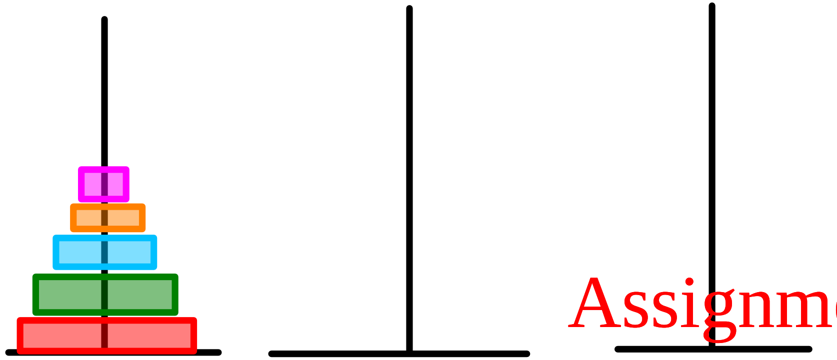
Towers of Hanoi.



Problem: How many moves (at least) do you need in order to complete the puzzle?

Monks move one disk per day...

Legend says: when the puzzle will be completed, that will be the end of the world!

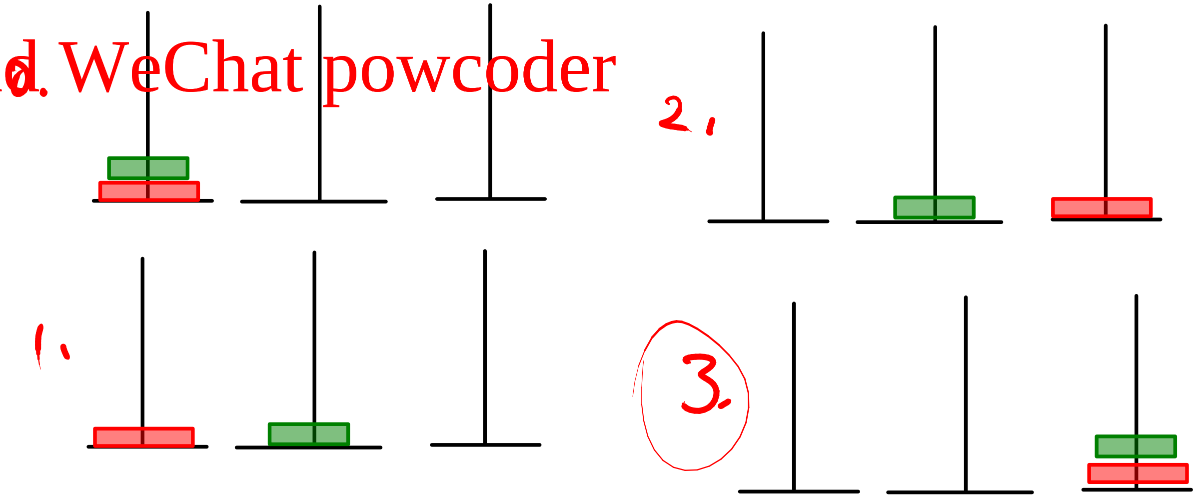
Let's find the smallest number of steps necessary to solve the puzzle; denote h_n

where n = The number of disks...

Basic cases: $h_0 = 0$ (degenerate case)

$h_1 = 1$

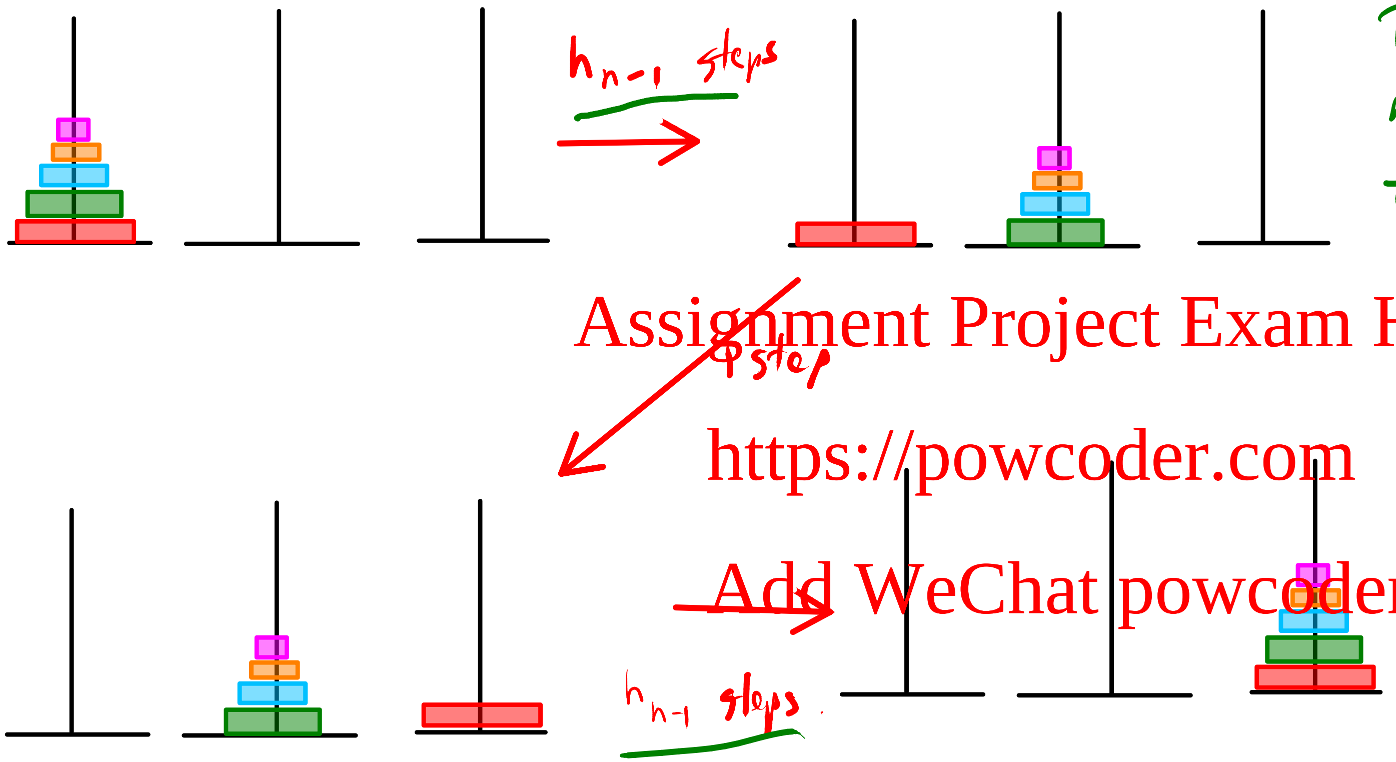
$h_2 = ?$



$h_2 = 3$

$\vdots$

① Recurrence relation:



This state is necessary in order to be able to move the largest disk to the 3rd pole.

By definition of h_{n-1} ...

$$\Rightarrow \quad h_n = h_{n-1} + 1 + h_{n-1} = 2h_{n-1} + 1$$

$$\boxed{h_0 = 0} \Rightarrow h_1 = 2(0) + 1 = h_1 = 1$$

$$\Rightarrow h_2 = 2(1) + 1 = 3 \checkmark$$

Let's solve the recurrence with GFs!

$$\begin{cases} h_n = 2h_{n-1} + 1 \\ h_0 = 0 \end{cases}$$

Let $H(x) = \sum_{n=0}^{\infty} h_n x^n$

Goal: Find a short expression for $H(x)$

Hope: This should be helpful for finding a short expression for h_n

2 We have:

$$\underbrace{H(x) = \sum_{n=0}^{\infty} h_n x^n}_{\text{Generic} \to} = \underbrace{h_0 x^0}_{\text{Basic case(s)}} + \underbrace{\sum_{n=1}^{\infty} h_n x^n}_{\text{Use recurrence to rewrite this.}}$$

$$= h_0 x^0 + \sum_{n=1}^{\infty} h_n x^n$$

$$= 0 + \sum_{n=1}^{\infty} (2h_{n-1}+1)x^n = 2\left(\sum_{n=1}^{\infty} h_{n-1} x^n\right) + \left(\sum_{n=1}^{\infty} x^n\right)$$

* Table: $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$

$$= 2x \sum_{n=1}^{\infty} h_{n-1} x^{n-1} + \left(\left(\sum_{n=0}^{\infty} x^n\right) - 1\right)$$

$(k = n-1)$

$$= 2x \sum_{k=0}^{\infty} h_k x^k + \left(\frac{1}{1-x} - 1\right)$$

$$H(x) = 2x\, H(x) + \frac{x}{1-x}$$

* Implicit equation for $H(x)$!!

Let's solve for $H(x)$ ...

$$H(x) - 2x H(x) = \frac{x}{1-x}$$

$$H(x)(1-2x) = \frac{x}{1-x}$$

$$H(x) = \frac{x}{(1-x)(1-2x)}.$$

short form

Let's use partial fractions! (Like in calculus)

③ Find the coefficient h_n in front of x^n...

$$H(x) = \frac{x}{(1-x)(1-2x)} = \frac{A}{1-x} + \frac{B}{1-2x}$$

Find A and B.—

$$\frac{x}{\cancel{(1-x)(1-2x)}} = \frac{A(1-2x) + B(1-x)}{\cancel{(1-x)(1-2x)}}$$

$$1x + 0 = A - 2Ax + B - Bx$$

When comparing the coefficients

$$\begin{cases} 1 = -2A - B \\ 0 = A + B \end{cases} \qquad A = -B$$

$$\Rightarrow 1 = -2A + A = -A$$

$$\boxed{\begin{aligned} A &= -1 \\ B &= 1 \end{aligned}}$$

That means:

$$H(x) = \frac{-1}{1-x} + \frac{1}{1-2x}.$$

Use the table to expand...

$$= -\sum_{n=0}^{\infty} x^n + \sum_{n\geq 0}^{\infty} (2x)^n$$

$$= \sum_{n=0}^{\infty} \underbrace{(-1+2^n)}_{h_n} x^n$$

Answer: $\boxed{h_n = 2^n - 1}$

- Ex2: Fibonacci...

$$f_0 = 0$$

$$f_1 = 1$$

$$(n \geq 2) \quad f_n = f_{n-1} + f_{n-2}$$

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ...

Let $F(x) = \sum_{n=0}^{\infty} f_n x^n$.

① Find an implicit equation for $F(x)$, by using the recurrence.

$$F(x) = \sum_{n=0}^{\infty} f_n x^n = f_0 x^0 + f_1 x^1 + \sum_{n=2}^{\infty} f_n x^n$$

$$= 0 + x + \sum_{n=2}^{\infty} (f_{n-1} + f_{n-2}) x^n$$

$$= x + \sum_{n=2}^{\infty} f_{n-1} x^n + \sum_{n=2}^{\infty} f_{n-2} x^n$$

$$= x + x\sum_{n=2}^{\infty} f_{n-1} x^{n-1} + x^2 \sum_{n=2}^{\infty} f_{n-2} x^{n-2}$$

$* \; k = n-1$

$* \; j = n-2$

$$= x + x\sum_{k=1}^{\infty} f_k x^k + x^2 \sum_{j=0}^{\infty} f_j x^j$$

$$= x + x\left(\sum_{k=0}^{\infty} f_k x^k - f_0 x^0\right) + x^2 F(x)$$

$$F(x) = x + x\left(F(x) - 0\right) + x^2 F(x)$$

$$F(x) = x + xF(x) + x^2 F(x).$$

② Solve the implicit equation

$$F(x) - xF(x) - x^2F(x) = x$$

$$F(x)(1 - x - x^2) = x$$

$$F(x) = \frac{x}{1 - x - x^2}$$ (Short form)

③ Partial fractions...

Factorize: $1 - x - x^2 = -(x-\alpha)(x-\beta)$ where α, β are the roots

Quadratic formula: $\frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{1 \pm \sqrt{1-4(-1)}}{2(-1)} = -\left(\frac{1 \pm \sqrt{5}}{2}\right)$

* Note: $\varphi = \dfrac{1+\sqrt{5}}{2}$ is called "golden ratio"

Its conjugate is $\bar{\varphi} = \dfrac{1-\sqrt{5}}{2}$.

$\Rightarrow \alpha = -\varphi$

$\beta = -\bar{\varphi}$

$\Rightarrow 1 - x - x^2 = -(x+\varphi)(x+\bar{\varphi})$

$x^2 + x - 1 = (x+\varphi)(x+\bar{\varphi})$.

$$\Rightarrow \quad F(x) = \frac{-x}{(x+\varphi)(x+\bar{\varphi})} = \frac{A}{x+\varphi} + \frac{B}{x+\bar{\varphi}}$$

$$-x = A(x+\bar\varphi) + B(x+\varphi)$$

$$0 - x = Ax + A\bar\varphi + Bx + B\varphi$$

$$0 = A\bar\varphi + B\varphi \rightarrow 0 = A\left(\frac{1-\sqrt5}{\cancel{2}}\right) + B\left(\frac{1+\sqrt5}{\cancel{2}}\right)$$

$$-1 = A + B$$

$$0 = A - \sqrt5 A + B + \sqrt5 B$$

$$0 = A + B - \sqrt5 A + \sqrt5 B$$

$$0 = -1 - \sqrt5 A + \sqrt5 B$$

$$1 = \sqrt5 (B - A)$$

$$\frac{1}{\sqrt5} = B - A$$

Add:

$$-1 + \frac{1}{\sqrt5} = 2B$$

$$B = \frac{-\sqrt5 + 1}{2\sqrt5} = \frac{\bar\varphi}{\sqrt5}$$

$$A = -1 - B = \frac{-2\sqrt5 + \sqrt5 - 1}{2\sqrt5} = \frac{-(1+\sqrt5)}{2\sqrt5} = \frac{-\varphi}{\sqrt5}$$

$$\Rightarrow F(x) = \frac{-\varphi}{\sqrt{5}\,(x+\varphi)} + \frac{\bar{\varphi}}{\sqrt{5}\,(x+\bar{\varphi})}$$

* Note:

$\varphi\bar{\varphi} = \frac{1-5}{4} = -1$

$-\varphi = \frac{1}{\bar{\varphi}}$ and $-\bar{\varphi} = \frac{1}{\varphi}$

* $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$

$$= \frac{-\varphi}{\sqrt{5}}\left(\frac{1}{x+\varphi}\right) + \frac{\bar{\varphi}}{\sqrt{5}}\left(\frac{1}{x+\bar{\varphi}}\right)$$

$$= \frac{-\cancel{\varphi}}{\sqrt{5}\,\cancel{\varphi}}\left(\frac{1}{-\bar{\varphi}x+1}\right) + \frac{\cancel{\bar{\varphi}}}{\sqrt{5}\,\cancel{\bar{\varphi}}}\left(\frac{1}{-\varphi x+1}\right)$$

$$= \frac{1}{\sqrt{5}}\left(\sum_{n=0}^{\infty} -(\bar{\varphi}x)^n + \sum_{n=0}^{\infty}(\varphi x)^n\right)$$

$$= \sum_{n=0}^{\infty} \underbrace{\frac{1}{\sqrt{5}}\left(-\bar{\varphi}^n + \varphi^n\right)}\, x^n$$

$$\Rightarrow f_n = \frac{-\bar{\varphi}^n + \varphi^n}{\sqrt{5}} = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$

"neglectable" (For large n)

* Note: Fibonacci numbers should be integers.

$$|\bar{\varphi}| = \left|\frac{1-\sqrt{5}}{2}\right| < 1$$

$$\bar{\varphi}^n \to 0 \quad \text{as } n \to \infty.$$

(Removes the decimal part of $\frac{\varphi^n}{\sqrt{5}}$ to make it an integer.)

$$f_n \approx \frac{\varphi^n}{\sqrt{5}} \quad \Rightarrow \quad f_n = \left\{\frac{\varphi^n}{\sqrt{5}}\right\} = \text{Closest integer to } \frac{\varphi^n}{\sqrt{5}}$$

* Note: If you did Math 240, you probably knew that, but you would have found it with another method, that only works for "linear recurrences of "finite order"...

Next time, we shall see an example of a recurrence that is not linear and not of finite order, but that we can still solve with the same method!