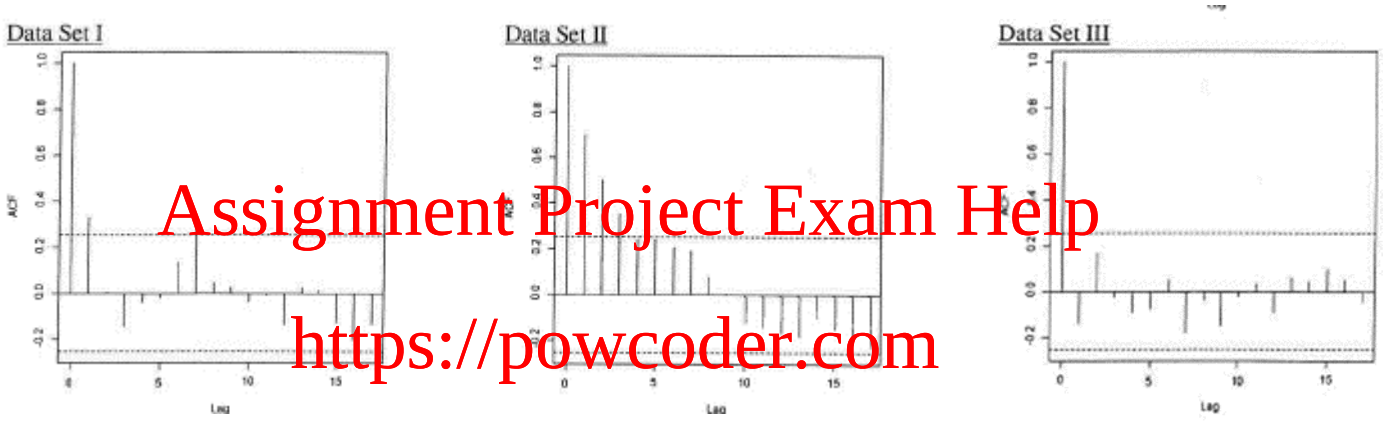
PSTAT 174/274, Fall 2022: Homework # 2.

Note: $\{Z_t\} \sim WN(0, \sigma_Z^2)$ denotes white noise.

1. Bellow, you are given the following graphs of autocorrelation functions for three separate data sets, each with n observations. The dotted lines in each graph correspond to 95% confidence intervals. Determine which of the above data sets exhibit statistically significant autocorrelations. Explain how you came to this conclusion.

A. I only; B. II only; C. III only; D. I, II and III; E. The answer is not given by (A), (B), (C), or (D).



2. For each of the two time series models, check stationarity and invertibility. Fully justify your answer.

(2.a) $X_t = Z_t - \frac{2}{3}Z_{t-1} - \frac{1}{3}Z_{t-2}$.
(2.b) $X_t = \frac{2}{3}X_{t-1} + \frac{1}{3}X_{t-2} + Z_t$.

3. (3.a) For a MA(3) process with coefficients $\theta_1 = 2, \theta_2 = 0.5$, and $\theta_3 = -0.1$, (i) write the mathematical equation for MA(3) model with these coefficients, and (ii) calculate the autocorrelation function at lags 1, 2, 3, 4: $\rho(1), \rho(2), \rho(3)$ and $\rho(4)$.

(3.b) For an AR(1) process with coefficient $\phi_1 = -0.5$, (i) write the mathematical equation for AR(1)) model with these coefficients, and (ii) calculate the autocorrelation function at lags 1, 2, 3, 4: $\rho(1), \rho(2), \rho(3)$ and $\rho(4)$.

4. You are given the following process: $X_t = 3 + Y + Z_t$, where Y is a mean zero random variable with variance σ_Y^2, independent of the white noise $\{Z_t\}$. Determine whether the process X is stationary and find its autocovariance and autocorrelation functions.

5. Let $X_t = Z_t + 2Z_{t-1} - 8Z_{t-2}$.
(i) Identify the model as the model as MA(q) or AR(p), specify q or p respectively.
(ii) Is the model stationary and invertible? Explain fully and show calculations where needed.
(*Hint: review 4 from homework 1!*)
(iii) Find $\rho_X(2)$. Use R to simulate 300 values of $\{X_t\}$ and use your simulated values to plot sample acf. Compare your sample estimate of $\rho_X(2)$ to its true value found by calculations. Redo this part using 10,000 simulated values of X_t.

The following problems are for students enrolled in PSTAT 274 ONLY

G1 Let $\{Z_t\} \sim WN(0,1)$ and $\{X_t\}$ be given by $X_t = Z_t + \theta Z_{t-2}$.
(a) Find the autocovariance and autocorrelation function for this process when $\theta = 0.8$.
(b) Compute the variance of the sample mean $(X_1 + X_2 + X_3 + X_4)/4$ when $\theta = 0.8$.
(c) Repeat (b) when $\theta = -0.8$ and compare your answer with the result obtained in (b).

G2 Provide at least two examples of AR(2) models with autocovariance functions exhibiting very different behavior pattern. Include plots of corresponding theoretical acfs and the corresponding R code.

G3 Let $X_t = Z_t + \theta Z_{t-1}$, $t = 1, 2, \ldots$, where $Z_t \sim IID(0, \sigma_Z^2)$. Show that X_t is both weakly and strictly stationary.
(*Hint: for the last part express the joint moment generating function* $E\exp(\sum_{i=1}^n \lambda_i X_i)$ *in terms of function* $m(\lambda) = E\exp(\lambda Z_i)$.)