Some counting problems

Jérôme Fortier

McGill University

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Sum and product principles

Problem 1

How many possible canadian postal codes are there?

$|L| = 26$ $|D| = 10$

L D L D L D

Letter Digit

Ex: H3N 9D0

H0H 0H0

Product principle: Cart. product Cardinality

$$|A_1 \times A_2 \times \cdots \times A_n| = |A_1| \times |A_2| \times \cdots \times |A_n|$$

Finite sets. product

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i\}$$

$$\Rightarrow |\text{Postal Codes}| = 26 \times 10 \times 26 \times 10 \times 26 \times 10 = 26^3 \cdot 10^3 = 17576000$$

Problem 2

Show that for any finite set A, we have: $|\mathcal{P}(A)| = 2^{|A|}$. ✓

* Recall: $\mathcal{P}(A) = \{ X \mid X \text{ is a subset of } A \}$.

→ Ex: $A = \{1, 2, 3\}$.

$\mathcal{P}(A) = \{ \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, A, \emptyset \}$

$8 = 2^3$

→ Proof: To form a set $X \subseteq A$, you need to choose, for each $a \in A$ whether $a \in X$ or $a \notin X$ ...

$A = \{a_1, a_2, \ldots, a_n\}$

$|A| = n$

Yes No
2 options

Y N   Y N

$\underline{\quad}_{a_1}$ $\underline{\quad}_{a_2}$ $\underline{\quad}_{a_3}$ $\underline{\quad}_{a_4}$ ... $\underline{\quad}_{a_n}$

$|\mathcal{P}(A)| = |\text{ways to make that choice}|$
$= 2^n$ (product principle).

Problem 3 (Sum principle)

We must choose a president for something. Among the candidates, there are 3 men and 4 women. How many choices do we have?

Obviously: 3 men + 4 women = 7 candidates.

Answer = 7.

⇒ Sum rule: If $A_1, A_2, \ldots, A_n$ are pairwise disjoint sets

($A_i \cap A_j = \emptyset$ when $i \neq j$).

then $|A_1 \cup A_2 \cup \ldots \cup A_n| = |A_1| + |A_2| + \cdots + |A_n|$.

Problem 4

A password on some website has to contain at least 6 and at most 8 characters. There are 62 admissible characters (a-z,A-Z,0-9). How many admissible passwords are there?

Passwords = Passwords with 6 chars $\rightsquigarrow P = P_6 \cup P_7 \cup P_8$

$\cup$ Passwords with 7 chars

$\cup$ Passwords with 8 chars.

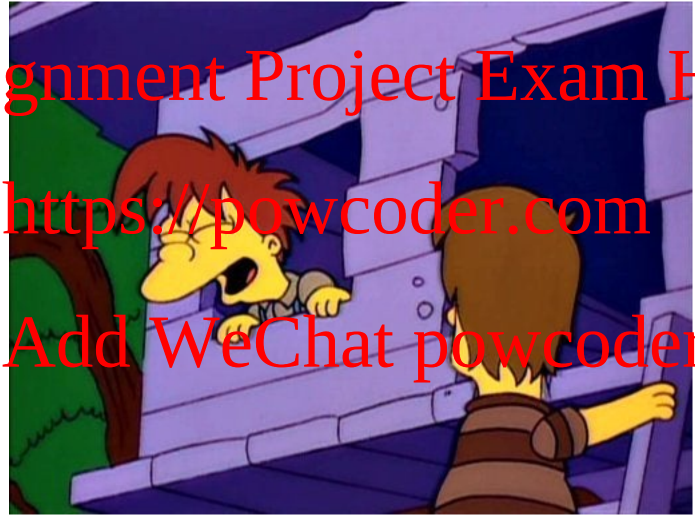
$$\Rightarrow |P| = |P_6| + |P_7| + |P_8|$$

By the product principle: $|P_n| = 62^n$

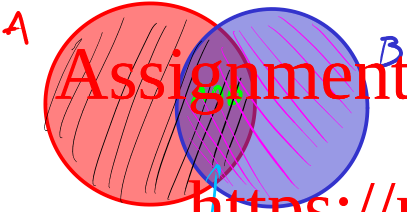
$$\Rightarrow |P| = 62^6 + 62^7 + 62^8 = \text{Big} \ldots$$

$$\approx 2.22 \times 10^{14}$$

The Inclusion-Exclusion Principle



The Inclusion-Exclusion Principle



$|A \cup B| = |A| + |B| - |A \cap B|$

2 times − 1 time = 1 time

Problem 5

2 characters: 0 and 1

"Union of sets"

How many bit strings of length 8 start with 1 or end with 00?

A = strings of length 8 that start with 1

B = strings of length 8 that end with 00.

Goal: Compute $|A \cup B| = |A| + |B| - |A \cap B|$.

A: 1 (2 options) $\Rightarrow |A| = 2^7$

B: 0 0 $\Rightarrow |B| = 2^6$

$A \cap B$: 1 0 0 $\Rightarrow |A \cap B| = 2^5$

Answer: $|A \cup B| = 2^7 + 2^6 - 2^5 = 2^5(4+2-1) = 5 \times 2^5$.

Problem 6

How many integers from 1 to 1000 are divisible by 4 or by 7?

$A = \{x \in [1..1000] : 4 \mid x\} \to |A| = \left\lfloor \frac{1000}{4} \right\rfloor = 250$

$B = \{x \in [1..1000] : 7 \mid x\} \to |B| = 142$

$A \cap B = \{x \in [1..1000] : 28 \mid x\} \to |A \cap B| = \left\lfloor \frac{1000}{28} \right\rfloor = 35$

$|A \cup B| = |A| + |B| - |A \cap B|$
$= 250 + 142 - 35$
$= 357$

Least common factor
LCM

Counting those sets?

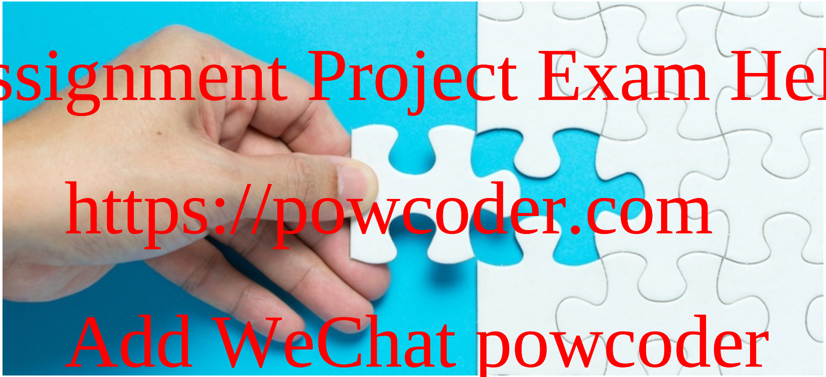
B:

$\left\lfloor \frac{1000}{7} \right\rfloor = \lfloor 142.\cancel{86} \rfloor = 142.$

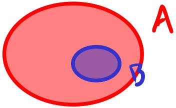
$142 \times 7 = 994$

Def: $\lfloor x \rfloor$ = "Floor function"
= Largest integer $\le x$.

The Complement Principle



$B \subseteq A$. Then $|A \setminus B| = |A - B| = |A| - |B|$



$A = B \cup (A \setminus B)$

$|A| = |B| + |A \setminus B| \Rightarrow |A \setminus B| = |A| - |B|$.

Problem 7

How many 6-characters license plates do **not** contain the word 'SEX'?

_ _ _ _ _ _

Letter or digit. (36 options for each character).

Answer: $36^6 - (36^3 \times 4 - 1)$

$\Rightarrow$ Total number of plates: 36^6

$= 36^6 - 4 \times 36^3 + 1$

Want: # plates that do contain 'SEX' (we will subtract it from the total).

$A \cap D \neq \emptyset$: SEXSEX.

Possible formats:

SEX_ _ _ : A

_ SEX _ _ : B

_ _ SEX _ : C

_ _ _ SEX : D

Answer: $|A| + |B| + |C| + |D| - 1$

$|A \cap D|$

$= 36^3 \times 4 - 1$

The pigeonhole principle

2 pigeons.

The pigeonhole principle

If you store N objects (pigeons) in k boxes (holes) and $N > k$,

then there must be at least one box that holds at least 2 objects...

$\rightarrow \frac{N}{k} > 1 \Rightarrow \lceil \frac{N}{k} \rceil \geq 2$

More general version:

N objects, k boxes. Then there is at least one box that contains at least $\lceil \frac{N}{k} \rceil$ objects.

Proof: Contradiction.

Suppose we can fit $< \lceil \frac{N}{k} \rceil$ objects per box. $\Rightarrow \leq \lceil \frac{N}{k} \rceil - 1$

$N = \#$ Total objects $\leq k\left(\lceil \frac{N}{k} \rceil - 1\right) < k\left(\frac{N}{k}\right) = N$

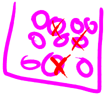
$N < N$: Contradiction!

Ceiling function: $\lceil x \rceil =$ smallest integer $\geq x$.

$\lceil x \rceil - 1 < x$

Problem 8

You have an urn with 10 balls inside of it. Balls are numbered with numbers from 1 to 10. We draw 3 balls (3 distinct numbers) and add them up. How many times do we need to do this to guarantee that the same number appears twice?



Add up the numbers ...

Objects: A repetition of this experiment.

Boxes: Result (Sum of three balls)

$\{$ Smallest: $1+2+3 = 6$
Largest: $10+9+8 = 27$

Boxes: $[6..27] \Rightarrow k = 27-6+1 = 22$.

① Count the number of boxes k.

② Add 1: $N = k+1 = \boxed{23}$.

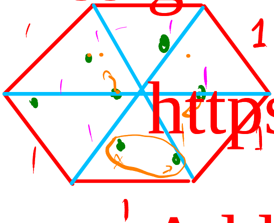
③ Conclusion follows from the pigeonhole principle!

Ans: $N = 23$ times.

Problem 9

regular

Take a ~~rectangular~~ hexagon with side 1. Put 7 points inside the hexagon (at random). Show that at least two of them have a distance ≤ 1.



Find 6 boxes ...

Such that

same box $\Rightarrow$ Distance ≤ 1.

6 triangles (boxes)

side 1 $\Rightarrow$ Distance ≤ 1.

By the Pigeonhole principle, at least one triangle contains at least 2 points.