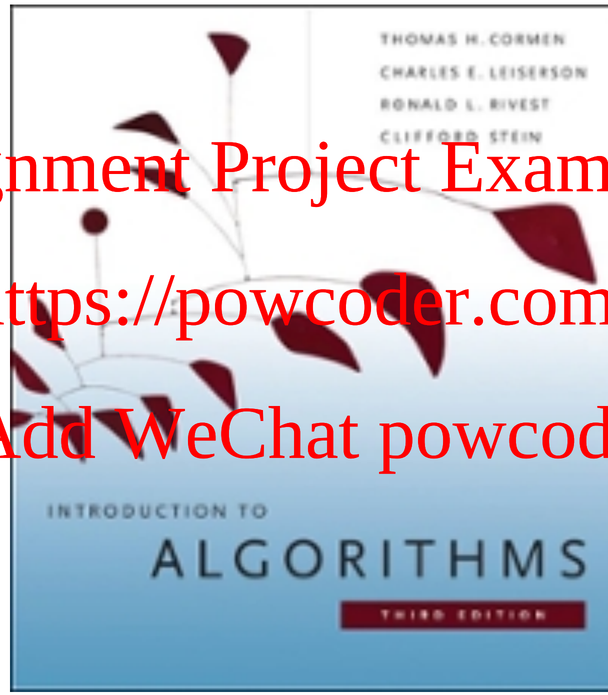
CS146 Data Structures and Algorithms



Chapter 22: Elementary Graph Algorithm

Graphs

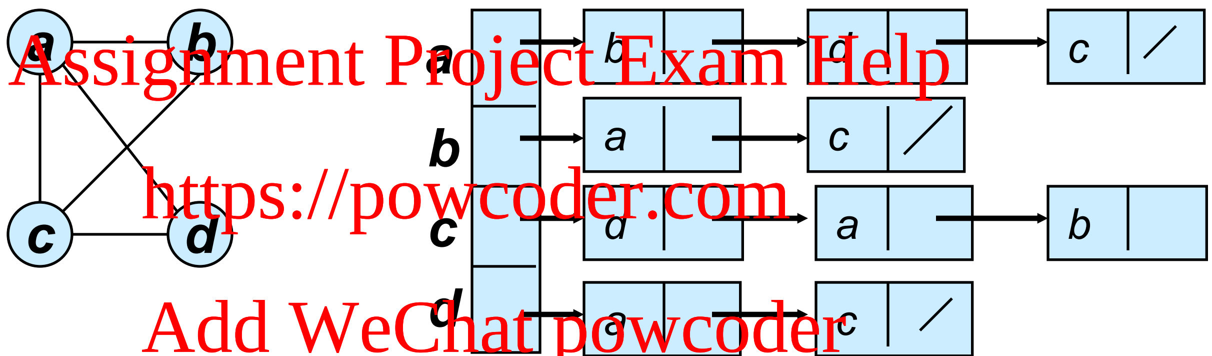
- *Graph G* = (*V*, *E*)
  - V = set of vertices
  - E = set of edges $\subseteq (V \times V)$
- Types of graphs
  - Undirected: edge $(u, v) = (v, u)$; for all v, $(v, v) \notin E$ (No self loops.)
  - Directed: (u, v) is edge from u to v, denoted as $u \rightarrow v$. Self loops are allowed.
  - Weighted: each edge has an associated weight, given by a weight function $w : E \rightarrow \mathbf{R}$.
  - Dense: $|E| \approx |V|^2$.
  - Sparse: $|E| << |V|^2$.
- $|E| = O(|V|^2)$

Graphs

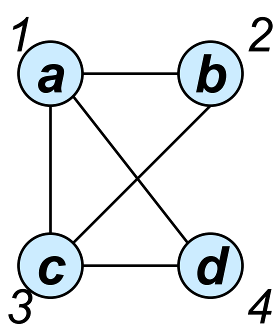
- If $(u, v) \in E$, then vertex v is adjacent to vertex u.
- Adjacency relationship is:
  - Symmetric if G is undirected.
  - Not necessarily so if G is directed.
- If G is connected:
  - There is a path between every pair of vertices.
  - $|E| \geq |V| - 1$.
  - Furthermore, if $|E| = |V| - 1$, then G is a tree.
- Other definitions in Appendix B (B.4 and B.5) as needed.

Representation of Graphs$_1$

- Two standard ways.
  - Adjacency Lists.



  - Adjacency Matrix.



	1	2	3	4
1	0	1	1	1
2	1	0	1	0
3	1	1	0	1
4	1	0	1	0

Representation of Graphs_2

- Undirected graph

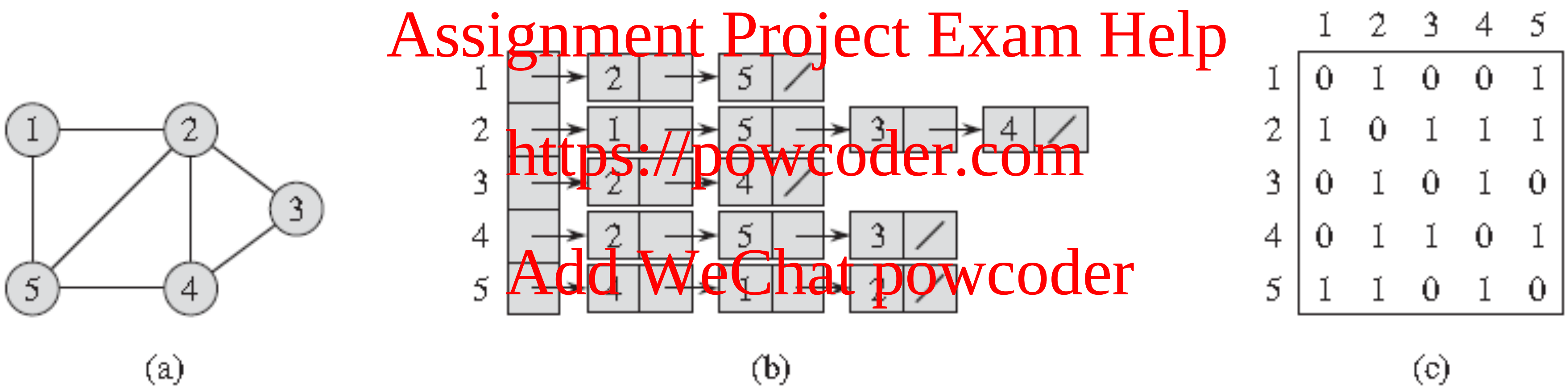
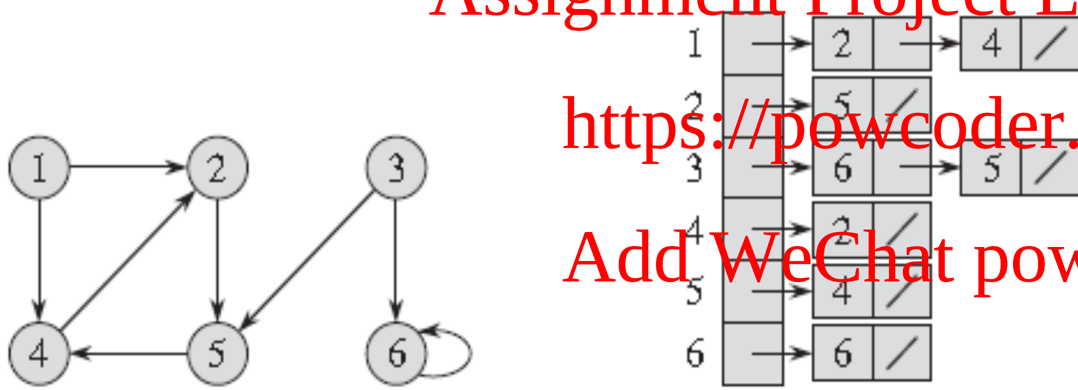


Figure 22.1 Two representations of an undirected graph. **(a)** An undirected graph G with 5 vertices and 7 edges. **(b)** An adjacency-list representation of G. **(c)** The adjacency-matrix representation of G.

Representation of Graphs$_3$

- Directed Graph



(a) (b)

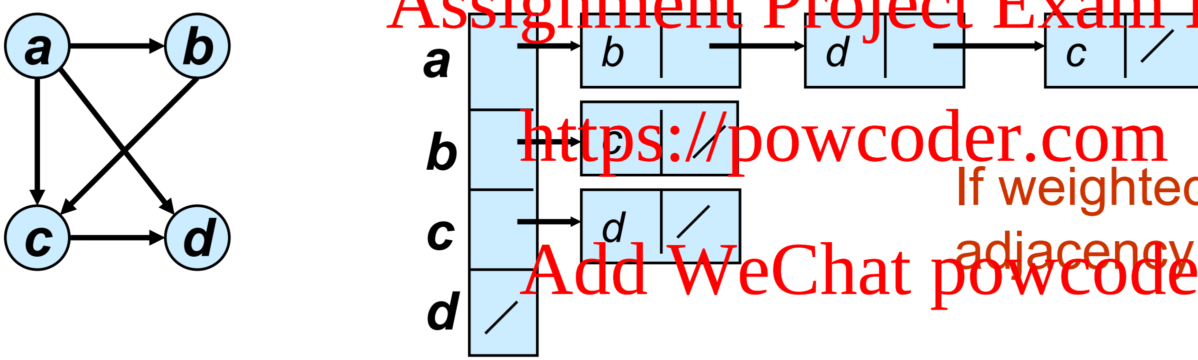
	1	2	3	4	5	6
1	0	1	0	1	0	0
2	0	0	0	0	1	0
3	0	0	0	0	1	1
4	0	1	0	0	0	0
5	0	0	0	1	0	0
6	0	0	0	0	0	1

(c)

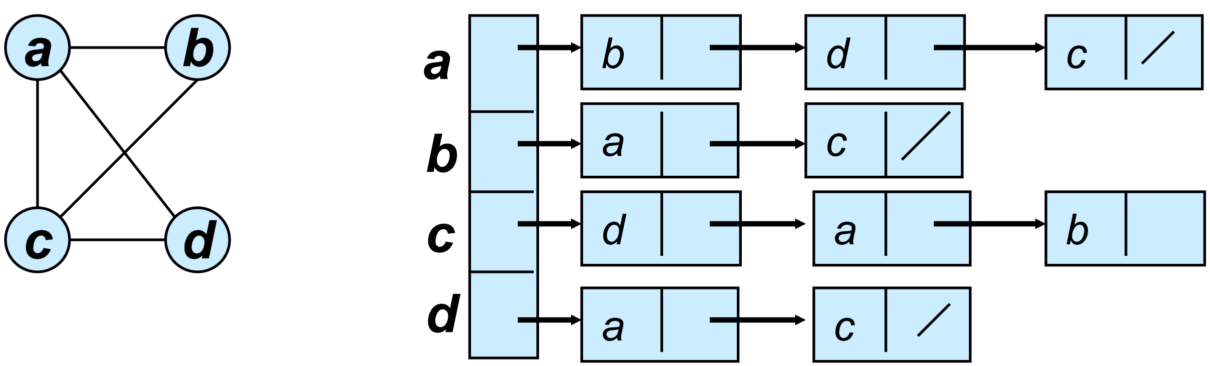
Figure 22.2 Two representations of a directed graph. **(a)** A directed graph G with 6 vertices and 8 edges. **(b)** An adjacency-list representation of G. **(c)** The adjacency-matrix representation of G.

Adjacency Lists

- Consists of an array *Adj* of $|V|$ lists.
- One list per vertex.
- For $u \in V$, $Adj[u]$ consists of all vertices adjacent to u.

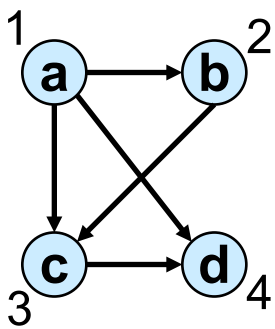


If weighted, store weights also in adjacency lists.

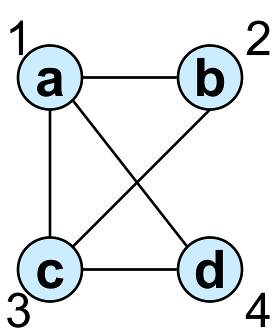


Adjacency Matrix

- $|V| \times |V|$ matrix A.
- Number vertices from 1 to $|V|$ in some arbitrary manner.
- A is then given by: $A[i, j] = a_{ij} = \begin{cases} 1 & \text{if } (i, j) \in E \\ 0 & \text{otherwise} \end{cases}$



	1	2	3	4
1	0	1	1	1
2	0	0	1	0
3	0	0	0	1
4	0	0	0	0



	1	2	3	4
1	0	1	1	1
2	1	0	1	0
3	1	1	0	1
4	1	0	1	0

$A = A^T$ for undirected graphs.

Graph-searching Algorithms

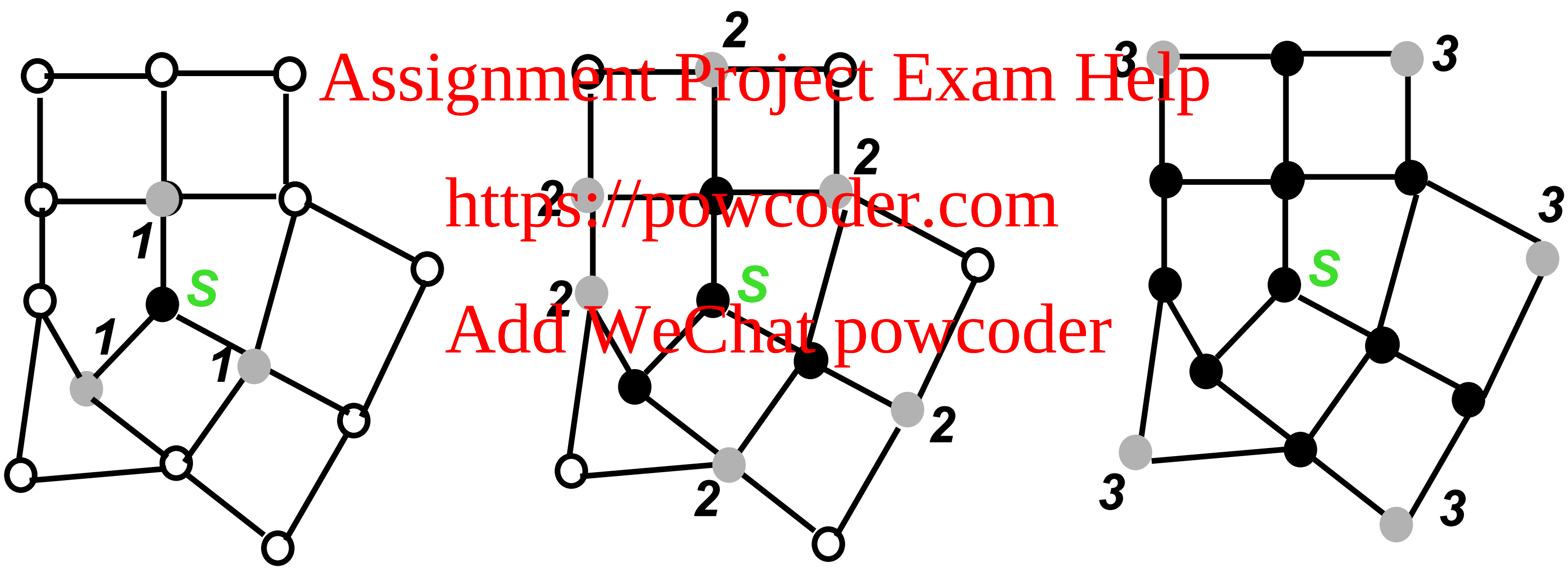
- Searching a graph:
  - Systematically follow the edges of a graph to visit the vertices of the graph.
- Used to discover the structure of a graph.
- Standard graph-searching algorithms.
  - Breadth-first Search (BFS).
  - Depth-first Search (DFS).

Breadth-first Search

- **Input:** Graph $G = (V, E)$, either directed or undirected, and ***source vertex*** $s \in V$.
- **Output:**
  - $d[v]$ = distance (smallest # of edges, or shortest path) from s to v, for all $v \in V$. $d[v] = \infty$ if v is not reachable from s.
  - $\pi[v] = u$ such that (u, v) is last edge on shortest path $s \rightsquigarrow v$.
    - u is v's predecessor.
  - Builds breadth-first tree with root s that contains all reachable vertices.

Definitions:
Path between vertices u and v: Sequence of vertices $(v_1, v_2, \ldots, v_k)$ such that $u=v_1$ and $v=v_k$, and $(v_i,v_{i+1}) \in E$, for all $1 \leq i \leq k-1$.
Length of the path: Number of edges in the path.
Path is simple if no vertex is repeated.

BFS for Shortest Paths



Finished ***Discovered*** ***Undiscovered***

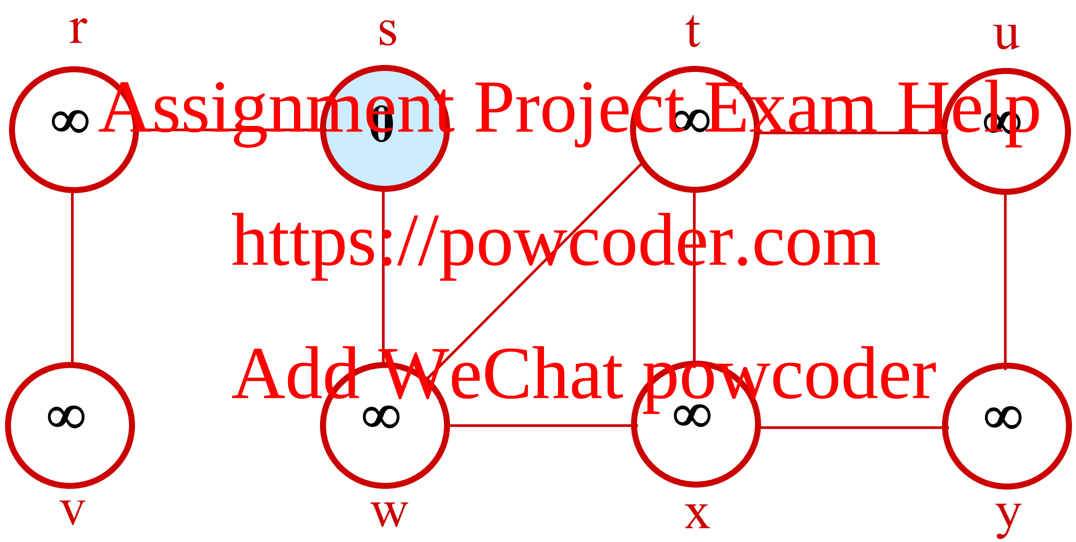
BFS(G,s)

```
for each vertex u in V[G] – {s}
    do color[u] ← white
       d[u] ← ∞
       π[u] ← nil
color[s] ← gray
d[s] ← 0
π[s] ← nil
Q ← Φ
enqueue(Q,s)
while Q ≠ Φ
    do u ← dequeue(Q)
        for each v in Adj[u]
            do if color[v] = white
                then color[v] ← gray
                     d[v] ← d[u] + 1
                     π[v] ← u
                     enqueue(Q,v)
        color[u] ← black
```

white: undiscovered
gray: discovered
black: finished

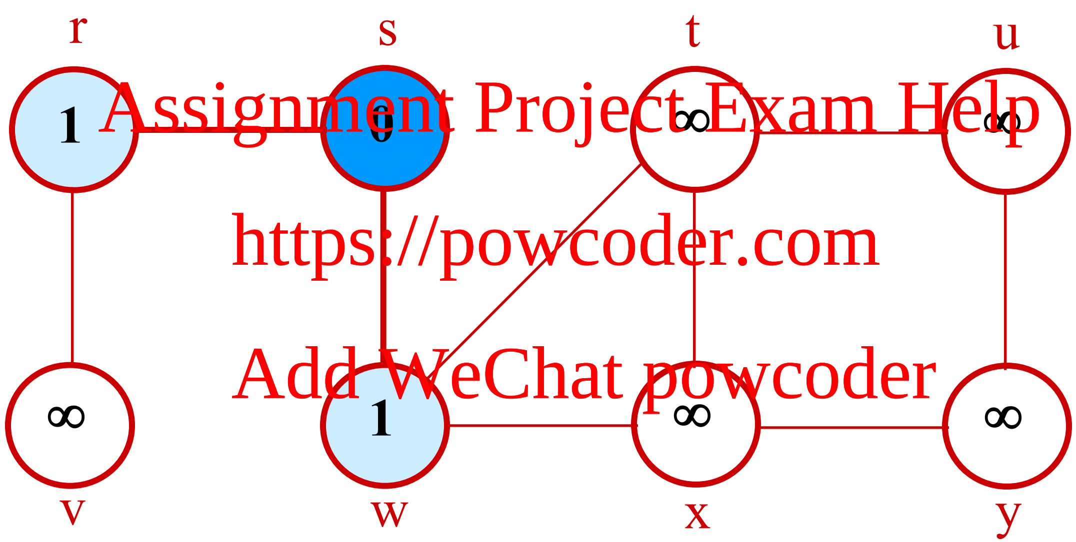
Q: a queue of discovered vertices
color[v]: color of v
d[v]: distance from s to v
π[u]: predecessor of v

Example (BFS)



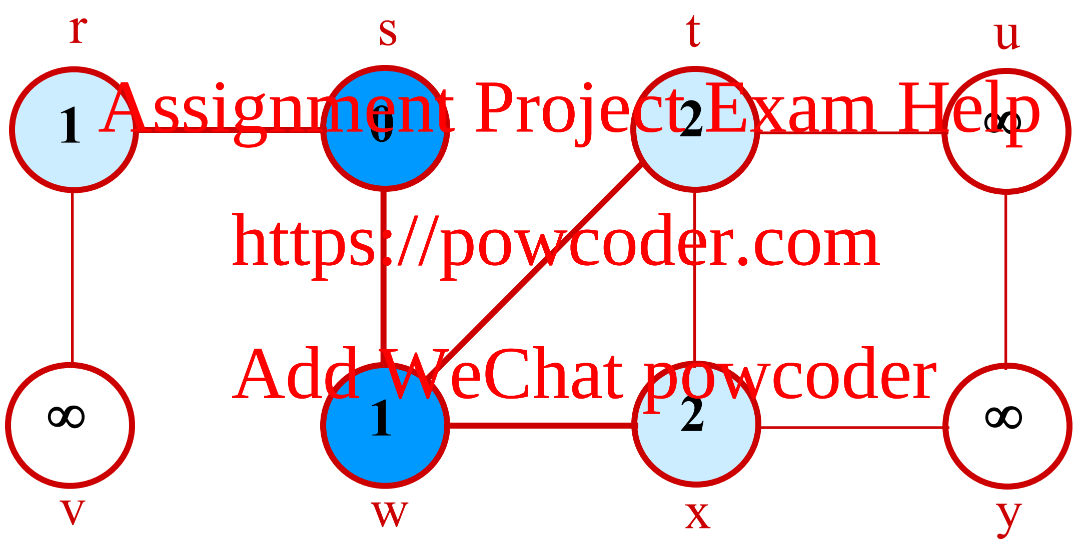
Q: s
0

Example (BFS)



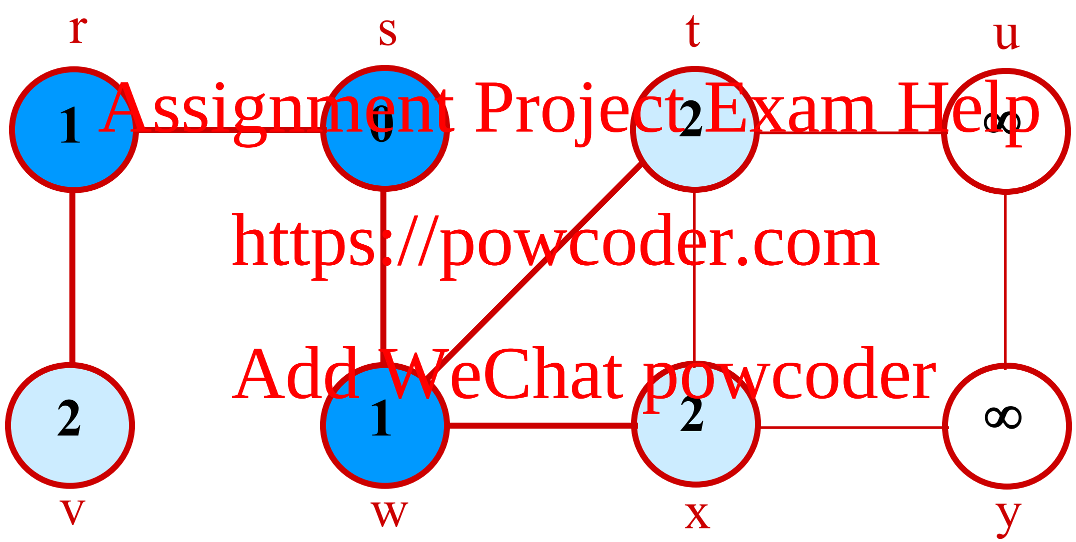
Q:	w	r
	1	1

Example (BFS)



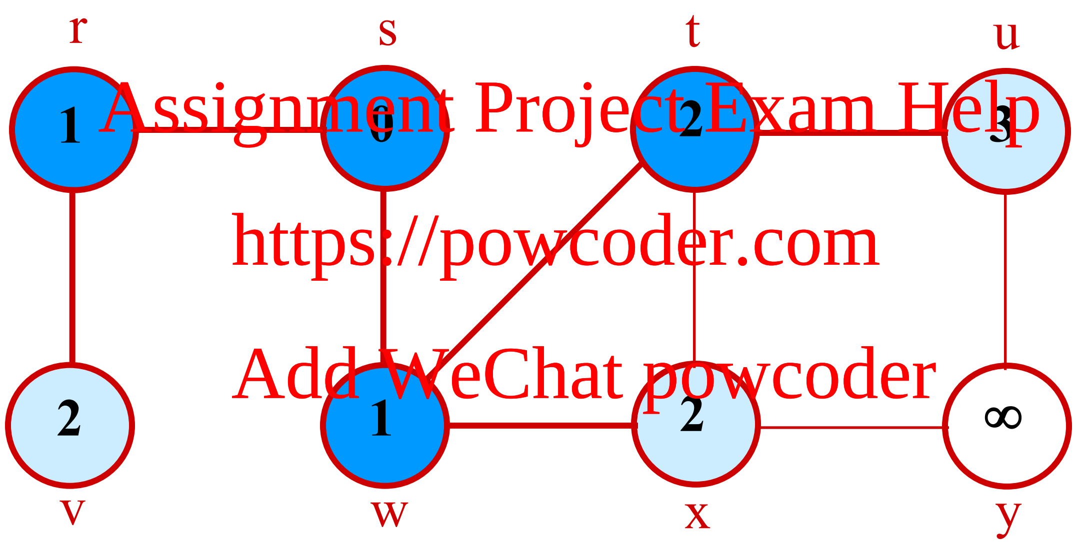
Q:	r	t	x
	1	2	2

Example (BFS)



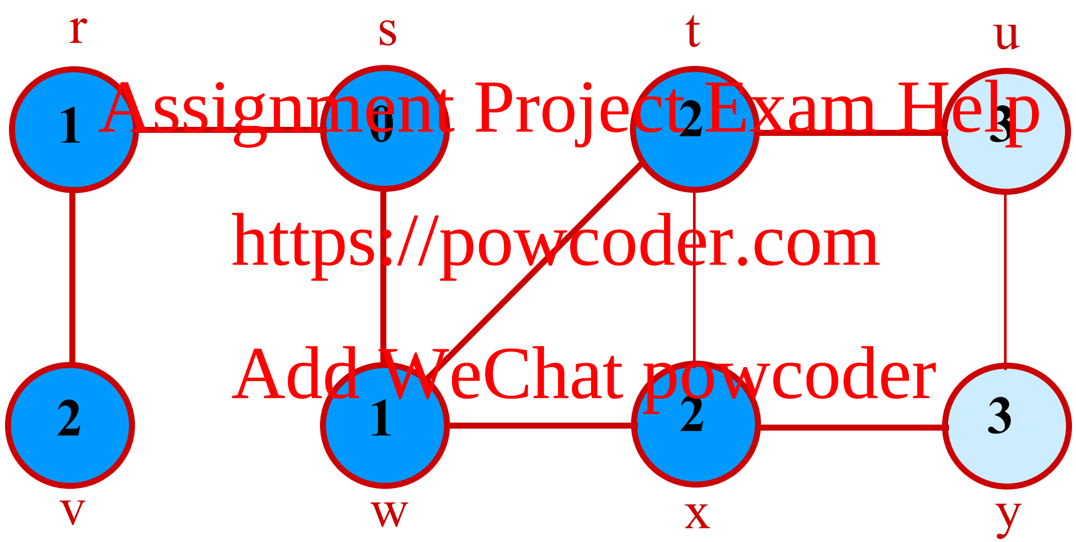
Q:	t	x	v
	2	2	2

Example (BFS)



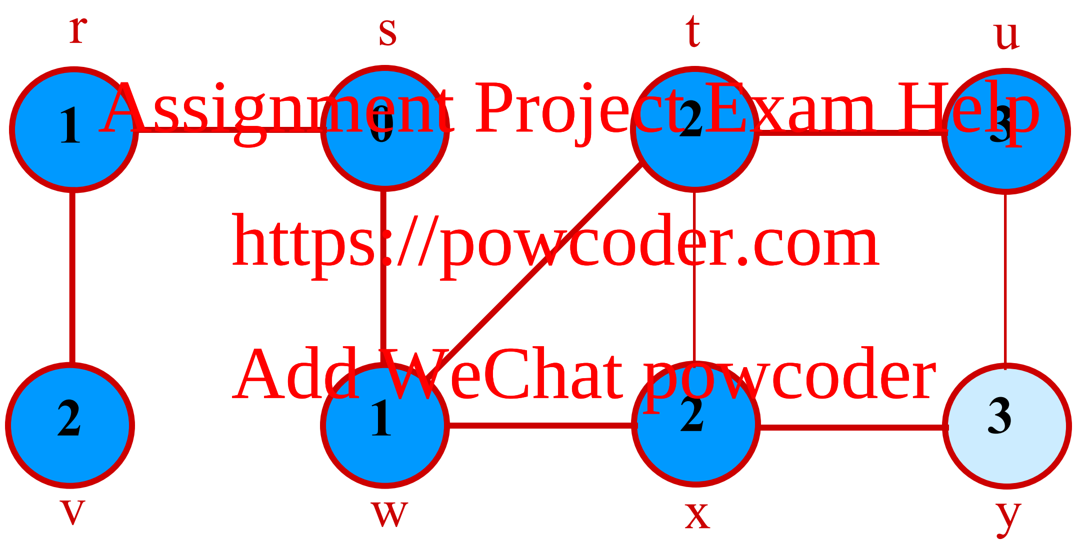
Q:	x	v	u
	2	2	3

Example (BFS)



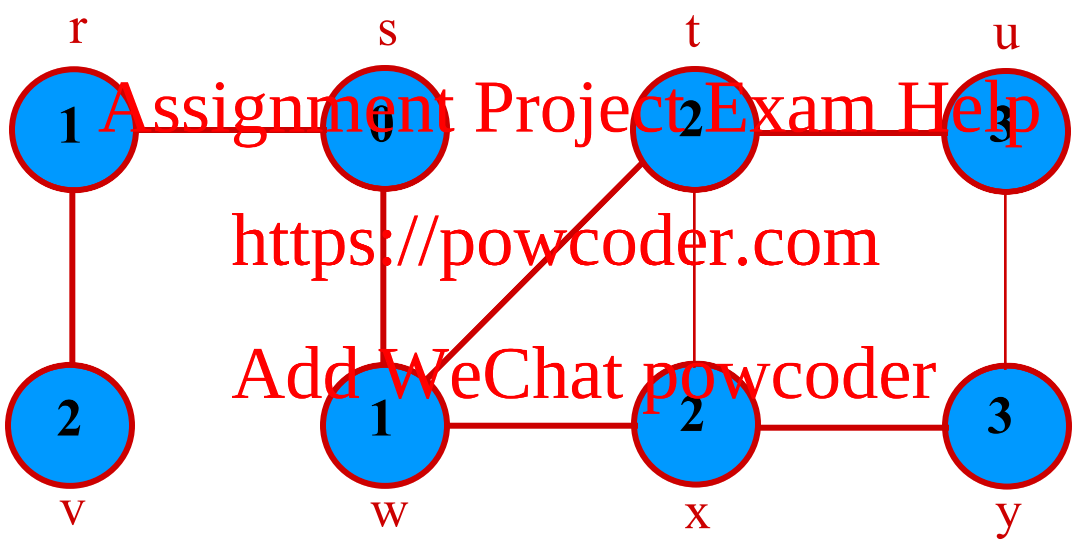
Q:	u	y
	3	3

Example (BFS)



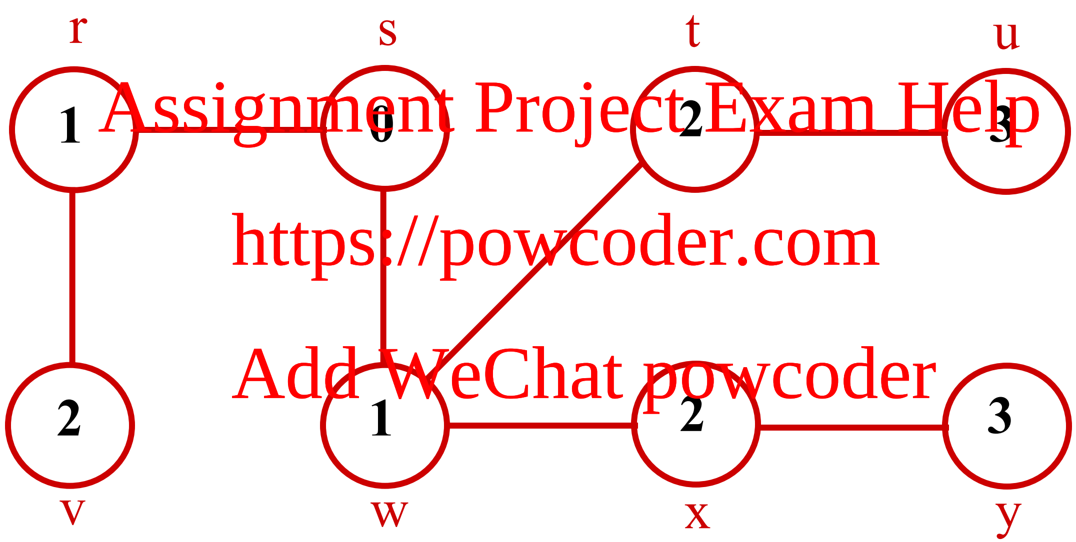
Q:	y
	3

Example (BFS)



Q: $\emptyset$

Example (BFS)



BF Tree

Analysis of BFS

- Initialization takes $O(V)$.
- Traversal Loop
  - After initialization, each vertex is enqueued and dequeued at most once, and each operation takes $O(1)$. So, total time for queuing is $O(V)$.
  - The adjacency list of each vertex is scanned at most once. The sum of lengths of all adjacency lists is $\Theta(E)$.
- Summing up over all vertices => total running time of BFS is $O(V+E)$, linear in the size of the adjacency list representation of graph.

Depth-First Search

- *Depth-first search* is another strategy for exploring a graph
  - Explore "deeper" in the graph whenever possible
  - Edges are explored out of the most recently discovered vertex v that still has unexplored edges
  - When all of v's edges have been explored, backtrack to the vertex from which v was discovered

Depth-first Search (DFS)

- Explore edges out of the most recently discovered vertex *v*.
- When all edges of *v* have been explored, backtrack to explore other edges leaving the vertex from which *v* was discovered (its *predecessor*).
- "Search as deep as possible first."
- Continue until all vertices reachable from the original source are discovered.
- If any undiscovered vertices remain, then one of them is chosen as a new source and search is repeated from that source.

Depth-first Search

- **Input:** $G = (V, E)$, directed or undirected. No source vertex given!
- **Output:**
  - 2 **timestamps** on each vertex. Integers between 1 and 2|V|.
    - o $d[v]$ = ***discovery time*** (v turns from white to gray)
    - o $f[v]$ = ***finishing time*** (v turns from gray to black)
  - $\pi[v]$: predecessor of $v = u$, such that v was discovered during the scan of u's adjacency list.
- Uses the same coloring scheme for vertices as BFS.

Pseudo-code

DFS(G)

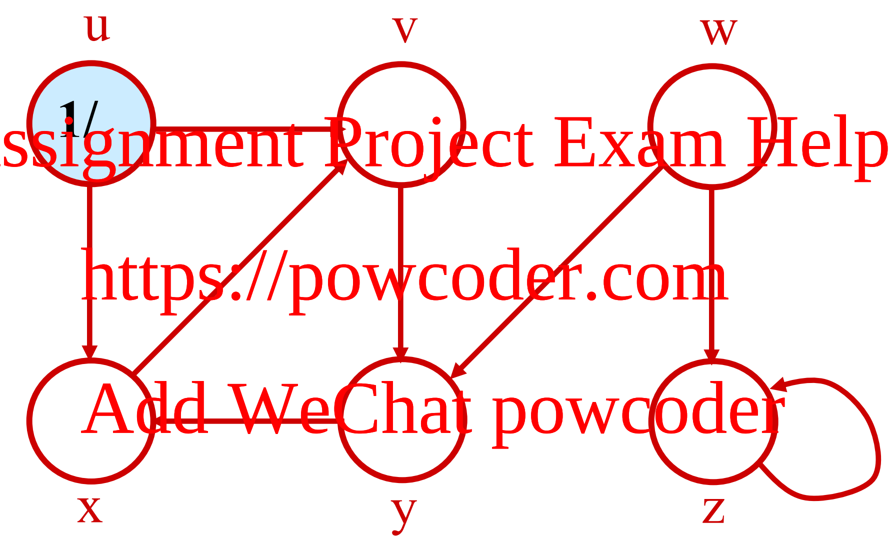
```
1. for each vertex u ∈ V[G]
2.      do color[u] ← white
3.           π[u] ← NIL
4. time ← 0
5. for each vertex u ∈ V[G]
6.      do if color[u] = white
7.           then DFS-Visit(u)
```

Uses a global timestamp **time**.

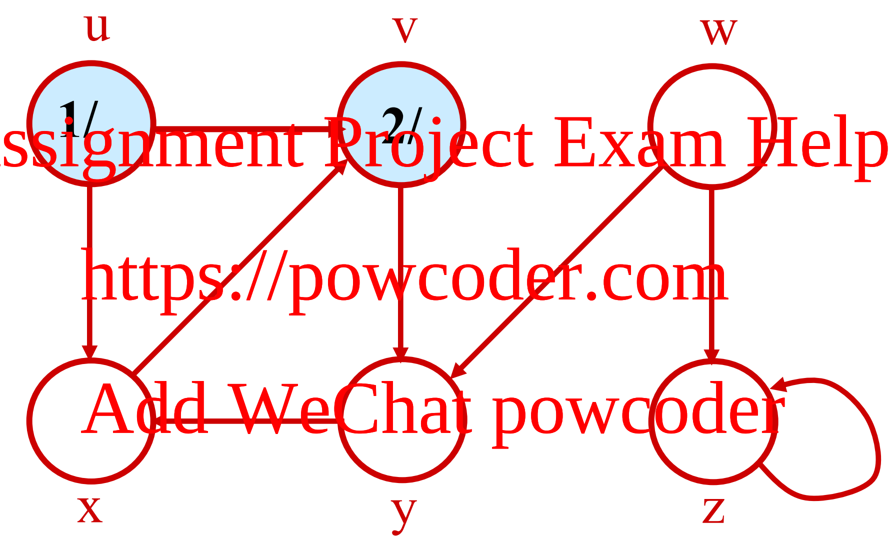
DFS-Visit(u)

```
1.   color[u] ← GRAY  ∇ White vertex u
     has been discovered
2.   time ← time + 1
3.   d[u] ← time
4.   for each v ∈ Adj[u]
5.        do if color[v] = WHITE
6.             then π[v] ← u
7.                  DFS-Visit(v)
8.   color[u] ← BLACK    ∇ Blacken u;
     it is finished.
9.   f[u] ← time ← time + 1
```

Example (DFS)



Example (DFS)



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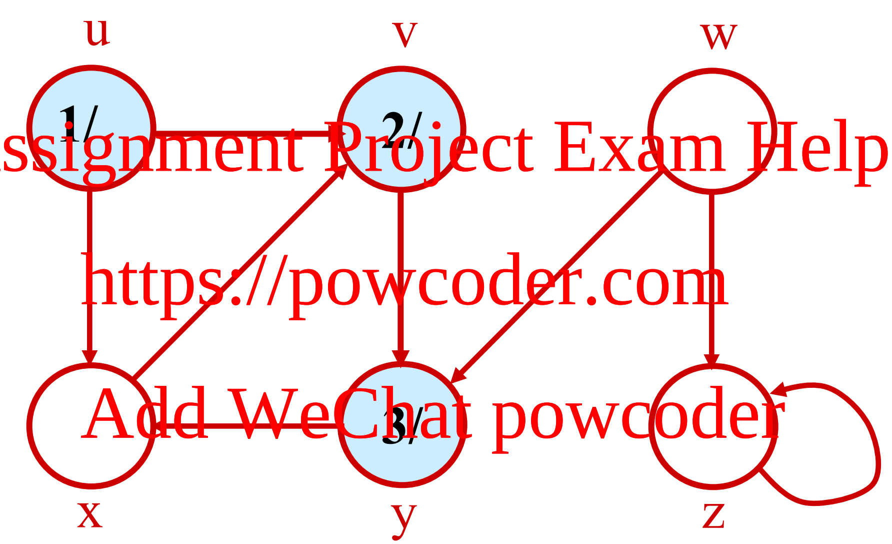

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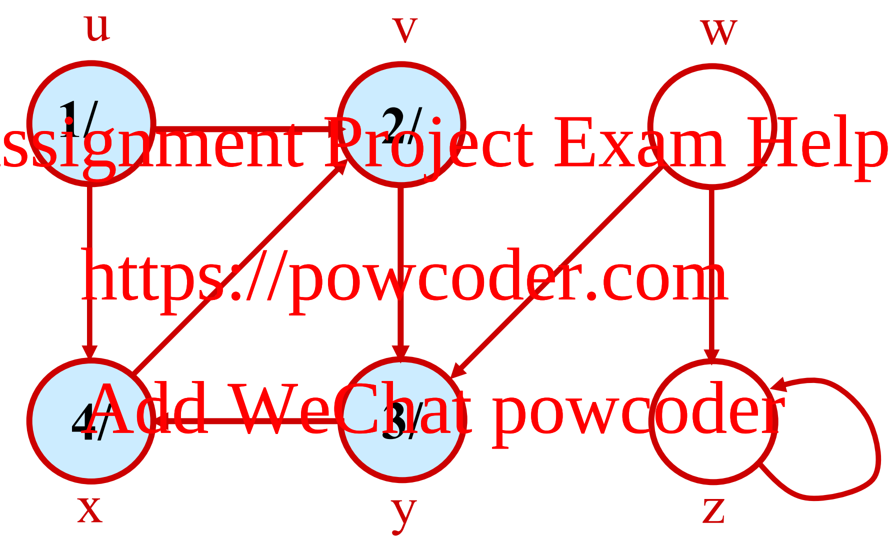
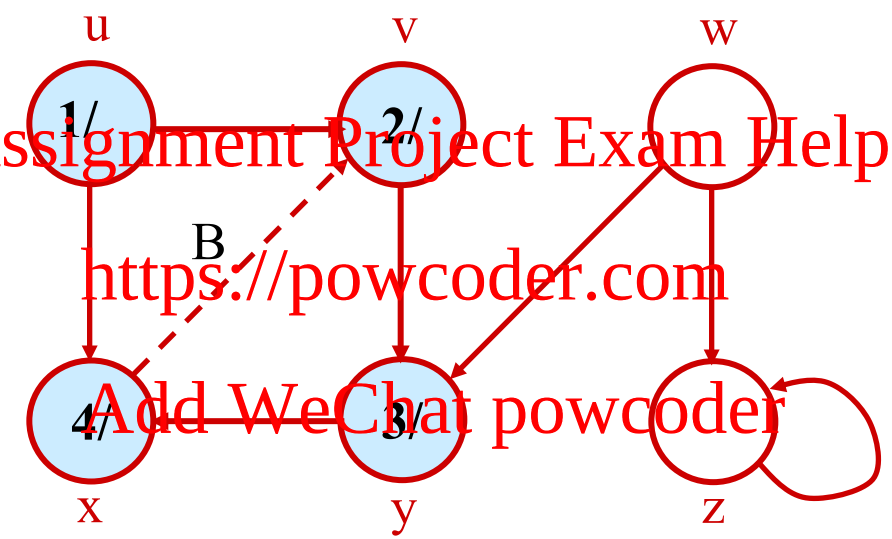
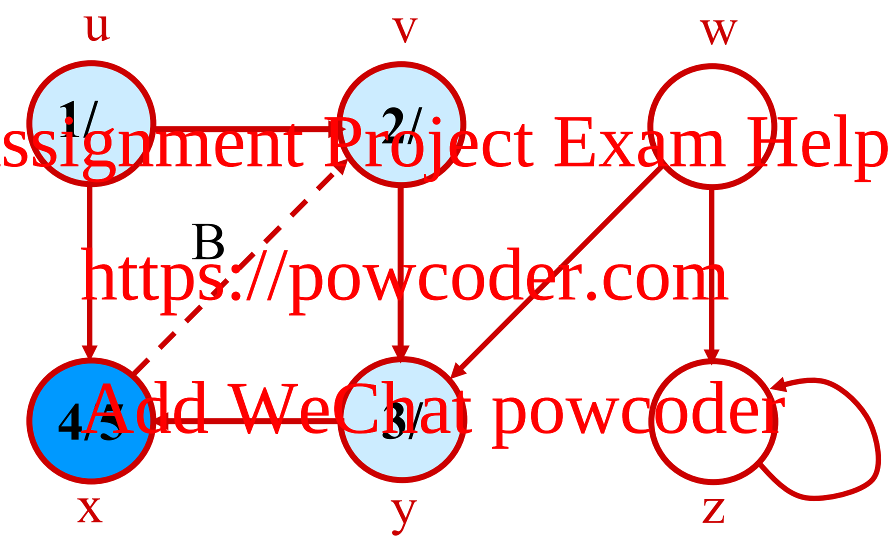


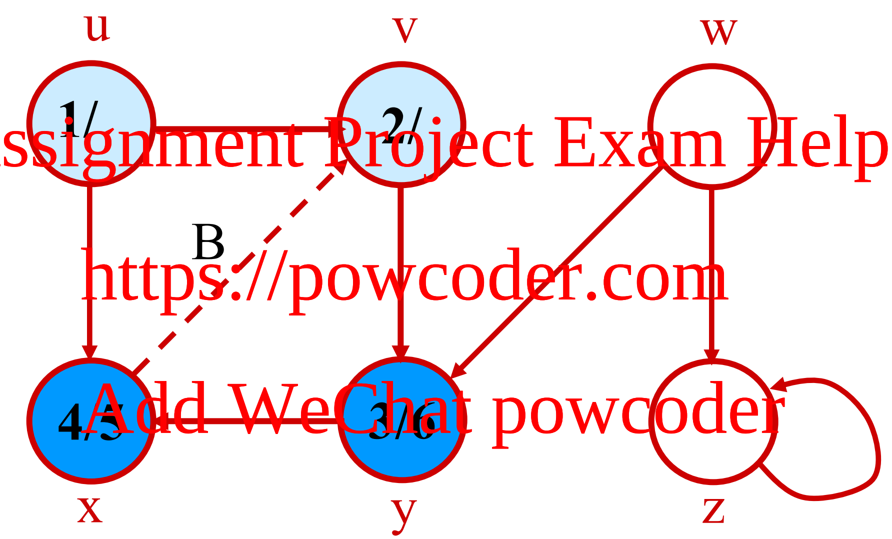
Example (DFS)



Example (DFS)



Example (DFS)



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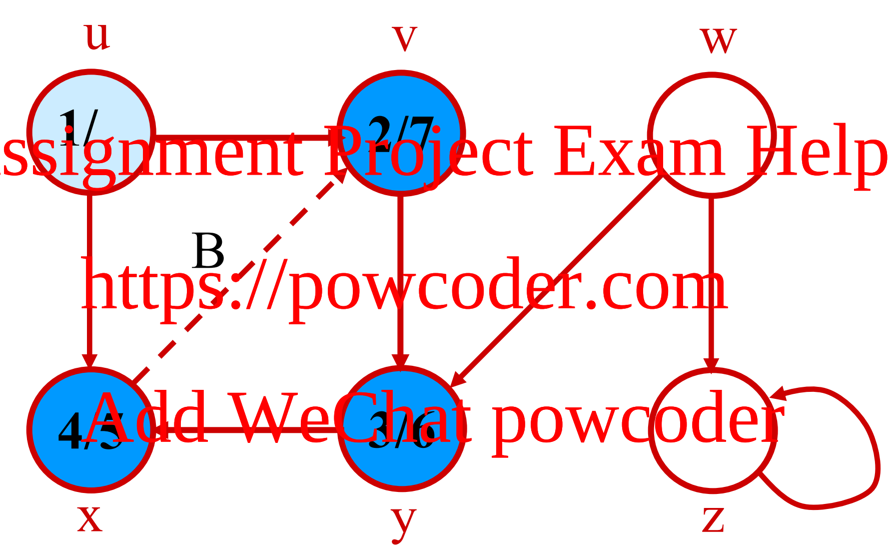

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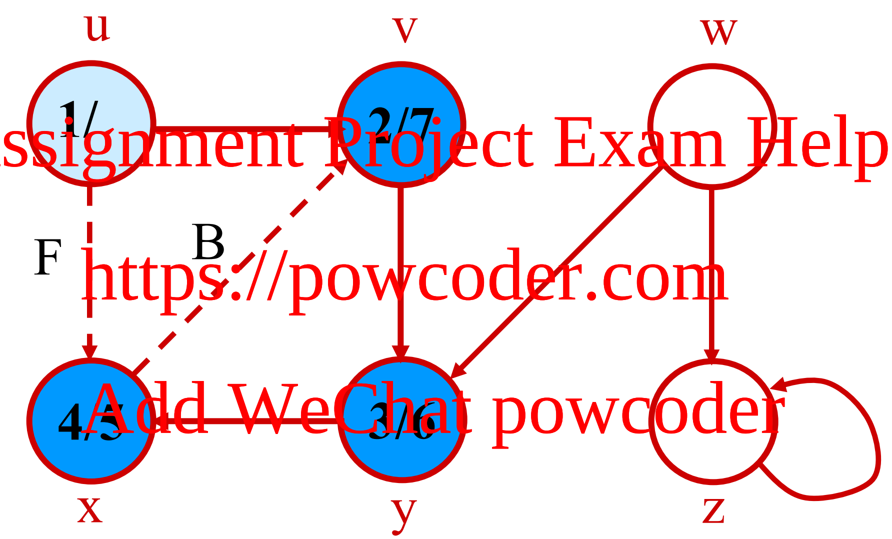

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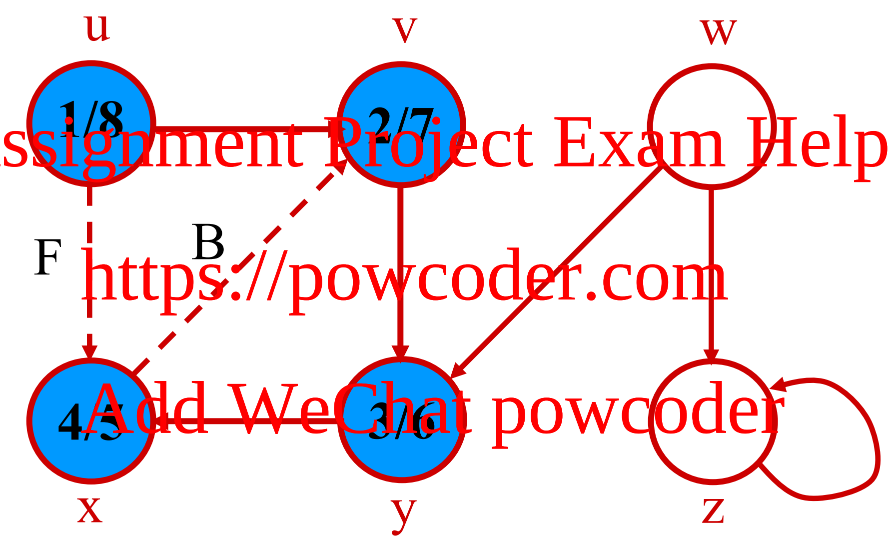

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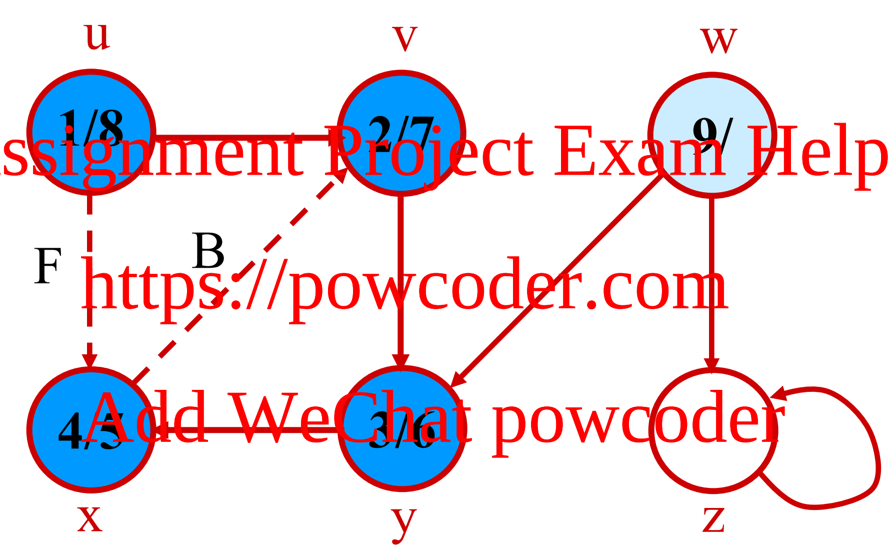

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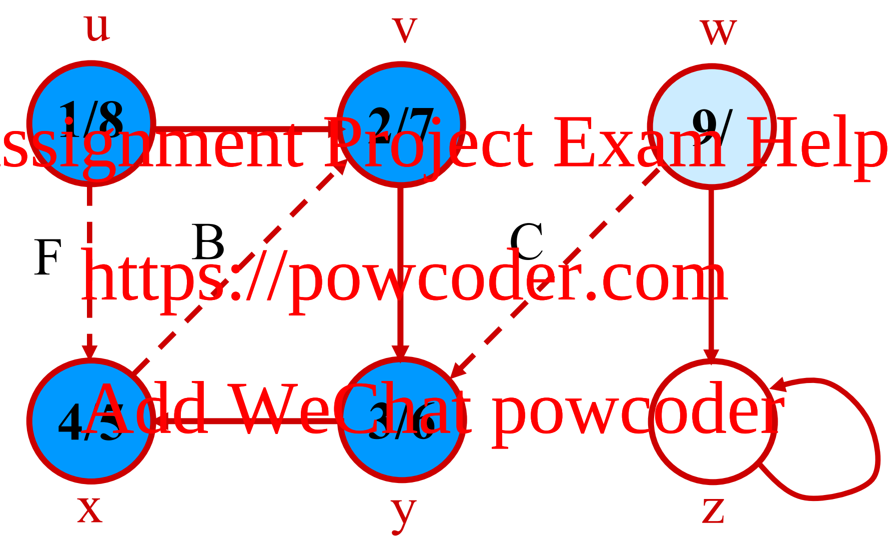

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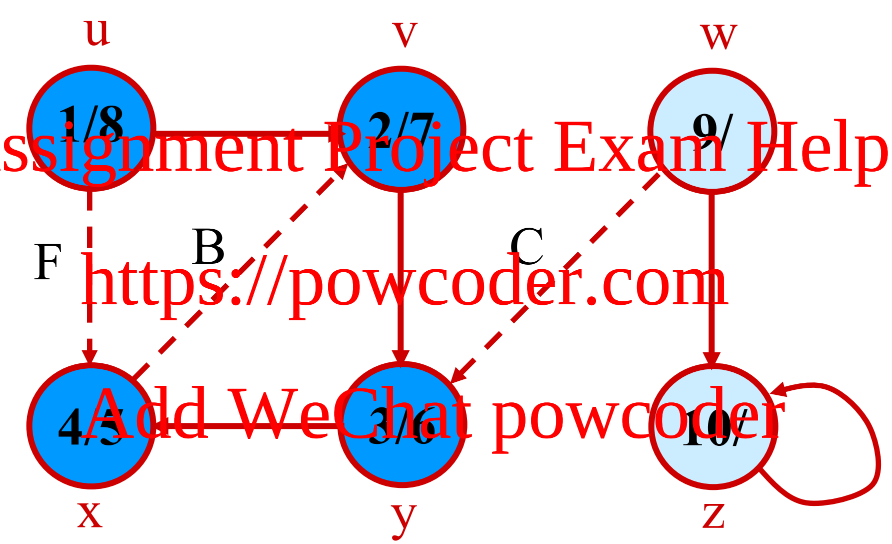

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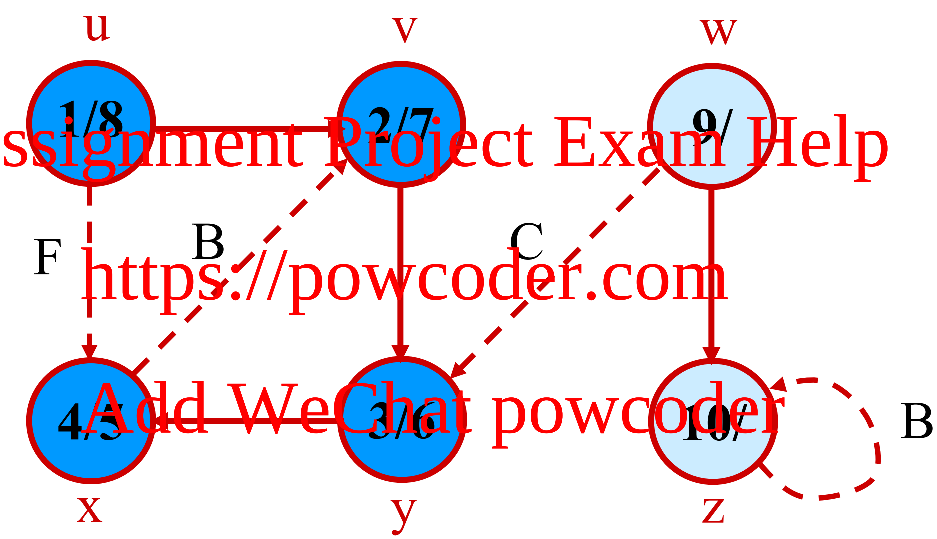
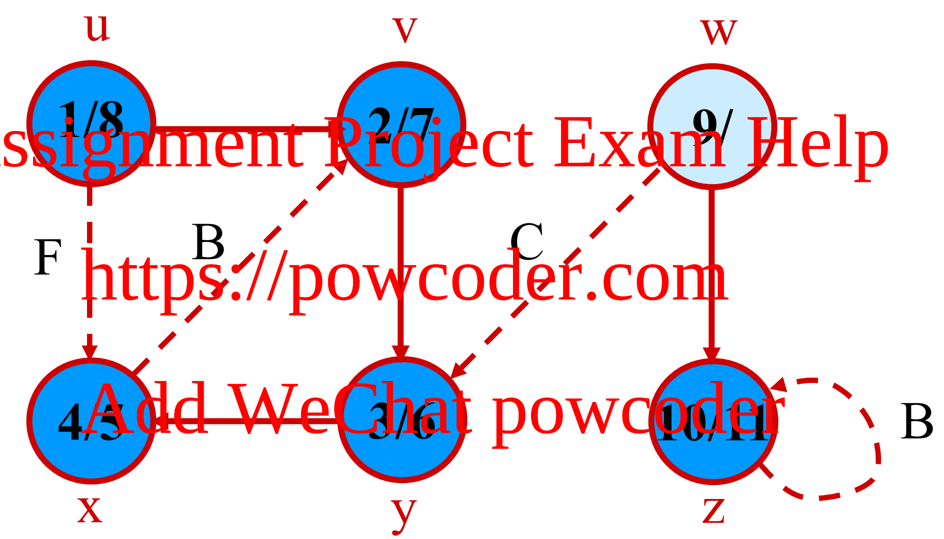


Example (DFS)



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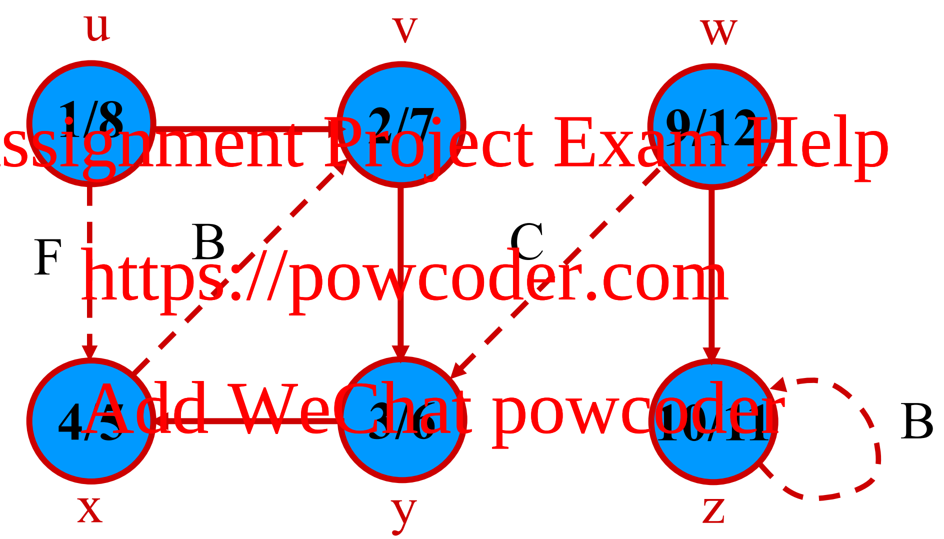
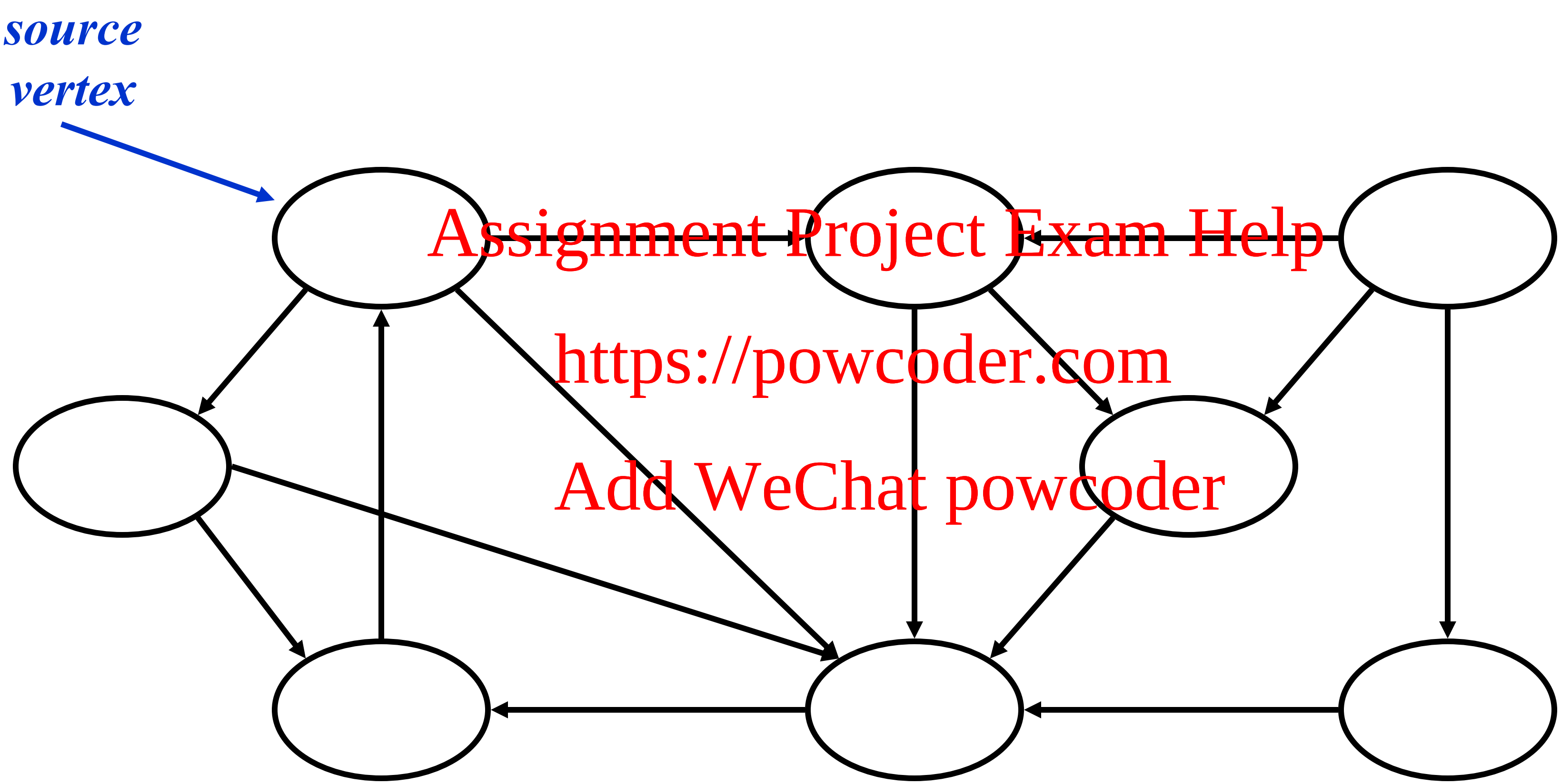
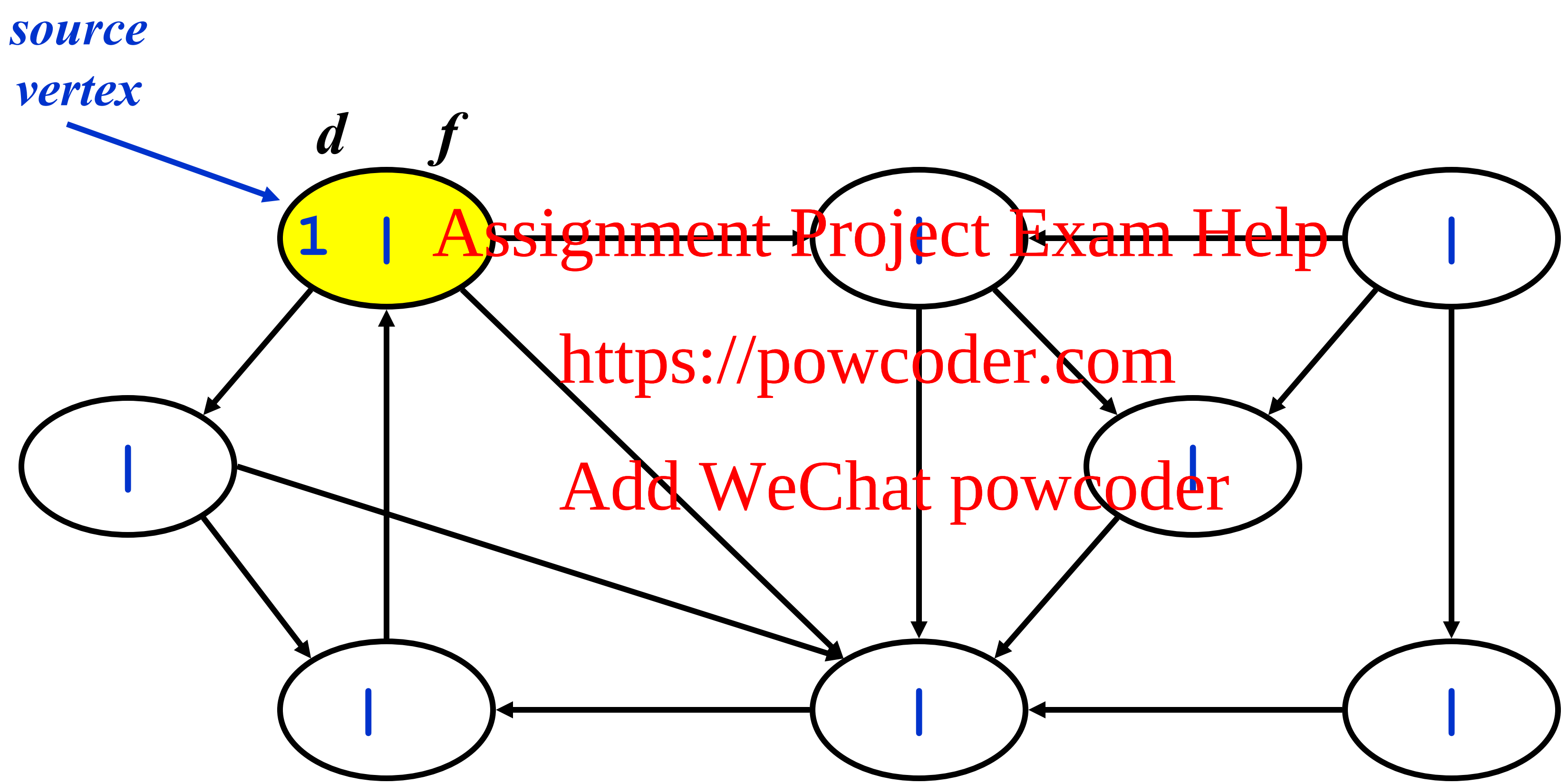


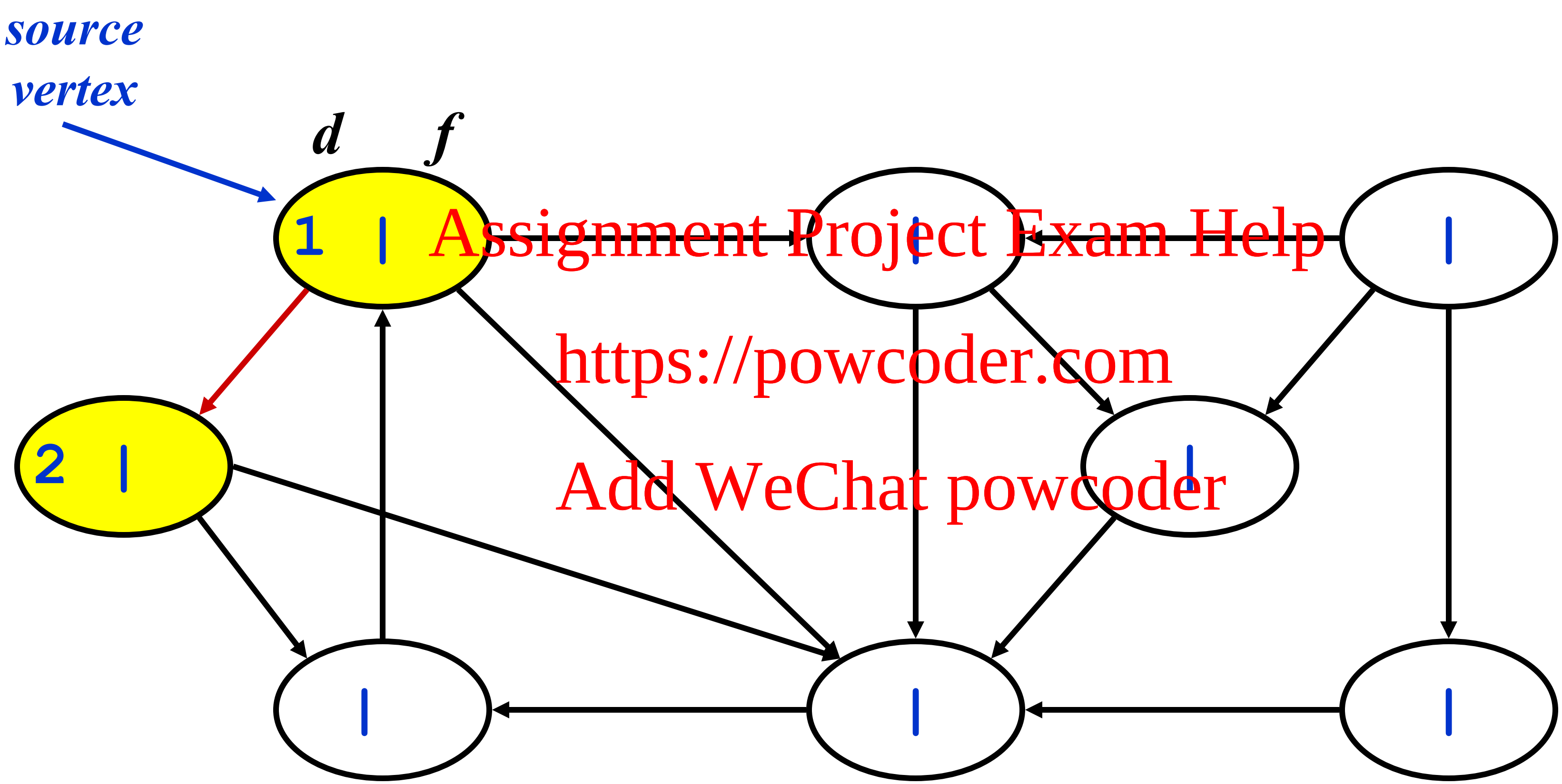
Another DFS Example



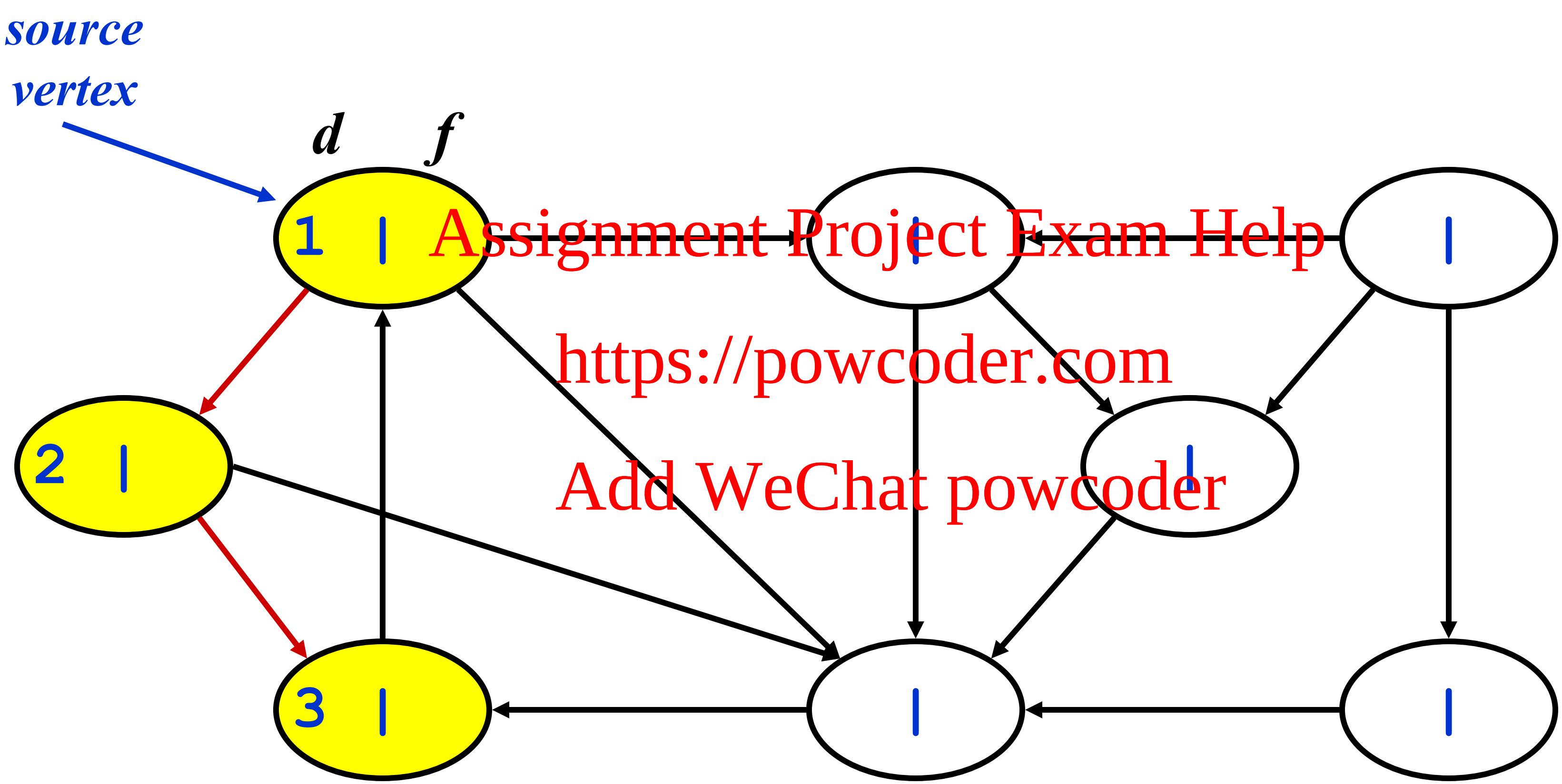
DFS Example



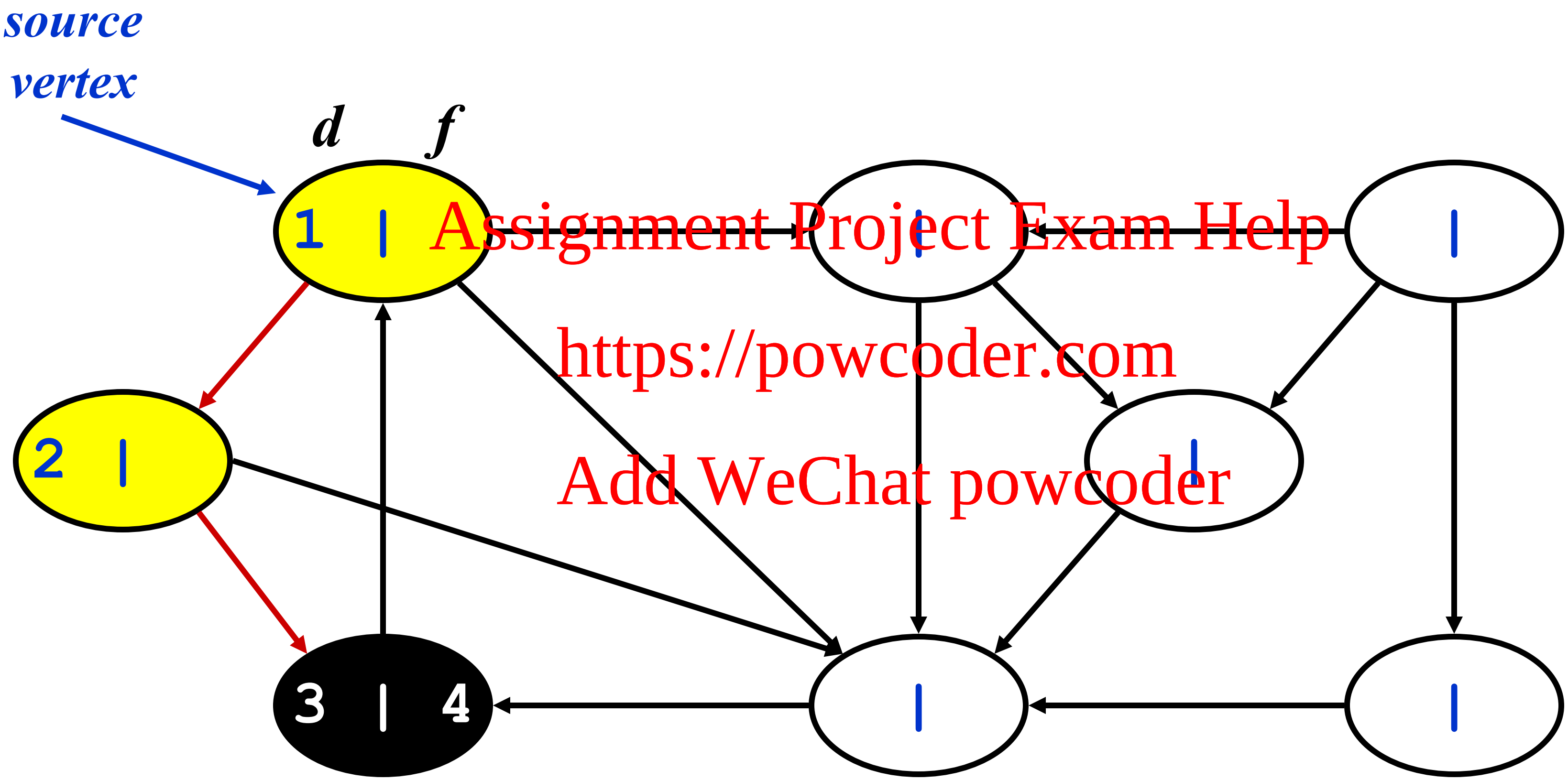
DFS Example



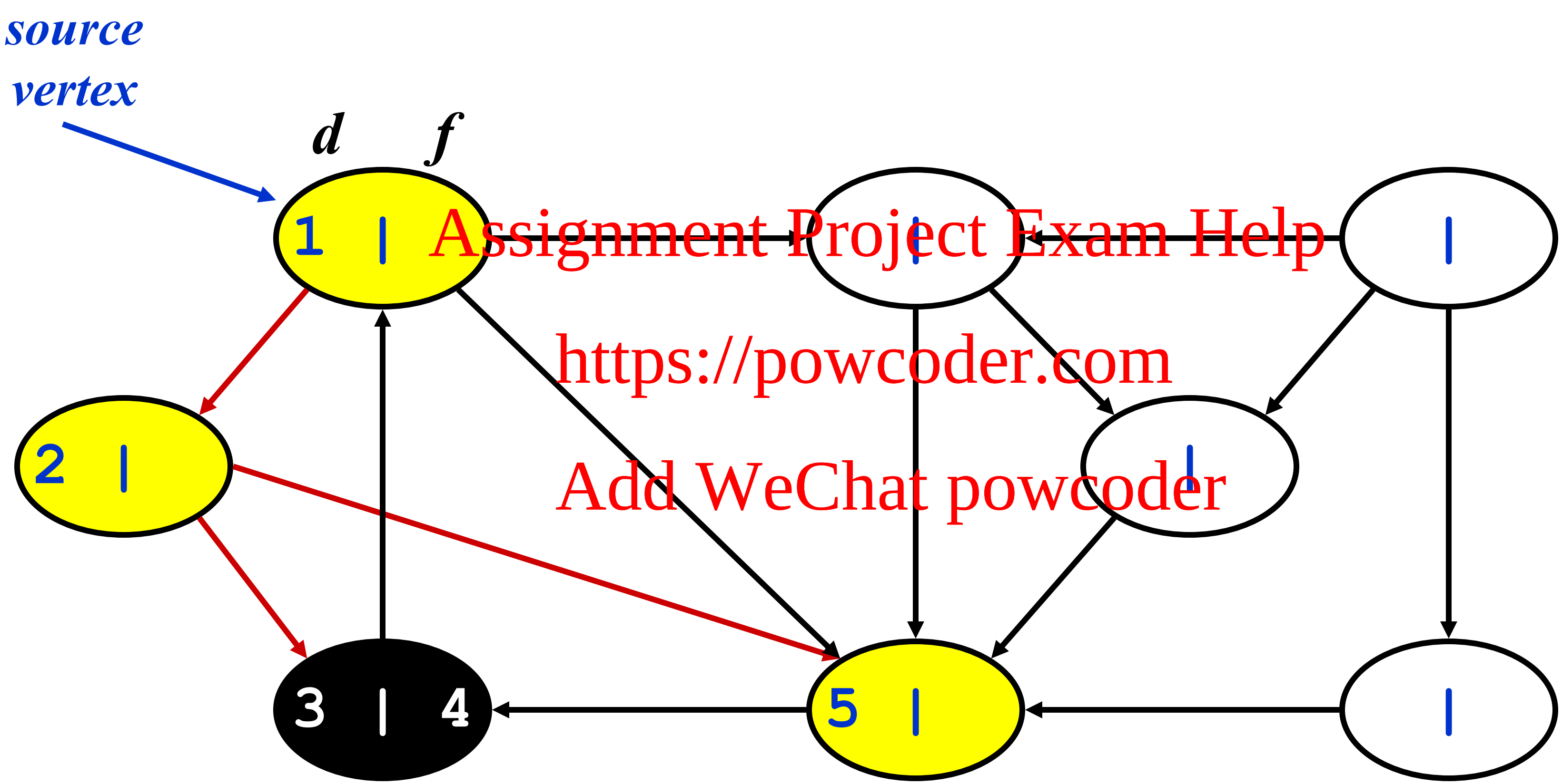
DFS Example



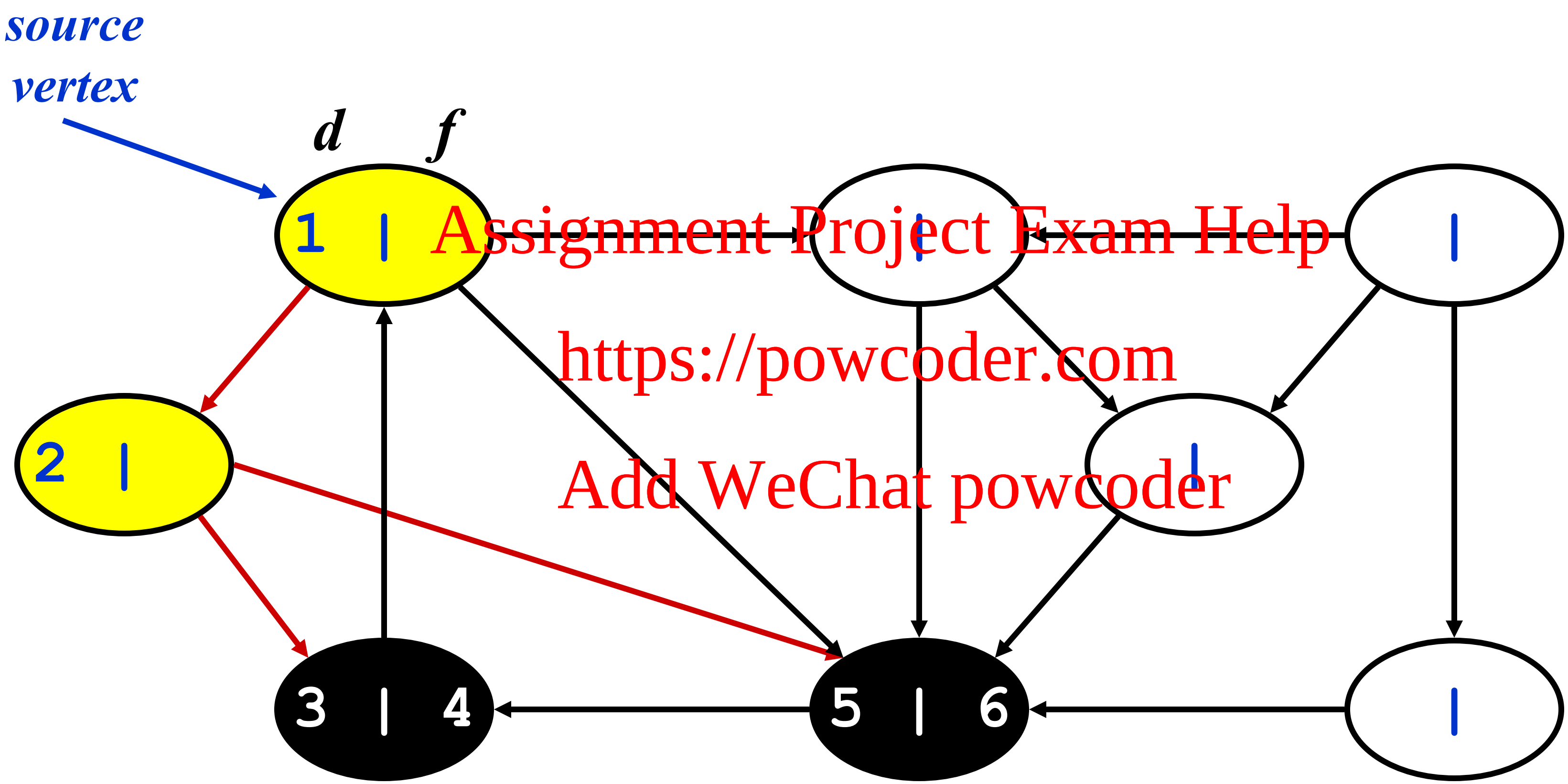
DFS Example



DFS Example



DFS Example



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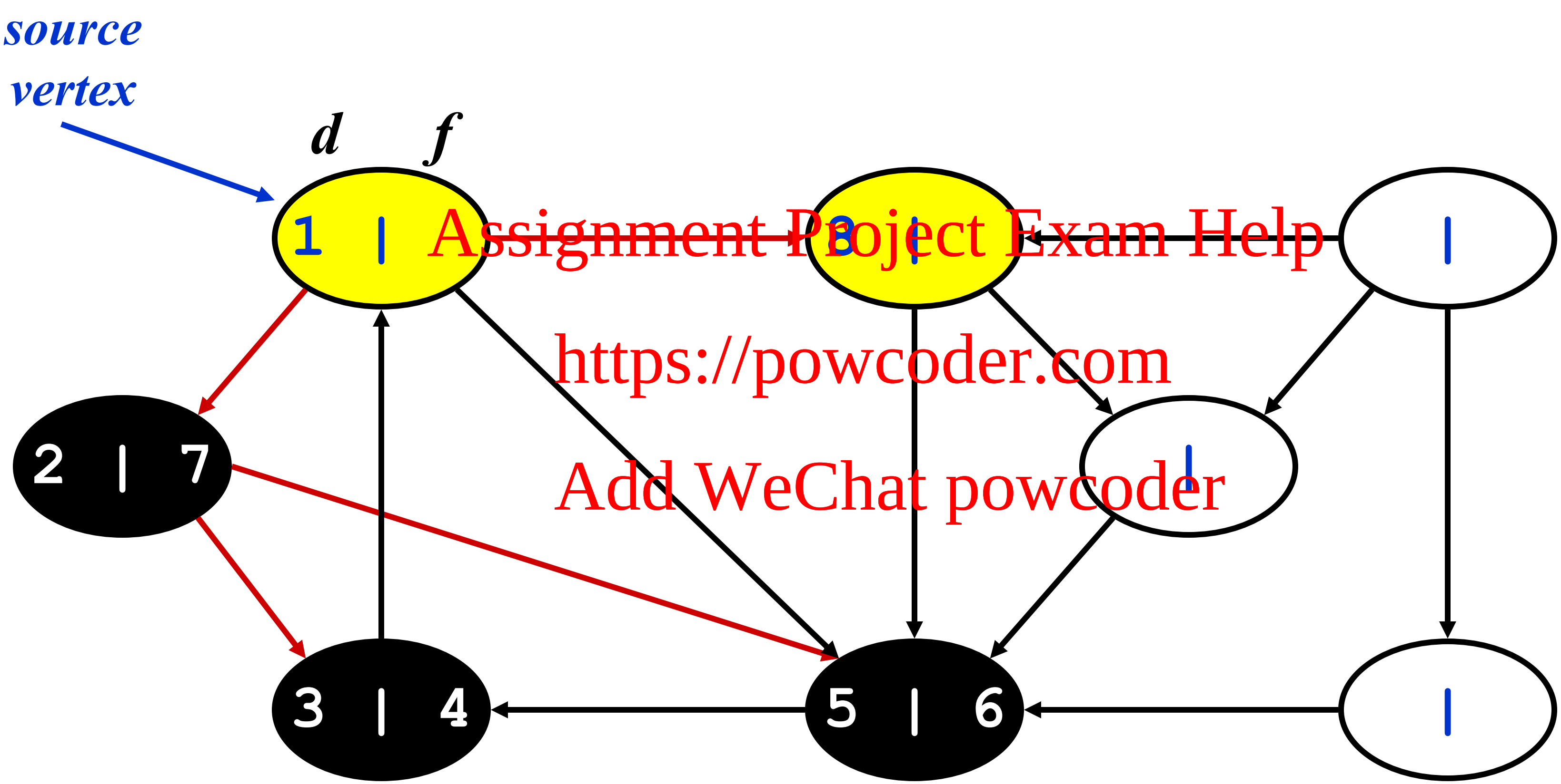
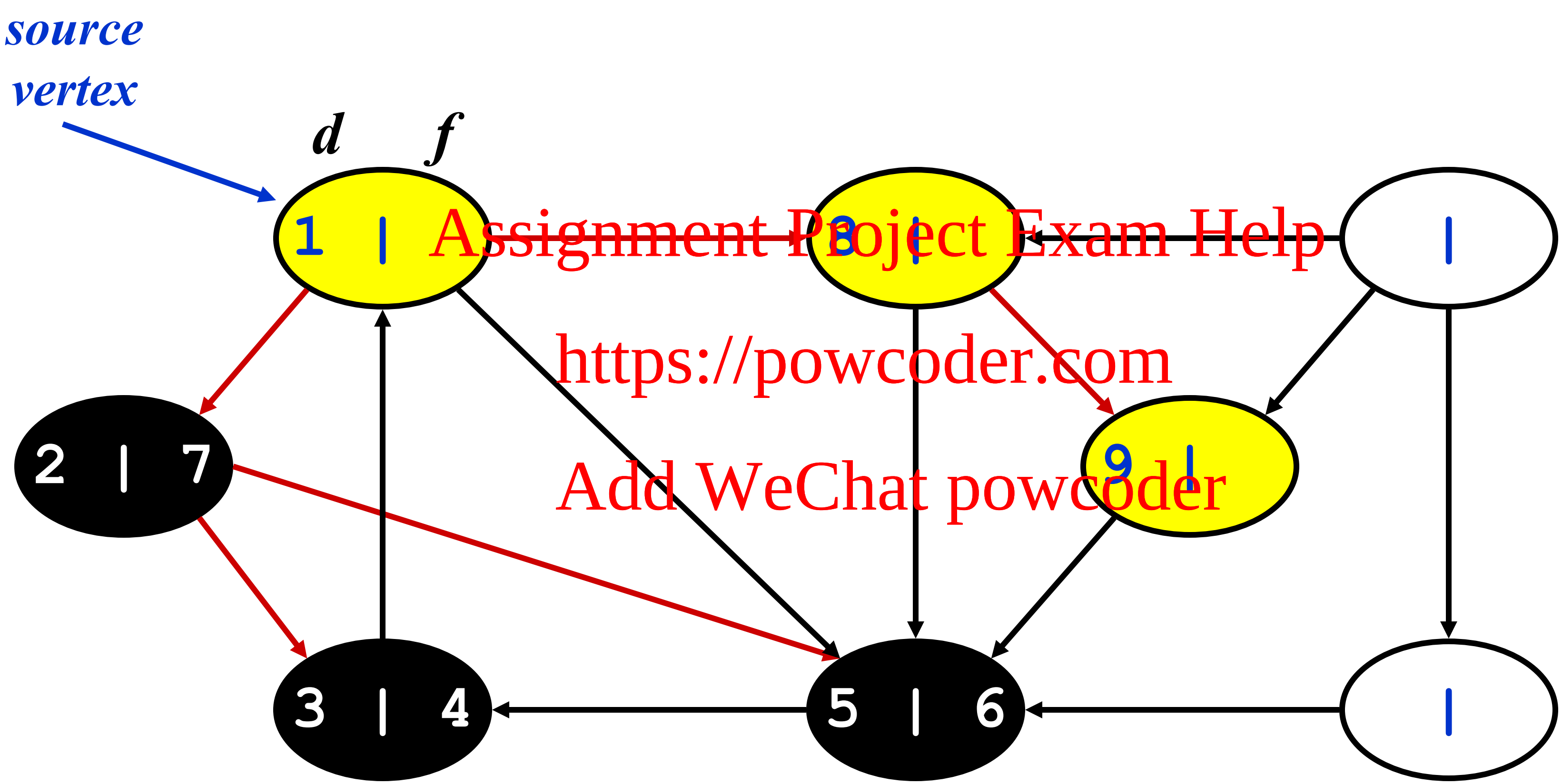
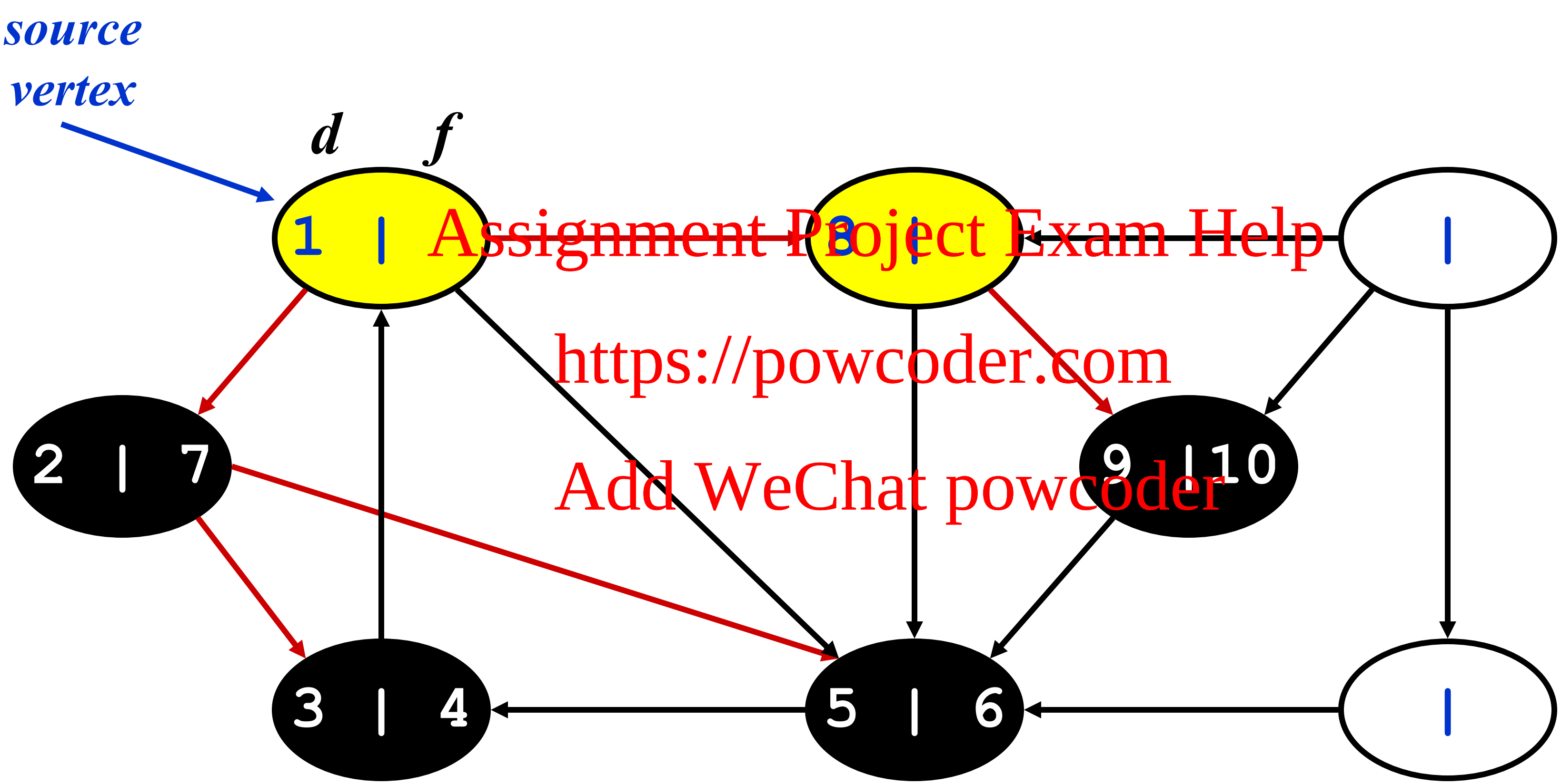


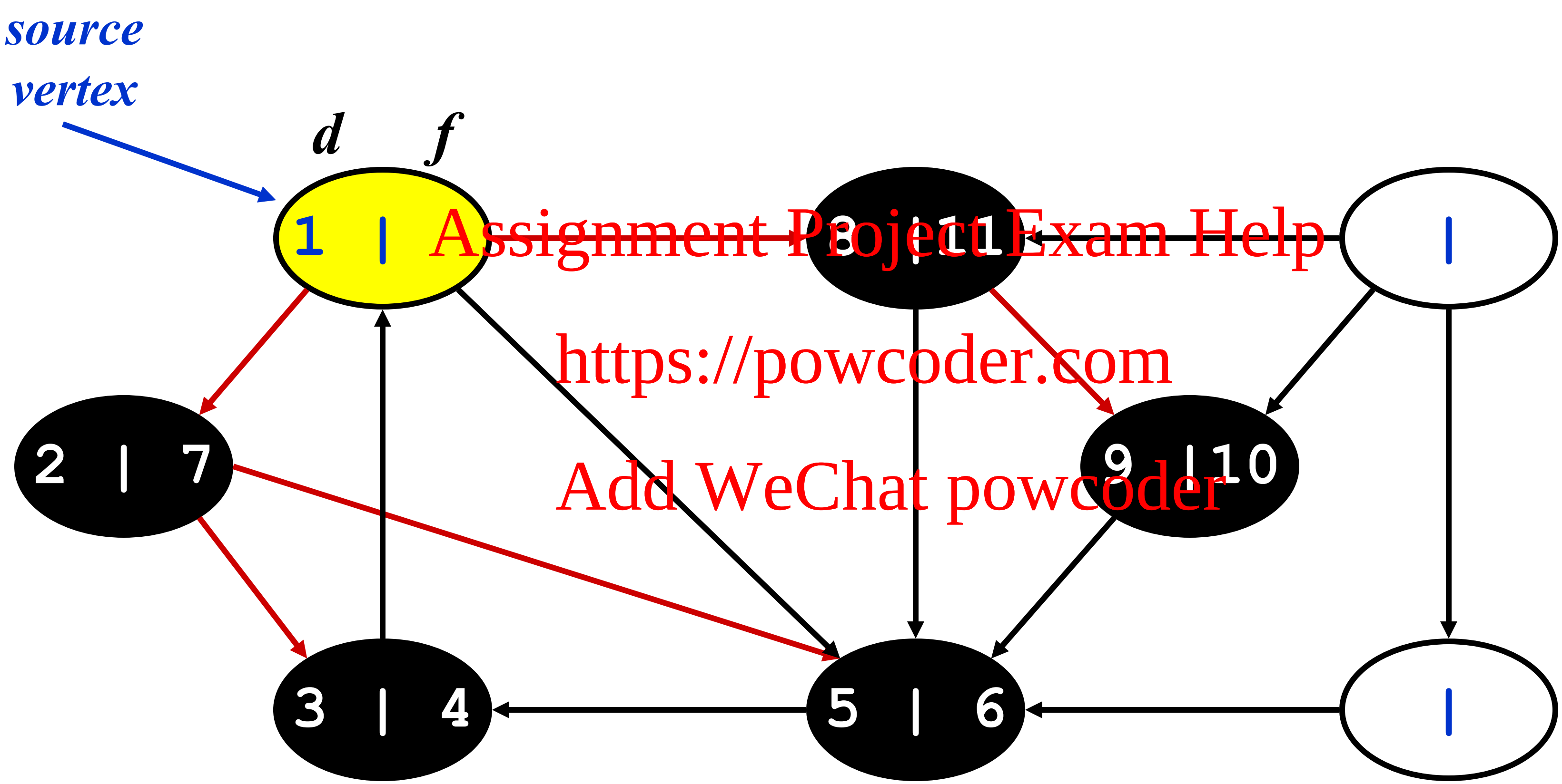
DFS Example



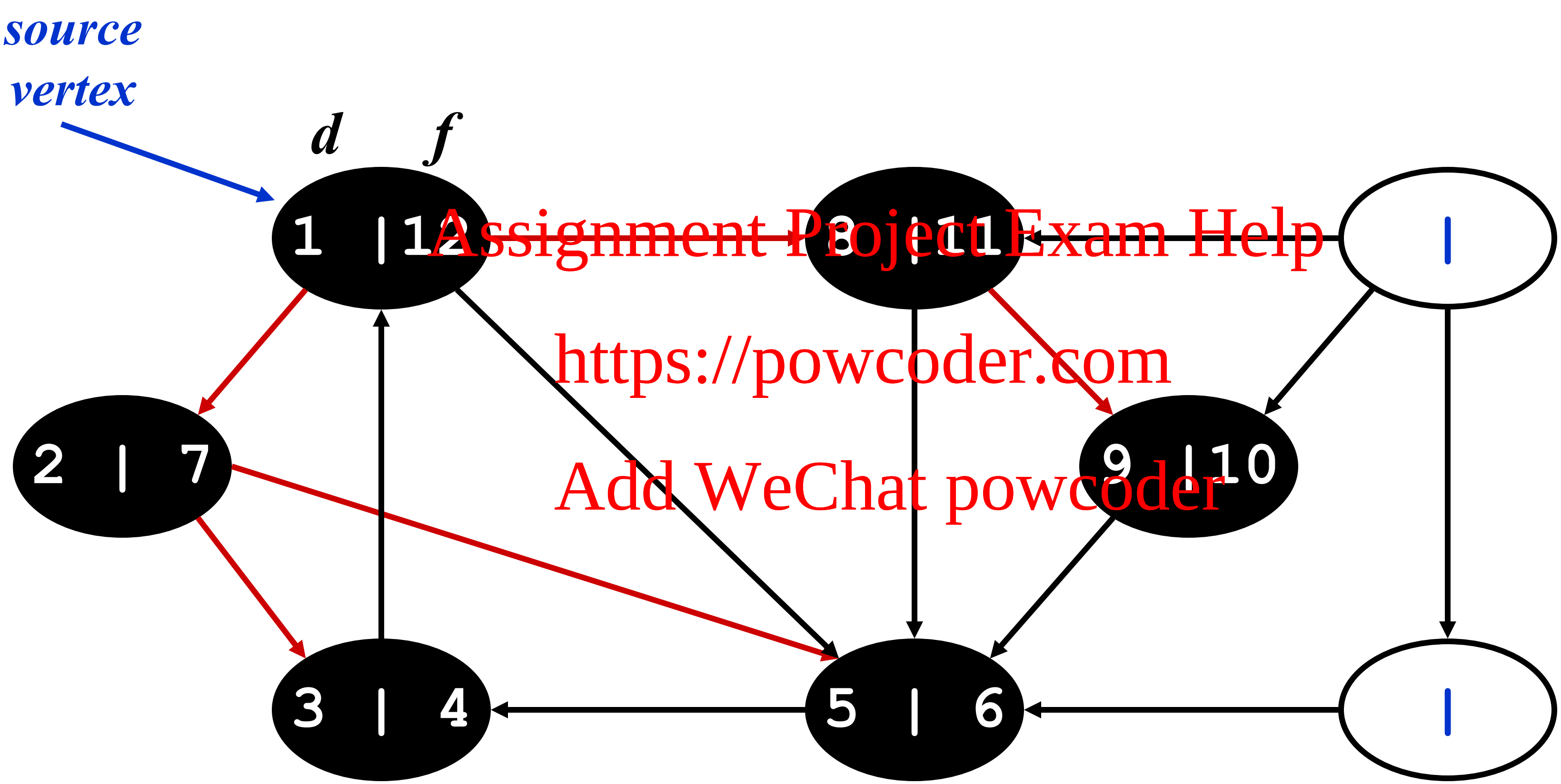
DFS Example



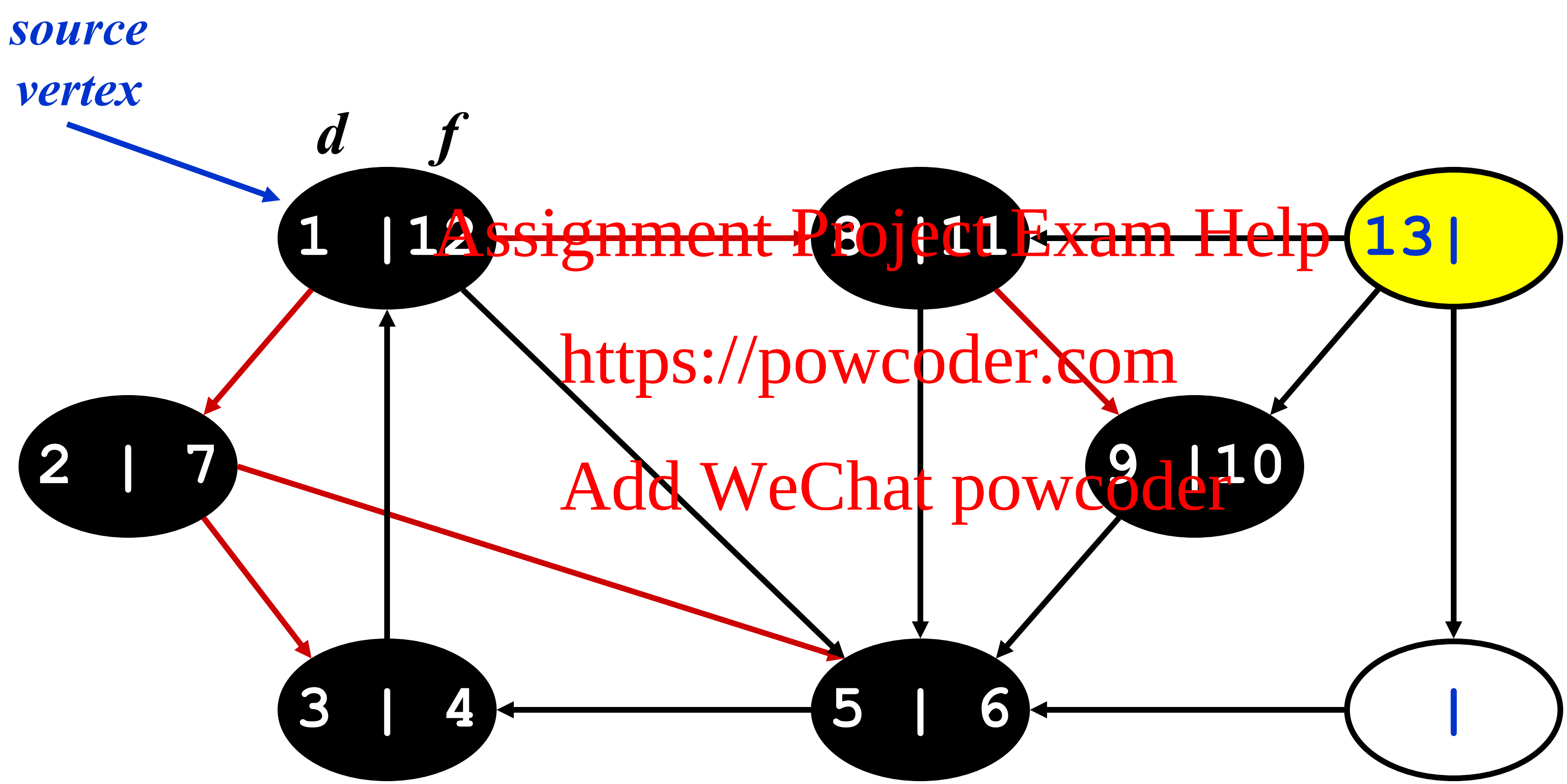
DFS Example



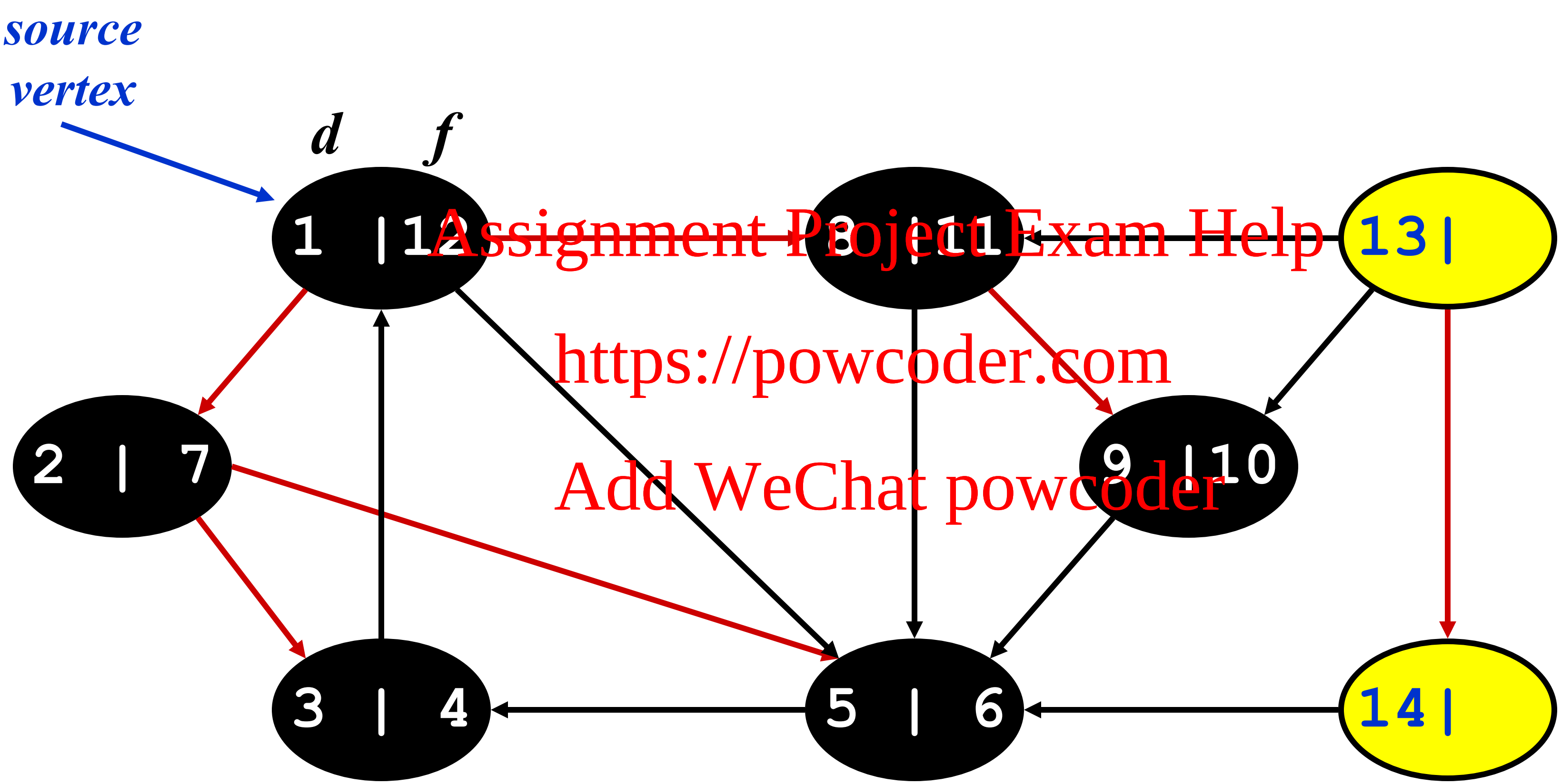
DFS Example



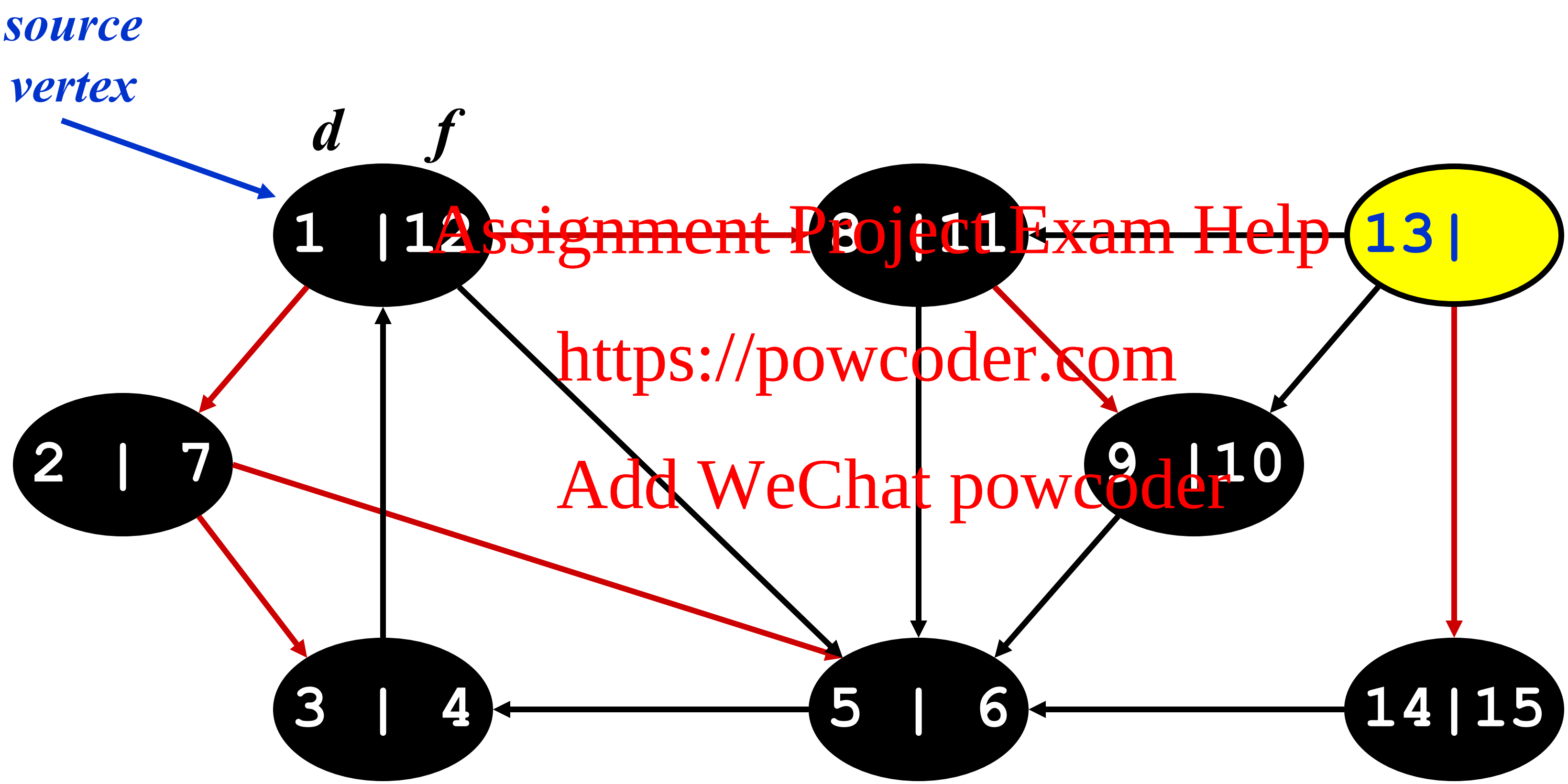
DFS Example



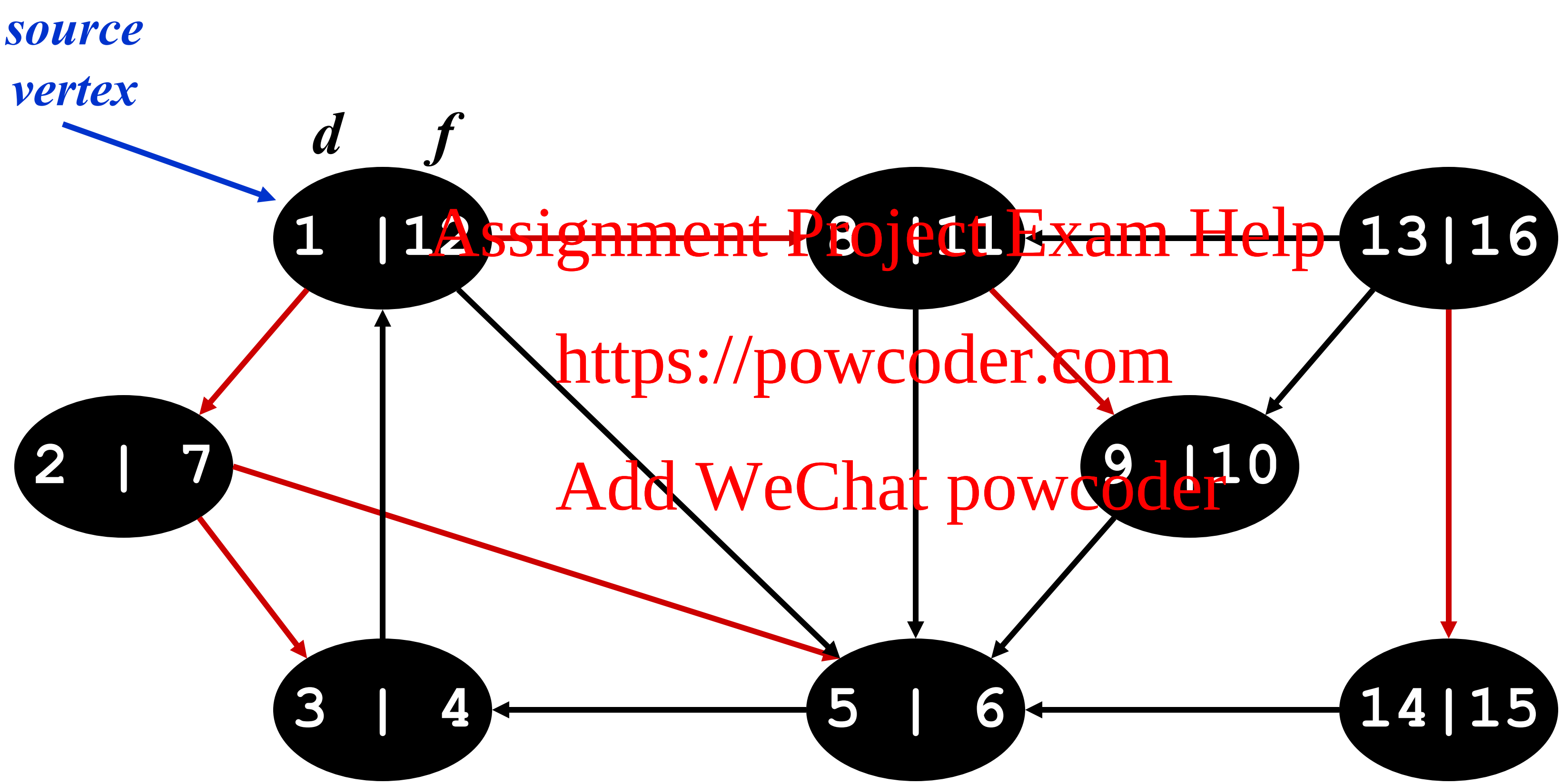
DFS Example



DFS Example



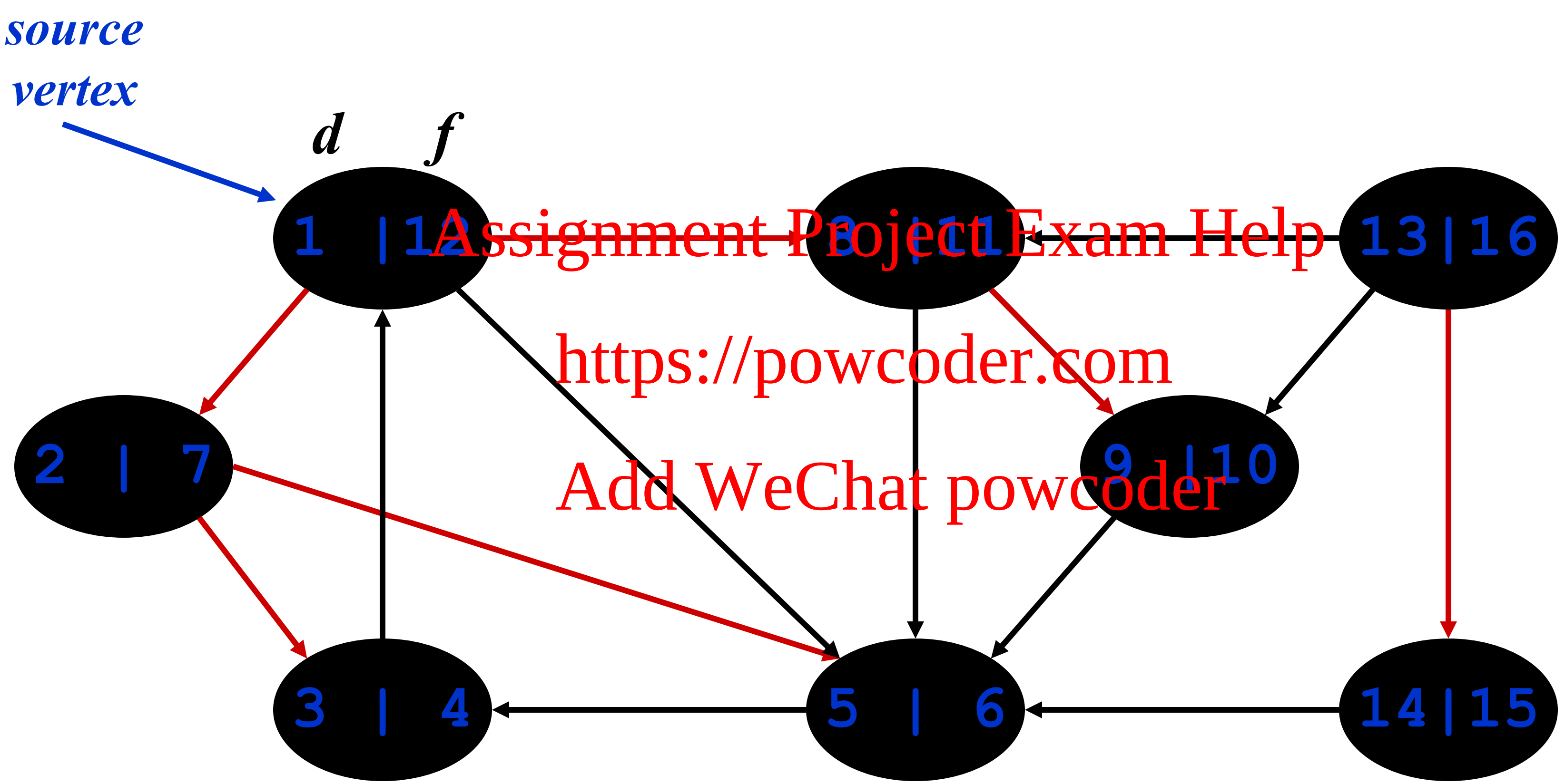
DFS Example



DFS: Kinds of edges

- DFS introduces an important distinction among edges in the original graph:
  - *Tree edge*: encounter new (white) vertex
    - The tree edges form a spanning forest
    - *Can tree edges form cycles? Why or why not?*

DFS Example

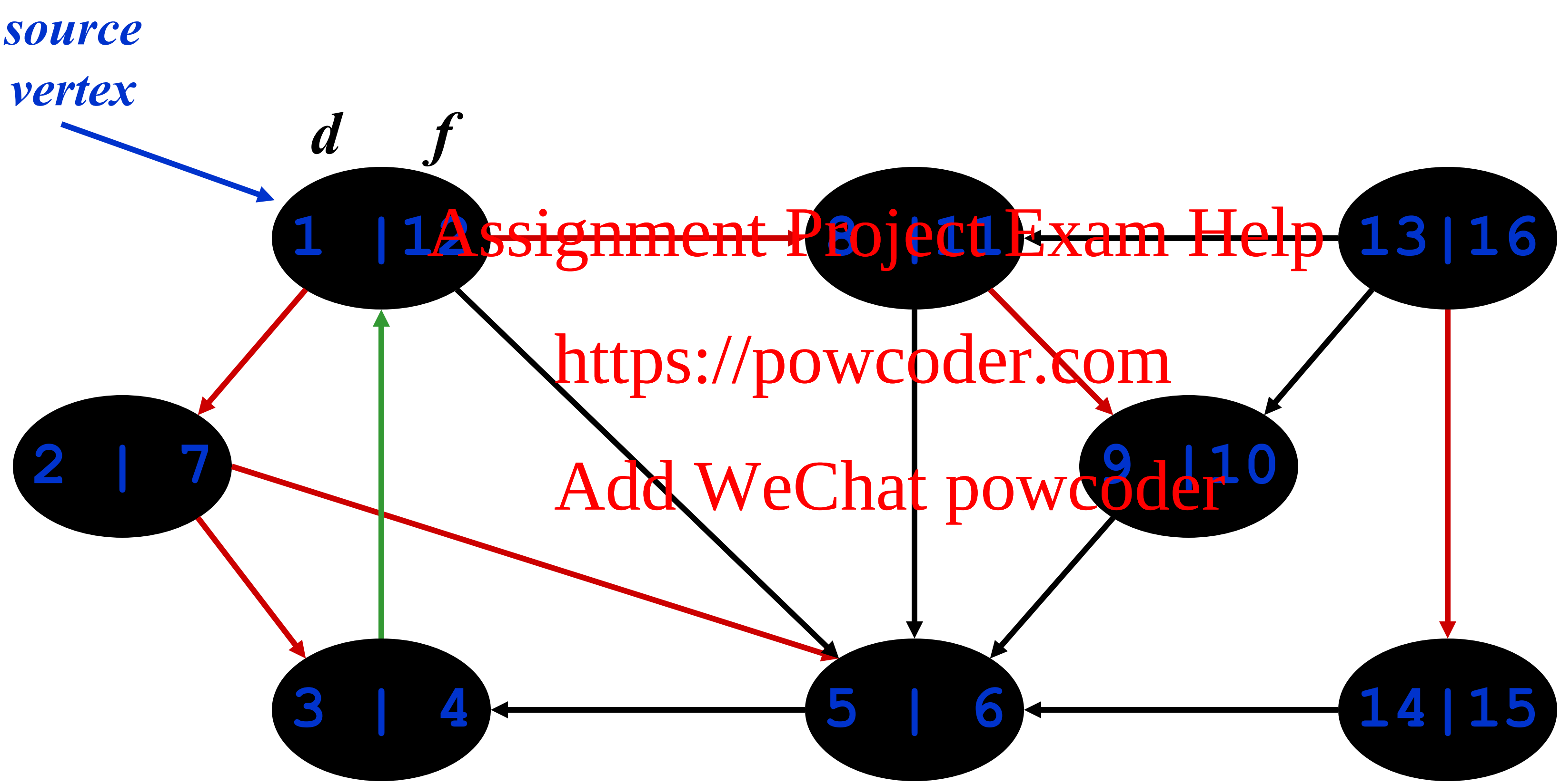


Tree edges

DFS: Kinds of edges

- DFS introduces an important distinction among edges in the original graph:
  - *Tree edge*: encounter new (white) vertex
  - *Back edge*: from descendent to ancestor
    - Encounter a yellow vertex (yellow to yellow)

DFS Example

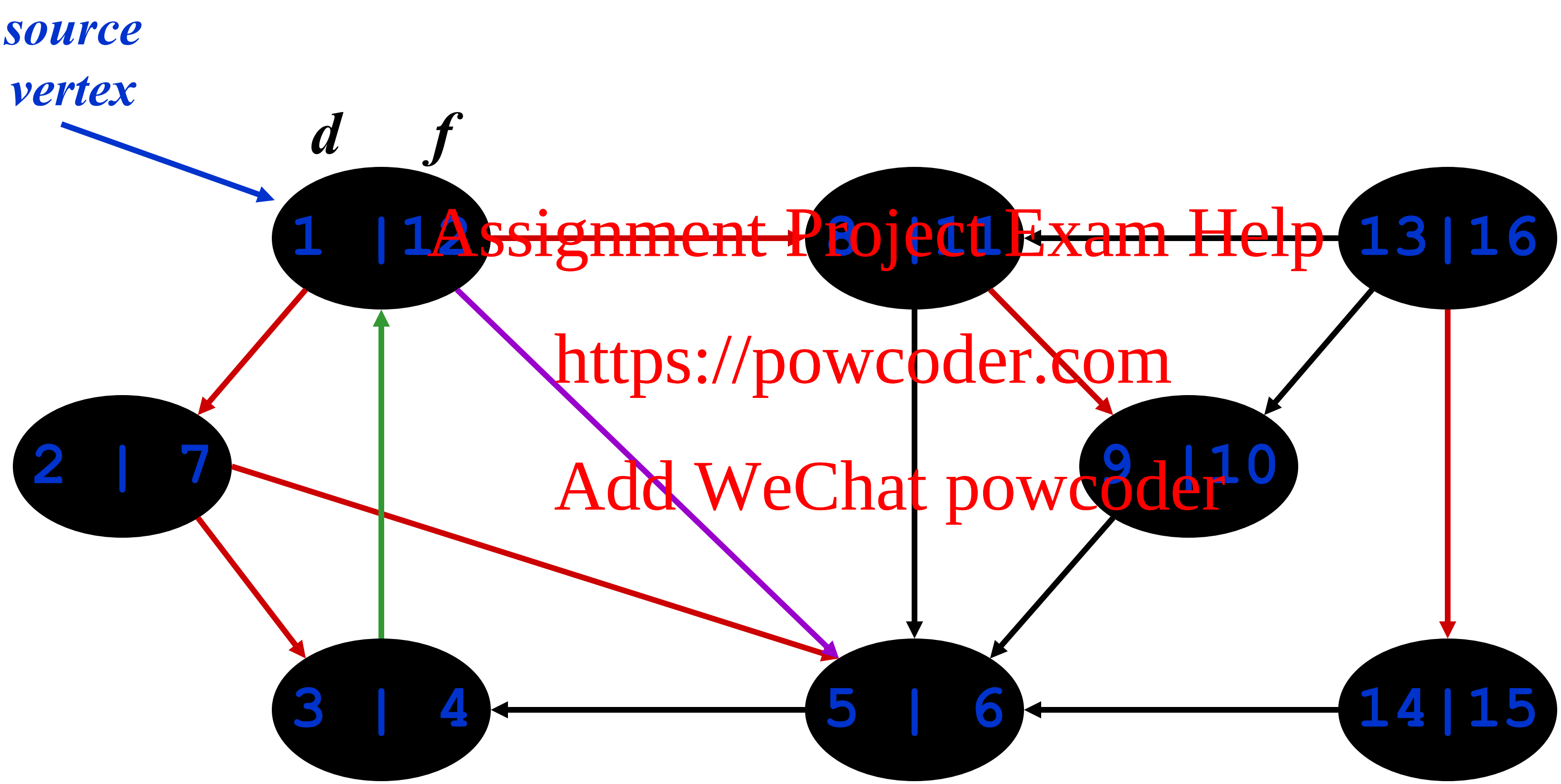


Tree edges *Back edges*

DFS: Kinds of edges

- DFS introduces an important distinction among edges in the original graph:
  - *Tree edge*: encounter new (white) vertex
  - *Back edge*: from descendent to ancestor
  - *Forward edge*: from ancestor to descendent
    - Not a tree edge, though
    - From yellow node to black node

DFS Example

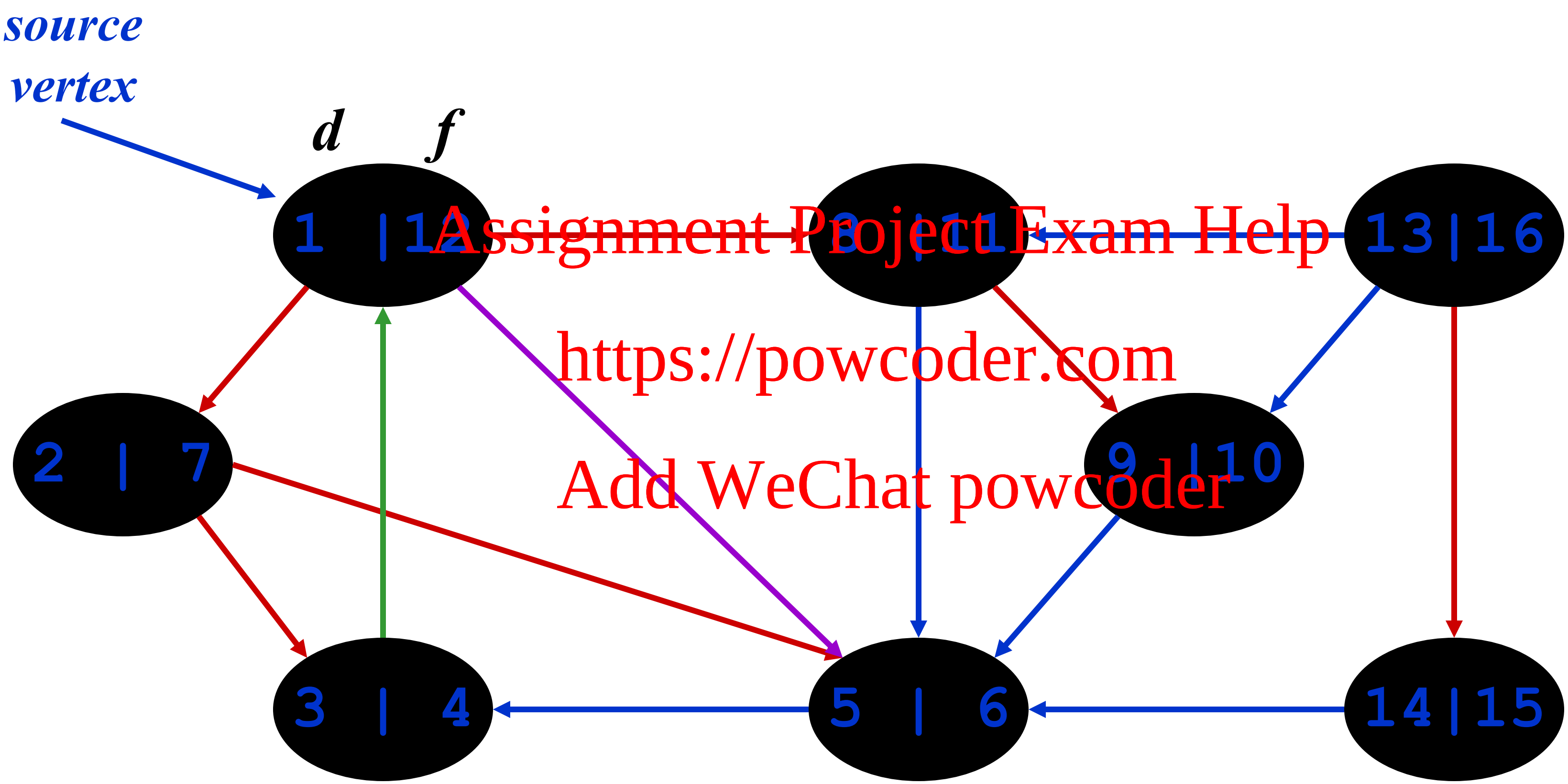


Tree edges *Back edges* *Forward edges*

DFS: Kinds of edges

- DFS introduces an important distinction among edges in the original graph:
  - *Tree edge*: encounter new (white) vertex
  - *Back edge*: from descendent to ancestor
  - *Forward edge*: from ancestor to descendent
  - *Cross edge*: between a tree or subtrees
    - From a yellow node to a black node

DFS Example



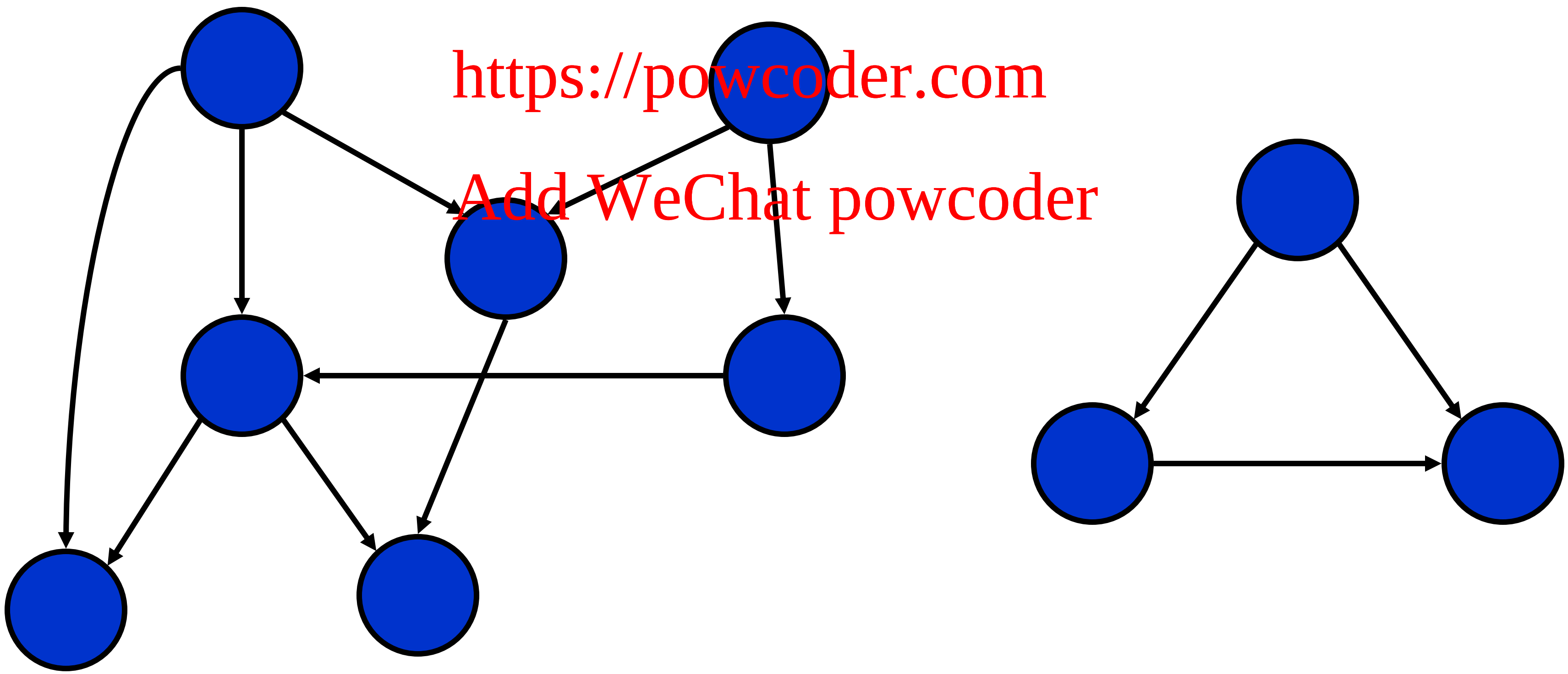
Tree edges ***Back edges*** ***Forward edges*** ***Cross edges***

DFS: Kinds of edges

- DFS introduces an important distinction among edges in the original graph:
  - *Tree edge*: encounter new (white) vertex
  - *Back edge*: from descendent to ancestor
  - *Forward edge*: from ancestor to descendent
  - *Cross edge*: between a tree or subtrees
- Note: tree & back edges are important; most algorithms don't distinguish forward & cross

Directed Acyclic Graphs

- A *directed acyclic graph* or *DAG* is a directed graph with no directed cycles:



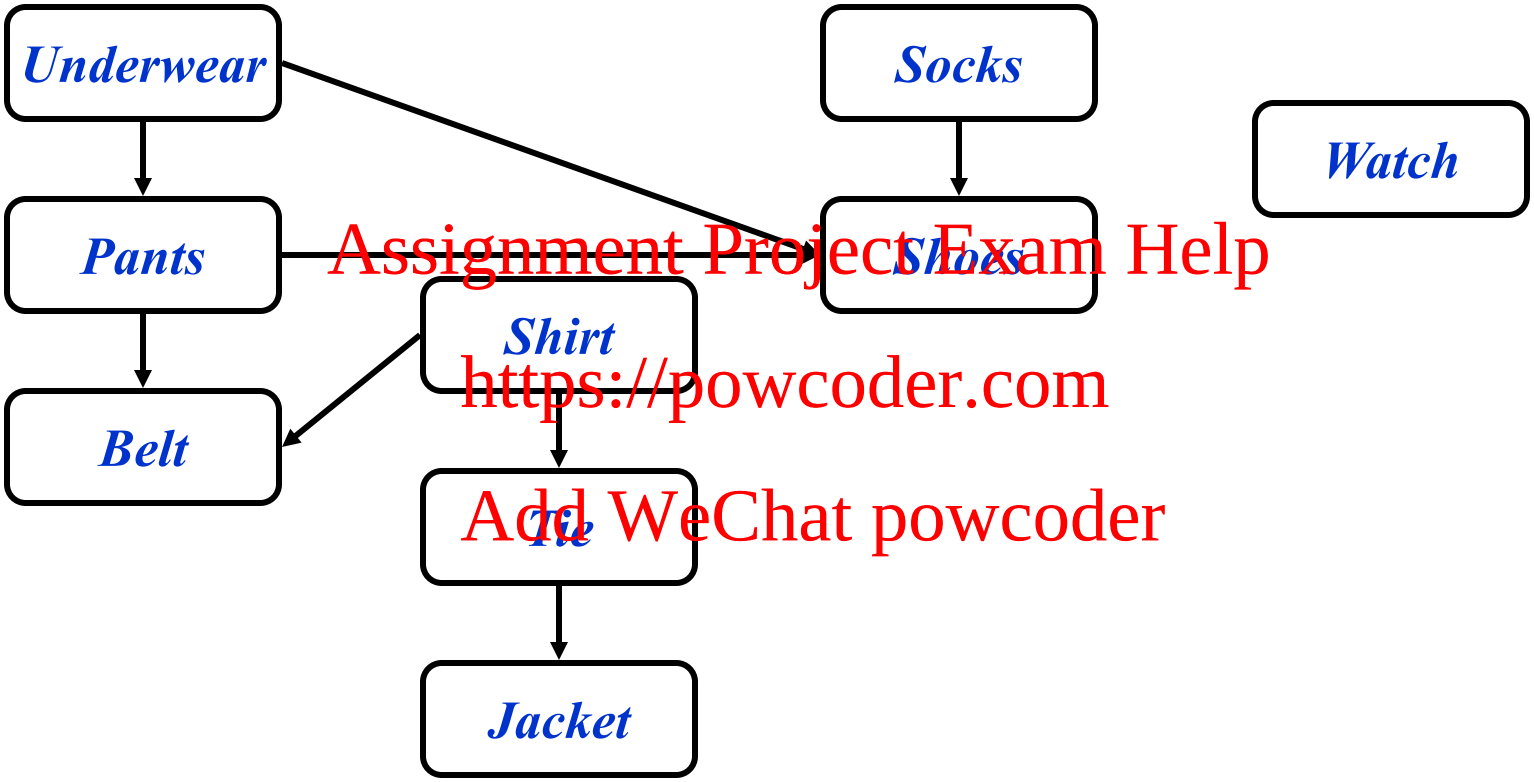
DFS and DAGs

- Argue that a directed graph G is acyclic iff a DFS of G yields no back edges:
  - Forward: if G is acyclic, will be no back edges
    - Trivial: a back edge implies a cycle
  - Backward: if no back edges, G is acyclic
    - Argue contrapositive: G has a cycle $\Rightarrow \exists$ a back edge
      - Let v be the vertex on the cycle first discovered, and u be the predecessor of v on the cycle
      - When v discovered, whole cycle is white
      - Must visit everything reachable from v before returning from DFS-Visit()
      - So path from u→v is yellow→yellow, thus (u, v) is a back edge

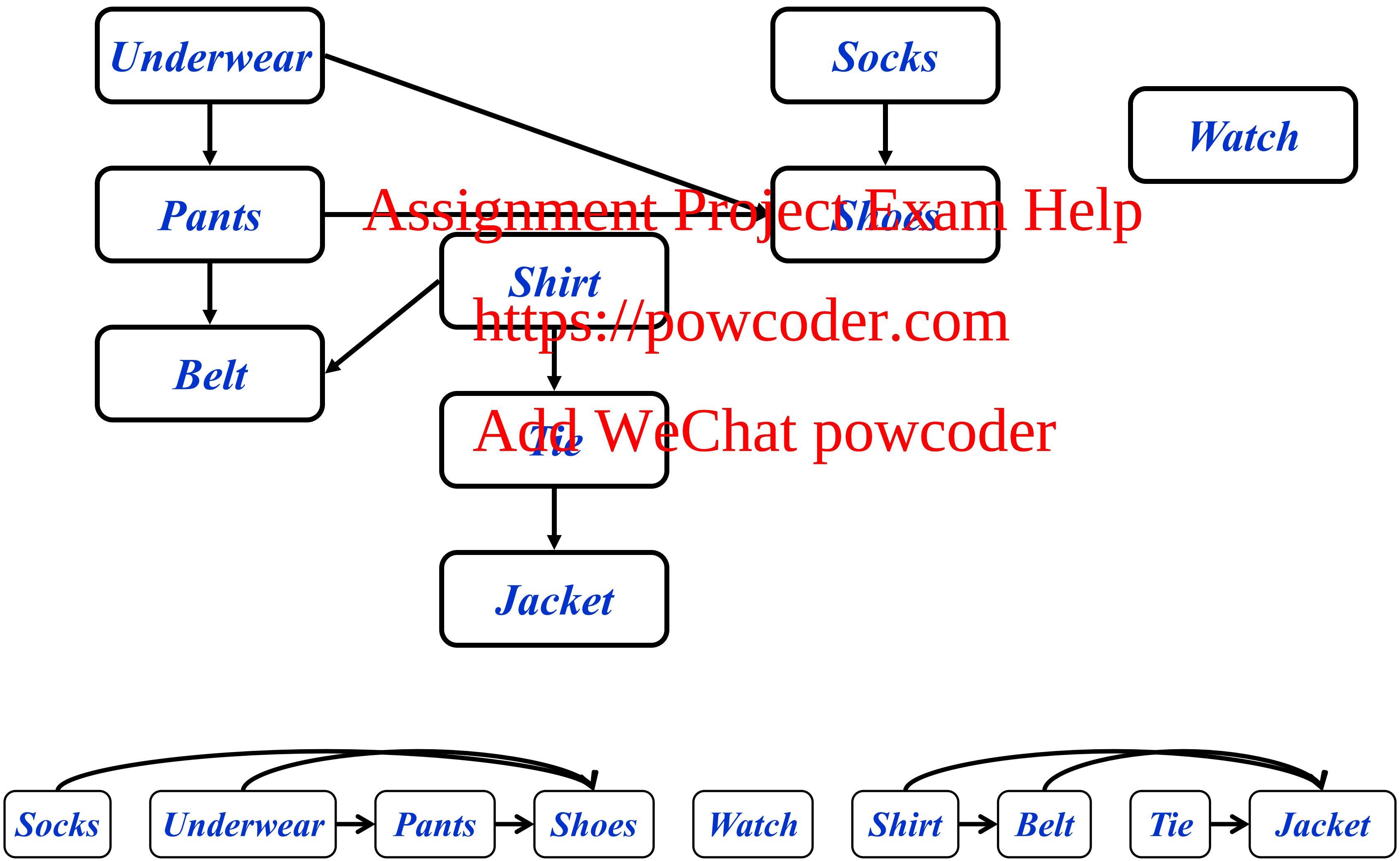
Topological Sort

- *Topological sort* of a DAG:
  - Linear ordering of all vertices in graph G such that vertex u comes before vertex v if edge $(u, v) \in G$
- Real-world example: getting dressed

Getting Dressed



Getting Dressed



Topological Sort Algorithm

```
Topological-Sort()
{
    Run DFS
    When a vertex is finished, output it
    Vertices are output in reverse
      topological order
}
```


- Time: O(V+E)
- Correctness: Want to prove that

$$(u,v) \in G \Rightarrow u\rightarrow f > v\rightarrow f$$

Correctness of Topological Sort

- Claim: $(u,v) \in G \Rightarrow u \rightarrow f > v \rightarrow f$
  - When (u,v) is explored, u is yellow
    - v = yellow $\Rightarrow (u,v)$ is back edge. Contradiction (*Why?*)
    - v = white $\Rightarrow v$ becomes descendent of $u \Rightarrow v \rightarrow f < u \rightarrow f$ (since must finish v before backtracking and finishing u)
    - v = black $\Rightarrow v$ already finished $\Rightarrow v \rightarrow f < u \rightarrow f$

The End