Math 340 Tutorial 1 Solutions
Jordan Barrett

1. Consider the set of n digit numbers. How many of these numbers

   (a) have an even number as the last digit?

   (b) have at least one repeating digit?

   (c) have exactly k digits that are 9s?

   Solution:

   (a) There are 5 ways to choose the last digit, and 10 ways to choose the remaining $n-1$ digits. Hence, the answer is $5 \cdot 10^{n-1}$.

   (b) It is easier to count the opposite, i.e. n digit numbers with no repeating digits. This is just $P(10, n)$, which is $\frac{10!}{(10-n)!}$ if $n \leq 10$, or 0 if $n > 10$. Since we counted the opposite, we must subtract it from the total, meaning the answer is

$$10^n - P(10, n).$$

   (c) There are n digits to be filled, and we must fill k of them with the digit 9. The number of ways to do this is $\binom{n}{k}$. With the 9s chosen, we fill the remaining $n-k$ digits, and there are 9 choices for each digit, so the number of ways to do this is 9^{n-k}. Hence, the answer is

$$\binom{n}{k} \cdot 9^{n-k}.$$

(1 continued:)

2. Consider functions f from the set $\{1, 2, \ldots, n\}$ to the set $\{0, 1\}$.

   (a) How many functions are there?

   (b) How many functions have $f(1) = 1$?

   (c) How many functions have $f(k) \neq f(k+1)$ for all $1 \leq k < n$?

   (d) How many functions have $f(k) = 0$ for exactly 3 distinct values of k?

   Solution:

   (a) There are 2 options for $f(k)$ for each $k \in \{1, 2, \ldots, n\}$. Hence, the number of possible functions is 2^n.

   (b) Since $f(1) = 1$, we only need to choose $f(2)$ through $f(n)$, and so the number of choices is 2^{n-1}.

   (c) Suppose $f(1) = 0$. Then $f(2)$ must be 1, and $f(3)$ must be 0, and so on. So once $f(1)$ is decided, the rest of the function is determined. Hence, there are only 2 such functions.

   (d) We need to choose the 3 values of k that will give $f(k) = 0$. There are n values to choose from, so there are $\binom{n}{3}$ ways to do this. Notice that all other values of k must have $f(k) = 1$. Hence, the number of functions is $\binom{n}{3}$.

(2 continued:)

3. Recall the stars and bars counting principle.

   (a) How many ordered sets of non-negative integers (a, b, c) are there such that $a + b + c = 15$? What about positive integers?

   (b) How many ways can you make a buffet platter with 20 portions of food, given that your options for each portion are; chicken wings, mozza sticks, egg rolls, and potato skins? (Note: the presentation of the food doesn't matter, i.e. the order doesn't matter).

Solution:

(a) One such set is $(5, 6, 4)$, and we can imagine this as $A, A, A, A, A, B, B, B, B, B, B, C, C, C, C$. In general, if we choose an *unorded set* of size 15, where each option is A, B or C, this corresponds uniquely to an *ordered set* of non-negative integers (a, b, c), where a is the number of As in our set, b is the number of Bs in our set, and c is the number of Cs in our set. Hence, by the stars and bars principle, the number of sets is

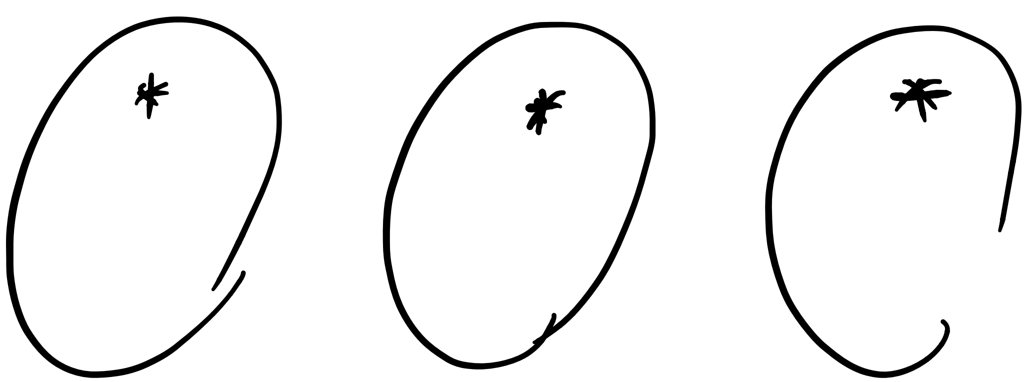
$$\binom{15+3-1}{3-1} = \binom{17}{2}.$$

If a, b and c must be positive integers, we can instead count the number of ordered sets of non-negative integers (a', b', c') where $a' + b' + c' = 12$. Think of $a' = a - 1$, and similarly for b' and c'. So by the same logic as (a), the number of sets is

$$\binom{12+3-1}{3-1} = \binom{14}{2}.$$

(b) This word problem is intentionally designed to distract you. Look through the buffet platter and realize that this is just a stars and bars problem with 20 stars and $4 - 1$ bars. So the answer is

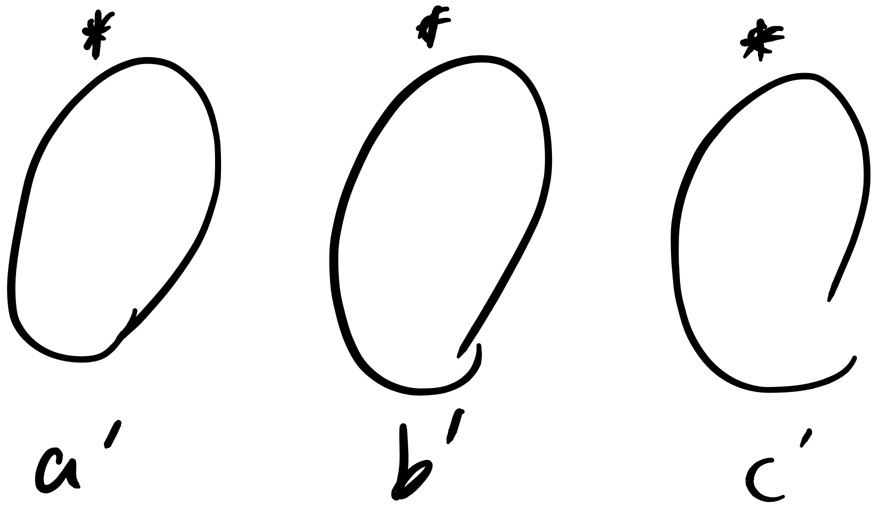
$$\binom{20}{3}^{+3}\Big).$$



a b c

$a + b + c = 15$

(3 continued:)



$a' + b' + c' = 12$

$a + b + c = 15$

$a \geq 1$

$b \geq 3$

$c \geq 2$

$(a'+1) + (b'+3) + (c'+2) = 15$

$\Rightarrow a' + b' + c' = 15 - 6 = 9$

4. An urn contains k red balls, k blue balls, and k yellow balls. How many balls must you take from the urn to guarantee

   (a) 2 balls of the same colour?

   (b) k balls of the same colour?

   (c) 2 balls of different colour?

   (d) 3 balls of different colour?

   (e) At least 1 red ball and 1 blue ball?

   Solution:

   (a) If you take 3 balls, there could be 1 of each colour. If you take 4 balls, then you must have at least 2 balls of the same colour, since there are only 3 colours. Hence, the answer is 4.

   (b) You could pick out $k-1$ balls of each colour, which is $3k-3$ balls total. Hence, if you pick $3k-2$ balls, you must have k balls of a single colour. So the answer is $3k-2$.

   (c) You could pick k balls and they all happen to be red. However, if you pick $k+1$ balls, you must have 2 balls of different colour. So the answer is $k+1$.

   (d) You could pick $2k$ balls and they are all red and blue. So similarly to the previous solution, the answer is $2k+1$.

   (e) In the worst case scenario, the first k balls are all yellow. Then, the next k balls could all be red. So you can pick $2k$ balls without finding at least 1 red and 1 blue. Hence, you need at least $2k+1$ balls, since you are guaranteed to find a ball of every colour.

(*4 continued:*)

5. Prove the following identities (note that they can be proved algebraically, combinatorially, or via Pascal's identity):

   (a) Vandermonde's Identity:

$$\binom{m+n}{k} = \sum_{i=0}^{k} \binom{m}{i}\binom{n}{k-i}.$$

   (b) The Hockey Stick Identity:

$$\binom{n+1}{r+1} = \sum_{j=r}^{n} \binom{j}{r}.$$

Solution:

(a) Suppose we are given m red balls and n blue balls, and the balls are numbered so that they are all distinct. How many ways can we choose k balls from this set? Well the obvious answer is $\binom{m+n}{k}$. Alternatively, we could first decide how many red balls we will include in our set. If we include i red balls, then we must include $k-i$ blue balls. In this case, the total number of choices is $\binom{m}{i}\binom{n}{k-i}$. Summing over all possible i gives us

$$\sum_{i=0}^{k} \binom{m}{i}\binom{n}{k-i}.$$

Since we have counted the same thing twice, it must be the case that

$$\binom{m+n}{k} = \sum_{i=0}^{k} \binom{m}{i}\binom{n}{k-i}.$$

1 2 3 4 5 1 2 3 4

$$\binom{5}{3} + \binom{5}{2}\cdot\binom{4}{1} + \binom{5}{1}\binom{4}{2} + \binom{4}{3}$$

(*5 continued:*)

(b) Let's count the number of ways to choose $r+1$ elements from $\{1, 2, \ldots, n+1\}$. The obvious solution is $\binom{n+1}{r+1}$. Now consider breaking it into cases based on the largest integer in our set. For example, if the largest integer we choose is $r+1$, then there is only one possible way to choose the remaining set. On the other hand, if $n+1$ is the largest integer we choose, then there are $\binom{n}{r}$ ways to choose the remaining set. In general, if $j+1$ is the largest integer we choose, then the number of ways to choose the remaining set is $\binom{j}{r}$. Summing from $j+1 = r+1$ to $j+1 = n+1$ (i.e. j from r to n), we get

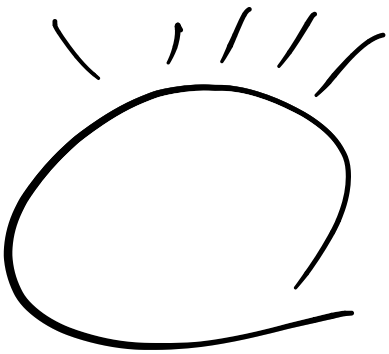
$$\sum_{j=r}^{n} \binom{j}{r}.$$

Since we have counted the same thing twice, it must be the case that

$$\binom{n+1}{r+1} = \sum_{j=r}^{n} \binom{j}{r}.$$

$$\binom{n+1}{r+1} = \sum_{j=r}^{n} \binom{j}{r}$$

$1, 2, 3, \ldots, n-1, n, n+1$



set of $r+1$ numbers.

(*5 continued:*)

Fix largest # in the set.

Case 1: If the largest # is $n+1$,

how many sets are there?

$1, 2, 3, \ldots\ldots, n-1, \cancel{n}, \cancel{n+1}$

$$\binom{n}{r} + \binom{n-1}{r} + \binom{n-2}{r} + \cdots + \binom{r}{r}$$

1, 2, 3, 4, 5

1, 2, 5

1, 3, 5

1, 4, 5

2, 3, 5

2, 4, 5

3, 4, 5

5 is largest

1, 2, 4

1, 3, 4

2, 3, 4

4 is the largest

1, 2, 3

3 is the largest

$$\binom{4}{2} + \binom{3}{2} + \binom{2}{2} = \binom{5}{3}$$