Problem 22

The Hatcheck Problem

At a party with a *very large* number of guests, everyone has to leave their hat at the entrance. When they leave the party, everyone is too drunk to remember what hat is theirs, so they just pick one at random. **What is the probability** that no one picks their own hat?

n

Probability $= \frac{?}{n!}$ {# ways that no one picks their own hat / {# ways to pick a hat...

Permutations of n hats with no fixpoint

(Derangement).

- **Def:** A derangement of a set with n elements is a permutation of that set that keeps no element in its original position.

- **Ex:** $X = \{1, 2, 3, 4, 5\}$.

$[2, 3, 4, 1, 5]$ That is a derangement.

1st 2nd 3rd 4th 5th

$[\boxed{1}, 3, 5, \boxed{4}, 2]$ Not a derangement.

1 4

Notation: $!n$: "subfactorial n"

= Number of derangements of n objects

Ex: $!0 = 0$

$!1 = 0$

$!2 = 1 \rightarrow$ ~~[1 2]~~ [2 1] ✓

$!3 = 2$

~~1 2 3~~

~~1 3 2~~

~~2 1 3~~

2 3 1 ✓

3 1 2 ✓

~~3 2 1~~

⋮

$!n = ?$ (General Formula)

Use inclusion-exclusion:

$X = \{\text{All permutations of } n \text{ objects}\}$

$A_i = \{\text{Permutations where } i \text{ is in position } i\}$.

Want to avoid this!

$D = \{\text{Derangements of } n \text{ objects}\}$

$D = X \setminus (A_1 \cup A_2 \cup \dots \cup A_n)$

Ex: $n = 5$

$|A_2 \cap A_4| = ?$

$\square\ 2\ \square\ 4\ \square$

3 options, 2 opt., 1 opt.

$\Rightarrow 3! = (5-2)!$

We know that: $|D| = \sum_{k=0}^{n} (-1)^k \binom{n}{k} N(k)$

where $N(k) = |A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_k}| = (n-k)!$

$N(0) = |X| = n! = (n-0)!$

$n-k$ objects (all except $i_1 \dots i_k$) in $n-k$ positions.

$$!n = |D| = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)!$$

$$= \sum_{k=0}^{n} (-1)^k \frac{n!}{k!\,\cancel{(n-k)!}} \cancel{(n-k)!}$$

$$= n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

Uses the MacLaurin series $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$\Rightarrow$ Answer to the problem:

$$\text{Prob.} = \lim_{n\to\infty} \frac{!n}{n!} = \lim_{n\to\infty} \frac{\cancel{n!}}{\cancel{n!}} \sum_{k=0}^{n} \frac{(-1)^k}{k!} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} = e^{-1} = \frac{1}{e}.$$

<u>Generating Functions</u> (Counting with power series)

- <u>Def:</u> Given a sequence $(a_n)_{n=0}^{\infty} = (a_0, a_1, a_2, \ldots)$, the <u>generating function</u> of that sequence is the (formal) power series

$$A(x) = \sum_{n=0}^{\infty} a_n x^n .$$

- <u>Ex:</u> $a_n = 2^n \implies A(x) = \sum_{n=0}^{\infty} 2^n x^n = \sum_{n=0}^{\infty} (2x)^n$

$b_n = n \implies B(x) = \sum_{n=0}^{\infty} n x^n$

- **Def**: Two formal power series are equal

$$A(x) = B(x)$$

$$\sum_{n=0}^{\infty} a_n x^n \qquad \sum_{n=0}^{\infty} b_n x^n$$

if $a_n = b_n$ for all n.

→ **Operations**:

Addition

$$A(x) + B(x) = \left(\sum_{n=0}^{\infty} a_n x^n\right) + \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

$$= \sum_{n=0}^{\infty} (a_n + b_n) x^n.$$

→ **Ex**: $\left(\sum_{n=0}^{\infty} 2^n x^n\right) + \left(\sum_{n=0}^{\infty} n x^n\right) = \sum_{n=0}^{\infty} (2^n + n) x^n$

Multiplication: (Cauchy's Formula):

$$A(x)B(x) = \left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right)x^n$$

Ex: $\left(\sum_{n=0}^{\infty} 2^n x^n\right)\left(\sum_{n=0}^{\infty} n x^n\right)$

$$= (1+2x+4x^2+8x^3+16x^4+\dots)(0+x+2x^2+3x^3+4x^4+\dots)$$

$$= 0 + 1x + (2+2)x^2 + (3+4+4+0)x^3 + (4+6+8+8+0)x^4 + \dots$$

$$= x + 4x^2 + 11x^3 + 26x^4 + \dots$$

* <u>Note</u>: With that structure $(+, \cdot)$, formal power series form a <u>ring</u>, denoted $R[[x]]$.

- <u>Ex 2</u>: $A(x) = \sum_{n=0}^{\infty} x^n \quad (a_n = 1, \forall n)$

$$A(x)^2 = A(x)\,A(x)$$

$$= \left(\sum_{n=0}^{\infty} x^n\right)\left(\sum_{n=0}^{\infty} x^n\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k a_{n-k}\right) x^n$$

$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} 1\right) x^n = \sum_{n=0}^{\infty} (n+1) x^n$$

Ex3. $A(x) = \sum_{n=0}^{\infty} 1x^n$ $\quad B(x) = 1 - x + 0x^2 + 0x^3 + 0x^4 + \cdots$

"Finite power series"

$b_0 = 1$

$b_1 = -1$

$b_n = 0 ; (n>1)$

$$A(x)B(x) = \left(\sum_{n=0}^{\infty} x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right) x^n = 1x^0 + 0x + 0x^2 + \cdots$$

$$= 1$$

$\Rightarrow$ $A(x)$ and $B(x)$ are inverses!

?

$n=0$: $a_0 b_0 = 1$

$n=1$: $a_0 b_1 + a_1 b_0 = 1(-1) + (-1)1 = 0$

$\vdots$

$n \geq 1$: $a_0 b_n + a_1 b_{n-1} + \cdots 0 + a_{n-1} b_1 + a_n b_0$

$-1 + 1 = 0$

$\Rightarrow \qquad A(x) = (B(x))^{-1}$

$$\sum_{n=0}^{\infty} x^n = (1-x)^{-1} = \frac{1}{1-x}$$

Equality between formal power series

* Recall $\sum_{n=0}^{N} x^n = \frac{1-x^{N+1}}{1-x}$

As $\sum_{n=0}^{\infty} x^n = \lim_{N\to\infty} \frac{1-x^{N+1}}{1-x} = \frac{1}{1-x}$

Equality between real numbers $(-1 < x < 1)$

Def: Let $F(x) = \sum_{n=0}^{\infty} a_n x^n$ be a formal power series and $f : (a,b) \to \mathbb{R}$ $(a < 0 < b)$.

If there is a number $R > 0$ such that for all $r \in (-R, R)$ the numerical series $F(x)\big|_{x=r} = F(r)$ converges to the value $f(r)$, then we say that $F(x)$ represents the function $f(x)$ and write

$$F(x) \approx f(x)$$

we will actually write = later...

Note that $\approx$ is compatible with $+$ and $\cdot$.

$$F(x) + G(x) \approx f(x) + g(x)$$

$$F(x)\, G(x) \approx f(x)\, g(x).$$

$\Rightarrow$ Algebra is the same. We can go from

one "world" to the other...

Formal Power Series $\rightleftarrows$ Real Functions.

We will then allow ourselves to write $=$ instead of $\approx$.

* **Recall:** Taylor's Theorem.

If $f(x) = \sum_{n=0}^{\infty} a_n x^n$ then f is infinitely differentiable and,

$$\boxed{a_n = \frac{f^{(n)}(0)}{n!}}$$

$\Rightarrow$ Power series representations of a function are unique!

Moreover, for the "common calculus functions" (powers, exp, log, trig, ...) $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$ does represent $f(x)$.

$\Rightarrow$ Some power series we may use ...

$$\frac{1-x^{N+1}}{1-x} = 1 + x + x^2 + \ldots + x^N \qquad \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$x \mapsto ax$ $\qquad$ $x \mapsto x^r$

$$\frac{1}{1-ax} = \sum_{n=0}^{\infty} a^n x^n \qquad \frac{1}{1-x^r} = \sum_{n=0}^{\infty} x^{rn}$$

$\binom{1/2}{2} = \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!} = \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2} = \frac{-1}{8}$

* Binomial Theorem (Extends)

$$(1+x)^r = \sum_{n=0}^{\infty} \binom{r}{n} x^n \text{ where } \binom{m}{k} = \frac{m(m-1)\cdots(m-k+1)}{k!}$$

Works for any real number!

$$\frac{1}{(1-x)^r} = \sum_{n=0}^{\infty} \binom{n+r-1}{n} x^n \qquad \sqrt{1+x} = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{(-1)^{n+1}}{4^n(2n-1)} x^n$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \qquad \sinh(x) = \frac{e^x - e^{-x}}{2} = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$