Problem 19

Use the Binomial Theorem to prove the following identities...

(a) $\sum_{j=0}^{n} \binom{n}{j} = 2^n$

(b) $\sum_{k=1}^{n} (-1)^k \binom{n}{k} = -1$

$+(-1)^0\binom{n}{0}$

$\leftrightarrow \sum_{k=0}^{n} (-1)^k \binom{n}{k} = -1+1 = 0$

Key: $2 = 1+1$

$(1+1)^n \overset{BT}{=} \sum_{j=0}^{n} \binom{n}{j} 1^{n-j} 1^j$

$= \sum_{j=0}^{n} \binom{n}{j}$

Key: $0 = 1-1$

$0 = 0^n = (1-1)^n$

$\overset{BT}{=} \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} (-1)^k$

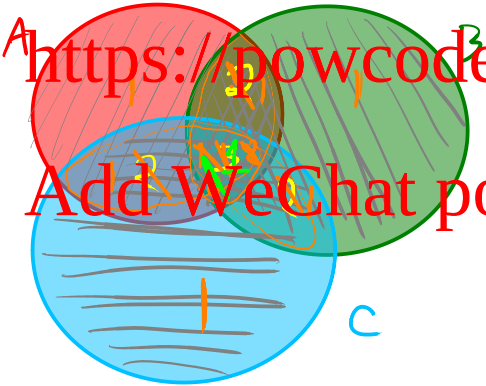
$= \sum_{k=0}^{n} \binom{n}{k} (-1)^k.$ $\square$

The General Inclusion-Exclusion Principle

Inclusion-Exclusion on 3 sets

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$



Now everything is counted exactly once!

Problem 20

How many permutations of the 26 letters of the alphabet do not contain any of the strings "fish", "rat" or "bird"?

Use complement.

X: All permutations of 26 letters. $|X| = 26!$

F: Permutations that contain "fish" → $|F| = 23!$

R: Permutations that contain "rat" → $|R| = 24!$

B: Permutations that contain "bird" → $|B| = 23!$

Goal: Count $|X \setminus (F \cup R \cup B)| = 26! - |F \cup R \cup B|$

$|F| = 23!$ (place "fish" and 22 other things (letters) in order)

$|F \cap R|$: Place "fish", "rat" and 19 other letters ⇒ $21!$ permutations.

Letter in common:

$|F \cap B| = 0$

$|R \cap B| = 0$

$|F \cap R \cap B| = 0$

⇒ Answer: $26! - (23! + 24! + 23! - 21! - 0 - 0 + 0)$

$= 26! - 2 \times 23! - 24! + 21!$ ← Fun simplifying

Inclusion-Exclusion on n sets

x is counted:

$$\underbrace{|A_1 \cup A_2 \cup \ldots \cup A_n|}\ =\ |A_1| + |A_2| + \ldots |A_n|$$

$+k$ times

$$-|A_1 \cap A_2| - |A_1 \cap A_3| - \ldots - |A_{n-1} \cap A_n|$$

$-\binom{k}{2}$ times

$$+|A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + \ldots + |A_{n-2} \cap A_{n-1} \cap A_n|$$

$+\binom{k}{3}$ times

$$- \ldots + (-1)^{n+1}|A_1 \cap A_2 \cap \ldots \cap A_n|$$

$- \ldots$

$+\binom{k}{n}$

$$= \sum_{k=1}^{n} (-1)^{k+1} \sum_{1 \le i_1 < \ldots < i_k \le n} |A_{i_1} \cap \ldots \cap A_{i_k}|$$

→ **Proof:**

We have to make sure that any $x \in A_1 \cup A_2 \cup \ldots \cup A_n$ is counted exactly once in the RHS.

Take x arbitrary. x appears in exactly k of the sets A_i.

$\Rightarrow$ x is counted: $\displaystyle\sum_{j=1}^{n} \binom{k}{j}(-1)^{j+1} = \sum_{j=1}^{k} \binom{k}{j}(-1)^{j+1}$ because $\binom{k}{j} = 0$ if $j > k$

$$= -\sum_{j=1}^{k} \binom{k}{j}(-1)^{j} = -(-1) = 1 \quad \square$$

Prob 19

Inclusion-Exclusion on n sets

Special case! When

$$N(k) = |A_{i_1} \cap \ldots \cap A_{i_k}|$$

is a number that just depends on k then the formula becomes:

$$\begin{aligned} |A_1 \cup A_2 \cup \ldots \cup A_n| &= \sum_{k=1}^{n} (-1)^{k+1} \sum_{1 \le i_1 < \cdots < i_k \le n} N(k) \\ &= \sum_{k=1}^{n} (-1)^{k+1} N(k) \sum_{1 \le i_1 < \cdots < i_k \le n} 1 \\ &= \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} N(k) \end{aligned}$$

Problem 21

Let $|A| = m$ and $|B| = n$. Count the number of **surjective** (onto) functions $f : A \to B$.

*Recall:

$\forall y \in B \ \exists x \in A \ \ f(x) = y$

$\Rightarrow m = |A| \geq |B| = n$

(otherwise, answer = 0. No surjective func.)

We want to AVOID: $\exists y \in B, \forall x \in A, f(x) \neq y$.

X = All functions $f : A \to B$

for $y \in B$: F_y = All functions that avoid the value y (Range $\subseteq B \setminus \{y\}$).

A function is surjective iff it is in none of the F_y's.

$\Leftrightarrow f \in X \setminus \left(\bigcup_{y \in B} F_y \right) = X \setminus (F_1 \cup F_2 \cup \dots \cup F_n)$

assuming $B = \{1, 2, 3, \dots n\}$

Let's count $|X \setminus (F_1 \cup F_2 \cup \dots \cup F_n)| = |X| - |F_1 \cup F_2 \cup \dots \cup F_n|$.

Note: $|F_i| = $ $\boxed{\begin{array}{l} f(x): \ \not{i} \quad \overset{(n-1)}{} \ (n-1) \text{ options} \\ x: \ 1 \quad 2 \quad 3 \quad \dots \quad m \end{array}}$ $\Rightarrow |F_i| = (n-1)^n$.

$(i \neq j)$ $|F_i \cap F_j| = (n-2)^m$: $(n-2)$ options of $f(x)$, for each x.

$\vdots$

$N(k) = |F_{i_1} \cap \dots \cap F_{i_k}| = (n-k)^m$

By incl-excl: $|F_1 \cup \dots \cup F_n| = \sum_{k=1}^{n} \binom{n}{k} (-1)^{k+1} N(k)$

$N(0) = \boxed{|X| = n^m}$ (n options for each x). $\quad = \sum_{k=1}^{n} \binom{n}{k} (-1)^{k+1} (n-k)^m$

Answer: $n^m + \sum_{k=1}^{n} \binom{n}{k} (-1)^{k\cancel{+1}} (n-k)^n = \sum_{k=0}^{n} \binom{n}{k} (n-k)^m (-1)^k$

* <u>Remark</u>: When we really want to count the <u>complement</u> of a certain union,

$$|X \setminus (A_1 \cup \cdots \cup A_n)| = |X| - |A_1 \cup \cdots \cup A_n|$$

$$= N(0) + \sum_{k=1}^{n} \binom{n}{k} (-1)^{k} N(k)$$

$$= \sum_{k=0}^{n} \binom{n}{k} (-1)^{k} N(k)$$

Shorter formula for the complement of the union!

* <u>Def</u>: $N(0) = |X|$.

Problem 22

The Hatcheck Problem

At a party with a *very large* number of guests, everyone has to leave their hat at the entrance. When they leave the party, everyone is too drunk to remember what hat is theirs, so they just pick one at random. **What is the probability** that no one picks their own hat?