Math 340 Tutorial 3 Solutions

1. Use generating functions to find a closed formula for the following sums:

   (a) $\sum_{k=0}^{n} k$.

   (b) $\sum_{k=0}^{n}(k+1)^2$.

   (c) $\sum_{k=0}^{n} 2^k$.

   Solution:

   (a) Let $a_n = \sum_{k=0}^{n} k$. Then $a_n = a_{n-1} + n$, with $a_0 = 0$. We can solve this recurrence using generating functions. Let $A(x) = \sum_{n\geq 0} a_n x^n$. Let's write the first few terms in the recurrence:

$$
\begin{aligned}
a_1 &= a_0 + 1\\
a_2 &= a_1 + 2\\
a_3 &= a_2 + 3\\
a_4 &= a_3 + 4\\
a_5 &= a_4 + 5\\
&\vdots
\end{aligned}
$$

Now we will multiply each equation by a power of x like so:

$$
\begin{aligned}
a_1 x &= a_0 x + x\\
a_2 x^2 &= a_1 x^2 + 2x^2\\
a_3 x^3 &= a_2 x^3 + 3x^3\\
a_4 x^4 &= a_3 x^4 + 4x^4\\
a_5 x^5 &= a_4 x^5 + 5x^5\\
&\vdots
\end{aligned}
$$

If we add these equations together, then the sum on the left hand side is

$$a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots = A(x) - a_0 = A(x),$$

and the right hand side is

$$\left(a_0 x + a_1 x^2 + a_2 x^3 + a_3 x^4 + a_4 x^5 + \dots\right) + \left(x + 2x^2 + 3x^3 + 4x^4 + 5x^5 + \dots\right)$$
$$= xA(x) + B(x)$$

where $B(x)$ is the generating function that corresponds to the sequence $0, 1, 2, 3, 4, 5, \dots$. This sequence can be found by taking the derivative of $\frac{1}{1-x}$, then multiplying by

x. So $B(x) = \frac{x}{(1-x)^2}$. Now we can solve for $A(x)$:

$$\begin{aligned} A(x) &= xA(x) + \frac{x}{(1-x)^2} \\ (1-x)A(x) &= \frac{x}{(1-x)^2} \\ A(x) &= \frac{x}{(1-x)^3}. \end{aligned}$$

So we have $A(x)$, but our ultimate goal is to solve for a_n. a_n is the coefficient of x^n in $A(x)$, so we just need to figure out the coefficients of $A(x)$. Starting from $\frac{1}{1-x}$, taking the derivative twice gives us $\frac{2}{(1-x)^3}$. Then we divide by 2 and multiply by x to get $\frac{x}{(1-x)^3}$. The corresponding sequence is:

$$0, \frac{1 \cdot 2}{2}, \frac{2 \cdot 3}{2}, \frac{3 \cdot 4}{2}, \frac{4 \cdot 5}{2}, \ldots, \frac{n(n+1)}{2}, \ldots$$

Hence, $a_n = \frac{n(n+1)}{2}$.

(a) *continued:*

(b) Let $a_n = \sum_{k=0}^{n}(k+1)^2$. Then $a_n = a_{n-1} + (n+1)^2$, with $a_0 = 1$. Letting $A(x) = \sum_{n\geq 0} a_n x^n$, by the same method as before, we get

$$A(x) - 1 = xA(x) + B(x),$$

where $B(x)$ corresponds to the sequence $0, 2^2, 3^2, 4^2, 5^2, 6^2, \ldots$. To get this sequence, we first take the derivative of $\frac{1}{1-x}$ twice to get $\frac{2}{(1-x)^3}$, which corresponds to the sequence $1\cdot 2, 2\cdot 3, 3\cdot 4, 4\cdot 5, \ldots, (n+1)(n+2), \ldots$. This is close, but we need to subtract $(n+1)$ from each of these terms to get $(n+1)^2$. Hence, we need to subtract the sequence $1, 2, 3, 4, 5, 6, \ldots, (n+1), \ldots$, and we know this corresponds to $\frac{1}{(1-x)^2}$. So $\frac{2}{(1-x)^3} - \frac{1}{(1-x)^2}$ gives us $1^2, 2^2, 3^2, 4^2, \ldots$. Subtracting 1 gives us the sequence for $B(x)$. Hence,

$$B(x) = \frac{2}{(1-x)^3} - \frac{1}{(1-x)^2} - 1 = \frac{2-(1-x)}{(1-x)^3} - 1 = \frac{2+x}{(1-x)^3} - 1.$$

Now we can solve for $A(x)$:

$$\begin{aligned} A(x) - 1 &= xA(x) + \frac{1+x}{(1-x)^3} - 1 \\ (1-x)A(x) &= \frac{1+x}{(1-x)^3} \\ A(x) &= \frac{1+x}{(1-x)^4} \end{aligned}$$

So we need for figure out the sequence that corresponds to $\frac{x+x^2}{(1-x)^4}$. We should really think of this as $\frac{1}{(1-x)^4} + \frac{x}{(1-x)^4}$. Now $\frac{6}{(1-x)^4}$ is the third derivative of $\frac{1}{1-x}$, which corresponds to $1\cdot 2\cdot 3, 2\cdot 3\cdot 4, \ldots, (n+1)(n+2)(n+3), \ldots$, but we need to divide this by 6. Lastly, we need to multiply by x for the second term to get $\frac{x}{(1-x)^4}$, corresponding to $0, 1\cdot 2\cdot 3/6, 2\cdot 3\cdot 4/6, \ldots, n(n+1)(n+2)/6, \ldots$. Putting this all together, we get

$$\begin{aligned} a_n &= \frac{n(n+1)(n+2) + (n+1)(n+2)(n+3)}{6} \\ &= \frac{(n+1)(n+2)(2n+3)}{6}. \end{aligned}$$

(b) *continued:*

(c) Same strategy as before. We have

$$A(x) - 1 = xA(x) + B(x)$$

where $B(x)$ corresponds to the sequence $0, 2^1, 2^2, 2^3, 2^4, \ldots, 2^n, \ldots$. We know that $\frac{1}{1-2x}$ corresponds to $2^0, 2^1, 2^2, 2^3, 2^4, \ldots$. So we just need to subtract 1 to get $B(x)$. Now we can solve for $A(x)$:

$$\begin{aligned} A(x) - 1 &= xA(x) + \frac{1}{1-2x} - 1 \\ (1-x)A(x) &= \frac{1}{1-2x} \\ A(x) &= \frac{1}{(1-x)(1-2x)} \\ &= \frac{-1}{1-x} + \frac{2}{1-2x}. \end{aligned}$$

Hence, $a_n = -1 + 2(2^n)$. Notice that this is equal to $\frac{1-2^{n+1}}{1-2}$, which agrees with the geometric formula.

2. Solve the following recurrences using generating functions:

   (a) $a_0 = 1, a_1 = 1, a_n = 2a_{n-1} - a_{n-2}$.

   (b) $a_0 = 1, a_1 = 6, a_n = 3a_{n-1} - 2a_{n-2}$.

   (c) $a_0 = 1, a_1 = 2, a_2 = 3, a_n = 3a_{n-1} - 3a_{n-2} + a_{n-3}$.

   (d) $a_0 = 1, a_1 = 2, a_n = 2a_{n-1} - a_{n-2} + n$.

Solution:

(a) Let $A(x) = \sum_{n \geq 0} a_n x^n$. Just like we did in question 1, we will write out a bunch of the equations in the recurrence like so:

$$\begin{aligned} a_2 &= 2a_1 - a_0 \\ a_3 &= 2a_2 - a_1 \\ a_4 &= 2a_3 - a_2 \\ a_5 &= 2a_4 - a_3 \\ &\vdots \end{aligned}$$

Now multiply be an appropriate power of x like so:

$$\begin{aligned} a_2x^2 &= 2a_1x^2 - a_0x^2 \\ a_3x^3 &= 2a_2x^3 - a_1x^3 \\ a_4x^4 &= 2a_3x^4 - a_2x^4 \\ a_5x^5 &= 2a_4x^5 - a_3x^5 \\ &\vdots \end{aligned}$$

If we add these equations together, we have the following equation:

$$A(x) - a_1x - a_0 = 2x\big(A(x) - a_0\big) - x^2A(x).$$

We know $a_0 = 1$ and $a_1 = 1$, so let's plug those values in:

$$A(x) - x - 1 = 2x\big(A(x) - 1\big) - x^2A(x).$$

Now solve for $A(x)$:

$$A(x) = \frac{1-x}{1-2x+x^2}.$$

$1 - 2x + x^2 = (1-x)^2$, so we have

$$A(x) = \frac{1-x}{(1-x)^2} = \frac{1}{1-x}.$$

So as it turns out, $a_n = 1$ for all n.

(a) *continued:*

(b) Using the same strategy as before, we get

$$A(x) - a_1 x - a_0 = 3x\big(A(x) - a_0\big) - 2x^2 A(x).$$

Plugging in $a_1 = 6$ and $a_0 = 1$ gives

$$A(x) - 6x - 1 = 3x\big(A(x) - 1\big) - 2x^2 A(x).$$

Solving for $A(x)$ gives

$$A(x) = \frac{3x+1}{1 - 3x + 2x^2} = \frac{3x+1}{(1-x)(1-2x)}.$$

Now we can split $\frac{3x+1}{(1-x)(1-2x)}$ using partial fractions. We end up with

$$\frac{3x+1}{(1-x)(1-2x)} = \frac{-4}{1-x} + \frac{5}{1-2x}.$$

So

$$A(x) = \frac{-4}{1-2x} + \frac{5}{1-x}$$

Hence, $a_n = 5(2^n) - 4$.

(b) *continued:*

(c) Because this is a 3 step recurrence and not a 2 step, we'll derive the formula for $A(x)$ again:

$$
\begin{aligned}
a_3 &= 3a_2 - 3a_1 + a_0 \\
a_4 &= 3a_3 - 3a_2 + a_1 \\
a_5 &= 3a_4 - 3a_3 + a_2 \\
a_6 &= 3a_5 - 3a_4 + a_3 \\
a_7 &= 3a_6 - 3a_5 + a_4 \\
&\vdots
\end{aligned}
$$

Now multiply by the appropriate powers of x like so:

$$
\begin{aligned}
a_3x^3 &= 3a_2x^3 - 3a_1x^3 + a_0x^3 \\
a_4x^4 &= 3a_3x^4 - 3a_2x^4 + a_1x^4 \\
a_5x^5 &= 3a_4x^5 - 3a_3x^5 + a_2x^5 \\
a_6x^6 &= 3a_5x^6 - 3a_4x^6 + a_3x^6 \\
a_7x^7 &= 3a_6x^7 - 3a_5x^7 + a_4x^7 \\
&\vdots
\end{aligned}
$$

Adding the equations together, we get

$$A(x) - a_2x^2 - a_1x - a_0 = 3x\Big(A(x) - a_1x - a_0\Big) - 3x^2\Big(A(x) - a_0\Big) + x^3A(x).$$

Plugging in $a_0 = 1, a_1 = 2, a_2 = 3$, we get

$$A(x) - 3x^2 - 2x - 1 = 3x\Big(A(x) - 2x - 1\Big) - 3x^2\Big(A(x) - 1\Big) + x^3A(x).$$

Solving for $A(x)$ gives us

$$A(x) = \frac{1-x}{1-3x+3x^2-x^3} = \frac{1-x}{(1-x)^3} = \frac{1}{(1-x)^2}.$$

Since $\frac{1}{(1-x)^2}$ corresponds to the sequence $1, 2, 3, 4, 5, \ldots$, it follows that $a_n = n+1$.

(c) *continued:*

(d) Now we have an inhomogeneous factor in our recurrence. Not to worry though, as we can perform the exact same steps as before:

$$\begin{aligned}
a_2 &= 2a_1 - a_0 + 2\\
a_3 &= 2a_2 - a_1 + 3\\
a_4 &= 2a_3 - a_2 + 4\\
a_5 &= 2a_4 - a_3 + 5\\
a_6 &= 2a_5 - a_4 + 6\\
&\vdots
\end{aligned}$$

Multiply by the appropriate powers of x:

$$\begin{aligned}
a_2x^2 &= 2a_1x^2 - a_0x^2 + 2x^2\\
a_3x^3 &= 2a_2x^3 - a_1x^3 + 3x^3\\
a_4x^4 &= 2a_3x^4 - a_2x^4 + 4x^4\\
a_5x^5 &= 2a_4x^5 - a_3x^5 + 5x^5\\
a_6x^6 &= 2a_5x^6 - a_4x^6 + 6x^6\\
&\vdots
\end{aligned}$$

Adding the equations together gives

$$A(x) - a_1x - a_0 = 2x\Big(A(x) - a_0\Big) - x^2A(x) + B(x),$$

where $B(x)$ corresponds to the sequence $0, 0, 2, 3, 4, 5, 6, dots$. To get here, we take the derivative of $\frac{1}{1-x}$, subtract 1, and multiply by x. So

$$A(x) - a_1x - a_0 = 2x\Big(A(x) - a_0\Big) - x^2A(x) + x\left(\frac{1}{(1-x)^2} - 1\right).$$

Now plug in $a_0 = 1, a_1 = 2$:

$$A(x) - 2x - 1 = 2x\Big(A(x) - 1\Big) - x^2A(x) + x\left(\frac{1}{(1-x)^2} - 1\right).$$

Solve for $A(x)$:

$$\begin{aligned}
A(x) &= \frac{1}{1-2x+x^2} + \frac{x}{1-2x+x^2}\left(\frac{1}{(1-x)^2} - 1\right)\\
&= \frac{1}{(1-x)^2} + \frac{x}{(1-x)^4} - \frac{x}{(1-x)^2}\\
&= \frac{1}{1-x} + \frac{x}{(1-x)^4}.
\end{aligned}$$

$\frac{x}{(1-x)^4}$ is found by taking the derivative of $\frac{1}{1-x}$ 3 times to get $\frac{6}{(1-x)^4}$, then multiplying by $x/6$. So the corresponding sequence is

$$0, 1 \cdot 2 \cdot 3/6, 2 \cdot 3 \cdot 4/6, \ldots, n(n+1)(n+2)/6, \ldots$$

Therefore, $a_n = 1 + \frac{n(n+1)(n+2)}{6}$.

(d) *continued:*

3. Suppose that $a_n = b_{n-1} + \sum_{i=1}^{n} a_{n-i}$ for some sequence b_n. Show that $a_n = 2a_{n-1} + b_{n-1} - b_{n-2}$ for $n \geq 2$.

   Hint: this fact will be useful for problems 4 (b) and 5(b).

   Solution

   Observe that

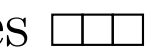
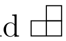
$$a_n - a_{n-1} = b_{n-1} + \sum_{i=1}^{n} a_{n-i} - (b_{n-2} + \sum_{i=1}^{n} a_{n-1-i})$$

$$= b_{n-1} - b_{n-2} + a_{n-1} + (\sum_{i=2}^{n} a_{n-i} - \sum_{i=1}^{n-1} a_{n-1-i}) = a_{n-1} + b_{n-1} - b_{n-2},$$

   and so $a_n = 2a_{n-1} + b_{n-1} - b_{n-2}$ as desired.

4. Tilings

(a) Let g_n be the number of ways to cover a $1 \times n$ board using only 1×1 red tiles, 1×1 blue tiles, and 1×2 yellow tiles. Find a recurrence for g_n and solve this recurrence using generating functions.

(b) A *tromino* consists of 3 connected unit squares, i.e. the shapes  and  along with their rotations. How many ways are there to tile a $2 \times n$ unit rectangle with trominoes?

Solution to 4a: Suppose we have an empty $1 \times n$ board and we start filling it with tiles. How many ways can we fill the left most tile? We could use any of the 3 tiles. If we use either of the 1×1 tiles, we are left to fill in a $1 \times n - 1$ board, and if we use the 1×2 tiles, we are left to fill in a $1 \times n - 2$ board. The number of ways to fill the $1 \times n$ board is just the sum over all 3 cases of the number of ways to fill the remaining board. Hence,

$$g_n = 2g_{n-1} + g_{n-2}.$$

For the base cases, $g_0 = 1$ (1 way to do nothing) and $g_1 = 2$ since there are 2 1×1 tiles. Now we can solve this recurrence using generating functions. Let $G(x) = \sum_{n \geq 0} g_n x^n$. Then

$$G(x) - g_1 x - g_0 = 2x\Big(G(x) - a_0\Big) + x^2 G(x).$$

Putting $g_1 = 2$ and $g_0 = 1$:

$$G(x) - 2x - 1 = 2x\Big(G(x) - 1\Big) + x^2 G(x)$$

Solving for $G(x)$:

$$\begin{aligned}
G(x) &= \frac{1}{1 - 2x - x^2} \\
&= \frac{1}{\Big(1 - (1 + \sqrt{2})x\Big)\Big(1 - (1 - \sqrt{2})x\Big)} \\
&= \left(\frac{2 + \sqrt{2}}{4}\right) \frac{1}{1 - (1 + \sqrt{2})x} + \left(\frac{2 - \sqrt{2}}{4}\right) \frac{1}{1 - (1 - \sqrt{2})x}
\end{aligned}$$

Therefore,

$$g_n = \left(\frac{2 + \sqrt{2}}{4}\right)(1 + \sqrt{2})^n + \left(\frac{2 - \sqrt{2}}{4}\right)(1 - \sqrt{2})^n$$

Solution to 4b:

Let T_n be the number of ways to tile a $2 \times n$ board with trominoes.

First, we consider the different ways of starting our tiling. There are 3 cases:

Case 1: The lower left corner of the board is filled with a piece. In this case the top left corner must also be filled with a piece, and so the start of our tiling is . Let L_n be the number of tilings of a $2 \times n$ board that start this way and notice that $L_n = T_{n-3}$.

Case 2: The lower left corner is filled with a piece. Notice that the only way to extend this start is to next place 0 up to k pieces followed by a piece. For example, with 0 pieces this position looks like , and with 1 it looks like . Let $A_{n,k}$ be the number of ways to tile a $2 \times n$ board that start with a followed by k pieces followed by a piece, and notice that $A_{n,k} = T_{n-3(k+1)}$.

Case 3: The lower left corner is filled with a piece. The case analysis here is the same as in case 2, but with all of the pieces flipped upside down. Let $B_{n,k}$ be the number of ways to tile a $2 \times n$ board in by starting in the relevant way, and notice that $B_{n,k} = T_{n-3(k+1)}$.

With the starting case analysis done, notice that $T_n = 0$ if n is not a multiple of 3, and otherwise we see that

$$T_{3m} = L_{3m} + \sum_{k=0}^{m-1}(A_{3m,k} + B_{3m,k}) = T_{3(m-1)} + \sum_{k=0}^{m-1} 2T_{3m-3(k+1)}.$$

Now, considering $T_{3m} - T_{3(m-1)}$, by the same argument from problem 3 we see that

$$T_{3m} = 4T_{3(m-1)} - T_{3(m-2)}.$$

Now, to make our lives easier let $a_n = T_{3n}$ and consider $A(x) = \sum a_n x^n$. By our usual tricks we see that

$$A(x) = a_0 + a_1 x + 4x(A(x) - a_0) - x^2 A(x).$$

Plugging in the initial conditions $a_0 = 1$ and $a_1 = 3$ and solving for $A(x)$ we get

$$A(x) = \frac{1 - x}{1 - 4x + x^2}.$$

Using partial fractions:

$$A(x) = \frac{1/6(3 + \sqrt{3}}{1 - (2 + \sqrt{3}x} + \frac{1/6(3 - \sqrt{3}}{1 - (2 - \sqrt{3}x}.$$

Finally, using our formula for the sequence corresponding to $\frac{c}{1-\alpha x}$ we see

$$a_n = \frac{1}{6}(3+\sqrt{3})(2+\sqrt{3})^n + \frac{1}{6}(3-\sqrt{3})(2-\sqrt{3})^n.$$

5. Lattice walks

(a) Consider walks on the integer lattice that start from $(0,0)$ and use steps of the form $(1,0)$, $(0,1)$, or $(-1,0)$ (i.e., from the point (a,b) we can reach any of $(a+1,b)$, $(a,b+1)$, or $(a-1,b)$ in a single step). How many length n walks are there? How many length n walks are there that do not intersect themselves?

(b) Consider walks on the integer lattice that start from $(0,0)$ and use steps of the form $(1,0)$ or $(0,a)$ for $a \in \{1,...,n\}$. How many such walks end on a point (x,y) such that $x+y=n$?

Solution to 5a: We partition based off of the last two step.

If the last step was up, then this could have been preceded by any other type of step, and so there are a_{n-1} ways to end with an up step.

Similarly, note that there are a_{n-1} paths that end in any of left-left, right-right, or up-left (we take any path of length $n-1$ then do the last forced step).

Finally, there a_{n-2} walks that end with an up-right move.

As this covers all of the ways to end a walk, we see that $a_n = 2a_{n-1} + a_{n-2}$. Therefore, letting $A(x) = \sum_{n\geq 0} a_n x^n$ and using our standard tricks we see that

$$A(x) = a_0 + a_1 x + 2x(A(x) - a_0) + x^2 A(x).$$

Next, plugging in $a_0 = 1$ and $a_1 = 3$ and solving for $A(x)$, we see that

$$A(x) = \frac{1+x}{1-2x-x^2}.$$

Using partial fractions, we find

$$A(x) = \frac{.5(1+\sqrt{2})}{1-(1+\sqrt{2})x} + \frac{.5(1-\sqrt{2}}{1-(1-\sqrt{2})x}$$

So, by our formula for the generating function of $\frac{c}{1-\alpha x}$, we have

$$a_n = \frac{1+\sqrt{2}}{2}(1+\sqrt{2})^n + \frac{1-\sqrt{2}}{2}(1-\sqrt{2})^n$$

Solution to problem 5b: As usual, we set up our recurrence by partitioning based on the last move. If we end with a $(1, 0)$ move then the second to last point (x, y) in our walk satisfies $x + y = n - 1$. Similarly, If we end with a $(0, a)$ move then the second to last point (x, y) in our walk satisfies $x + y = n - a$.

Consequently, we see that $a_n = a_{n-1} + \sum_{i=1}^{n} a_{n-i}$. Therefore, by problem 3. we know $a_n = 3a_{n-1} - a_{n-2}$.

Now, rather than finishing the proof by hand we'll actually just show that $a_n = F_{2n}$, where F_{2n} is the $2n$'th Fibonacci number. Since we already know the formula for the Fibonacci numbers from class this will save us some time.

As we know $F_0 = a_0 = 1$ and $F_2 = a_1 = 3$, it suffices to show that $F_{2n} = 3F_{2(n-1} - F_{2(n-2)}$. To do this we just use the fact that $F_n = F_{n-1} + F_{n-2}$ twice followed by the fact that $F_{2n-3} = F_{2n-2} - F_{2n-4}$ once:

$$F_{2n} = F_{2n-1} + F_{2n-2} = (F_{2n-2} + F_{2n-3}) + F_{2n-2} = 3F_{2n-2} - F_{2n-4}$$

as desired.