Winter 2020

Midterm Exam

Name: ____________________ McGill ID: ____________

- **Do not turn this page over until we tell you that the exam starts!** You have two hours from that point to complete the exam.
- The exam counts 5 questions worth 20 marks each, adding-up to 100 marks.
- This is a closed book examination. But you may use the table of power series below.
- Calculators are allowed, given that you can't write notes in them. Smarter devices are obviously forbidden.
- Answer all problems in the booklet provided with this exam. Show all your work.
- Please **return this exam** with the booklet in the end.
- Good luck!

$\dfrac{1-x^{N+1}}{1-x} = 1 + x + x^2 + \ldots + x^N$	$\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$
$(1+x)^r = \sum_{n=0}^{\infty} \binom{r}{n} x^n$	$\dfrac{1}{(1-x)^r} = \sum_{n=0}^{\infty} \binom{n+r-1}{n} x^n$
$\sqrt{1+x} = \sum_{n=0}^{\infty} \binom{2n}{n} \dfrac{(-1)^{n+1}}{4^n(2n-1)} x^n$	$e^x = \sum_{n=0}^{\infty} \dfrac{x^n}{n!}$
$\cosh(x) = \dfrac{e^x + e^{-x}}{2} = \sum_{n=0}^{\infty} \dfrac{x^{2n}}{(2n)!}$	$\sinh(x) = \dfrac{e^x - e^{-x}}{2} = \sum_{n=0}^{\infty} \dfrac{x^{2n+1}}{(2n+1)!}$

Question 1
How many integers $1 \le n \le 1000$ are neither perfect squares, perfect cubes, nor multiples of 7?

Question 2

(a) Find an algebraic proof the following identity.

$$n2^{n-1} = \sum_{k=1}^{n} k\binom{n}{k}$$

(b) Find a combinatorial proof of the same identity.

Question 3
Use the Generating Functions Method to solve the recurrence $a_n = 3a_{n-1} + 4^n$ with initial condition $a_0 = 1$.

Question 4
Use an Exponential Generating Function to count the number of strings of length n over the alphabet $\{0, 1, 2\}$ with an even number of '0' and an odd number of '1'.

Question 5
A function $f : \{0, 1 \dots n\} \to \{0, 1 \dots n\}$ is called *nilpotent* if for all $x \in \{0, 1 \dots n\}$, there is a number k_0 such that for all $k \ge k_0$, $f^k(x) = 0$. Here f^k denotes the k-th iterate: $f^k = f \circ f \circ \cdots \circ f$ (k times).

(a) Show that the function f defined by the following table is nilpotent.

x	0	1	2	3	4	5
$f(x)$	0	4	0	4	2	2

(b) Show that if $f : \{0, 1 \dots n\} \to \{0, 1 \dots n\}$ is nilpotent, then we can simply take $k_0 = n$ in the definition, i.e. show that for all x and for all $k \ge n$, $f^k(x) = 0$.

Hint: *One of the key steps here might be the Pigeonhole Principle...*

(c) Apply Joyal's bijection to the function in part **(a)** and draw the corresponding vertebrate.

(d) Generalize this example to show that the number of nilpotent functions on $\{0, 1 \dots n\}$ is $(n + 1)^{(n-1)}$.