Problem 10

(no jokers)

How many cards, from a regular deck, do we need to pick up in order to guarantee that at least 3 of them are of the same suit?

Objects: Cards

Boxes: Suits: ♡ ♢ ♣ ♠

$k = 4$ possibilities

Pigeonhole principle tells us $\left\lceil \frac{N}{k} \right\rceil$ objects in one box.

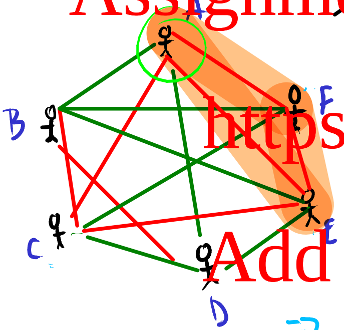
Find N (minimal) s.t. $\left\lceil \frac{N}{k} \right\rceil \geq 3 \iff \frac{N}{k} > 2$

$N > 2k = 8$

$N \geq 9.$

Problem 11

Assume that in a group of six people, each pair of two people consists either of 2 friends or 2 enemies. Show that there are either 3 mutual friends or 3 mutual enemies in the group.



That is just one case (need a general argument)

Using the pigeonhole principle!

Objects: B, C, D, E, F $N=5$

Boxes: • Friends of A. $k=2$ $\lceil \frac{5}{2} \rceil = 3$

• Enemies of A

$\Rightarrow$ There is a box that contains at least 3 people.

WLOG (By symmetry), assume: 3 enemies.

"Without Loss of Generality"

$\Rightarrow$ 2 cases:

① They are mutual friends $\Rightarrow$ Done.

② At least 2 of them are enemies: $\Rightarrow$ Done (Form a triangle with A).

Permutations and Combinations

Problem 12

From a class of 20 students, how many ways are there to form a committee of 3 students with named positions (President, VP, Secretary)?

20   19   18

Pres   VP   Secr

$\Rightarrow$ Number of committees is

$20 \times 19 \times 18 = 6840$

$= P(20,3)$

The president elect cannot be VP (everyone else could).

Permutations

Definition

A *k-permutation* of a set with n elements is an ordered arrangement of k elements from that set. The number of such k-permutations is denoted $P(n, k)$ or sometimes nPk. An n-permutation is simply called a permutation.

$$P(n, k) = n \cdot (n-1) \cdot (n-2) \cdots \underbrace{(n-(k-1))}_{(n-k+1)}$$

$$= \frac{n!}{(n-k)!}$$

Special case: $P(n, n) = \frac{n!}{(n-n)!} = n!$

$* \; 0! = 1$

Problem 13

How many permutations of the set $S = \{1, 2, 3, 4, 5\}$ are there?

$n = 5$

$\Rightarrow \quad P(5,5) = 5!$

Problem 14

From a class of 20 students, how many ways are there to form a committee of 3 students without named positions?

Same 3 people: A, B, C those are considered the same committee:

Pres	VP	S
A	B	C
A	C	B
B	A	C
B	C	A
C	A	B
C	B	A

→ Same committee is "repeated" 6 times = 3!

→ The actual number of different committees is:

$$\frac{P(20,3)}{3!} = 1140$$

Combinations

Definition

An *k-combination* of a set with n elements is k-elements subset from that set. The number of such k-combinations is denoted $C(n, k)$, or sometimes nCk or $\binom{n}{k}$ (also called binomial coefficient).

!

"n choose k"

$$\binom{n}{k} = C(n,k) = \frac{P(n,k)}{k!} = \frac{n!}{(n-k)!\,k!}$$

→ Ex: Poker hands (5 cards) that can be drawn from 52 cards.

$$\binom{52}{5} = \frac{52!}{47!\,5!} = 2\,598\,960 \text{ hands.}$$

Problem 15

→ Note: $\binom{n}{k} = \binom{n}{n-k}$

A chocolate box contains chocolates in 7 flavours: black, white, cherry, milk, nuts, orange and truffles. Assuming that there are at least 4 of each, in how many different ways can we pick 4 chocolates? (The order does not count.)

"Stars and bars" technique ...

$* \mid ** \mid \mid \mid \mid * \mid \mid$

black | white | cherry | milk | nuts | orange | Truffles

$* \mid * \; * \mid \mid \mid * \mid \mid$

1 2 3 4 5 6 7 8 9 10

That string means: 1 black, 2 white, 1 nuts.

Want to count such strings with 4 "$*$" and 6 "$\mid$"

≈ Binary strings of length $4+6=10$ with four "$*$"

Choose $\frac{4}{6}$ of the 10 positions to hold $\frac{*}{\mid}$: $\binom{10}{4} = \frac{10!}{6!\,4!} = \binom{10}{6}$

Problem 16

Prove the following combinatorial identity: $\binom{n}{k} = \binom{n}{n-k}$.

① Direct (algebraic) proof:

$$\binom{n}{k} = \binom{n}{n-k}$$

$$\frac{\cancel{n!}}{\cancel{(n-k)!}\,k!} = \frac{\cancel{n!}}{(n-(n-k))!\,\cancel{(n-k)!}}$$

$$\frac{1}{k!} = \frac{1}{k!} \quad ✓$$

② Combinatorial proof: Argue that the two sides of = count the same objects.

$\binom{n}{k}$ counts the number of binary strings with k zeroes

|| Same.

$\binom{n}{n-k}$ counts the number of bin. strings with $(n-k)$ ones.

Problem 17

Give combinatorial proof of Pascal's Identity: $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$.

2 cases ...

LHS:

$\binom{n+1}{k}$ is the number of committees of k students from a class of $(n+1)$ students.

RHS: Class = $\{0, 1, 2, 3, \ldots, n\}$

Two cases: (1) 0 is on the committee.

Need to choose the $k-1$ other members $\Rightarrow \binom{n}{k-1}$ options

(2) 0 is not on the committee.

Choose k members from $\{1, 2, \ldots, n\}$. $\Rightarrow \binom{n}{k}$ options.

By the sum principle, the number of committees is $\boxed{\binom{n}{k-1} + \binom{n}{k}}$

Pascal's Triangle

Previous row.

n: rows

k: columns

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

$$\binom{n}{0} = \binom{n}{n} = 1$$

$$\binom{0}{0}$$

$$\binom{1}{0} \quad + \quad \binom{1}{1}$$

$$\binom{2}{0} \quad + \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad + \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$

$$\binom{6}{0} \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad \binom{6}{6}$$

$$\binom{7}{0} \quad \binom{7}{1} \quad \binom{7}{2} \quad \binom{7}{3} \quad \binom{7}{4} \quad \binom{7}{5} \quad \binom{7}{6} \quad \binom{7}{7}$$

$$\binom{8}{0} \quad \binom{8}{1} \quad \binom{8}{2} \quad \binom{8}{3} \quad \binom{8}{4} \quad \binom{8}{5} \quad \binom{8}{6} \quad \binom{8}{7} \quad \binom{8}{8}$$

$$\binom{9}{0} \quad \binom{9}{1} \quad \binom{9}{2} \quad \binom{9}{3} \quad \binom{9}{4} \quad \binom{9}{5} \quad \binom{9}{6} \quad \binom{9}{7} \quad \binom{9}{8} \quad \binom{9}{9}$$

$$\binom{10}{0} \quad \binom{10}{1} \quad \binom{10}{2} \quad \binom{10}{3} \quad \binom{10}{4} \quad \binom{10}{5} \quad \binom{10}{6} \quad \binom{10}{7} \quad \binom{10}{8} \quad \binom{10}{9} \quad \binom{10}{10}$$

?

⋮

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

7 21 35 35 21 7 1

Binomial Theorem

$$
\begin{aligned}
(x+y)^2 &= x^2 + 2xy + y^2 \\
(x+y)^3 &= x^3 + 3x^2y + 3xy^2 + y^3 \\
(x+y)^4 &= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 \\
&\vdots \\
(x+y)^n &= \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k \\
&= \binom{n}{0} x^n y^0 + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \ldots + \binom{n}{n} x^0 y^n
\end{aligned}
$$

$\rightarrow$ Proof: $(x+y)^n = (x+y)(x+y)(x+y)\cdots(x+y)$ (n times).

To perform this product, multiply one term from each bracket together: gives $x^{n-k}y^k$ (if you choose y k times).

Then add this up: $x^{n-k}y^k$ will appear $\binom{n}{k}$ times. $\Rightarrow$ Coefficient.

Problem 18

What is the coefficient of $x^{12}y^{13}$ in the expression of $(2x+y)^{25}$?

$$(2x+y)^{25} = \sum_{k=0}^{25} \binom{25}{k} (2x)^{25-k} y^{k}$$

When $k = 13$, you get:

$$\binom{25}{13}(2x)^{12} y^{13} = \boxed{\binom{25}{13} 2^{12}} x^{12} y^{13}$$

Coefficient. (answer)