PSTAT 173 QUIZ 1
RISK THEORY
OCTOBER 4, 2022
HAL W. PEDERSEN

The quiz is released October 4th 11:30am Pacific. The submission window closes at 11:30am Pacific on October 5th. You have 24 hours to complete this quiz and successfully upload your solutions to GauchoSpace. Leave sufficient time to complete your upload. Provide sufficient reasoning to back up your answer but do not write more than necessary. Show all work, do not just write an answer. For questions that use a calculator, make sure to write down the formula you are using that is typed into the calculator. Please make your final results easy to read. Good luck!

Problem 1. You are given the following information on the survival of a population.[1]

time	survived
0	50
1	26
2	0

Assume that deaths/termination/mortality is uniformly distributed between dates.

(1) Determine the cumulative distribution, probability density, and hazard rate functions.

(2) A study suggests that a new hazard function $h^*(t)$ is effective to this population prior to time 1, such that

$$h^*(t) = \alpha + 0.1t, \quad \text{for} \quad t < 1.$$

What is the value of α?

Problem 2. The random variable X has parameters α, θ and support $0 < x < \infty$ with the cumulative distribution function

$$F_X(x) = e^{-(\theta/x)^\alpha}.$$

What is the density function of X^{-1}?

[1] In words, there are 50 people starting at time 0, at time 1 there are 26 remaining, and at time 2 no one remains.

Problem 3. The severity of individual claims has a Pareto distribution with parameters $\alpha = 3$ and $\theta = 6000$. Use the central limit theorem to approximate the probability that the sum of 200 independent claims will exceed 800,000.

Problem 4. X $\sim$ log-logistic($\gamma = 2$, $\theta = 8$). Determine the 75th percentile of X.

Problem 5. The following information is available regarding the random variables X and Y:

- X follows a Pareto distribution with $\alpha = 2$ and $\theta = 100$.
- $Y = ln(1 + X/\theta)$

Calculate the variance of Y.

Problem 6. You are given a distribution with the following density function,

$$f(x) = \theta(1-x)^{\theta-1}, \quad 0 \leq x \leq 1, \quad \theta > 0$$

Calculate the closed form expression for $e(d)$, the mean excess loss for deductible d, in terms of d and θ.

Standard Normal Distribution Table

$$\Phi(z) = \mathbb{P}(Z \leq z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx, \qquad \Phi(-z) = 1 - \Phi(z)$$

z	0	+0.01	+0.02	+0.03	+0.04	+0.05	+0.06	+0.07	+0.08	+0.09
0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

Solutions

Problem 1. (1)

$$\begin{aligned} F(t) &= 0.48t,\ 0 \le t < 1 \\ &= 0.52t - 0.04,\ 1 \le t < 2 \\ &= 1,\ 2 \le t \end{aligned}$$

Differentiating with respect to t yields:

$$\begin{aligned} f(t) &= 0.48,\ 0 \le t < 1 \\ &= 0.52,\ 1 \le t < 2 \\ &= 0,\ 2 \le t \end{aligned}$$

(2)

$$\begin{aligned} S(t) = 1 - F(t) &= 1 - 0.48t,\ 0 \le t < 1 \\ &= 1.04 - 0.52t,\ 1 \le t < 2 \\ &= 0,\ 2 \le t \end{aligned}$$

$$\begin{aligned} h(t) = \frac{f(t)}{S(t)} &= \frac{0.48}{1 - 0.48t},\ 0 \le t < 1 \\ &= \frac{0.52}{1.04 - 0.52t},\ 1 \le t < 2 \end{aligned}$$

(3)

$$\begin{aligned} S(t) &= e^{-\int_0^t h(x)\,dx} \\ S^*(1) &= e^{-\int_0^1 h^*(x)\,dx} \\ \frac{26}{50} &= e^{-\int_0^1 a+0.1x\,dx} \\ -\ln(0.52) &= \int_0^1 a + 0.1t\,dt \\ &= at + 0.05t^2|_0^1 \\ &= a + 0.05 \\ a &= -\ln(0.52) - 0.05 \approx 0.6039 \end{aligned}$$

$\square$

Problem 2.

$$F_X(x) = e^{-(\theta/x)^\alpha}$$

is the Distribution Function for the Fréchet/Inverse Weibull Distribution. Taking the Inverse of an Inverse gives us the original Weibull Distribution (however, note that the parameter θ we give is actually the reciprocal of the parameter listed in the textbook), with the density:

$$f(y) = \frac{\alpha(y\theta)^\alpha e^{-(y\theta)^\alpha}}{y}$$

This can also be found with the CDF transformation method and differentiation as follows. Let Y equal the inverse Random Variable to X as defined by the given CDF. We assume the CDFs are only defined on strictly positive Real Numbers We therefore have:

$$\begin{aligned} F_Y(y) &= \mathbb{P}(Y < y) = \mathbb{P}\left(\frac{1}{X} < y\right) = \mathbb{P}\left(\frac{1}{y} < X\right) \\ &= 1 - e^{-(\theta y)^\alpha} \end{aligned}$$

Differentiating with respect to y gives us:

$$\begin{aligned} f_Y(y) = \frac{\partial}{\partial y} F_Y(y) &= \frac{\partial}{\partial y} 1 - e^{-(\theta y)^\alpha} \\ &= -e^{-(\theta y)^\alpha}(-\alpha(\theta y)^{\alpha-1})\theta \\ &= \frac{\alpha(y\theta)^\alpha e^{-(y\theta)^\alpha}}{y} \end{aligned}$$

exactly the same as knowing the distributional relationship. □

Problem 3. As per the solution to Book exercise 3.22, a single claim has mean according to the formula:

$$\begin{aligned} \mathbb{E}(X^p) &= \frac{\theta^p p!}{(a-1)\ldots(a-p)} \\ \mathbb{E}(X) &= \frac{6000^1 1!}{(3-1)} \\ &= \frac{6000}{2} = 3000 \end{aligned}$$

And a single claim has variance:

$$\begin{aligned} \mathbb{E}(X^2) &= \frac{6000^2 2!}{(3-1)} \\ &= \frac{36,000,000 * 2}{2} = 36,000,000 \end{aligned}$$

$$\begin{aligned}\mathsf{Var}(X) &= \mathbb{E}(X^2) - (\mathbb{E}(X))^2 \\ &= 36,000,000 - 3,000^2 = 27,000,000\end{aligned}$$

The sum of 200 claims has mean 600,000 and variance 5,400,000,000, which is a standard deviation of 73484.69.

The probability of exceeding 800,000 is approximately

$$\begin{aligned}&= \Phi\left(\frac{600,000 - 800,000}{73484.69}\right) \\ &= \Phi(-2.721) \\ &= 1 - \Phi(2.721) \\ &= 1 - 0.9968 \text{ according to the table} \\ &= 0.0032\end{aligned}$$

$\square$

Problem 4. $X_{0.75}$, the 75th percentile of x, satisfies

$$\mathrm{F}(\mathrm{x}{=}X_{0.75}) = 0.75$$

Based on textbook Appendix A,

$$\begin{aligned}F(x) &= \frac{(x/\theta)^\gamma}{1+(x/\theta)^\gamma} \\ \frac{3}{4} &= \frac{(x/8)^2}{1+(x/8)^2} \\ 3(1+(x/8)^2) &= 4(x/8)^2 \\ 3 &= (x/8)^2 \\ x &= 8\sqrt{3} \approx 13.8564\end{aligned}$$

$\square$

Problem 5. We need to determine the distribution of $Y = \ln\left(1 + \frac{X}{\theta}\right)$ where we know the distribution of X. The most direct approach is to compute it using the CDF.

$$\mathbb{P}\left(Y \le y\right) = \mathbb{P}\left(\ln\left(1+\tfrac{X}{\theta}\right) \le y\right) = \mathbb{P}\left(1+\tfrac{X}{\theta} \le e^y\right) = \mathbb{P}\left(X \le (e^y - 1)\theta\right)$$

From Appendix A, page 494 of Klugman et al. we know that when $X \sim \mathsf{Pareto}(\alpha, \theta)$ we have

$$\mathbb{P}(X \le x) = 1 - \left(\frac{\theta}{x+\theta}\right)^\alpha$$

and therefore

$$(\star) \qquad \mathbb{P}\Big(X \leq (e^y - 1)\theta\Big) = 1 - \left(\frac{\theta}{(e^y-1)\theta + \theta}\right)^{\alpha} = 1 - e^{-y\alpha}.$$

In summary, we have shown that if $X \sim \mathsf{Pareto}(\alpha, \theta)$ and $Y = \ln\left(1 + \frac{X}{\theta}\right)$ then

$$\mathbb{P}\left(Y \leq y\right) = 1 - e^{-y\alpha}$$

Therefore $Y \sim \mathsf{Exponential}\left(1/\alpha\right)$.

In this problem we are told $X \sim \mathsf{Pareto}(2, 100)$ so we know that $Y \sim \mathsf{Exponential}\left(1/2\right)$.

Alternatively, we might have arrived at $(\star)$ and then noted that since $X \sim \mathsf{Pareto}(2, 100)$ we have

$$\mathbb{P}\Big(X \leq (e^y - 1)\theta\Big) = 1 - e^{-2y} = 1 - e^{-y/(1/2)}$$

and concluded that $Y \sim \mathsf{Exponential}\left(1/2\right)$.

For any random variable,

$$\mathsf{Var}(Y) = \mathbb{E}[Y^2] - \mathbb{E}[Y]^2.$$

When $Y \sim \mathsf{Exponential}\left(\beta\right)$ we know from Appendix A, page 499 of Klugman et al. that

$$\mathbb{E}[Y] = \beta \quad \text{and} \quad \mathbb{E}[Y^2] = 2\beta^2$$

Therefore, when $Y \sim \mathsf{Exponential}\left(1/2\right)$ we have

$$\mathsf{Var}(Y) = \mathbb{E}[Y^2] - \mathbb{E}[Y]^2 = 2\left(\tfrac{1}{2}\right)^2 - \left(\tfrac{1}{2}\right)^2 = \left(\tfrac{1}{2}\right)^2 = \tfrac{1}{4}.$$

$$\text{Answer} = \tfrac{1}{4}$$

We could also determine the distribution of Y by computing its moment generating function. We know from Appendix A, page 494 of Klugman et al. we know that when $X \sim \mathsf{Pareto}(\alpha, \theta)$ we have

$$f(x) = \frac{\alpha\theta^{\alpha}}{(x+\theta)^{\alpha+1}}, \quad x > 0$$

Since $Y = \ln\left(1 + \frac{X}{\theta}\right)$, the moment generating function for Y is

$$\mathbb{E}\left[e^{tY}\right] = \mathbb{E}\left[e^{t\ln\left(1+\frac{X}{\theta}\right)}\right] = \mathbb{E}\left[\left(1 + \tfrac{X}{\theta}\right)^t\right] = \int_0^{\infty} \left(1 + \tfrac{x}{\theta}\right)^t \frac{\alpha\theta^{\alpha}}{(x+\theta)^{\alpha+1}}\, dx$$

$$= \int_0^{\infty} \left(\tfrac{x+\theta}{\theta}\right)^t \frac{\alpha\theta^{\alpha}}{(x+\theta)^{\alpha+1}}\, dx = \int_0^{\infty} \alpha\theta^{\alpha-t} \frac{1}{(x+\theta)^{\alpha+1-t}}\, dx$$

$$= \int_0^{\infty} \alpha\theta^{\alpha-t}\left(\frac{d}{dx}\left[\frac{-1/(\alpha-t)}{(x+\theta)^{\alpha-t}}\right]\right) dx = \alpha\theta^{\alpha-t}\,\frac{1}{\alpha-t}\,\frac{1}{\theta^{\alpha-t}} = \frac{1}{1-\frac{t}{\alpha}}$$

Therefore, we have shown that $M_X(t) = \left(1 - t/\alpha\right)^{-1}$ which is the MGF for an Exponential $\left(1/\alpha\right)$ random variable as per Appendix A, page 499 of Klugman et al. We would now compute the variance in the same was as we have already demonstrated for an exponential random variable. □

Problem 6. We can solve this by a direct or brute force method or by using the more refined formulas we developed for computing $e(d)$. First, we apply the direct or brute force approach from the basic definition of $e(d)$.

$$\begin{aligned} e(d) = \mathbb{E}[X - d | X > d] &= \frac{\int_d^\infty (x-d) f(x)\, dx}{\mathbb{P}(X > d)} = \frac{\int_d^\infty x f(x)\, dx - d\int_d^\infty f(x)\, dx}{\mathbb{P}(X > d)} \\ &= \frac{\int_d^\infty x f(x)\, dx - d\,\mathbb{P}(X > d)}{\mathbb{P}(X > d)} = \frac{\int_d^\infty x f(x)\, dx}{\mathbb{P}(X > d)} - d \end{aligned}$$

Since the support for X is the interval $[0, 1]$ (*i.e.* the random variable X only takes on values between 0 and 1), we can change the upper limit of ∞ in this last integral to 1 giving us the formula

$$e(d) = \frac{\int_d^1 x f(x)\, dx}{\mathbb{P}(X > d)} - d.$$

Now use the expression we were given for $f(x)$ to compute the necessary terms in this formula.

$$\int_d^1 x f(x)\, dx = \int_d^1 x\theta(1-x)^{\theta-1}\, dx$$

Apply the substitution: $\quad y = 1 - x \iff x = 1 - y, \quad dy = -dx$

$$\begin{aligned} &= \int_{1-d}^0 (1-y)\theta y^{\theta-1}\,(-dy) = \int_0^{1-d} (1-y)\theta y^{\theta-1}\, dy \\ &= \int_0^{1-d} \theta y^{\theta-1}\, dy - \int_0^{1-d} \theta y^{\theta}\, dy = \int_0^{1-d} \left(y^\theta\right)'\, dy - \int_0^{1-d} \frac{\theta}{\theta+1}\left(y^{\theta+1}\right)'\, dy \\ &= (1-d)^\theta - \frac{\theta}{\theta+1}(1-d)^{\theta+1} = (1-d)^\theta\left[1 - \frac{\theta(1-d)}{\theta+1}\right] \\ &= (1-d)^\theta\left[\frac{\theta + 1 - \theta(1-d)}{\theta+1}\right] = (1-d)^\theta\left[\frac{1+\theta d}{\theta+1}\right] \end{aligned}$$

In summary

$$\int_d^1 x f(x)\, dx = (1-d)^\theta\left[\frac{1+\theta d}{\theta+1}\right]$$

$$\mathbb{P}(X > d) = \int_d^1 f(x)\, dx = \int_d^1 \theta(1-x)^{\theta-1}\, dx = \int_d^1 \left[-(1-x)^\theta\right]'\, dx = (1-d)^\theta$$

Therefore

$$e(d) = \frac{(1-d)^\theta \left[\frac{1+\theta d}{\theta+1}\right]}{(1-d)^\theta} - d = \frac{1+\theta d}{\theta+1} - d = \frac{1+\theta d - d(\theta+1)}{\theta+1} = \frac{1-d}{\theta+1}$$

$$\text{Answer} = \frac{1-d}{\theta+1}$$

A more efficient solution is to apply the result

$$e(d) = \frac{1}{S(d)} \int_0^\infty S(x)\,dx = \frac{1}{S(d)} \int_0^1 S(x)\,dx$$

where we can replace ∞ with 1 since the random variable can only take values between 0 and 1.

$$S(x) = \int_x^1 f(x)\,dx = \int_x^1 \theta(1-u)^{\theta-1}\,du = \int_x^1 \left[-(1-u)^\theta\right]'\,du = (1-x)^\theta$$

$$\int_0^1 S(x)\,dx = \int_0^1 (1-x)^\theta\,dx = \int_0^1 \left[\frac{-1}{\theta+1}(1-x)^{\theta+1}\right]'\,dx = \frac{(1-d)^{\theta+1}}{\theta+1}$$

Therefore

$$e(d) = \frac{1}{S(d)} \int_0^1 S(x)\,dx = \frac{1}{(1-d)^\theta}\,\frac{(1-d)^{\theta+1}}{\theta+1} = \frac{(1-d)}{\theta+1}$$

giving the same asnwer as the direct or brute force method. □