Lambda calculus
(Advanced Functional Programming)

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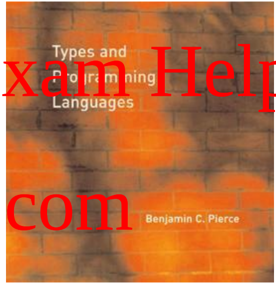
Course outline

Books



OCaml from the very beginning
John Whittington
Coherent Press (2013)

Real World OCaml
Yaron Minsky,
Anil Madhavapeddy &
Jason Hickey
O'Reilly Media (2013)

Types and Programming Languages
Benjamin C. Pierce
MIT Press (2002)

Tooling

OPAM
OCaml package manager

Linux / OSX / VirtualBox

IOCaml

$F\omega$ interpreter

Philosophy and approach

- **practical**: with theory as necessary for understanding
- **real-world**: patterns and techniques from real applications
- **reusable**: general, widely applicable techniques
- **current**: topics of ongoing research

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- **practical**: with theory as necessary for understanding
- **real-world**: patterns and techniques from real applications
- **reusable**: general, widely applicable techniques
- **current**: topics of ongoing research
- **opinionated** (but you don't have to agree)

Mailing list

cl-acs-28@lists.cam.ac.uk

Announcements, questions and discussion. Feel free to post!

Have a question but feeling shy? Mail me directly and I'll anonymise and post your question:

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Exercises assessed and unassessed

Unassessed exercises:

Useful preparation for the assessed exercises, so we recommend that you work through them. Hand in for feedback, discuss freely on the mailing list.

Assessed exercises:

Mon 25 Jan	Thu 11 Feb	Mon 7 Mar
$\downarrow$	$\downarrow$	$\downarrow$
Mon 8 Feb	Thu 25 Feb	Fri 25 Apr

Course structure

- **Technical background**
  Lambda calculus; type inference
- **Themes**
  Propositions as types; parametricity and abstraction
- **(Fancy) types**
  Higher-rank and higher-kinded polymorphism; modules and functors; generalised algebraic types
- **Patterns and techniques**
  Monads, applicatives, arrows, etc.; datatype-generic programming; staged programming
- **Applications**
  Functional programming at scale with unikernels; concurrency and reagents

Motivation & background

Function composition in OCaml:

```
fun f g x -> f (g x)
```


Function composition in System Fω:

```
Λα::*.
   Λβ::*.
      Λγ::*.
         λf:α → β.
            λg:γ → α.
               λx::γ.f (g x)
```

What's the point of System Fω?

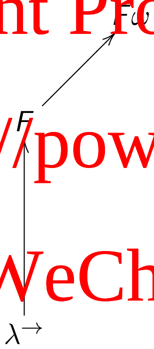
A framework for understanding language features and programming patterns

- the elaboration language for type inference
- the proof system for reasoning with propositional logic

- the background for parametricity properties
- the language underlying higher-order polymorphism in OCaml

- the elaboration language for modules
- the core calculus for GADTs

Roadmap



Inference rules

$$\cfrac{\begin{array}{c}\text{premise 1}\\ \text{premise 2}\\ \cdots\\ \text{premise N}\end{array}}{\text{conclusion}}\ \text{rule name}$$

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$$\frac{\begin{array}{c}\text{all } M \text{ are } P\\ \text{all } S \text{ are } M\end{array}}{\text{all } S \text{ are } P}\ \text{modus barbara}$$

Inference rules

$$\frac{\begin{array}{c}\text{premise 1}\\ \text{premise 2}\\ \ldots\\ \text{premise N}\end{array}}{\text{conclusion}}\ \text{rule name}$$

$$\frac{\begin{array}{c}\text{all } M \text{ are } P\\ \text{all } S \text{ are } M\end{array}}{\text{all } S \text{ are } P}\ \text{modus barbara}$$

$$\frac{\begin{array}{c}\text{all programs are buggy}\\ \text{all functional programs are programs}\end{array}}{\text{all functional programs are buggy}}\ \text{modus barbara}$$

Typing rules

$$\frac{\Gamma \vdash M : A \to B \qquad \Gamma \vdash N : A}{\Gamma \vdash M\ N : B} \to\text{-elim}$$

Kinds: $K, K_1, K_2, \ldots$

K is a kind

Environments: Γ

Γ is an environment

Types: $A, B, C, \ldots$

$\Gamma \vdash A :: K$

Terms: $L, M, N, \ldots$

$\Gamma \vdash M : A$

$\lambda^{\rightarrow}$

(simply typed lambda calculus)

$\lambda^{\rightarrow}$ by example

In $\lambda^{\rightarrow}$:	**In OCaml:**
λx:A.x	`fun x -> x`
λf:B $\rightarrow$ C. λg:A $\rightarrow$ B. λx:A.f (g x)	`fun f g x -> f (g x)`

Kinds in $\lambda^{\rightarrow}$

$$\frac{}{\ast \text{ is a kind}}\ \ast\text{-kind}$$

Kinding rules (type formation) in $\lambda^{\rightarrow}$

$$\frac{}{\Gamma \vdash \mathcal{B} :: *} \text{kind-}\mathcal{B}$$

$$\frac{\Gamma \vdash A :: * \qquad \Gamma \vdash B :: *}{\Gamma \vdash A \to B :: *} \text{kind-}\to$$

A kinding derivation

$$
\dfrac{
  \dfrac{
    \dfrac{}{\Gamma \vdash \mathcal{B} :: *}\ \textsf{kind-}\mathcal{B}
    \qquad
    \dfrac{}{\Gamma \vdash \mathcal{B} :: *}\ \textsf{kind-}\mathcal{B}
  }{\Gamma \vdash \mathcal{B} \to \mathcal{B} :: *}\ \textsf{kind-}\to
  \qquad
  \dfrac{}{\Gamma \vdash \mathcal{B} :: *}\ \textsf{kind-}\mathcal{B}
}{\Gamma \vdash (\mathcal{B} \to \mathcal{B}) \to \mathcal{B} :: *}\ \textsf{kind-}\to
$$

Environment formation rules

$$\frac{}{\cdot \text{ is an environment}}\ \Gamma\text{-}\cdot$$

$$\frac{\Gamma \text{ is an environment} \qquad \Gamma \vDash A :: *}{\Gamma, x{:}A \text{ is an environment}}\ \Gamma\text{-}:$$

Typing rules (term formation) in $\lambda^{\rightarrow}$

$$\frac{x : A \in \Gamma}{\Gamma \vdash x : A}\ \text{tvar}$$

$$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x{:}A.M : A \rightarrow B}\ \rightarrow\text{-intro} \qquad \frac{\Gamma \vdash M : A \rightarrow B \quad \Gamma \vdash N : A}{\Gamma \vdash M\ N : B}\ \rightarrow\text{-elim}$$

A typing derivation for the identity function

$$\dfrac{\cdot, x : A \vdash x : A}{\cdot \vdash \lambda x{:}A.x : A \to A} \to\text{-intro}$$

Products by example

In $\lambda^{\rightarrow}$ with products:

λp:(A $\rightarrow$ B) $\times$ A.
  fst p (snd p)

λx:A.$\langle$x, x$\rangle$

λf:A $\rightarrow$ C.
  λg.B $\rightarrow$ C.
    λp.A $\times$ B.
      $\langle$f (fst p),
       g (snd p)$\rangle$

λp.A $\times$ B.$\langle$snd p, fst p$\rangle$

In OCaml:

```
fun (f,p) -> f p

fun x -> (x, x)

fun f g (x,y) -> (f x, g y)

fun (x,y) -> (y,x)
```

Kinding and typing rules for products

$$\frac{\Gamma \vdash A :: * \qquad \Gamma \vdash B :: *}{\Gamma \vdash A \times B :: *}\ \text{kind-}\times$$

$$\frac{\Gamma \vdash M : A \qquad \Gamma \vdash N : B}{\Gamma \vdash \langle M, N \rangle : A \times B}\ \times\text{-intro}$$

$$\frac{\Gamma \vdash M : A \times B}{\Gamma \vdash \mathsf{fst}\ M : A}\ \times\text{-elim-1}$$

$$\frac{\Gamma \vdash M : A \times B}{\Gamma \vdash \mathsf{snd}\ M : B}\ \times\text{-elim-2}$$

Sums by example

In $\lambda^{\rightarrow}$ with sums:

```
λf:A → C.
  λg:B → C.
    λs:A + B.
      case s of
        x.f x
      | y.g y
```

```
λs:A + B.
  case s of
    x.inr [B] x
  | y.inl [A] y
```

In OCaml:

```
fun f g s ->
  match s with
    Inl x -> f x
  | Inr y -> g y
```

```
function
  Inl x -> Inr x
| Inr y -> Inl y
```

Kinding and typing rules for sums

$$\frac{\Gamma \vdash A :: * \qquad \Gamma \vdash B :: *}{\Gamma \vdash A + B :: *}\ \text{kind-}+$$

$$\frac{\Gamma \vdash M : A}{\Gamma \vdash \mathsf{inl}\,[B]\,M : A + B}\ +\text{-intro-1}$$

$$\frac{\Gamma \vdash N : B}{\Gamma \vdash \mathsf{inr}\,[A]\,N : A + B}\ +\text{-intro-2}$$

$$\frac{\begin{array}{c}\Gamma \vdash L : A + B \\ \Gamma, x : A \vdash M : C \\ \Gamma, y : B \vdash N : C\end{array}}{\Gamma \vdash \mathsf{case}\ L\ \mathsf{of}\ x.M \mid y.N : C}\ +\text{-elim}$$

System F

(polymorphic lambda calculus)

$\Lambda\alpha{::}*.\lambda x{:}\alpha.x$

$\Lambda\alpha{::}*.$
$\quad\Lambda\beta{::}*.$
$\quad\quad\Lambda\gamma{::}*.$
$\quad\quad\quad\lambda f{:}\beta\to\gamma.$
$\quad\quad\quad\quad\lambda g{:}\alpha\to\beta.$
$\quad\quad\quad\quad\quad\lambda x{:}\alpha.f\ (g\ x)$

$\Lambda\alpha{::}*.\Lambda\beta{::}*.\lambda p{:}(\alpha\to\beta)\times\alpha.\mathtt{fst}\ p\ (\mathtt{snd}\ p)$

New kinding rules for System F

$$\frac{\Gamma, \alpha{::}K \vdash A :: *}{\Gamma \vdash \forall \alpha{::}K.A :: *}\ \text{kind-}\forall \qquad \frac{\alpha{::}K \in \Gamma}{\Gamma \vdash \alpha :: K}\ \text{tyvar}$$

New environment rule for System F

$$\frac{\Gamma \text{ is an environment} \qquad K \text{ is a kind}}{\Gamma, \alpha : K \text{ is an environment}}\ \Gamma\text{-}\cdot$$

New typing rules for System F

$$\frac{\Gamma, \alpha{::}K \vdash M : A}{\Gamma \vdash \Lambda\alpha{::}K.M : \forall\alpha{::}K.A}\ \forall\text{-intro}$$

$$\frac{\Gamma \vdash M : \forall\alpha{::}K.A \qquad \Gamma \vdash B :: K}{\Gamma \vdash M\ [B] : A[\alpha{::}{=}B]}\ \forall\text{-elim}$$

Existential types

What's the point of existentials?

- $\forall$ and $\exists$ in logic are closely connected to polymorphism and existentials in type theory

- As in logic, $\forall$ and $\exists$ for types are closely related to each other
- Module types can be viewed as a kind of existential type
- OCaml's variant types now support existential variables

Existential intuition

Existentials
correspond to
abstract types

Kinding rules for existentials

$$\frac{\Gamma, \alpha{::}K \vdash A :: *}{\Gamma \vdash \exists\alpha{::}K.A :: *}\ \text{kind-}\exists$$

Typing rules for existentials

$$\frac{\Gamma \vdash M : A[\alpha{::=}B] \qquad \Gamma \vdash \exists\alpha{::}K.A :: *}{\Gamma \vdash \textsf{pack}\ B, M\ \textsf{as}\ \exists\alpha{::}K.A : \exists\alpha{::}K.A}\ \exists\text{-intro}$$

$$\frac{\begin{array}{c}\Gamma \vdash M : \exists\alpha{::}K.A \\ \Gamma, \alpha :: K, x : A \vdash M' : B\end{array}}{\Gamma \vdash \textsf{open}\ M\ \textsf{as}\ \alpha, x\ \textsf{in}\ M' : B}\ \exists\text{-elim}$$