A Table of Power Series

$$\frac{1-x^{N+1}}{1-x} = 1 + x + x^2 + \ldots + x^N \qquad \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\frac{1}{1-ax} = \sum_{n=0}^{\infty} a^n x^n \qquad \frac{1}{1-x^r} = \sum_{n=0}^{\infty} x^{rn}$$

$$(1+x)^r = \sum_{n=0}^{\infty} \binom{r}{n} x^n \text{ where } \binom{m}{k} = \frac{m(m-1)\cdots(m-k+1)}{k!}$$

$$\frac{1}{(1-x)^r} = \sum_{n=0}^{\infty} \binom{n+r-1}{n} x^n \qquad \sqrt{1+x} = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{(-1)^{n+1}}{4^n(2n-1)} x^n$$

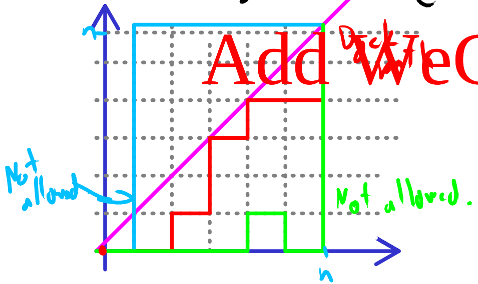
$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

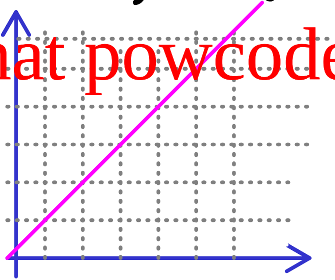
$$\cosh(x) = \frac{e^x + e^{-x}}{2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \qquad \sinh(x) = \frac{e^x - e^{-x}}{2} = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

Dyck Paths.

Dyck paths are grid paths such that:

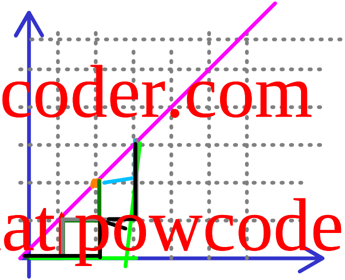
- The only allowed steps are → ↑
- start at $(0,0)$ and end at (n,n)
- Never go above the diagonal $y=x$.





Problem: Count C_n = The number of Dyck paths that end at (n,n).

Basic cases:



C_0 : (empty path)

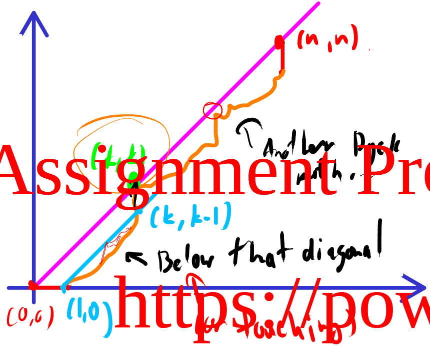
C_1 : 1

C_2 : $\rightarrow$ 2.

C_3 : $\rightarrow$ 5.

$\vdots$

C_n : Find a recurrence...



Must start with →

end with ↑

(k,k) = First point on the diagonal that the path touches (after (0,0)).

$k \in \{1, 2, \ldots, n\}$

If a path touches at (k,k) for the first time, then it is made out of:

1. a Dyck path from (1,0) to (k,k-1): C_{k-1} of those.
2. a Dyck path from (k,k) to (n,n): C_{n-k} of those.

Put the pieces together:

(Product principle)

$$C_n = \sum_{k=1}^{n} \underbrace{C_{k-1}\, C_{n-k}} = \sum_{l=0}^{n-1} C_l\, C_{n-(l+1)}$$

$l = k-1$

$k = l+1$

$n-(l+1) = n-l-1$

(Sum principle)

That is a recurrence that involves:

(1) Products of c_ℓ's $\Rightarrow$ Not linear!

(2) All of the previously calculated c_ℓ's $\Rightarrow$ Infinite order!

Let's solve it!

Let $C(x) = \sum_{n=0}^{\infty} c_n x^n$

$= c_0 + \sum_{n=1}^{\infty} c_n x^n$

$= 1 + \sum_{n=1}^{\infty} \left(\sum_{l=0}^{n-1} c_l c_{n-l-1} \right) x^n$

$= 1 + \sum_{m=0}^{\infty} \left(\sum_{l=0}^{m} c_l c_{m-l} \right) x^{m+1}$

$m = n-1$

$\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right) x^n$

Looks like Cauchy product...

$$= 1 + x \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} c_l c_{n-l} \right) x^n$$

$$= 1 + x \left(\sum_{n=0}^{\infty} c_n x^n \right)^2$$

$$\Rightarrow \quad C(x) = 1 + x \left(C(x) \right)^2$$

$0 = x C^2 - C + 1$: Degree 2 equation for variable C ...

Quadratic formula:

$$C = \frac{1 \pm \sqrt{1 - 4x}}{2x}$$

For the moment: Two possibilities.

Now expand!

We know: $\sqrt{1+y} = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{(-1)^{n+1}}{4^n(2n-1)} y^n$

Replace y by $-4x$ ...

$$C = \frac{1}{2x} \pm \frac{1}{2x} \sum_{n=0}^{\infty} \binom{2n}{n} \frac{(-1)^{n+1}}{4^n(2n-1)} (-4x)^n$$

$$= \frac{1}{2x} \pm \left(\frac{1}{2x} + \frac{1}{2x} \sum_{n=1}^{\infty} \binom{2n}{n} \frac{(-1)^{n+1}}{4^n(2n-1)} (-4x)^n \right)$$

If we chose $+$, the series would start in $\frac{1}{x}$

→ Not a power series (contrary to assumption)

⇒ The correct closed form is the one with the − sign!

$$C(x) = \cancel{\frac{1}{2x}} - \left(\cancel{\frac{1}{2x}} + \frac{1}{2x}\sum_{n=1}^{\infty}\binom{2n}{n}\frac{(-1)^{n-1}}{4^n(2n-1)}(-4x)^n\right)$$

$$= \frac{1}{2x}\sum_{n=1}^{\infty}\binom{2n}{n}\frac{\cancel{(-1)}^{n+1}}{\cancel{4^n}(2n-1)}(\cancel{-4}x)^n$$

$$= \frac{1}{2x}\sum_{n=1}^{\infty}\binom{2n}{n}\frac{x^n}{2n-1}$$

$$= \sum_{n=1}^{\infty}\binom{2n}{n}\frac{x^{n-1}}{2(2n-1)}$$

$$C(x) = \sum_{k=0}^{\infty} \binom{2(k+1)}{k+1} \frac{x^k}{2(2(k+1)-1)}$$

$(k = n-1)$

$$\Rightarrow C_k = \binom{2(k+1)}{k+1} \frac{1}{2(2(k+1)-1)}$$

Let's simplify...

$$C_n = \binom{2n+2}{n+1} \frac{1}{2(2n+1)}$$

$$= \frac{(2n+2)!}{(n+1)!\,(n+1)!} \cdot \frac{1}{2(2n+1)}$$

$$= \frac{(2n+2)\cancel{(2n+1)}(2n)!}{(n+1)!\,(n+1)!\,2\cancel{(2n+1)}}$$

$$= \frac{\cancel{(2n+2)}(2n)!}{(n+1)!\,\cancel{(n+1)n!\,2}}$$

$$= \boxed{\frac{(2n)!}{n!\,n!}}\frac{1}{(n+1)}$$

$$C_n = \binom{2n}{n}\frac{1}{n+1} \quad : \text{Catalan Numbers!}$$

Counting with GFs.

Say that we have a (multi)set A and are interested in the numbers (repetitions are allowed)

a_n = The number of ways to choose n elements from A.

Ex: A contains 5 balls numbered 1, 2, 3, 4, 5

$\Rightarrow a_n = \binom{5}{n}$

B contains 5 identical balls:

$$b_n = \begin{cases} 1 & \text{if } n \le 5 \\ 0 & \text{if } n > 5. \end{cases}$$

The generating functions for each a sequence are called <u>enumerators</u> for the set...

- Ex: $A(x) = \binom{5}{0}x^0 + \binom{5}{1}x + \binom{5}{2}x^2 + \binom{5}{3}x^3 + \binom{5}{4}x^4 + \binom{5}{5}x^5$

$$= (1+x)^5$$

$$B(x) = 1 + x + x^2 + x^3 + x^4 + x^5 + 0 + \cdots$$

$$= \frac{1 - x^6}{1 - x}$$

- <u>Thm</u>: If A and B are disjoint,

then the product of GFs: $A(x) B(x)$

is the enumerator for $A \cup B$.

- **Proof:**

$$A(x)B(x) = \left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right) x^n$$

This counts the # of ways to pick k elements from A and $(n-k)$ from B, for all $k \in \{0, \ldots, n\}$:

Same as choosing n elements from $A \cup B$. $\square$

Ex: Make a fruit salad by choosing 6 fruit pieces from a buffet that has enough [illegible] Apples, Bananas and Cherries.

How many different fruit salads are there?

a_n = # ways to choose n apples (indistinguishable) $= 1$

Sol: $A(x) = 1 + 1x + 1x^2 + 1x^3 + 1x^4 + \dots = \dfrac{1}{1-x}$

[illegible]

Same for $B(x) = C(x) = \dfrac{1}{1-x}$.

$\Rightarrow$ For mixing the 3 types of fruit, the GF is:

$$A(x)\,B(x)\,C(x) = \left(\frac{1}{1-x}\right)^3 = \frac{1}{(1-x)^3}$$

We want the x^6 coefficient.

From table: $$\frac{1}{(1-x)^3} = \sum_{n=0}^{\infty} \binom{n+2}{n} x^n$$

$\Rightarrow$ At $n=6$: $\binom{6+2}{6} = \binom{8}{6}$ salads.