,ss}}{1+\tau_{c,ss}} \frac{\hat{\tau}_{c,t+1} - \hat{\tau}_{c,t}}{\sigma} - \frac{\beta A_{ss} \alpha K_{ss}^{\alpha-1} L^{1-\alpha}}{\sigma} \{\hat{A}_{t+1} + (\alpha-1) \hat{K}_{t+1}\}.$$

Here, we use $1 = \beta [A_{ss} \alpha K_{ss}^{\alpha-1} L^{1-\alpha} + (1-\delta)]$.

For the market clearing condition, we use (3) as follows:

$$\hat{Y}_t = \frac{C_{ss}}{C_{ss} + I_{ss} + G_{ss}} \hat{C}_t + \frac{I_{ss}}{C_{ss} + I_{ss} + G_{ss}} \hat{I}_t + \frac{G_{ss}}{C_{ss} + I_{ss} + G_{ss}} \hat{G}_t$$

$$\therefore Y_{ss} \hat{Y}_t = C_{ss} \hat{C}_t + I_{ss} \hat{I}_t + G_{ss} \hat{G}_t.$$

<u>Question 2 (40 points)</u>

Consider a version of the NK model with the following preference. The model is identical to the baseline model discussed in the lecture, except that the household's per period utility is given by

$$\sum_{t=1}^{\infty} \beta^{t-1}\left[\frac{1}{1-\gamma}\left(C_t - \frac{N_t^{1+\chi_n}}{1+\chi_n}\right)^{1-\gamma}\right]$$

- Derive the private-sector equilibrium conditions of the model.
- Assuming that (i) the policy rate is determined by the standard Taylor rule, (ii) the inflation target is zero (that is, $\Pi^{targ}=1$), and (iii) $\tau=\frac{1}{\theta-1}$, analytically compute the standard steady state of the model.

Answer to Question 2

Part A:

The household's maximization problem can be written as

$$\max_{\{C_t, N_t, B_t^h\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \beta^{t-1}\left[\frac{1}{1-\gamma}\left(C_t - \frac{N_t^{1+\chi_n}}{1+\chi_n}\right)^{1-\gamma}\right]$$

subject to the budget constraint

$$C_t + R_t^{-1}\frac{B_t}{P_t} \le w_t N_t + \frac{B_{t-1}}{P_t} + \Phi_t + T_t$$

Its Lagrange function is given as

$$L := \sum_{t=1}^{\infty} \beta^{t-1}\left[\left(\frac{1}{1-\gamma}\left(C_t - \frac{N_t^{1+\chi_n}}{1+\chi_n}\right)^{1-\gamma}\right) - \lambda_t\left(C_t + R_t^{-1}\frac{B_t}{P_t} - w_t N_t - \frac{B_{t-1}}{P_t} - \Phi_t - T_t\right)\right]$$

Take a derivative of the Lagrange function for each variable:

$$\frac{\partial L}{\partial C_t} : \left(C_t - \frac{N_t^{1+\chi_n}}{1+\chi_n}\right)^{-\gamma} - \lambda_t = 0$$

$$\frac{\partial L}{\partial B_t} : -\frac{\lambda_t}{R_t P_t} + \beta\frac{\lambda_{t+1}}{P_{t+1}} = 0$$

$$\frac{\partial L}{\partial N_t}: -N_t^{\chi_n}\left(C_t - \frac{N_t^{1+\chi_n}}{1+\chi_n}\right)^{-\gamma} + \lambda_t w_t = 0$$

Multiply the second equation with P_{t+1}:

$$\frac{-\lambda_t}{R_t}\Pi_{t+1} + \beta\lambda_{t+1} = 0$$

$$\Rightarrow \lambda_t = \beta R_t \lambda_{t+1}\Pi_{t+1}^{-1}$$

Combine the first equation and the third equation:

$$-N_t^{\chi_n}\left(C_t - \frac{N_t^{1+\chi_n}}{1+\chi_n}\right)^{-\gamma} + \lambda_t w_t = 0$$

$$\Rightarrow \left(C_t - \frac{N_t^{1+\chi_n}}{1+\chi_n}\right)^{-\gamma} w_t = N_t^{\chi_n}\left(C_t - \frac{N_t^{1+\chi_n}}{1+\chi_n}\right)^{-\gamma}$$

$$\Rightarrow w_t = N_t^{\chi_n}$$

Therefore, the new private-sector equilibrium conditions of the model are

$$\lambda_t = \left(C_t - \frac{N_t^{1+\chi_n}}{1+\chi_n}\right)^{-\gamma}$$

$$\lambda_t = \beta R_t \lambda_{t+1}\Pi_{t+1}^{-1}$$

$$w_t = N_t^{\chi_n}$$

$$\frac{Y_t}{\lambda_t}\left[\varphi(\Pi_t - 1)\Pi_t - (1+\tau)(1-\theta) - \theta w_t\right] = \beta\frac{Y_{t+1}}{\lambda_{t+1}}\varphi(\Pi_{t+1}-1)\Pi_{t+1}$$

$$Y_t = C_t + \frac{\varphi}{2}\left[\Pi_t - 1\right]^2 Y_t$$

$$Y_t = N_t$$

Now, we want to find the steady state of the model. We are assuming that (i) the policy rate is determined by the standard Taylor rule, (ii) the inflation target is zero (that is, $\Pi^{targ} = 1$), and (iii) $\tau = \frac{1}{\theta - 1}$.

Then, the standard steady state of the model satisfies:

$$\lambda_{ss}=\left(C_{ss}-\frac{N_{ss}^{1+\chi_n}}{1+\chi_n}\right)^{-\gamma}$$

$$\lambda_{ss}=\beta R_{ss}\lambda_{ss}\Pi_{ss}^{-1}$$

$$w_{ss}=N_{ss}^{\chi_n}$$

$$\frac{Y_{ss}}{\lambda_{ss}}\left[\varphi(\Pi_{ss}-1)\Pi_{ss}-(1+\tau)(1-\theta)-\theta w_{ss}\right]=\beta\frac{Y_{ss}}{\lambda_{ss}}\varphi(\Pi_{ss}-1)\Pi_{ss}$$

$$Y_{ss}=C_{ss}+\frac{\varphi}{2}[\Pi_{ss}-1]^2 Y_{ss}$$

$$Y_{ss}=N_{ss}$$

$$R_{ss}=\frac{\Pi^{targ}}{\beta}\left(\frac{\Pi_{ss}}{\Pi^{targ}}\right)^{\Phi_\pi}$$

Manipulating the second equation, we get

$$\lambda_{ss}=\beta R_{ss}\lambda_{ss}\Pi_{ss}^{-1}$$

$$\Rightarrow 1-\beta R_{ss}\Pi_{ss}^{-1}=0$$

$$\Rightarrow R_{ss}=\frac{1}{\beta}\Pi_{ss}$$

Because $\Pi^{targ}=1$, comparing this equation with the Taylor rule, we get

$$\frac{1}{\beta}\Pi_{ss}=\frac{\Pi^{targ}}{\beta}\left(\frac{\Pi_{ss}}{\Pi^{targ}}\right)^{\Phi_\pi}$$

$$\Rightarrow \frac{1}{\beta}=\frac{1}{\beta}\Pi_{ss}^{\Phi_\pi-1}$$

$$\Rightarrow \Pi_{ss}^{\Phi_\pi-1}=1$$

$$\Rightarrow \Pi_{ss}=1$$

From the fourth equation,

$$\frac{Y_{ss}}{\lambda_{ss}}\left[\varphi(\Pi_{ss}-1)\Pi_{ss}-(1+\tau)(1-\theta)-\theta w_{ss}\right]=\beta\frac{Y_{ss}}{\lambda_{ss}}\varphi(\Pi_{ss}-1)\Pi_{ss}$$

Using $\Pi_{ss}=1$ and $\tau=\frac{1}{\theta-1}$,

$$-(1+\tau)(1-\theta)-\theta w_{ss}=0$$

$$\Rightarrow w_{ss}=\frac{-1}{\theta}(1+\tau)(1-\theta)$$

$$\Rightarrow w_{ss}=\frac{-1}{\theta}\left(1+\frac{1}{\theta-1}\right)(1-\theta)$$

$$\Rightarrow w_{ss}=\frac{-1}{\theta}\left(\frac{\theta}{\theta-1}\right)(1-\theta)$$

$$\Rightarrow w_{ss}=1$$

Because $\Pi_{ss}=1$, the fourth equation becomes

$$Y_{ss}=C_{ss}+\frac{\varphi}{2}\left[\Pi_{ss}-1\right]^2 Y_{ss}$$

$$\Rightarrow Y_{ss}=C_{ss}$$

Because $w_{ss}=1$ and $Y_{ss}=C_{ss}=N_{ss}$, the third equation becomes

$$w_{ss}=N_{ss}^{\chi_n}$$

$$\Rightarrow N_{ss}=1$$

To sum up, the solution is

$$Y_{ss}=C_{ss}=N_{ss}=w_{ss}=\Pi_{ss}=1, R_{ss}=\frac{1}{\beta}$$

Question 3 (60 points)

Consider the following two-period model with a static PC with the ELB. There is a labor income tax that is distributed to households as a lump-sum transfer at time one. Key equilibrium conditions of this model are given by the following equations.

At t=1:

$$y_1 = y_2 - \sigma\left(i_1 - \pi_2 - r_1^n\right)$$
$$(5)$$

$$\pi_1 = \kappa\left(y_1 - \Gamma\,\tau_{w,1}\right)$$
$$(6)$$

$$i_1 \geq 0$$

At t=2:

$$y_2 = -\sigma\left(i_2 - r^{¿}\right)$$
$$(7)$$

$$\pi_2 = \kappa\left(y_2 - \Gamma\,\tau_{w,2}\right)$$
$$(8)$$

$$i_2 \geq 0.$$

The utility function for the government at each time is given by the standard quadratic objective function. That is,

$$u\left(\pi_t, y_t\right) = \frac{-1}{2}\left[\pi_t^2 + \lambda\, y_t^2\right]$$

for each t=1,2.

<u>Part A:</u>

Now, assume that the government is optimizing under discretion.

Assume also that the value of r_1^n is such that $i_1 = 0$ but $i_2 > 0$.

Assume that the government can optimally choose labor income taxes at time one and time two.

- Formulate the optimization problem(s) of the central bank.
- Define the Markov-Perfect equilibrium.
- Solve the model analytically.

<u>Part B:</u>

Assume that the government is optimizing under commitment.

Assume also that the value of r_1^n is such that $i_1 = 0$ but $i_2 > 0$.

Assume that the government can optimally choose labor income taxes at time one and time two.

- Formulate the optimization problem of the central bank.
- Define the Ramsey equilibrium.
- Solve the model analytically.

<u>Answer to Question 3</u>

<u>Part A:</u>

The optimization problem of the central bank under discretion at t = 2,

$$V_2 = \max_{\pi_2, y_2, i_2, \tau_{w,2}} u(\pi_2, y_2)$$

subject to

$$y_2 = -\sigma(i_2 - r^{¿})$$

$$\pi_2 = \kappa(y_2 - \Gamma \tau_{w,2})$$

$$i_2 \geq 0.$$

The optimization problem of the central bank under discretion at t = 1,

$$V_1 = \max_{\pi_1, y_1, i_1, \tau_{w,1}} u(\pi_1, y_1)$$

subject to

$$y_1 = y_2 - \sigma(i_1 - \pi_2 - r_1^n)$$

$$\pi_1 = \kappa(y_1 - \Gamma \tau_{w,1})$$

$$i_1 \geq 0$$

taking y_2 and π_2 as given.

A Markov-Perfect equilibrium is defined as a vector $\backslash\{ y_1, \pi_1, i_1, \tau_{w,1}, y_2, \pi_2, i_2, \tau_{w,2} \backslash\}$ **that solves the two optimization problems at t=1 and t=2.**

First, we set up a Lagrange function to solve the optimization problem at t=2:

$$L \coloneqq u(\pi_2, y_2) + \phi_{EE,2}\left(y_2 + \sigma\left(i_2 - r^{¿}\right)\right) + \phi_{PC,2}\left(\pi_2 - \kappa\left(y_2 - \Gamma\,\tau_{w,2}\right)\right) + \phi_{ELB,2}\, i_2 .$$

Take a derivative of the Lagrange function for each variable:

$$\frac{\partial L}{\partial y_2} = -\lambda\, y_2 + \phi_{EE,2} - \kappa\, \phi_{PC,2} = 0, \tag{9}$$

$$\frac{\partial L}{\partial \pi_2} = -\pi_2 + \phi_{PC,2} = 0, \tag{10}$$

$$\frac{\partial L}{\partial i_2} = \sigma\, \phi_{EE,2} + \phi_{ELB,2} = 0, \tag{11}$$

$$\frac{\partial L}{\partial \tau_{w,2}} = \kappa\Gamma\, \phi_{PC,2} = 0. \tag{12}$$

Since we assume $i_2 > 0$, $\phi_{ELB,2} = 0$ should hold. By (10) and (12), $\pi_2 = 0$ holds. Moreover, substituting $\phi_{PC,2} = 0$ and $\phi_{EE,2} = 0$ into (9) yields $y_2 = 0$. Therefore, $i_2 = r^{¿}$ and $\tau_{w,2} = 0$ are implied by (7) and (8), respectively.

Next, we set up a Lagrange function to solve the optimization problem at t=1:

$$L \coloneqq u(\pi_1, y_1) + \phi_{EE,1}\left(y_1 - y_2 + \sigma\left(i_1 - \pi_2 - r_1^n\right)\right) + \phi_{PC,1}\left(\pi_1 - \kappa\left(y_1 - \Gamma\,\tau_{w,1}\right)\right) + \phi_{ELB,1}\, i_1 .$$

Take a derivative of the Lagrange function for each variable:

$$\frac{\partial L}{\partial y_1} = -\lambda\, y_1 + \phi_{EE,1} - \kappa\, \phi_{PC,1} = 0, \tag{13}$$

$$\frac{\partial L}{\partial \pi_1} = -\pi_1 + \phi_{PC,1} = 0, \quad (14)$$

$$\frac{\partial L}{\partial i_1} = \sigma \phi_{EE,1} + \phi_{ELB,1} = 0, \quad (15)$$

$$\frac{\partial L}{\partial \tau_{w,1}} = \kappa \Gamma \phi_{PC,1} = 0. \quad (16)$$

By assumption, we have $i_1 = 0$. By (14) and (16), $\pi_1 = 0$ holds. Moreover, it follows from (5) that $y_1 = \sigma r_1^n$ holds. Therefore, (6) implies $\tau_{w,1} = \sigma r_1^n / \Gamma$.

In summary, a Markov-Perfect equilibrium is characterized by $(y_1, \pi_1, i_1, \tau_{w,1}, y_2, \pi_2, i_2, \tau_{w,2}) = (\sigma r_1^n, 0, 0, \sigma r_1^n/\Gamma, 0, 0, r^{¿}, 0)$.

Part B:

The optimization problem of the central bank with commitment is

$$V_{ram} = \max_{\pi_1, y_1, i_1, \tau_{w,1}, \pi_2, y_2, i_2, \tau_{w,2}} u(\pi_1, y_1) + \beta u(\pi_2, y_2)$$

subject to

$$y_1 = y_2 - \sigma(i_1 - \pi_2 - r_1^n)$$

$$\pi_1 = \kappa(y_1 - \Gamma \tau_{w,1})$$

$$i_1 \geq 0$$

$$y_2 = -\sigma(i_2 - r^{¿})$$

$$\pi_2 = \kappa(y_2 - \Gamma \tau_{w,2})$$

$$i_2 \geq 0.$$

A Ramsey equilibrium is defined as a vector $\{y_1, \pi_1, i_1, \tau_{w,1}, y_2, \pi_2, i_2, \tau_{w,2}\}$ **that solves the optimization problem above.**

We set up a Lagrange function to solve the optimization problem:

$$L \coloneqq u(\pi_1, y_1) + \phi_{EE,1}\left(y_1 - y_2 + \sigma\left(i_1 - \pi_2 - r_1^n\right)\right) + \phi_{PC,1}\left(\pi_1 - \kappa\left(y_1 - \Gamma \tau_{w,1}\right)\right) + \phi_{ELB,1} i_1 + \beta\left\{u(\pi_2, y_2) + \phi_{EE,2}\left(y_2 + \sigma\left(i_2 - r^{¿}\right)\right) + \right.$$

Take a derivative of the Lagrange function for each variable:

$$\frac{\partial L}{\partial y_1} = -\lambda y_1 + \phi_{EE,1} - \kappa \phi_{PC,1} = 0,$$

(17)

$$\frac{\partial L}{\partial \pi_1} = -\pi_1 + \phi_{PC,1} = 0,$$

(18)

$$\frac{\partial L}{\partial i_1} = \sigma \phi_{EE,1} + \phi_{ELB,1} = 0,$$

(19)

$$\frac{\partial L}{\partial \tau_{w,1}} = \kappa \Gamma \phi_{PC,1} = 0,$$

(20)

$$\frac{\partial L}{\partial y_2} = -\phi_{EE,1} + \beta\left(-\lambda y_2 + \phi_{EE,2} - \kappa \phi_{PC,2}\right) = 0,$$

(21)

$$\frac{\partial L}{\partial \pi_2} = -\sigma \phi_{EE,1} + \beta\left(-\pi_2 + \phi_{PC,2}\right) = 0,$$

(22)

$$\frac{\partial L}{\partial i_2} = \beta\left(\sigma \phi_{EE,2} + \phi_{ELB,2}\right) = 0,$$

(23)

$$\frac{\partial L}{\partial \tau_{w,2}} = \beta \kappa \Gamma \phi_{PC,2} = 0.$$

(24)

By assumption, we have $i_1 = 0$. Moreover, since we assume $i_2 > 0$, $\phi_{ELB,2} = 0$ should hold. By (20), (23), and (24), $\phi_{PC,1} = \phi_{EE,2} = \phi_{PC,2} = 0$. Thus, (18) implies $\pi_1 = 0$. Now, (17), (21), and (22) imply $y_1 = \phi_{EE,1}/\lambda$, $y_2 = -\phi_{EE,1}/(\beta\lambda)$, and $\pi_2 = -\sigma\phi_{EE,1}/\beta$, respectively. Substituting them into (5) yields

$$\phi_{EE,1}=\frac{\beta\lambda\sigma}{1+\beta+\lambda\sigma^2}r_1^n,$$

which implies

$$y_1=\frac{\beta\sigma}{1+\beta+\lambda\sigma^2}r_1^n,$$

$$y_2=\frac{-\sigma}{1+\beta+\lambda\sigma^2}r_1^n,$$

$$\pi_2=\frac{-\lambda\sigma^2}{1+\beta+\lambda\sigma^2}r_1^n.$$

Then, by (6), (7), and (8), we have

$$\tau_{w,1}=\frac{\beta\sigma}{\Gamma\left(1+\beta+\lambda\sigma^2\right)}r_1^n,$$

$$i_2=r^{¿}+\frac{1}{1+\beta+\lambda\sigma^2}r_1^n,$$

$$\tau_{w,2}=\frac{\sigma\left(\lambda\sigma-\kappa\right)}{\Gamma\kappa\left(1+\beta+\lambda\sigma^2\right)}r_1^n.$$

In summary, a Ramsey equilibrium is characterized by

$$y_1=\frac{\beta\sigma}{1+\beta+\lambda\sigma^2}r_1^n,$$

$$\pi_1=0,i_1=0,$$

$$\tau_{w,1}=\frac{\beta\sigma}{\Gamma\left(1+\beta+\lambda\sigma^2\right)}r_1^n,$$

$$y_2=\frac{-\sigma}{1+\beta+\lambda\sigma^2}r_1^n,$$

$$\pi_2=\frac{-\lambda\sigma^2}{1+\beta+\lambda\sigma^2}r_1^n,$$

$$i_2=r^{¿}+\frac{1}{1+\beta+\lambda\sigma^2}r_1^n,$$

$$\tau_{w,2} = \frac{\sigma\left(\lambda\sigma - \kappa\right)}{\Gamma\kappa\left(1+\beta+\lambda\sigma^2\right)} r_1^n .$$

<u>Question 4 (20 points)</u>

Consider the following two-period loglinearized New Keynesian model with a <u>static</u> Phillips curve and with a time-one demand shock ($r_1^n < 0$ and $r_2^n = r^{¿} > 0$). The policy rule is given by a nominal-income targeting rule.

At t=1,

$$y_1 = y_2 - \sigma\left(i_1 - \pi_2 - r_1^n\right)$$

$$\pi_1 = \kappa\, y_1$$

$$p_1 = p_0 + \pi_1$$

$$i_1 = max\left[0, r^{¿} + \phi\left(p_1 + y_1\right)\right]$$

At t=2,

$$y_2 = -\sigma\left(i_2 - r^{¿}\right)$$

$$\pi_2 = \kappa\, y_2$$

$$p_2 = p_1 + \pi_2$$

$$i_2 = max\left[0, r^{¿} + \phi\left(p_2 + y_2\right)\right]$$

- Assuming that (i) $p_0 = 0$ and and (ii) the shock size is such that the policy rate is zero and positive at time one and two, respectively, solve the model analytically.

<u>Answer to Question 4</u>

Since $i_2 > 0$ and $p_0 = 0$, at t=2,

$$y_2 = -\sigma\left(i_2 - r^{¿}\right)$$

$$\Rightarrow y_2 = -\sigma\phi\left(p_2 + y_2\right) = -\sigma\phi\left(p_1 + \pi_2 + y_2\right)$$

$$¿ -\sigma\phi\left(\pi_1 + \pi_2 + y_2\right) = -\kappa\sigma\phi\, y_1 - \left(\kappa + 1\right)\sigma\phi\, y_2$$

$$\Rightarrow y_2 = \frac{-\kappa\sigma\phi}{1 + \left(\kappa + 1\right)\sigma\phi} y_1$$

Since $i_1 = 0$, at t=1,

$$y_1 = y_2 - \sigma\left(-\pi_2 - r_1^n\right) = \left(1 + \sigma\kappa\right) y_2 + \sigma r_1^n$$

$$\Rightarrow \frac{1 + \left(\kappa + 1\right)\sigma\phi + \left(1 + \sigma\kappa\right)\kappa\sigma\phi}{1 + \left(\kappa + 1\right)\sigma\phi} y_1 = \sigma r_1^n$$

$$\Rightarrow y_1 = \frac{\left\{1 + \left(\kappa + 1\right)\sigma\phi\right\}\sigma}{1 + \left(2\kappa + 1\right)\sigma\phi + \kappa^2\sigma^2\phi} r_1^n$$

$$\pi_1 = \kappa\, y_1 = \frac{\left\{1 + \left(\kappa + 1\right)\sigma\phi\right\}\kappa\sigma}{1 + \left(2\kappa + 1\right)\sigma\phi + \kappa^2\sigma^2\phi} r_1^n$$

$$p_1 = p_0 + \pi_1 = \frac{\left\{1 + \left(\kappa + 1\right)\sigma\phi\right\}\kappa\sigma}{1 + \left(2\kappa + 1\right)\sigma\phi + \kappa^2\sigma^2\phi} r_1^n.$$

Then, at t=2,

$$y_2 = \frac{-\kappa\sigma\phi}{1 + \left(\kappa + 1\right)\sigma\phi} y_1 = \frac{-\kappa\sigma^2\phi}{1 + \left(2\kappa + 1\right)\sigma\phi + \kappa^2\sigma^2\phi} r_1^n$$

$$\pi_2 = \kappa\, y_2 = \frac{-\kappa^2\sigma^2\phi}{1 + \left(2\kappa + 1\right)\sigma\phi + \kappa^2\sigma^2\phi} r_1^n$$

$$p_2 = p_1 + \pi_2 = \frac{\left\{1 + \left(\kappa + 1\right)\sigma\phi\right\}\kappa\sigma}{1 + \left(2\kappa + 1\right)\sigma\phi + \kappa^2\sigma^2\phi} r_1^n - \frac{\kappa^2\sigma^2\phi}{1 + \left(2\kappa + 1\right)\sigma\phi + \kappa^2\sigma^2\phi} r_1^n$$

$$\Rightarrow p_2 = \frac{\left(1 + \sigma\phi\right)\kappa\sigma}{1 + 2\kappa\sigma\phi + \kappa^2\sigma^2\phi} r_1^n$$

$$i_2 = r^{¿} + \frac{\kappa\sigma\phi}{1+(2\kappa+1)\sigma\phi+\kappa^2\sigma^2\phi} r_1^n$$

To sum up, the solution is

$$y_1 = \frac{\{1+(\kappa+1)\sigma\phi\}\sigma}{1+(2\kappa+1)\sigma\phi+\kappa^2\sigma^2\phi} r_1^n$$

$$\pi_1 = \frac{\{1+(\kappa+1)\sigma\phi\}\kappa\sigma}{1+(2\kappa+1)\sigma\phi+\kappa^2\sigma^2\phi} r_1^n$$

$$i_1 = 0$$

$$p_1 = \frac{\{1+(\kappa+1)\sigma\phi\}\kappa\sigma}{1+(2\kappa+1)\sigma\phi+\kappa^2\sigma^2\phi} r_1^n$$

$$y_2 = \frac{-\kappa\sigma^2\phi}{1+(2\kappa+1)\sigma\phi+\kappa^2\sigma^2\phi} r_1^n$$

$$\pi_2 = \frac{-\kappa^2\sigma^2\phi}{1+(2\kappa+1)\sigma\phi+\kappa^2\sigma^2\phi} r_1^n$$

$$i_2 = r^{¿} + \frac{\kappa\sigma\phi}{1+(2\kappa+1)\sigma\phi+\kappa^2\sigma^2\phi} r_1^n$$

$$p_2 = \frac{(1+\sigma\phi)\kappa\sigma}{1+2\kappa\sigma\phi+\kappa^2\sigma^2\phi} r_1^n$$