Problem 1. Losses occur with the following distribution.

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 - e^{-x/\theta} & \text{otherwise} \end{cases}$$

Suppose we have an ordinary deductible of $\$d$. Please find the following:

(1) The per-payment Distribution Function
(2) The per-loss Survival Function
(3) The Expected Cost per **loss**

Problem 2. Use the same loss distribution as the prvious problem, with $\theta = 40$ but that our policy has **no deductible**, a limit of \$200 and a coinsurance of 75%. We assume an annual inflation rate of 2% What is the variance of each loss? You may assume that $\Gamma(3; 200/1.02) = 1$.

Problem 3. Determine the first four probabilities for an ETNB distribution with $r = 2$ and $\beta = 1$. Do this both for the truncated version and for the modified version, with $p_0^M = 0.1$

Problem 4. Consider a Poisson random variable with parameters $\lambda = 2$. Determine the first four probabilities for this random variable. Then determine the corresponding probabilities for the zero-truncated and zero-modified (with $p_0^M = 0.2$) versions.

Standard Normal Distribution Table

$$\Phi(z) = \mathbb{P}(Z \leq z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx, \qquad \Phi(-z) = 1 - \Phi(z)$$

z	0	+0.01	+0.02	+0.03	+0.04	+0.05	+0.06	+0.07	+0.08	+0.09
0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

Solutions

This is an Exponential(θ) distribution

Problem 1.

(1) Per-Payment Distribution Function:

$$\begin{aligned}
F_{Y^p}(y) &= \frac{F_X(y+d) - F_X(d)}{1 - F_X(d)} \\
&= \frac{e^{-d/\theta} - e^{-(y+d)/\theta}}{e^{-d/\theta}} \\
&= \frac{e^{-d/\theta} - e^{-y/\theta}e^{-d/\theta}}{e^{-d/\theta}} \\
&= 1 - e^{-y/\theta}
\end{aligned}$$

(2) Per-Loss Survival Function:

$$\begin{aligned}
S_{Y^L}(y) &= S_X(y+d) \\
&= e^{-(y+d)/\theta}
\end{aligned}$$

(3) Expected Cost per Loss for ordinary deductible:

$$\begin{aligned}
\mathbb{E}(X) - \mathbb{E}(X \wedge d) &= \theta - \theta(1 - e^{-d/\theta}) \\
&= \theta e^{-d/\theta}
\end{aligned}$$

$\square$

Problem 2.

(1) Per-Payment **density** Function:

$$\begin{aligned}
f_{Y^P}(y) &= \frac{f_X(y)}{S_X(d)},\ y > d \\
&= \frac{e^{-y/\theta}e^{d/\theta}}{\theta} = \frac{e^{(d-y)/\theta}}{\theta}
\end{aligned}$$

(2) Per-Loss hazard rate Function:

$$h_{Y^L}(y) = \begin{cases} 0 & 0 < y < d \\ h_X(y) & y > d \end{cases}$$

$$(\text{assume } y > d) = \frac{e^{-y/\theta}e^{y/\theta}}{\theta} = 1$$

(3) Expected Cost per payment for franchise deductible:

$$\begin{aligned}\frac{\mathbb{E}(X)-\mathbb{E}(X\wedge d)}{1-F(d)}+d &= (\theta-\theta(1-e^{-d/\theta}))e^{d/\theta}+d \\ &= \theta(e^{-d/\theta}e^{d/\theta})+d=\theta+d\end{aligned}$$

□

Problem 3.

(1) The Loss Elimination Ratio:

$$\frac{\mathbb{E}(X\wedge d)}{\mathbb{E}(X)}=\frac{\theta(1-e^{-d/\theta})}{\theta}=1-e^{-d/\theta}$$

(2) The expected cost per loss is defined as:

$$\begin{aligned}(1+r)\left\{\mathbb{E}[X]-\mathbb{E}[X\wedge d/(1+r)]\right\} &= (1.02)\{\mathbb{E}[X]-\mathbb{E}[X\wedge d/1.02]\} \\ &= 1.02(\theta-\theta(1-e^{-(d/1.02\theta)})) \\ &= 1.02\theta e^{-(d/1.02\theta)}\end{aligned}$$

□

Problem 4.

Based on our loss distribution and policy details, we have the following per-loss random variable:

$$Y^L=\begin{cases}\alpha(1+r)X & 0\le X<\frac{u}{1.02}\\ \alpha u & X\ge\frac{u}{1.02}\end{cases}$$

We need the 1st and 2nd moments of our new variable Y^L. Note that because $d=0$, $\mathbb{E}[X\wedge d]=\mathbb{E}[X\wedge 0]=0$ because we always have $0\le X$. It follows that:

$$\begin{aligned}\mathbb{E}[Y^L] &= \alpha(1+r)\left(\mathbb{E}[X\wedge\frac{u}{1.02}]\right) \\ &= \alpha(1+r)\theta(1-e^{-(u/1.02\theta)})=30.373\end{aligned}$$

and (again, because $d=0$)

$$\begin{aligned}\mathbb{E}[{Y^L}^2] &= \alpha^2(1+r)^2\mathbb{E}[(X\wedge(u/1.02))^2] \\ &= \alpha^2(1+r)^2\theta^2\Gamma(3)\Gamma(3;u/(1.02\theta))+(u/1.02\theta)^2e^{-(u/1.02\theta)}=2032.625\end{aligned}$$

Combining these, we have:

$$\mathsf{Var}(Y^L)=2032.625-30.373^2=1110.132$$

□