Problem 1. For a continuous random variable X you are told that $P(X > 0) = 1$. Additionally, you are told that

$$e_X(d) = \frac{d + 100}{2}, \quad \text{for all} \quad d > 0.$$

Compute $P(X > 10)$.

Problem 2. Let $X \sim Gamma(\alpha = 1, \theta = 1)$. Find the following:

(1) $\mathsf{VaR}_{0.99}(X)$ (3 points)
(2) $e_X(d)$ for d an arbitrary value (3 points)
(3) $\mathsf{TVaR}_{0.99}(X)$. (4 points) [If you had trouble with either previous computation, leave $\mathsf{VaR}_{0.99}(X)$ or $e_X(d)$ as algebraic expressions in your answer for partial credit.]

Problem 3. The random variable X has parameters α, θ and support $0 < x < \infty$ with the cumulative distribution functino (CDF)

$$F_X(x) = e^{-(\theta/x)^\alpha}$$

What is the density function of X^{-1}?

Problem 4. The severities of individual claims have a Pareto distribution with parameters $\alpha = 3$ and $\theta = 6000$. Use the central limit theorem to approximate the probability that the sum of 100 independent claims will exceed 400,000.

Standard Normal Distribution Table

$$\Phi(z) = \mathbb{P}(Z \leq z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx, \qquad \Phi(-z) = 1 - \Phi(z)$$

z	0	+0.01	+0.02	+0.03	+0.04	+0.05	+0.06	+0.07	+0.08	+0.09
0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

Solutions

Problem 1. The question is solved by writing $e_X(d)$ out in terms of the survival function and then differentiating with respect to d to obtain an identity the survival function can be computed from. The general formula is offered in the text and we discussed it in class (if you use this formula directly, providing you have it correct, that is fine).

$$e_X(d) = \frac{1}{S(d)} \int_d^\infty S(x)\, dx.$$

$$\frac{d+100}{2} \cdot S(d) = \int_d^\infty S(x)\, dx.$$

Differentiating with respect to d yields

$$\frac{1}{2} \cdot S(d) + \left(\frac{d+100}{2}\right) S'(d) = -S(d)$$

$$\left(\frac{d+100}{2}\right) S'(d) = -\frac{3}{2} \cdot S(d)$$

$$\frac{-S'(d)}{S(d)} = \frac{3}{d+100}$$

$$-\left(\log S(d)\right)' = \frac{3}{d+100}$$

$$-\log S(d) + \log S(0) = \int_0^d \frac{3}{x+100}\, dx = 3 \log(x+100)\Big|_0^d = 3 \log\left(\frac{d+100}{100}\right)$$

We were told that $S(0) = P(X > 0) = 1$ and thus

$$S(d) = \left(\frac{100}{d+100}\right)^3$$

Therefore,

$$S(10) = \left(\frac{100}{110}\right)^3 = 0.7513$$

$$\text{Answer} = 0.7513.$$

□

Problem 2. *Gamma*$(1, 1)$ is an exponential distribution with parameter 1, so this problem is very straightforward.

(1) $\mathsf{VaR}_{0.99}(X) = -\ln(0.01)$

(2)

$$e_X(d) = \frac{\mathbb{E}[X] - \mathbb{E}[X \wedge d]}{S(d)} = \frac{1 - \left(1 - e^{-d}\right)}{e^{-d}} = 1$$

(3) We therefore have:

$$\mathsf{TVaR}_{0.99}(X) = -\ln(0.01) + 1$$

□

Problem 3.

$$F_X(x) = e^{-(\theta/x)^\alpha}$$

is the Distribution Function for the Fréchet/Inverse Weibull Distribution. Taking the Inverse of an Inverse gives us the original Weibull Distribution (however, note that the parameter θ we give is actually the reciprocal of the parameter listed in the textbook), with the density:

$$f(y) = \frac{\alpha(y\theta)^\alpha e^{-(y\theta)^\alpha}}{y}$$

This can also be found with the CDF transformation method and differentiation as follows. Let Y equal the inverse Random Variable to X as defined by the given CDF. We assume the CDFs are only defined on strictly positive Real Numbers We therefore have:

$$\begin{aligned} F_Y(y) &= \mathbb{P}(Y < y) = \mathbb{P}\left(\frac{1}{X} < y\right) = \mathbb{P}\left(\frac{1}{y} < X\right) \\ &= 1 - e^{-(\theta y)^\alpha} \end{aligned}$$

Differentiating with respect to y gives us:

$$\begin{aligned} f_Y(y) = \frac{\partial}{\partial y} F_Y(y) = \frac{\partial}{\partial y} 1 - e^{-(\theta y)^\alpha} \\ &= -e^{-(\theta y)^\alpha}(-\alpha(\theta y)^{\alpha-1})\theta \\ &= \frac{\alpha(y\theta)^\alpha e^{-(y\theta)^\alpha}}{y} \end{aligned}$$

exactly the same as knowing the distributional relationship. □

Problem 4. As per the solution to Book exercise 3.22, a single claim has mean according to the formula:

$$\begin{aligned}\mathbb{E}(X^p) &= \frac{\theta^p p!}{(a-1)...(a-p)} \\ \mathbb{E}(X) &= \frac{6000^1 1!}{(3-1)} \\ &= \frac{6000}{2} = 3000\end{aligned}$$

And a single claim has variance:

$$\begin{aligned}\mathbb{E}(X^2) &= \frac{6000^2 2!}{(3-1)} \\ &= \frac{36,000,000 * 2}{2} = 36,000,000\end{aligned}$$

$$\begin{aligned}\mathsf{Var}(X) &= \mathbb{E}(X^2) - (\mathbb{E}(X))^2 \\ &= 36,000,000 - 3,000^2 = 27,000,000\end{aligned}$$

The sum of 100 claims has mean 300,000 and variance 2,700,000,000, which is a standard deviation of 51961.52. The probability of exceeding 400,000 is approximately

$$\begin{aligned}&= 1 - \Phi\left(\frac{400,000 - 300,000}{51961.52}\right) \\ &= 1 - \Phi(1.9245) = 1 - 0.97 = 0.03\end{aligned}$$

□