Final Exam

Question 1 (40 points)

Consider the neoclassical growth model. The equilibrium conditions of the model are given by

$$K_{t+1}=I_t+K_t(1-\delta)$$

$$Y_t=K_t^{\alpha}L^{1-\alpha}$$

$$C_t^{-\sigma}=\beta C_{t+1}^{-\sigma}[\alpha K_{t+1}^{\alpha-1}L^{1-\alpha}+(1-\delta)]$$

$$I_t=Y_t-C_t$$

- Log-linearize these four equations around the steady state.

Answer to Question 1

For the left hand side (LHS) and right hand side (RHS) of the capital accumulation equation:

$$K_{t+1}=I_t+K_t(1-\delta)$$

$$LHS=f(K_{t+1})$$

$$RHS=g(I_t,K_t)$$

Take a derivative of them for each variable:

$$f_{K,t+1}=1$$

$$g_I=1$$

$$g_{K,t}=1-\delta$$

Loglinearize the LHS and RHS around the steady state:

$$LHS\approx f(K_{ss})+f_{K,t+1,ss}K_{ss}\widehat{K}_{t+1}$$

$$\text{¿} f\left(K_{ss}\right)+K_{ss}\widehat{K}_{t+1}$$

$$RHS\approx g\left(I_{ss},K_{ss}\right)+g_{I,ss}I_{ss}\hat{I}_t+g_{K,ss}K_{ss}\widehat{K}_t$$

$$\text{¿} g\left(I_{ss},K_{ss}\right)+I_{ss}\hat{I}_t+(1-\delta)K_{ss}\widehat{K}_t$$

Combine the LHS and RHS:

$$f\left(K_{ss}\right)+K_{ss}\widehat{K}_{t+1}=g\left(I_{ss},K_{ss}\right)+I_{ss}\hat{I}_t+(1-\delta)K_{ss}\widehat{K}_t$$

$$\Rightarrow K_{ss}\widehat{K}_{t+1}=I_{ss}\hat{I}_t+(1-\delta)K_{ss}\widehat{K}_t$$

For the LHS and RHS of the production function:

$$Y_t=K_t^{\alpha}L^{1-\alpha}$$

$$LHS=f\left(Y_t\right)$$

$$RHS=g\left(K_t\right)$$

Take a derivative of them for each variable:

$$f_Y=1$$

$$g_K=\alpha K_t^{\alpha-1}L^{1-\alpha}$$

Loglinearize the LHS and RHS:

$$LHS\approx f\left(Y_{ss}\right)+f_{Y,ss}Y_{ss}\widehat{Y}_t$$

$$\text{¿} f\left(Y_{ss}\right)+Y_{ss}\widehat{Y}_t$$

$$RHS\approx g\left(K_{ss}\right)+g_{K,ss}K_{ss}\widehat{K}_t$$

$$\text{¿} g\left(K_{ss}\right)+\alpha K_{ss}^{\alpha-1}L^{1-\alpha}K_{ss}\widehat{K}_t$$

$$\text{¿} g\left(K_{ss}\right)+\alpha K_{ss}^{\alpha}L^{1-\alpha}\widehat{K}_t$$

Combine the LHS and RHS:

$$f\left(Y_{ss}\right)+Y_{ss}\widehat{Y}_t=g\left(K_{ss}\right)+\alpha K_{ss}^{\alpha}L^{1-\alpha}\widehat{K}_t$$

$$\Rightarrow \widehat{Y}_t=\alpha \widehat{K}_t$$

For the LHS and RHS of the Euler equation

$$C_t^{-\sigma}=\beta C_{t+1}^{-\sigma}\left[\alpha K_{t+1}^{\alpha-1}L^{1-\alpha}+\left(1-\delta\right)\right]$$

$$LHS=f\left(C_t\right)$$

$$RHS=g\left(C_{t+1},K_{t+1}\right)$$

Take a derivative of them for each variable:

$$f_{C,t}=-\sigma C_t^{-\sigma-1}$$

$$g_{C,t+1}=\beta\left(-\sigma C_{t+1}^{-\sigma-1}\right)\left[\alpha K_{t+1}^{\alpha-1}L^{1-\alpha}+\left(1-\delta\right)\right]$$

$$g_K=\beta C_{t+1}^{-\sigma}\left[\alpha\left(\alpha-1\right)K_{t+1}^{\alpha-2}L^{1-\alpha}\right]$$

Loglinearize the LHS and RHS:

$$LHS\approx f\left(C_{ss}\right)+f_{C,t,ss}C_{ss}\widehat{C}_t$$

$$¿ f\left(C_{ss}\right)-\sigma C_{ss}^{-\sigma-1}C_{ss}\widehat{C}_t$$

$$¿ f\left(C_{ss}\right)-\sigma C_{ss}^{-\sigma}\widehat{C}_t$$

$$RHS\approx g\left(C_{ss},K_{ss}\right)+g_{C,t+1,ss}C_{ss}\widehat{C}_{t+1}+g_{K,ss}K_{ss}\widehat{K}_{t+1}$$

$$¿ g\left(C_{ss},K_{ss}\right)+\beta\left(-\sigma C_{ss}^{-\sigma-1}\right)\left[\alpha K_{ss}^{\alpha-1}L^{1-\alpha}+\left(1-\delta\right)\right]C_{ss}\widehat{C}_{t+1}+\beta C_{ss}^{-\sigma}\left[\alpha\left(\alpha-1\right)K_{ss}^{\alpha-2}L^{1-\alpha}\right]K_{ss}\widehat{K}_{t+1}$$

$$¿ g\left(C_{ss},K_{ss}\right)+\beta\left(-\sigma C_{ss}^{-\sigma}\right)\left[\alpha K_{ss}^{\alpha-1}L^{1-\alpha}+\left(1-\delta\right)\right]\widehat{C}_{t+1}+\beta C_{ss}^{-\sigma}\left[\alpha\left(\alpha-1\right)K_{ss}^{\alpha-1}L^{1-\alpha}\right]\widehat{K}_{t+1}$$

Combine the LHS and RHS:

$$f\left(C_{ss}\right)-\sigma C_{ss}^{-\sigma}\widehat{C}_t=g\left(C_{ss},K_{ss}\right)+\beta\left(-\sigma C_{ss}^{-\sigma}\right)\left[\alpha K_{ss}^{\alpha-1}L^{1-\alpha}+\left(1-\delta\right)\right]\widehat{C}_{t+1}+\beta C_{ss}^{-\sigma}\left[\alpha\left(\alpha-1\right)K_{ss}^{\alpha-1}L^{1-\alpha}\right]\widehat{K}_{t+1}$$

$$\Rightarrow \widehat{C}_t=\beta\left[\alpha K_{ss}^{\alpha-1}L^{1-\alpha}+(1-\delta)\right]\widehat{C}_{t+1}-\frac{\beta}{\sigma}\left[\alpha(\alpha-1)K_{ss}^{\alpha-1}L^{1-\alpha}\right]\widehat{K}_{t+1}$$

For the LHS and RHS of the production function:

$$I_t=Y_t-C_t$$

$$LHS=f(I_t)$$

$$RHS=g(Y_t,C_t)$$

Take a derivative of them for each variable:

$$f_Y=1$$

$$g_Y=1$$

$$g_C=-1$$

Loglinearize the LHS and RHS:

$$LHS\approx f(I_{ss})+f_{I,ss}I_{ss}\hat{I}_t$$

$$RHS\approx g(Y_{ss},C_{ss})+g_{Y,ss}Y_{ss}\widehat{Y}_t+g_{C,ss}C_{ss}\widehat{C}_t$$

Combine the LHS and RHS:

$$f(I_{ss})+f_{I,ss}I_{ss}\hat{I}_t=g(Y_{ss},C_{ss})+g_{Y,ss}Y_{ss}\widehat{Y}_t+g_{C,ss}C_{ss}\widehat{C}_t$$

$$\Rightarrow I_{ss}\hat{I}_t=Y_{ss}\widehat{Y}_t-C_{ss}\widehat{C}_t$$

To sum up, we can write the equilibrium conditions of the neoclassical growth model in a loglinearized form as:

$$K_{ss}\widehat{K}_{t+1}=I_{ss}\hat{I}_t+(1-\delta)K_{ss}\widehat{K}_t$$

$$\widehat{Y}_t=\alpha\widehat{K}_t$$

$$\widehat{C}_t=\beta\left[\alpha K_{ss}^{\alpha-1}L^{1-\alpha}+(1-\delta)\right]\widehat{C}_{t+1}-\frac{\beta}{\sigma}\left[\alpha(\alpha-1)K_{ss}^{\alpha-1}L^{1-\alpha}\right]\widehat{K}_{t+1}$$

$$I_{ss}\hat{I}_t=Y_{ss}\hat{Y}_t-C_{ss}\widehat{C}_t$$

(Appendix)

For those who solved the steady state variables, we get

$$K_{ss}=\left(\frac{\alpha\beta}{1-\beta+\beta\delta}\right)^{\frac{1}{1-\alpha}}L$$

$$Y_{ss}=\left(\frac{\alpha\beta}{1-\beta+\beta\delta}\right)^{\frac{\alpha}{1-\alpha}}L$$

$$I_{ss}=\delta\left(\frac{\alpha\beta}{1-\beta+\beta\delta}\right)^{\frac{1}{1-\alpha}}L$$

$$C_{ss}=\left(\frac{\alpha\beta}{1-\beta+\beta\delta}\right)^{\frac{\alpha}{1-\alpha}}L-\delta\left(\frac{\alpha\beta}{1-\beta+\beta\delta}\right)^{\frac{1}{1-\alpha}}L$$

Then, the loglinearized neoclassical growth model becomes

$$\left(\frac{\alpha\beta}{1-\beta+\beta\delta}\right)^{\frac{1}{1-\alpha}}L\widehat{K}_{t+1}=\delta\left(\frac{\alpha\beta}{1-\beta+\beta\delta}\right)^{\frac{1}{1-\alpha}}L\hat{I}_t+(1-\delta)\left(\frac{\alpha\beta}{1-\beta+\beta\delta}\right)^{\frac{1}{1-\alpha}}L\widehat{K}_t$$

$$\Rightarrow\widehat{K}_{t+1}=\delta\hat{I}_t+(1-\delta)\widehat{K}_t$$

$$\hat{Y}_t=\alpha\widehat{K}_t$$

$$\widehat{C}_t=\beta\left[\alpha\left(\frac{\alpha\beta}{1-\beta+\beta\delta}\right)^{\frac{\alpha-1}{1-\alpha}}L^{\alpha-1}L^{1-\alpha}+(1-\delta)\right]\widehat{C}_{t+1}-\frac{\beta}{\sigma}\left[\alpha(\alpha-1)\left(\frac{\alpha\beta}{1-\beta+\beta\delta}\right)^{\frac{\alpha-1}{1-\alpha}}L^{\alpha-1}L^{1-\alpha}\right]\widehat{K}_{t+1}$$

$$\Rightarrow\widehat{C}_t=\beta\left[\alpha\frac{1-\beta+\beta\delta}{\alpha\beta}+(1-\delta)\right]\widehat{C}_{t+1}-\frac{\beta}{\sigma}\left[\alpha(\alpha-1)\frac{1-\beta+\beta\delta}{\alpha\beta}\right]\widehat{K}_{t+1}$$

$$\Rightarrow\widehat{C}_t=\widehat{C}_{t+1}-\frac{(\alpha-1)(1-\beta+\beta\delta)}{\sigma}\widehat{K}_{t+1}$$

$$\delta\left(\frac{\alpha\beta}{1-\beta+\beta\delta}\right)^{\frac{1}{1-\alpha}}L\hat{I}_t=\left(\frac{\alpha\beta}{1-\beta+\beta\delta}\right)^{\frac{\alpha}{1-\alpha}}L\hat{Y}_t-\left\{\left(\frac{\alpha\beta}{1-\beta+\beta\delta}\right)^{\frac{\alpha}{1-\alpha}}L-\delta\left(\frac{\alpha\beta}{1-\beta+\beta\delta}\right)^{\frac{1}{1-\alpha}}L\right\}\widehat{C}_t$$

$$\Rightarrow\delta\hat{I}_t=\left(\frac{\alpha\beta}{1-\beta+\beta\delta}\right)^{\alpha}\hat{Y}_t-\left\{\left(\frac{\alpha\beta}{1-\beta+\beta\delta}\right)^{\alpha}-\delta\right\}\widehat{C}_t$$

Therefore, we can write the four equations as:

$$\hat{K}_{t+1}=\delta \hat{I}_t+(1-\delta)\hat{K}_t$$

$$\hat{Y}_t=\alpha \hat{K}_t$$

$$\hat{C}_t=\hat{C}_{t+1}-\frac{(\alpha-1)(1-\beta+\beta\delta)}{\sigma}\hat{K}_{t+1}$$

$$\delta \hat{I}_t=\left(\frac{\alpha\beta}{1-\beta+\beta\delta}\right)^{\alpha}\hat{Y}_t-\left\{\left(\frac{\alpha\beta}{1-\beta+\beta\delta}\right)^{\alpha}-\delta\right\}\hat{C}_t$$

Question 2 (40 points)

Consider a version of the NK model with the following preference. The model is identical to the baseline model discussed in the lecture, except that the household's per period utility is given by

$$\sum_{t=1}^{\infty}\beta^{t-1}\left[\frac{1}{1-\gamma}\left(C_t-\frac{N_t^{1+\chi_n}}{1+\chi_n}\right)^{1-\gamma}\right]$$

- Derive the private-sector equilibrium conditions of the model.
- Assuming that (i) the policy rate is determined by the standard Taylor rule, (ii) the inflation target is zero (that is, $\Pi^{targ}=1$), and (iii) $\tau=\frac{1}{\theta-1}$, analytically compute the standard steady state of the model.

Answer to Question 2

Part A:

The household's maximization problem can be written as

$$\max_{\{C_t,N_t,B_t^h\}_{t=1}^{\infty}}\sum_{t=1}^{\infty}\beta^{t-1}\left[\frac{1}{1-\gamma}\left(C_t-\frac{N_t^{1+\chi_n}}{1+\chi_n}\right)^{1-\gamma}\right]$$

subject to the budget constraint

$$C_t + R_t^{-1}\frac{B_t}{P_t} \leq w_t N_t + \frac{B_{t-1}}{P_t} + \Phi_t + T_t$$

Its Lagrange function is given as

$$L \coloneqq \sum_{t=1}^{\infty} \beta^{t-1}\left[\left(\frac{1}{1-\gamma}\left(C_t - \frac{N_t^{1+\chi_n}}{1+\chi_n}\right)^{1-\gamma}\right) - \lambda_t\left(C_t + R_t^{-1}\frac{B_t}{P_t} - w_t N_t - \frac{B_{t-1}}{P_t} - \Phi_t - T_t\right)\right]$$

Take a derivative of the Lagrange function for each variable:

$$\frac{\partial L}{\partial C_t} : \left(C_t - \frac{N_t^{1+\chi_n}}{1+\chi_n}\right)^{-\gamma} - \lambda_t = 0$$

$$\frac{\partial L}{\partial B_t} : -\frac{\lambda_t}{R_t P_t} + \beta\frac{\lambda_{t+1}}{P_{t+1}} = 0$$

$$\frac{\partial L}{\partial N_t} : -N_t^{\chi_n}\left(C_t - \frac{N_t^{1+\chi_n}}{1+\chi_n}\right)^{-\gamma} + \lambda_t w_t = 0$$

Multiply the second equation with P_{t+1}:

$$-\frac{\lambda_t}{R_t}\Pi_{t+1} + \beta\lambda_{t+1} = 0$$

$$\Rightarrow \lambda_t = \beta R_t \lambda_{t+1}\Pi_{t+1}^{-1}$$

Combine the first equation and the third equation:

$$-N_t^{\chi_n}\left(C_t - \frac{N_t^{1+\chi_n}}{1+\chi_n}\right)^{-\gamma} + \lambda_t w_t = 0$$

$$\Rightarrow \left(C_t - \frac{N_t^{1+\chi_n}}{1+\chi_n}\right)^{-\gamma} w_t = N_t^{\chi_n}\left(C_t - \frac{N_t^{1+\chi_n}}{1+\chi_n}\right)^{-\gamma}$$

$$\Rightarrow w_t = N_t^{\chi_n}$$

Therefore, the new private-sector equilibrium conditions of the model are

$$\lambda_t = \left(C_t - \frac{N_t^{1+\chi_n}}{1+\chi_n}\right)^{-\gamma}$$

$$\lambda_t = \beta R_t \lambda_{t+1}\Pi_{t+1}^{-1}$$

$$w_t = N_t^{\chi_n}$$

$$\frac{Y_t}{\lambda_t}\left[\varphi\left(\Pi_t - 1\right)\Pi_t - \left(1+\tau\right)\left(1-\theta\right) - \theta\, w_t\right] = \beta \frac{Y_{t+1}}{\lambda_{t+1}} \varphi\left(\Pi_{t+1} - 1\right)\Pi_{t+1}$$

$$Y_t = C_t + \frac{\varphi}{2}\left[\Pi_t - 1\right]^2 Y_t$$

$$Y_t = N_t$$

Now, we want to find the steady state of the model. We are assuming that (i) the policy rate is determined by the standard Taylor rule, (ii) the inflation target is zero (that is, $\Pi^{targ} = 1$), and (iii) $\tau = \frac{1}{\theta - 1}$.

Then, the standard steady state of the model satisfies:

$$\lambda_{ss} = \left(C_{ss} - \frac{N_{ss}^{1+\chi_n}}{1+\chi_n}\right)^{-\gamma}$$

$$\lambda_{ss} = \beta R_{ss} \lambda_{ss} \Pi_{ss}^{-1}$$

$$w_{ss} = N_{ss}^{\chi_n}$$

$$\frac{Y_{ss}}{\lambda_{ss}}\left[\varphi\left(\Pi_{ss} - 1\right)\Pi_{ss} - \left(1+\tau\right)\left(1-\theta\right) - \theta\, w_{ss}\right] = \beta \frac{Y_{ss}}{\lambda_{ss}} \varphi\left(\Pi_{ss} - 1\right)\Pi_{ss}$$

$$Y_{ss} = C_{ss} + \frac{\varphi}{2}\left[\Pi_{ss} - 1\right]^2 Y_{ss}$$

$$Y_{ss} = N_{ss}$$

$$R_{ss} = \frac{\Pi^{targ}}{\beta}\left(\frac{\Pi_{ss}}{\Pi^{targ}}\right)^{\Phi_\pi}$$

Manipulating the second equation, we get

$$\lambda_{ss} = \beta R_{ss} \lambda_{ss} \Pi_{ss}^{-1}$$

$$\Rightarrow 1 - \beta R_{ss} \Pi_{ss}^{-1} = 0$$

$$\Rightarrow R_{ss} = \frac{1}{\beta}\Pi_{ss}$$

Because $\Pi^{targ}=1$, comparing this equation with the Taylor rule, we get

$$\frac{1}{\beta}\Pi_{ss}=\frac{\Pi^{targ}}{\beta}\left(\frac{\Pi_{ss}}{\Pi^{targ}}\right)^{\Phi_\pi}$$

$$\Rightarrow \frac{1}{\beta}=\frac{1}{\beta}\Pi_{ss}^{\Phi_\pi-1}$$

$$\Rightarrow \Pi_{ss}^{\Phi_\pi-1}=1$$

$$\Rightarrow \Pi_{ss}=1$$

From the fourth equation,

$$\frac{Y_{ss}}{\lambda_{ss}}\left[\varphi(\Pi_{ss}-1)\Pi_{ss}-(1+\tau)(1-\theta)-\theta w_{ss}\right]=\beta\frac{Y_{ss}}{\lambda_{ss}}\varphi(\Pi_{ss}-1)\Pi_{ss}$$

Using $\Pi_{ss}=1$ and $\tau=\frac{1}{\theta-1}$,

$$-(1+\tau)(1-\theta)-\theta w_{ss}=0$$

$$\Rightarrow w_{ss}=\frac{-1}{\theta}(1+\tau)(1-\theta)$$

$$\Rightarrow w_{ss}=\frac{-1}{\theta}\left(1+\frac{1}{\theta-1}\right)(1-\theta)$$

$$\Rightarrow w_{ss}=\frac{-1}{\theta}\left(\frac{\theta}{\theta-1}\right)(1-\theta)$$

$$\Rightarrow w_{ss}=1$$

Because $\Pi_{ss}=1$, the fourth equation becomes

$$Y_{ss}=C_{ss}+\frac{\varphi}{2}\left[\Pi_{ss}-1\right]^2 Y_{ss}$$

$$\Rightarrow Y_{ss}=C_{ss}$$

Because $w_{ss}=1$ and $Y_{ss}=C_{ss}=N_{ss}$, the third equation becomes

$$w_{ss}=N_{ss}^{\chi_n}$$

$$\Rightarrow N_{ss}=1$$

To sum up, the solution is

$$Y_{ss}=C_{ss}=N_{ss}=w_{ss}=\Pi_{ss}=1, R_{ss}=\frac{1}{\beta}$$

Question 3

A two-period model with a static PC with the ZLB. There is a government spending shock financed by a consumption tax at time one:

At t=1:

$$y_1-g_1=(y_2-g_2)-\sigma(\tau_{c,1}-\tau_{c,2})-\sigma(i_1-\pi_2-r_1^n)$$

$$\pi_1=\kappa(y_1-\Gamma_g g_1)$$

$$g_1=\tau_{c,1}$$

$$i_1\geq 0$$

At t=2:

$$y_2-g_2=-\sigma\tau_{c,2}-\sigma(i_2-r^{¿})$$

$$\pi_2=\kappa(y_2-\Gamma_g g_2)$$

$$g_2=\tau_{c,2}$$

$$i_2\geq 0$$

with $g_1>0$ and $g_2=0$. The payoff function for the central bank is given by the standard quadratic objective function. That is,

$$u(\pi_t, y_t)=\frac{-1}{2}[\pi_t^2+\lambda y_t^2]$$

for each t=1,2.

Part A:

Assume that the policy rate is determined by the truncated Taylor rule:

$$i_t = max\left[r^{¿} + \phi_\pi \pi_t, 0\right]$$

Assume also that r_1^n is sufficiently small so that $i_1 = 0$. Solve the model analytically.

Part B:

Now, assume that the government is optimizing under discretion.

- Formulate the optimization problem(s) of the central bank.
- Define the Markov-Perfect equilibrium.

Part C:

Assume that the government is optimizing under commitment.

- Formulate the optimization problem of the central bank.
- Define the Ramsey equilibrium.

<u>Answer to Question 3</u>

<u>Part A:</u>

If $i_2 > 0$, since $g_2 = \tau_{c,2} = 0$,

$$y_2 = -\sigma\left(i_2 - r^{¿}\right)$$

$$\Rightarrow y_2 = -\sigma\phi_\pi \pi_t$$

With $\pi_2 = \kappa y_2$, we get

$$y_2 = \pi_2 = 0, i_2 = r^{¿}$$

Using $i_1 = 0$, at t=1,

$$y_1 - g_1 = -\sigma\tau_{c,1} + \sigma r_1^n$$

$$\Rightarrow y_1 = (1-\sigma)\tau_{c,1} + \sigma r_1^n$$

$$\pi_1 = \kappa\left(y_1 - \Gamma_g g_1\right)$$

$$\Rightarrow \pi_1 = \kappa\left((1-\sigma)\tau_{c,1} + \sigma r_1^n - \Gamma_g \tau_{c,1}\right)$$

$$\Rightarrow \pi_1 = \kappa\left((1-\sigma-\Gamma_g)\tau_{c,1} + \sigma r_1^n\right)$$

To sum up, when $i_2 > 0$, the solution is

$$y_1 = (1-\sigma)\tau_{c,1} + \sigma r_1^n$$

$$\pi_1 = \kappa\left((1-\sigma-\Gamma_g)\tau_{c,1} + \sigma r_1^n\right)$$

$$i_1 = 0$$

$$y_2 = 0$$

$$\pi_2 = 0$$

$$i_2 = r^{¿}$$

If $i_2 = 0$, since $g_2 = \tau_{c,2} = 0$,

$$y_2 = -\sigma\left(i_2 - r^{¿}\right)$$

$$\Rightarrow y_2 = \sigma r^{¿}$$

With $\pi_2 = \kappa y_2$, we get

$$y_2 = \sigma r^{¿}, \pi_2 = \kappa\sigma r^{¿}, i_1 = 0$$

Using $i_1 = 0$, at t=1,

$$y_1 - g_1 = (y_2 - g_2) - \sigma(\tau_{c,1} - \tau_{c,2}) - \sigma\left(i_1 - \pi_2 - r_1^n\right)$$

$$\Rightarrow y_1 - g_1 = y_2 - \sigma\tau_{c,1} + \sigma\pi_2 + \sigma r_1^n$$

$$\Rightarrow y_1 = (1-\sigma)\tau_{c,1} + \sigma r^{¿} + \kappa\sigma^2 r^{¿} + \sigma r_1^n = (1-\sigma)\tau_{c,1} + (1+\kappa\sigma)\sigma r^{¿} + \sigma r_1^n$$

$$\pi_1 = \kappa\left(y_1 - \Gamma_g g_1\right)$$

$$\Rightarrow \pi_1 = \kappa\left((1-\sigma)\tau_{c,1} + (1+\kappa\sigma)\sigma r^{¿} + \sigma r_1^n - \Gamma_g\tau_{c,1}\right)$$

$$\Rightarrow \pi_1 = \kappa\left((1-\sigma-\Gamma_g)\tau_{c,1} + (1+\kappa\sigma)\sigma r^{¿} + \sigma r_1^n\right)$$

To sum up, when $i_2 = 0$, the solution is

$$y_1 = (1-\sigma)\tau_{c,1} + (1+\kappa\sigma)\sigma r^{¿} + \sigma r_1^n$$

$$\pi_1 = \kappa\left((1-\sigma-\Gamma_g)\tau_{c,1} + (1+\kappa\sigma)\sigma r^{¿} + \sigma r_1^n\right)$$

$$i_1 = 0$$

$$y_2 = \sigma r^{¿}$$

$$\pi_2 = \kappa\sigma r^{¿}$$

$$i_2 = 0$$

<u>Part B:</u>

The optimization problem of the central bank under discretion at t = 2,

$$V_2 = \max_{\pi_2, y_2, i_2} u(\pi_2, y_2)$$

subject to

$$y_2 - g_2 = -\sigma\tau_{c,2} - \sigma\left(i_2 - r^{¿}\right)$$

$$\pi_2 = \kappa\left(y_2 - \Gamma_g g_2\right)$$

$$g_2 = \tau_{c,2}$$

$$i_2 \geq 0$$

The optimization problem of the central bank under discretion at t = 1,

$$V_1 = \max_{\pi_1, y_1, i_1} u(\pi_1, y_1)$$

subject to

$$y_1 - g_1 = (y_2 - g_2) - \sigma(\tau_{c,1} - \tau_{c,2}) - \sigma\left(i_1 - \pi_2 - r_1^n\right)$$

$$\pi_1 = \kappa\left(y_1 - \Gamma_g g_1\right)$$

$$g_1 = \tau_{c,1}$$

$$i_1 \geq 0$$

taking V_2, y_2, and π_2 as given.

A Markov-Perfect equilibrium is defined as a vector $\{y_1, \pi_1, i_1, y_2, \pi_2, i_2\}$ that slves the two optimizations problem at t=1 and t=2.

Part C:

The optimization problem of the central bank with commitment is

$$V_{ram} = \max_{\pi_1, y_1, i_1, \pi_2, y_2, i_2} u(\pi_1, y_1) + \beta u(\pi_2, y_2)$$

subject to

$$y_1 - g_1 = (y_2 - g_2) - \sigma(\tau_{c,1} - \tau_{c,2}) - \sigma(i_1 - \pi_2 - r_1^n)$$

$$\pi_1 = \kappa(y_1 - \Gamma_g g_1)$$

$$g_1 = \tau_{c,1}$$

$$i_1 \geq 0$$

$$y_2 - g_2 = -\sigma \tau_{c,2} - \sigma(i_2 - r^{¿})$$

$$\pi_2 = \kappa(y_2 - \Gamma_g g_2)$$

$$g_2 = \tau_{c,2}$$

$$i_2 \geq 0$$

A Ramsey equilibrium is defined as a vector $\{y_1, \pi_1, i_1, y_2, \pi_2, i_2\}$ that slves the optimizations problem above.

Question 4

Consider the following two-period loglinearized New Keynesian model with a <u>static</u> Phillips curve and with a time-one demand shock ($r_1^n < 0$ and $r_2^n = r^{¿} > 0$). The policy rule is given by a price-level targeting rule.

At t=1,

$$y_1 = y_2 - \sigma\left(i_1 - \pi_2 - r_1^n\right)$$

$$\pi_1 = \kappa\, y_1$$

$$p_1 = p_0 + \pi_1$$

$$i_1 = max\left[0, r^{¿} + \phi\, p_1\right]$$

At t=2,

$$y_2 = -\sigma\left(i_2 - r^{¿}\right)$$

$$\pi_2 = \kappa\, y_2$$

$$p_2 = p_1 + \pi_2$$

$$i_2 = max\left[0, r^{¿} + \phi\, p_2\right]$$

- Assuming that (i) $p_0 = 0$ and and (ii) the shock size is such that the policy rate is zero and positive at time one and two, respectively, solve the model analytically.

<u>Answer to Question 4</u>

Since $i_2 > 0$ and $p_0 = 0$, at t=2,

$$y_2 = -\sigma\left(i_2 - r^{¿}\right)$$

$$\Rightarrow y_2 = -\sigma\phi\, p_2 = -\sigma\phi\left(p_1 + \pi_2\right) = -\sigma\phi\left(\pi_1 + \pi_2\right) = -\kappa\sigma\phi\left(y_1 + y_2\right)$$

$$\Rightarrow y_2 = \frac{-\kappa\sigma\phi}{1 + \kappa\sigma\phi}\, y_1$$

Since $i_1 = 0$, at t=1,

$$y_1 = y_2 - \sigma\left(-\pi_2 - r_1^n\right) = \left(1 + \sigma\kappa\right) y_2 + \sigma\, r_1^n$$

$$\Rightarrow \frac{1+\kappa\sigma\phi+(1+\sigma\kappa)\kappa\sigma\phi}{1+\kappa\sigma\phi} y_1=\sigma r_1^n$$

$$\Rightarrow y_1=\frac{(1+\kappa\sigma\phi)\sigma}{1+2\kappa\sigma\phi+\kappa^2\sigma^2\phi} r_1^n$$

$$\pi_1=\kappa y_1=\frac{(1+\kappa\sigma\phi)\kappa\sigma}{1+2\kappa\sigma\phi+\kappa^2\sigma^2\phi} r_1^n$$

$$p_1=p_0+\pi_1=\frac{(1+\kappa\sigma\phi)\kappa\sigma}{1+2\kappa\sigma\phi+\kappa^2\sigma^2\phi} r_1^n$$

Then, at t=2,

$$y_2=\frac{-\kappa\sigma\phi}{1+\kappa\sigma\phi} y_1=\frac{-\kappa\sigma^2\phi}{1+2\kappa\sigma\phi+\kappa^2\sigma^2\phi} r_1^n$$

$$\pi_2=\kappa y_2=\frac{-\kappa^2\sigma^2\phi}{1+2\kappa\sigma\phi+\kappa^2\sigma^2\phi} r_1^n$$

$$p_2=p_1+\pi_2=\frac{(1+\kappa\sigma\phi)\kappa\sigma}{1+2\kappa\sigma\phi+\kappa^2\sigma^2\phi} r_1^n-\frac{\kappa^2\sigma^2\phi}{1+2\kappa\sigma\phi+\kappa^2\sigma^2\phi} r_1^n$$

$$\Rightarrow p_2=\frac{\kappa\sigma}{1+2\kappa\sigma\phi+\kappa^2\sigma^2\phi} r_1^n$$

$$i_2=r^{¿}+\frac{\kappa\sigma\phi}{1+2\kappa\sigma\phi+\kappa^2\sigma^2\phi} r_1^n$$

To sum up, the solution is

$$y_1=\frac{(1+\kappa\sigma\phi)\sigma}{1+2\kappa\sigma\phi+\kappa^2\sigma^2\phi} r_1^n$$

$$\pi_1=\frac{(1+\kappa\sigma\phi)\kappa\sigma}{1+2\kappa\sigma\phi+\kappa^2\sigma^2\phi} r_1^n$$

$$i_1=0$$

$$p_1=\frac{(1+\kappa\sigma\phi)\kappa\sigma}{1+2\kappa\sigma\phi+\kappa^2\sigma^2\phi}r_1^n$$

$$y_2=\frac{-\kappa\sigma^2\phi}{1+2\kappa\sigma\phi+\kappa^2\sigma^2\phi}r_1^n$$

$$\pi_2=\frac{-\kappa^2\sigma^2\phi}{1+2\kappa\sigma\phi+\kappa^2\sigma^2\phi}r_1^n$$

$$i_2=r^{¿}+\frac{\kappa\sigma\phi}{1+2\kappa\sigma\phi+\kappa^2\sigma^2\phi}r_1^n$$

$$p_2=\frac{\kappa\sigma}{1+2\kappa\sigma\phi+\kappa^2\sigma^2\phi}r_1^n$$

<u>Question 5</u>

Consider the following two-period loglinearized New Keynesian model with a <u>static</u> Phillips curve and with a time-one demand shock ($r_1^n<0$ and $r_2^n=r^{¿}>0$). The policy rule is given by a Reifschneider-Williams rule.

At t=1,

$$y_1=y_2-\sigma(i_1-\pi_2-r_1^n)$$

$$\pi_1=\kappa y_1$$

$$i_1=max[0,i_1^{¿}-Z_1]$$

$$Z_1=Z_0+(i_0-i_0^{¿})$$

$$i_1^{¿}=r^{¿}+\phi\pi_1$$

At t=2,

$$y_2=-\sigma(i_2-r^{¿})$$

$$\pi_2=\kappa y_2$$

$$i_2=max[0,i_2^{¿}-Z_2]$$

$$Z_2 = Z_1 + \left(i_1 - i_1^{¿}\right)$$

$$i_2^{¿} = r^{¿} + \phi\,\pi_2$$

- Assuming that (i) $i_0 = i_0^{¿} = r^{¿}$ and $Z_0 = 0$ and (ii) the shock size is such that the policy rate is zero and positive at time one and two, respectively, solve the model analytically.

<u>Answer to Question 5</u>

Using $i_0 = i_0^{¿} = r^{¿}$ and $Z_0 = 0$, we get

$$Z_1 = Z_0 + \left(i_0 - i_0^{¿}\right) = 0$$

Since $i_1 = 0$, at t=1,

$$y_1 = y_2 - \sigma\left(-\pi_2 - r_1^n\right) = y_2 + \sigma\kappa\, y_2 + \sigma\, r_1^n$$

$$\Rightarrow y_1 = (1 + \sigma\kappa)\, y_2 + \sigma\, r_1^n$$

At t=2,

$$Z_2 = Z_1 + \left(i_1 - i_1^{¿}\right) = 0 + \left(0 - i_1^{¿}\right) = -i_1^{¿}$$

Since $i_2 > 0$,

$$i_2 = i_2^{¿} - Z_2 = i_2^{¿} + i_1^{¿}$$

Using this,

$$y_2 = -\sigma\left(i_2 - r^{¿}\right) = -\sigma\left(i_2^{¿} + i_1^{¿} - r^{¿}\right) = -\sigma\left(\left(r^{¿} + \phi\,\pi_2\right) + \left(r^{¿} + \phi\,\pi_1\right) - r^{¿}\right)$$

$$¿ - \sigma\left(r^{¿} + \phi\kappa\, y_2 + \phi\kappa\, y_1\right)$$

$$\Rightarrow \left(1 + \sigma\phi\kappa\right) y_2 = -\sigma\, r^{¿} - \sigma\phi\kappa\, y_1$$

Using $y_1 = (1 + \sigma\kappa)\, y_2 + \sigma\, r_1^n$,

$$\Rightarrow \left(1 + \sigma\phi\kappa\right) y_2 = -\sigma\, r^{¿} - \sigma\phi\kappa\left(\left(1 + \sigma\kappa\right) y_2 + \sigma\, r_1^n\right)$$

$$\Rightarrow \left[\left(1 + \sigma\phi\kappa\right) + \sigma\phi\kappa\left(1 + \sigma\kappa\right)\right] y_2 = -\sigma\, r^{¿} - \sigma^2\,\phi\kappa\, r_1^n$$

$$\Rightarrow y_2=\frac{-\sigma}{1+2\sigma\phi\kappa+\sigma^2\phi\kappa^2}r^{¿}-\frac{\sigma^2\phi\kappa}{1+2\sigma\phi\kappa+\sigma^2\phi\kappa^2}r_1^n$$

Plug this into other equations:

$$\pi_2=\kappa y_2=\frac{-\sigma\kappa}{1+2\sigma\phi\kappa+\sigma^2\phi\kappa^2}r^{¿}-\frac{\sigma^2\phi\kappa^2}{1+2\sigma\phi\kappa+\sigma^2\phi\kappa^2}r_1^n$$

$$y_1=(1+\sigma\kappa)y_2+\sigma r_1^n=(1+\sigma\kappa)\left(\frac{-\sigma}{1+2\sigma\phi\kappa+\sigma^2\phi\kappa^2}r^{¿}-\frac{\sigma^2\phi\kappa}{1+2\sigma\phi\kappa+\sigma^2\phi\kappa^2}r_1^n\right)+\sigma r_1^n$$

$$¿-\frac{\sigma+\sigma^2\kappa}{1+2\sigma\phi\kappa+\sigma^2\phi\kappa^2}r^{¿}-\frac{\sigma^2\phi\kappa+\sigma^3\phi\kappa^2-\left(\sigma+2\sigma^2\phi\kappa+\sigma^3\phi\kappa^2\right)}{1+2\sigma\phi\kappa+\sigma^2\phi\kappa^2}r_1^n$$

$$¿-\frac{\sigma+\sigma^2\kappa}{1+2\sigma\phi\kappa+\sigma^2\phi\kappa^2}r^{¿}+\frac{\sigma+\sigma^2\phi\kappa}{1+2\sigma\phi\kappa+\sigma^2\phi\kappa^2}r_1^n$$

$$\pi_1=\kappa y_1$$

$$\Rightarrow \pi_1=\frac{-\sigma\kappa+\sigma^2\kappa^2}{1+2\sigma\phi\kappa+\sigma^2\phi\kappa^2}r^{¿}+\frac{\sigma\kappa+\sigma^2\phi\kappa^2}{1+2\sigma\phi\kappa+\sigma^2\phi\kappa^2}r_1^n$$

$$i_1^{¿}=r^{¿}+\phi\pi_1=r^{¿}+\left(\frac{-\sigma\phi\kappa+\sigma^2\phi\kappa^2}{1+2\sigma\phi\kappa+\sigma^2\phi\kappa^2}r^{¿}+\frac{\sigma\phi\kappa+\sigma^2\phi^2\kappa^2}{1+2\sigma\phi\kappa+\sigma^2\phi\kappa^2}r_1^n\right)$$

$$¿\frac{1+\sigma\phi\kappa}{1+2\sigma\phi\kappa+\sigma^2\phi\kappa^2}r^{¿}+\frac{\sigma\phi\kappa+\sigma^2\phi^2\kappa^2}{1+2\sigma\phi\kappa+\sigma^2\phi\kappa^2}r_1^n$$

$$i_2^{¿}=r^{¿}+\phi\pi_2=r^{¿}+\phi\left(\frac{-\sigma\kappa}{1+2\sigma\phi\kappa+\sigma^2\phi\kappa^2}r^{¿}-\frac{\sigma^2\phi\kappa^2}{1+2\sigma\phi\kappa+\sigma^2\phi\kappa^2}r_1^n\right)$$

$$¿r^{¿}-\left(\frac{\sigma\phi\kappa}{1+2\sigma\phi\kappa+\sigma^2\phi\kappa^2}\right)r^{¿}-\left(\frac{\sigma^2\phi^2\kappa^2}{1+2\sigma\phi\kappa+\sigma^2\phi\kappa^2}\right)r_1^n$$

$$¿\frac{1+\sigma\phi\kappa+\sigma^2\phi\kappa^2}{1+2\sigma\phi\kappa+\sigma^2\phi\kappa^2}r^{¿}-\frac{\sigma^2\phi^2\kappa^2}{1+2\sigma\phi\kappa+\sigma^2\phi\kappa^2}r_1^n$$

$$Z_2 = -i_1^{¿}$$

$$\Rightarrow Z_2 = \frac{-1+\sigma\phi\kappa}{1+2\sigma\phi\kappa+\sigma^2\phi\kappa^2} r^{¿} - \frac{\sigma\phi\kappa+\sigma^2\phi^2\kappa^2}{1+2\sigma\phi\kappa+\sigma^2\phi\kappa^2} r_1^n$$

$$i_2 = i_2^{¿} + i_1^{¿}$$

$$¿\left(\frac{1+\sigma\phi\kappa+\sigma^2\phi\kappa^2}{1+2\sigma\phi\kappa+\sigma^2\phi\kappa^2} r^{¿} - \frac{\sigma^2\phi^2\kappa^2}{1+2\sigma\phi\kappa+\sigma^2\phi\kappa^2} r_1^n\right) + \left(\frac{1+\sigma\phi\kappa}{1+2\sigma\phi\kappa+\sigma^2\phi\kappa^2} r^{¿} + \frac{\sigma\phi\kappa+\sigma^2\phi^2\kappa^2}{1+2\sigma\phi\kappa+\sigma^2\phi\kappa^2} r_1^n\right)$$

$$¿\frac{2+2\sigma\phi\kappa+\sigma^2\phi\kappa^2}{1+2\sigma\phi\kappa+\sigma^2\phi\kappa^2} r^{¿} + \frac{\sigma\phi\kappa}{1+2\sigma\phi\kappa+\sigma^2\phi\kappa^2} r_1^n$$

To sum up, the solution is

$$y_1 = \frac{-\sigma+\sigma^2\kappa}{1+2\sigma\phi\kappa+\sigma^2\phi\kappa^2} r^{¿} + \frac{\sigma+\sigma^2\phi\kappa}{1+2\sigma\phi\kappa+\sigma^2\phi\kappa^2} r_1^n$$

$$\pi_1 = \frac{-\sigma\kappa+\sigma^2\kappa^2}{1+2\sigma\phi\kappa+\sigma^2\phi\kappa^2} r^{¿} + \frac{\sigma\kappa+\sigma^2\phi\kappa^2}{1+2\sigma\phi\kappa+\sigma^2\phi\kappa^2} r_1^n$$

$$i_1 = 0$$

$$i_1^{¿} = \frac{1+\sigma\phi\kappa}{1+2\sigma\phi\kappa+\sigma^2\phi\kappa^2} r^{¿} + \frac{\sigma\phi\kappa+\sigma^2\phi^2\kappa^2}{1+2\sigma\phi\kappa+\sigma^2\phi\kappa^2} r_1^n$$

$$Z_1 = 0$$

$$y_2 = \frac{-\sigma}{1+2\sigma\phi\kappa+\sigma^2\phi\kappa^2} r^{¿} - \frac{\sigma^2\phi\kappa}{1+2\sigma\phi\kappa+\sigma^2\phi\kappa^2} r_1^n$$

$$\pi_2 = \frac{-\sigma\kappa}{1+2\sigma\phi\kappa+\sigma^2\phi\kappa^2} r^{¿} - \frac{\sigma^2\phi\kappa^2}{1+2\sigma\phi\kappa+\sigma^2\phi\kappa^2} r_1^n$$

$$i_2 = \frac{2+2\sigma\phi\kappa+\sigma^2\phi\kappa^2}{1+2\sigma\phi\kappa+\sigma^2\phi\kappa^2} r^{¿} + \frac{\sigma\phi\kappa}{1+2\sigma\phi\kappa+\sigma^2\phi\kappa^2} r_1^n$$

$$i_2^{¿} = \frac{1+\sigma\phi\kappa+\sigma^2\phi\kappa^2}{1+2\sigma\phi\kappa+\sigma^2\phi\kappa^2} r^{¿} - \frac{\sigma^2\phi^2\kappa^2}{1+2\sigma\phi\kappa+\sigma^2\phi\kappa^2} r_1^n$$

$$Z_2 = \frac{-1+\sigma\phi\kappa}{1+2\sigma\phi\kappa+\sigma^2\phi\kappa^2} r^{¿} - \frac{\sigma\phi\kappa+\sigma^2\phi^2\kappa^2}{1+2\sigma\phi\kappa+\sigma^2\phi\kappa^2} r_1^n$$