A Table of Power Series

$$\frac{1-x^{N+1}}{1-x} = 1 + x + x^2 + \ldots + x^N \qquad \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\frac{1}{1-ax} = \sum_{n=0}^{\infty} a^n x^n \qquad \frac{1}{1-x^r} = \sum_{n=0}^{\infty} x^{rn}$$

$$(1+x)^r = \sum_{n=0}^{\infty} \binom{r}{n} x^n \text{ where } \binom{m}{k} = \frac{m(m-1)\cdots(m-k+1)}{k!}$$

$$\boxed{\frac{1}{(1-x)^{\boxed{r}}} = \sum_{n=0}^{\infty} \binom{n+r-1}{n} x^n} \qquad \sqrt{1+x} = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{(-1)^{n+1}}{4^n(2n-1)} x^n$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \qquad \sinh(x) = \frac{e^x - e^{-x}}{2} = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

- Formal differentiation

$$A(x) = \sum_{n=0}^{\infty} a_n x^n$$

(Same reason as: $\int_0^1 f(x)dx = \int_0^1 f(t)dt$)

- Def: The formal derivative of $A(x)$ is:

$$DA(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{k=0}^{\infty} (k+1) a_{k+1} x^k = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

or $n=0$

let $k=n-1$

- Ex: $A(x) = \sum_{n=0}^{\infty} 1 x^n = \frac{1}{1-x}$

Last Lecture

$$DA(x) = \sum_{n=0}^{\infty} (n+1) 1 x^n = A(x)^2 = \left(\frac{1}{1-x}\right)^2 = \frac{d}{dx}\left(\frac{1}{1-x}\right)$$

We note that the formal derivative matches with the "usual" notion of derivative for the function that $A(x)$ represents.

· Fact: $F(x) \simeq f(x) \Rightarrow DF(x) \simeq f'(x)$.

* Remark: $x D A(x) = x \sum_{n\geq 0}^{\infty} n a_n x^{n-1} = \sum_{n=0 \text{ or } n=1}^{\infty} n a_n x^n$

That operation introduces a factor n inside the sum...

- Ex: Find a short formula for the sum.

$$\sum_{k=0}^{n} k^2 = 0 + 1^2 + 2^2 + 3^2 + \dots + n^2$$

Expected answer (From calculus): $\sum k^2 = \underline{\dfrac{n(n+1)(2n+1)}{6}}$ ...

We will find this as a result of direct calculations involving power series (or generating functions)...

Let $a_n = \sum_{k=0}^{n} k^2$.

The generating function of the sequence is $A(x) = \sum_{n=0}^{\infty} a_n x^n$

$$A(x) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} k^2 \right) x^n$$

a_n is the coefficient in front of x^n ...

If we manage to find a short expression for $A(x)$, then it may be helpful for finding a short expression for a_n ...

$$A(x) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} k^2 \right) x^n$$

*Looks like Cauchy's rule: $\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right) x^n$

We then conclude:

$$A(x) = \left(\sum_{n=0}^{\infty} n^2 x^n\right)\left(\sum_{n=0}^{\infty} 1\, x^n\right)$$

$$= \; ? \; \left(\frac{1}{1-x}\right)$$

Question:

$$\left(\sum_{n=0}^{\infty} n x^n\right)\left(\sum_{n=0}^{\infty} n x^n\right) \neq \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} k(n-k)\right) x^n$$

To find $\boxed{?}$, we use formal differentiation...

$$xD\left(\frac{1}{1-x}\right) = xD\sum_{n=0}^{\infty} x^n$$

$$x\left(\frac{1}{(1-x)^2}\right) = \sum_{n=0}^{\infty} n x^n$$

$$xD\left(\frac{x}{(1-x)^2}\right) = xD\sum_{n=0}^{\infty} n x^n$$

$$x\left(\frac{1(1-x)^2 + x2(1-x)}{(1-x)^4}\right) = \sum_{n=0}^{\infty} n^2 x^n$$

$$\Rightarrow \sum_{n\geq 0}^{\infty} n^2 x^n = \frac{x\left(1 - \cancel{2x} + x^2 + \cancel{2x} - 2x^2\right)}{(1-x)^4}$$

$$= \frac{x\left(1 - x^2\right)}{(1-x)^4} = \frac{x(1+x)\cancel{(1-x)}}{(1-x)^{\cancel{4}}} = \frac{x(1+x)}{(1-x)^3}.$$

So:

$$\underline{A(x)} = \left(\sum_{n\geq 0}^{\infty} n^2 x^n\right)\left(\sum_{n\geq 0}^{\infty} 1\, x^n\right) = \left(\frac{x(1+x)}{(1-x)^3}\right)\left(\frac{1}{1-x}\right) = \frac{x+x^2}{(1-x)^4}$$

We have found a short form for $A(x)$!

We will use it to help us in finding an expression for a_n:

$$A(x) = \frac{x+x^2}{(1-x)^4} = (x+x^2)\cdot\frac{1}{(1-x)^4}$$

(From table)

$$= (x+x^2)\cdot\left(\sum_{n=0}^{\infty}\binom{n+3}{n}x^n\right)$$

$$= x\sum_{n=0}^{\infty}\binom{n+3}{n}x^n + x^2\sum_{n=0}^{\infty}\binom{n+3}{n}x^n$$

$$= \sum_{n=0}^{\infty}\binom{n+3}{n}x^{n+1} + \sum_{n=0}^{\infty}\binom{n+3}{n}x^{n+2}$$

$(k=n+1) \qquad\qquad\qquad\qquad (k=n+2)$

$$= \sum_{k=1}^{\infty} \binom{(k-1)+3}{k-1} x^k + \sum_{k=2}^{\infty} \binom{(k-2)+3}{k-2} x^k$$

$$= \binom{3}{0} x + \sum_{k=2}^{\infty} \underbrace{\left(\binom{k+2}{k-1} + \binom{k+1}{k-2} \right)}_{a_k} x^k$$

$$\sum a_n x^n + \sum b_n x^n = \sum (a_n + b_n) x^n$$

$\Rightarrow$ For $n \geq 2$, $a_n = \binom{n+2}{n-1} + \binom{n+1}{n-2}$ (That's our answer!)

$$a_n = \frac{(n+2)(n+1)n}{3!} + \frac{(n+1)(n)(n-1)}{3!} = \frac{(n+1)n}{6}\left((n+2)+(n-1)\right)$$

$$= \frac{n(n+1)(2n+1)}{6}.$$

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}$$

And: $\binom{n}{k} = \binom{n}{n-k}$

$$\binom{n+2}{n-1} = \binom{n+2}{(n+2)-(n-1)} = \binom{n+2}{3} = \frac{(n+2)(n+1)(n)}{3!}$$

Solving recurrences with G.F.

A sequence $(a_n)_{n\geq 0}$ is <u>recursive</u> if it is defined in two steps as follows:

1. <u>Base case</u>: Some starting value(s) of a_n, for small n, are given...

2. <u>Recurrence relation</u>: A general expression for a_n that depends on a_k for $k < n$...

These two steps together provide an algorithm for calculating the whole sequence...

Ex1: We define a sequence as follows...

(1) Base case: $a_0 = 1$

(2) Recurrence: $\boxed{a_n = 2a_{n-1}}$

We can then calculate subsequent values of a_n...

$a_1 = 2a_0 = 2(1) = 2$

$a_2 = 2a_1 = 2(2) = 4$

$a_3 = 2a_2 = 2(4) = 8$

$\vdots$

$a_n = \boxed{2^n}$ ... ← Solving the recurrence relation...

"Solving" the recurrence means: Finding a short expression for a_n that does not refer to the previous values of the sequence.

(we want a function of n...)

- Ex 2: Fibonacci Numbers:

$f_0 = 0, \; f_1 = 1$ (Two base cases)

$f_n = f_{n-1} + f_{n-2}$ (Recurrence of order 2)

$\Rightarrow f_2 = f_1 + f_0 = 1 + 0 = 1$

$f_3 = f_2 + f_1 = 1 + 1 = 2$

$f_4 = f_3 + f_2 = 2 + 1 = 3$

$f_5 = f_4 + f_3 = 3 + 2 = 5$

$f_6 = f_5 + f_4 = 5 + 3 = 8$

$$\Rightarrow (f_n)_{n\geq 0} = (0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots)$$

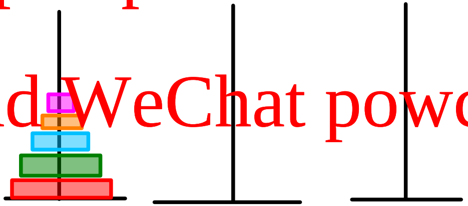
How do we solve the recurrence?

(for less obvious to figure out...)

We shall use GFs for that (next time)

- Ex3 : Towers of Hanoi . .

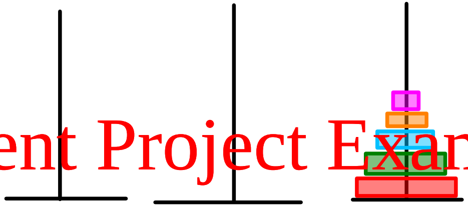
Story: There is a temple (in Hanoi?) where we can find 3 sticks with golden rings like that . . .



Initial state . . .

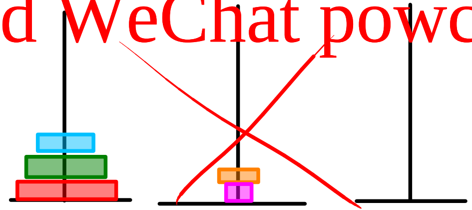
Goal:

Move all rings to the third stick...



Move one ring at a time.

Rule: At no point can a larger ring stand atop of a smaller one...



Forbidden

Problem: How many moves (at least) do you need in order to complete the puzzle?

Solution: next time