Math 340 Tutorial 2

1. *Inclusion/Exclusion*

   (a) How many numbers in $\{1, \ldots, 100\}$ are not divisible by 5 or 8?

   (b) How many numbers in $\{1, \ldots, n\}$ are not divisible by a or b where $1 < a < b < n$ and $\gcd(a, b) = 1$.

   (c) How many solutions does the equation $x_1 + x_2 + x_3 = 12$ have, where $x_i \in \{0, 1, ..., 4\}$ for each $i \leq 3$?

   (d) Call a natural number n *square free* if $n = p_1 \cdot ... \cdot p_i$ for some collection of disctinct primes $p_1, ..., p_i$. How many square free numbers are less than 50?

Solution:

(a) Let $A \subset \{1, \ldots, 100\}$ be the numbers divisible by 5 and $B \subset \{1, \ldots, 100\}$ be the numbers divisible by 8. Then $A \cap B$ are the numbers divisible by both 5 and 8, i.e. divisible by 40. It's clear that $|A| = 20$, and to count $|B|$, we consider the largest number up to 100 that is divisible by 8. This number is 96, and so $|B| = 96/8 = 12$. Lastly, $A \cap B = \{40, 80\}$ meaning $|A \cap B| = 2$. Therefore, by the principle of Inclusion/Exclusion, the number of elements in $\{1, \ldots, 100\}$ that are not divisible by 5 or 8 is

$$100 - |A| - |B| + |A \cap B| = 100 - 20 - 12 + 2 = 70.$$

(b) This question is just a generalization of (a). Let $A \subset \{1, \ldots, n\}$ be the numbers divisible by a and $B \subset \{1, \ldots, n\}$ be the numbers divisible by b. Then $A \cap B$ are the number divisible by both a and b. Since $\gcd(a, b) = 1$, it follows that $A \cap B$ are the numbers divisible by ab.

To compute $|A|$, first consider the case where a divides n. In this case, $A = \{a, 2a, 3a, \ldots, (n/a)a\}$, meaning $|A| = n/a$. If a does not divide n, then $n = ca + r$ for some $0 < r < a$. In this case, $A = \{a, 2a, 3a, \ldots, ca\}$, meaning $|A| = c = \lfloor \frac{n}{a} \rfloor$. So in general, $|A| = \lfloor \frac{n}{a} \rfloor$, and similarly for $|B|$ and $|A \cap B|$. Therefore, by the principle of Inclusion/Exclusion, the number of elements in $\{1, \ldots, n\}$ that are not divisible by a or b is

$$n - |A| - |B| + |A \cap B| = n - \left\lfloor \frac{n}{a} \right\rfloor - \left\lfloor \frac{n}{b} \right\rfloor + \left\lfloor \frac{n}{ab} \right\rfloor.$$

$$\left\lfloor \frac{n}{a} \right\rfloor = \left\lfloor \frac{ca+r}{a} \right\rfloor = c + \left\lfloor \frac{r}{a} \right\rfloor = c$$

(c) There is only 1 solution: $x_1 = x_2 = x_3 = 4$. How about instead we insist that $x_i \in \{0, 1, \ldots, 5\}$ for each i? In this case we will end up needing Inclusion/Exclusion.

We know that the total number of solutions to $x_1+x_2+x_3 = 12$ with $x_1, x_2, x_3 \geq 0$ is $\binom{12+2}{2}$. Now let A_1 be the set of solutions such that $x_1 > 5$, and similarly for A_2 and A_3.

To count $|A_1|$, consider the number of solutions to $x_1 + x_2 + x_3 = 12$ where $x_1 > 5$. We can instead count the number of solutions to $x'_1 + x_2 + x_3 = 6$, where $x'_1 = x_1 - 6$. Then $x'_1 \geq 0$. The number of solutions here is $\binom{6+2}{2}$, and the same argument works for $|A_2|$ and $|A_3|$. For $|A_1 \cap A_2|$, this is the case where $x_1 > 5$ and $x_2 > 5$. The only possibility here is that $x_1 = x_2 = 6$ and $x_3 = 0$, meaning $|A_1 \cap A_2| = 1$, and similarly for $|A_1 \cap A_3|$ and $|A_2 \cap A_3|$. Lastly, it is impossible for $x_1, x_2, x_3 > 5$ since then the sum is greater than 12. So $|A_1 \cap A_2 \cap A_3| = 0$.

Therefore by the principle of Inclusion/Exclusion, the number of solutions to $x_1 + x_2 + x_3 = 12$ with $x_i \in \{0, 1, \ldots, 5\}$ for each i is

$$\binom{12+2}{2} - 3 \cdot \binom{6+2}{2} + 3 \cdot 1.$$

(d) By this definition, a number is square free if and only if it is not divisible by p^2 for any prime p. Since $8^2 = 64$, a number in $\{1, \dots, 49\}$ is square free if and only if it is not divisible by $2^2, 3^3, 5^2$, or 7^2.

Let A_2 be the elements in $\{1, \dots, 49\}$ that are divisible by 2^2, and similarly for A_3, A_5 and A_7. Using our argument from (b), we can quickly find our solution to be:

$$49 - \left\lfloor \frac{49}{2^2} \right\rfloor - \left\lfloor \frac{49}{3^2} \right\rfloor - \left\lfloor \frac{49}{5^2} \right\rfloor - \left\lfloor \frac{49}{7^2} \right\rfloor + \left\lfloor \frac{49}{2^2 \cdot 3^2} \right\rfloor .$$

Notice that $A_2 \cap A_3$ is the only non-zero intersection. An element of $A_2 \cap A_5$ must be divisible by $2^2 \cdot 5^2 = 100$, and similarly for the other intersections. We could have written them in the solution as well, since the floor function would ensure that the terms are all zero.

$4^2 \mid K \Rightarrow 2^2 \mid K$

$6^2 \mid K \Rightarrow 2^2 \mid K$

$3^2 \mid K$

2. *Derangements*

   (a) Prove, either combinatorially or algebraically, that $D_n = (n-1)(D_{n-1} + D_{n-2})$.

   Bijections

   (b) Count the number of ~~functions~~ $f : \{1, \ldots, n\} \to \{1, \ldots, n\}$ such that exactly k elements of $\{1, \ldots, n\}$ are fixed by f.

   (c) (*8.6, question 24 in Rosen*) Prove the following identity:

$$n! = \binom{n}{0} D_n + \binom{n}{1} D_{n-1} + \cdots + \binom{n}{n} D_0.$$

Solution:

(a) Let's prove this combinatorially. Let $f : \{1, \dots, n\} \to \{1, \dots, n\}$ be a derangement function. The number of such functions is D_n, but we can instead count them by focusing on $f(1)$. Let $f(1) = i$, where $i \neq 1$. Then there are two cases to consider:

function = Bijective function in this argument.

Case 1: If $f(i) = 1$, then $f : \{2, \dots, i-1, i+1, \dots, n\} \to \{2, \dots, i-1, i+1, \dots, n\}$ is a derangement of $n-2$ elements. So the number of possibilities for case 1 is D_{n-2}.

Case 2: If $f(i) \neq 1$, then we can consider $f : \{2, \dots, n\} \to \{2, \dots, i-1, i', i+1, \dots, n\}$, where $i' = 1$. Writing it like this, it becomes clear that this is the same as a derangement of $n-1$ elements, since we know that $f(i) \neq i'$. Hence, the number of possibilities for case 2 is D_{n-1}.

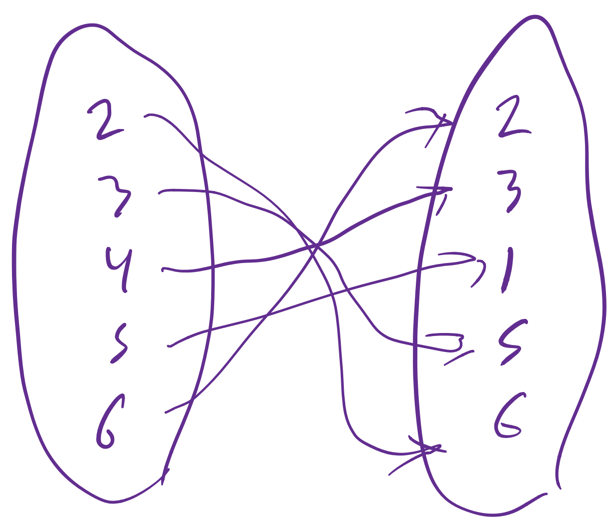
In total, the number of derangements with $f(1) = i$ is $D_{n-2} + D_{n-1}$. Summing over all $2 \leq i \leq n$, we get that the total number of derangements is $(n-1)(D_{n-2} + D_{n-1})$. Since we have counted the same thing twice, it must follow that

$$D_n = (n-1)(D_{n-1} + D_{n-2}).$$

1, 2, 3, 4, 5, 6

$f : \{2, 3, 5, 6\} \to \{2, 3, 5, 6\}$?

1, 2, 3, 4, 5, 6



(b) We first choose which k elements are fixed, and there are $\binom{n}{k}$ ways to do this. Then, the remaining function is a derangement of $n-k$ elements. Hence, the number of functions is

$$\binom{n}{k} D_{n-k}.$$

(c) Let's count the number of permutations of $(1, \ldots, n)$. The obvious solution is $n!$, but we can also break this up into cases based on the number of fixed elements in our permutation. By (b), if there are k fixed elements in the permutation, the number of such permutations is $\binom{n}{k} D_{n-k}$. Summing over all possible numbers of fixed elements, we get that the total number of permutations is

$$\sum_{k=0}^{n} \binom{n}{k} D_{n-k}.$$

Since we have counted the same thing twice, it must follow that

$$n! = \sum_{k=0}^{n} \binom{n}{k} D_{n-k}.$$

3. Given the sequence of coefficients, find the corresponding generating function:

(a) $0, 0, 1, 0, 1, 0, 0, 0, \ldots$

(b) $1, 1, 2, 1, 2, 1, 1, 1, \ldots$

(c) $-1, 0, 1, 2, 3, 4, 5, 6, 7, 8, \ldots$

(d) $0, 0, 1, 0, 0, 1, 0, 0, 1, \ldots$

(e) $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n-1}, \binom{n}{n}, 0, 0, 0, 0, \ldots$

(f) $0, 1, -2, 4, -8, 16, -32, 64, \ldots$

b) $1, 1, 2, 1, 2, 1, 1, 1, \ldots$

$= 1, 1, 1, 1, 1, 1, 1, 1, \ldots$

$+ 0, 0, 1, 0, 1, 0, 0, \ldots$

$$\frac{1}{1-x} + (x^2 + x^4)$$

$$\frac{1}{1-x} = \sum_{n \geq 0} x^n$$

Solution:

(a) $x^2 + x^4$.

(b) We can combine the sequences $0, 0, 1, 0, 1, 0, 0, 0, \ldots$ and $1, 1, 1, 1, 1, \ldots$ to get $1, 1, 2, 1, 2, 1, 1, 1, \ldots$. Hence, the generating function is

$$x^2 + x^4 + \frac{1}{1-x}.$$

a) $0, 0, 1, 0, 1, 0, 0, 0, \ldots$

$0 + 0x + 1 \cdot x^2 + 0x^3 + 1x^4 + 0x^5 + 0x^6 + \ldots$

$= x^2 + x^4$

(c) First, consider the power series for $\frac{1}{1-x}$:

$$\frac{1}{1-x} = \sum_{n \geq 0} x^n.$$

Now take the derivative:

$$\frac{1}{(1-x)^2} = \sum_{n \geq 1} n x^{n-1}.$$

Hence, $\frac{1}{(1-x)^2}$ corresponds to the sequence $1, 2, 3, 4, 5, 6, \ldots$. Multiplying this by x^2 gives us $0, 0, 1, 2, 3, 4, 5, 6, \ldots$, and adding -1 gives us the sequence in question. So the generating function is

$$\frac{x^2}{(1-x)^2} - 1.$$

$1, 1, 1, 1, \ldots \rightarrow (\text{derivative}) \rightarrow 1, 2, 3, 4, 5, 6, \ldots$

$\rightarrow (\text{multiply by } x^2) \rightarrow 0, 0, 1, 2, 3, 4, 5, 6, \ldots$

$\rightarrow (\text{add } -1) \rightarrow -1, 0, 1, 2, 3, 4, 5, 6, \ldots$

$$A(x) = a_0 + a_1 x + a_2 x^2 + \ldots$$

$$x^2 A(x) = a_0 x^2 + a_1 x^3 + a_2 x^4 + \ldots$$

(d) To "stretch" a sequence, we replace x by x^k where k is the amount of stretching we need. In this case, we need to stretch by a factor of 3, so our generating function is something like $\frac{1}{1-x^3}$. However, this gives $1, 0, 0, 1, 0, 0, 1, 0, 0, \ldots$, and so we need to slide over twice, meaning our generating function is

$$\frac{x^2}{1-x^3}.$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$$

$$\frac{1}{1-x^2} = 1 + (x^2) + (x^2)^2 + (x^2)^3 + \ldots$$

$$= 1 + x^2 + x^4 + x^6 + \ldots$$

$$\frac{1}{1-x^3} = 1 + x^3 + x^6 + x^9 + \ldots$$

$$\rightarrow \quad 1, 0, 0, 1, 0, 0, 1, 0, 0, \ldots$$

$$1, 0, 2, 0, 3, 0, 4, 0, \ldots \quad ?$$

$$\frac{1}{(1-(x^2))^2}$$

(e) The corresponding power series is

$$A(x) = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

From the Binomial Theorem, it follows that the generating function is

$$(x+1)^n.$$

Bin Thm: $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$

(f) Let's start with something easier: What is the generating function for $1, 2, 4, 8, 16, \ldots$? In this case, the coefficient of x^n is 2^n, so we can deduce that the generating function is $\frac{1}{1-2x}$. Then, to make the coefficients alternate parody, we have $\frac{1}{1+2x}$. Lastly, we multiply by x to slide everything over. So the generating function is

$$\frac{x}{1+2x}.$$

$$= \frac{1}{1-(-2x)}$$

$$A(x) = 1 + 2x + 2^2x^2 + 2^3x^3 + \cdots$$

$$= (2x)^0 + (2x)^1 + (2x)^2 + (2x)^3 + \cdots$$

4. Given the generating function, find the corresponding list of coefficients:

   (a) $x(x-1)(x-2)$

   (b) $\frac{x^4}{1-x}$

   (c) $\frac{1}{1+x^2}$

   (d) $\frac{1}{(1-x)^3}$

   (e) $\frac{1}{1+3x+2x^2}$

   (f) $\frac{1}{1-\alpha x}$ (for some constant α)

Solution:

(a) Expanding this gives $x(x^2-3x+2) = x^3-3x^2+2x$. So the sequence of coefficients is

$$0, 2, -3, 1, 0, 0, 0, 0, \ldots$$

(b) The sequence of $\frac{x^4}{1-x}$ is just the sequence of $\frac{1}{1-x}$ slid over 4 times. So it's

$$0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, \ldots$$

(c) x^2 stretches our sequence by a factor of 2, so $\frac{1}{1-x^2}$ gives the sequence $1, 0, 1, 0, 1, 0, \ldots$. The $+x^2$ instead of $-x^2$ causes our coefficients to alternate. So the sequence is

$$1, 0, -1, 0, 1, 0, -1, 0, \ldots$$

$$\frac{1}{1-(-x^2)} = 1 - (x^2) + (x^4) - (x^6) \ldots$$

(d) The derivative of $\frac{1}{(1-x)}$ is

$$\frac{1}{(1-x)^2} = \sum_{n\geq 1} nx^{n-1} = \sum_{n\geq 0}(n+1)x^n.$$

Taking the derivative again gives us

$$\frac{2}{(1-x)^3} = \sum_{n\geq 1}(n+1)nx^{n-1} = \sum_{n\geq 0}(n+2)(n+1)x^n.$$

Dividing by 2 gives us our generating function, and so the sequence is

$$\frac{2\cdot 1}{2}, \frac{3\cdot 2}{2}, \frac{4\cdot 3}{2}, \ldots, \frac{(n+2)(n+1)}{2}, \ldots$$

In fact, this turns out to be

$$\binom{2}{2}, \binom{3}{2}, \binom{4}{2}, \ldots$$

$A(x) = \frac{1}{1-x}$

$A(x)^3 = \frac{1}{(1-x)^3}$

(e) The best course of action is to use partial fractions to turn this into a sum of two generating functions that are easy to analyse.

$$\frac{1}{1+3x+2x^2} = \frac{1}{(1+2x)(1+x)} = \frac{A}{1+2x} + \frac{B}{1+x}.$$

We need $A+B=1$ and $A+2B=0$, and so $A=2$ and $B=-1$. So we get

$$\frac{1}{1+3x+2x^2} = \frac{2}{1+2x} + \frac{-1}{1+x}.$$

The corresponding sequences are

$$2, -4, 8, -16, 32, \ldots 2(-2)^n \ldots$$

and

$$-1, 1, -1, 1, -1, 1, \cdots - (-1)^n$$

respectively. So the overall sequence is

$$1, -3, 7, -15, 31, \ldots, 2(-2)^n - (-1)^n \ldots$$

(f) $1, \alpha, \alpha^2, \alpha^3, \ldots$

$$\frac{1}{1-\alpha x} = 1 + \alpha x + (\alpha x)^2 + (\alpha x)^3 + \ldots$$

5. Given the sequences of coefficients for $A(x)$ and $B(x)$, find the closed form of $A(x)B(x)$ using Cauchy's Theorem:

   (a) $A(x) : 1, 1, 1, 1, 1, \ldots$
   $B(x) : 1, -1, 0, 0, 0, 0, \ldots$

   (b) $A(x) : 1, 2, 3, 4, 5, \ldots$
   $B(x) : 1, 1, 1, 0, 0, 0, 0, \ldots$

   (c) $A(x) : 1, 0, 1, 0, 1, 0, \ldots$
   $B(x) : 0, 1, 0, 1, 0, 1, \ldots$

Solution:

(a) The coefficient of x^n is $\sum_{k=0}^{n} a_k b_{n-k}$. Notice that $b_0 = 1$, $b_1 = -1$, and $b_k = 0$ for all $k > 1$. So

$$\sum_{k=0}^{n} a_k b_{n-k} = a_n b_0 + a_{n-1} b_1 = a_n - a_{n-1}.$$

Hence, the coefficient of $A(x)B(x)$ is 0 for all $n \geq 1$. Furthermore, the coefficient of x^0 is 1, meaning $A(x)B(x) = 1$.

6. Given $A(x)$ and $B(x)$ as in the previous question, find the closed form of $A(x)B(x)$ by finding the closed forms of $A(x)$ and $B(x)$, then multiplying them together.

Solution:

(a) $1, 1, 1, 1, \ldots$ corresponds to $\frac{1}{1-x}$, and $1, -1, 0, 0, 0, 0, \ldots$ corresponds to $1 - x$. Hence,

$$A(x)B(x) = \frac{1-x}{1-x} = 1.$$