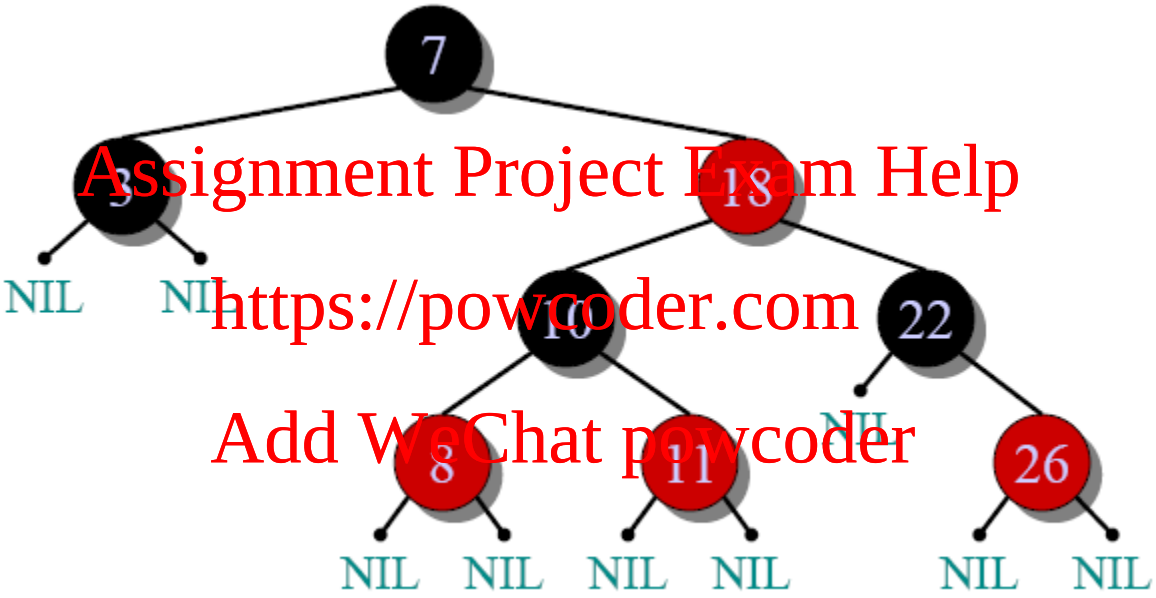
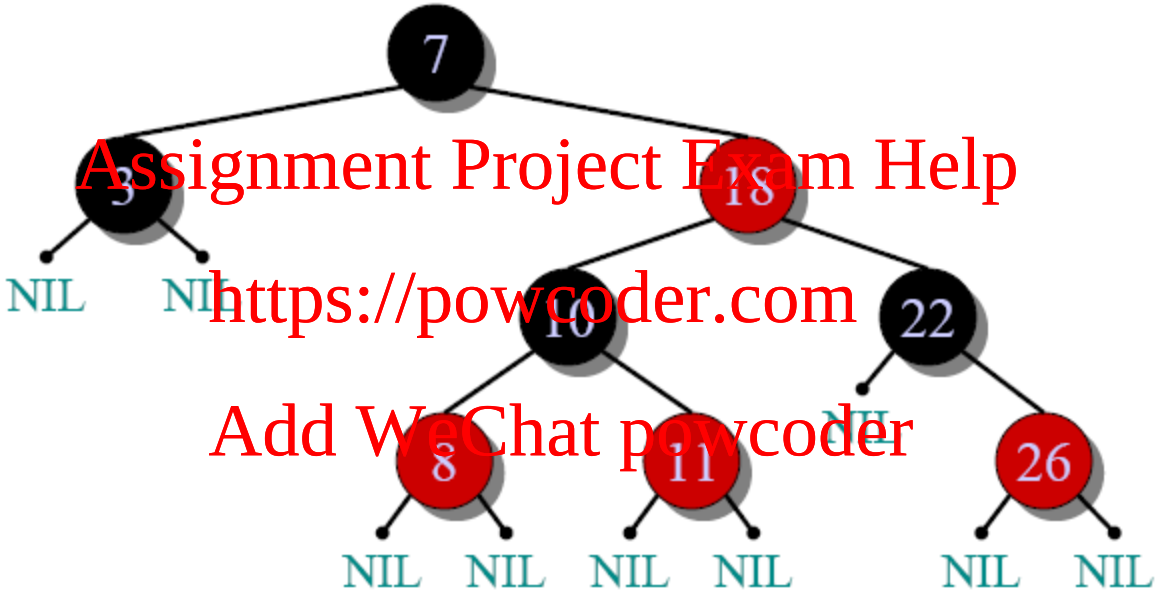
Example of a Red-black Tree



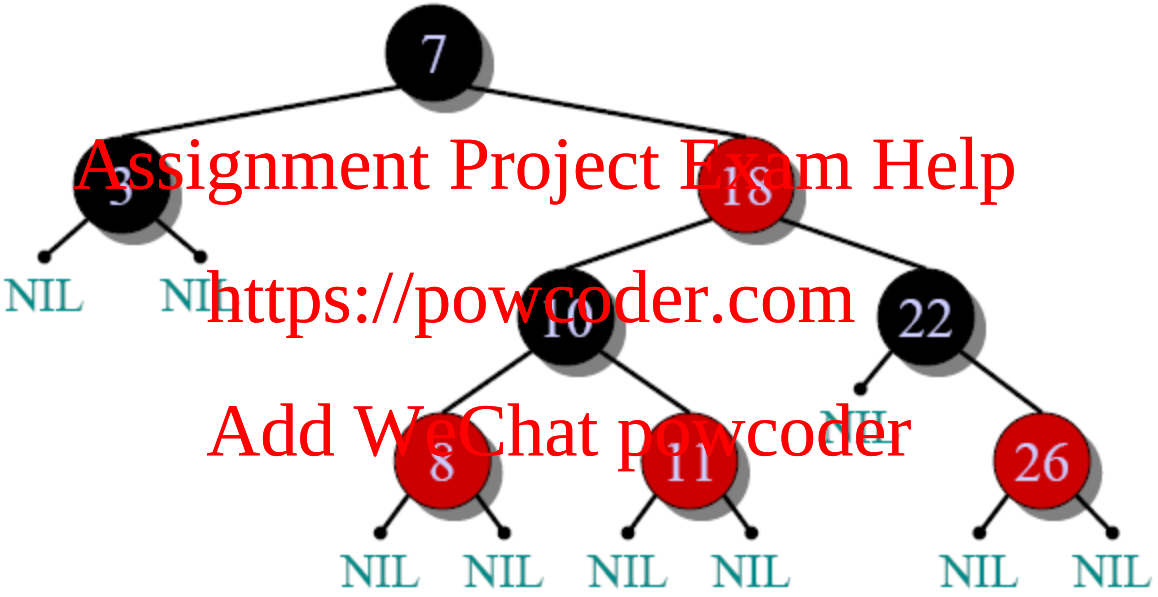
1. Every node is either red or **black**.

Example of a Red-black Tree



2. 3. The root and leaves (NIL's) are **black**.

Example of a Red-black Tree

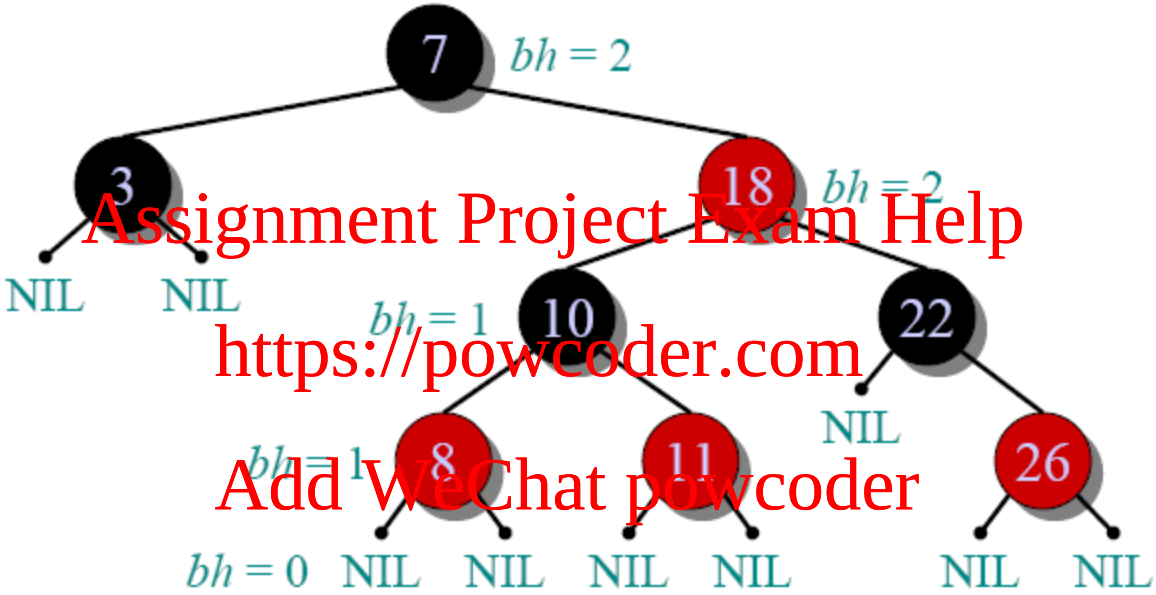


4. If a node is red, then its children are **black**.

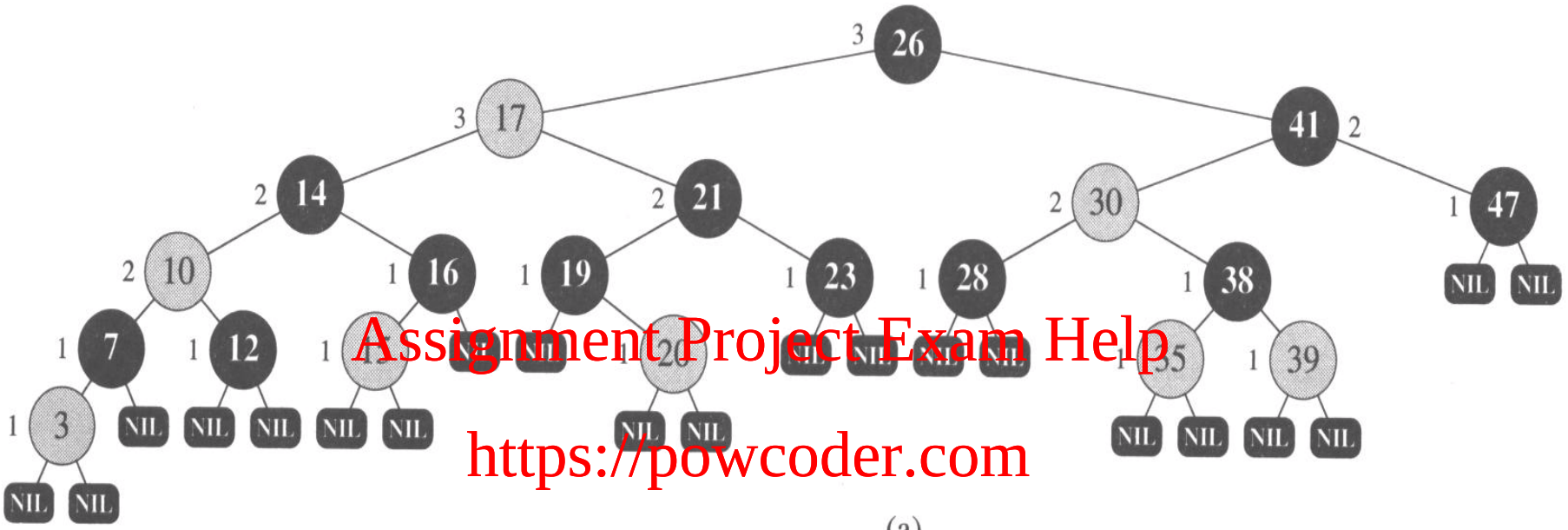
?? Means

If a node is red, then its parent is **black**.

Example of a Red-black Tree



5. All simple paths from any node x to a descendant leaf have the same number of black nodes = black-height(x).

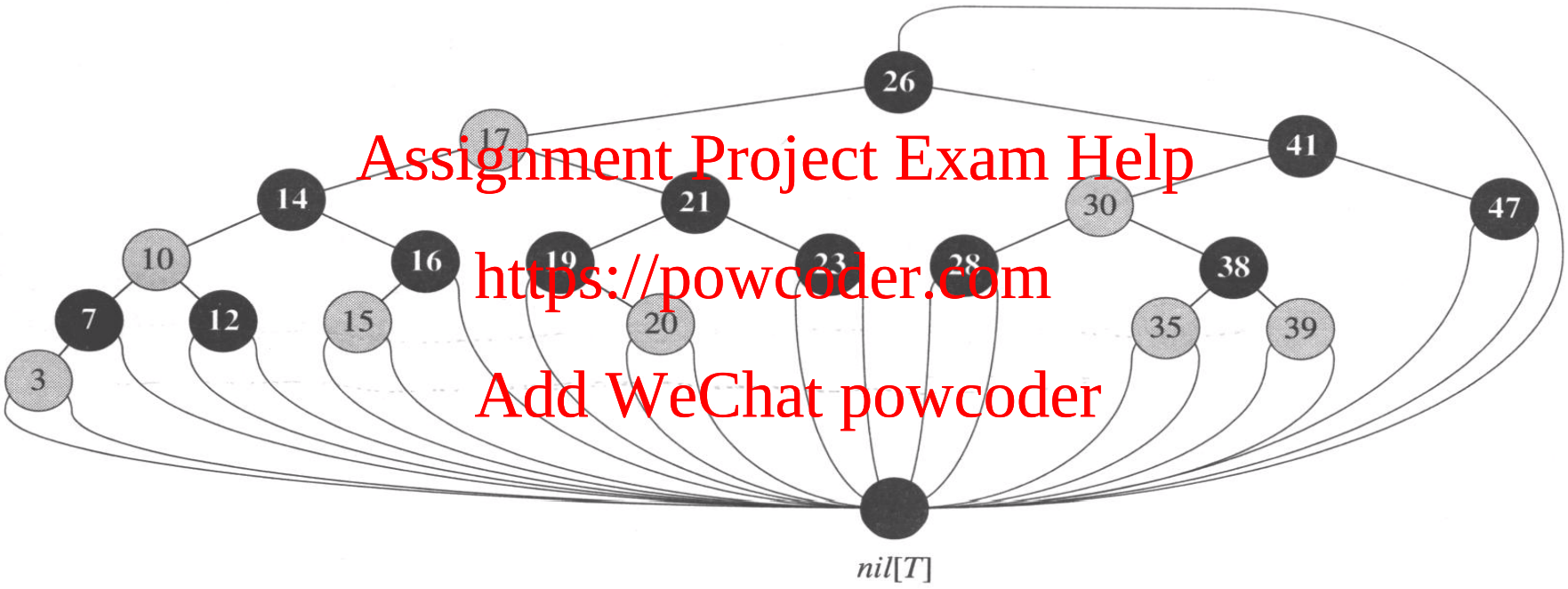


(a)

black-height of the node: bh(x)

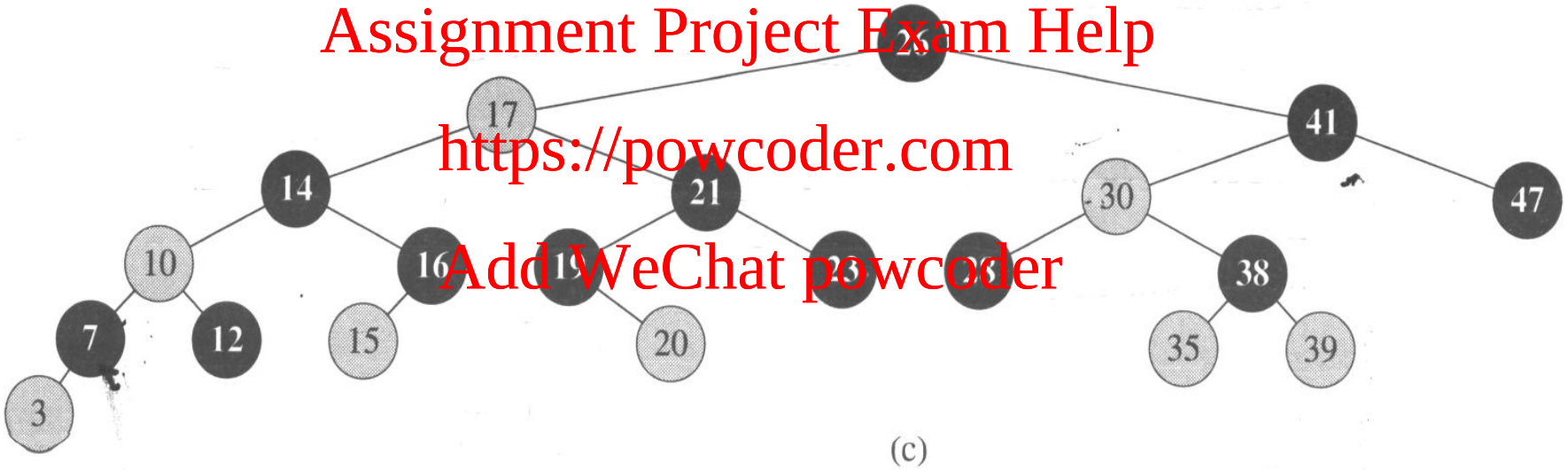
# of black nodes on any path from, but not including, a node x down to a leaf

black-height of a RB tree = black-height of its root



(b)

Leaves and the root's parent omitted entirely



We omit the leaves when we draw BR trees, because we generally confine our interest to the internal nodes, since they hold the key.

Height of a Red-black Tree

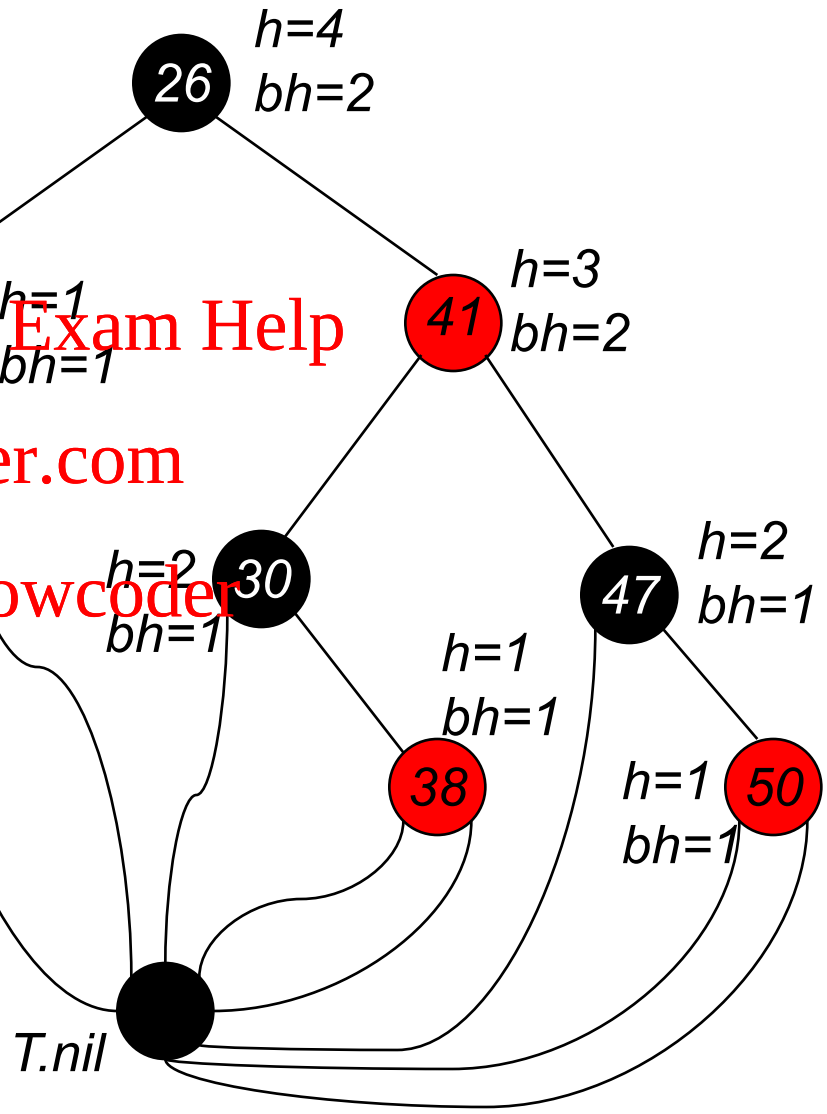
- Height of a node:
  - $h(x)$ = number of edges in a longest path to a leaf.
- Black-height of a node x, $bh(x)$:
  - $bh(x)$ = number of black nodes (including $T.nil$) on the path from x to leaf, not counting x.
- Black-height of a red-black tree is the black-height of its root.
  - By Property 5, black height is well defined.

Height of a Red-black Tree

- Example:
- Height of a node:

  $h(x)$ = # of edges in a longest path to a leaf.
- Black-height of a node

  $bh(x)$ = # of black nodes on path from x to leaf, not counting x.
- How are they related?
  - $bh(x) \le h(x) \le 2\,bh(x)$



Lemma "RB Height"

Consider a node x in an RB tree: The longest descending path from x to a leaf has length $h(x)$, which is at most twice the length of the shortest descending path from x to a leaf.

Proof:

black nodes on any path from $x = bh(x)$ (prop 5)

$\leq$ # nodes on shortest path from x, $s(x)$. (prop 1)

But, there are no consecutive red (prop 4),

and we end with black (prop 3), so $h(x) \leq 2\ bh(x)$.

Thus, $h(x) \leq 2\ s(x)$.

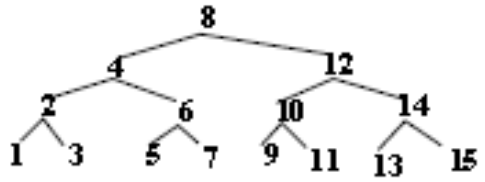
Bound on RB Tree Height

- **Lemma:** The subtree rooted at any node x has $\geq 2^{bh(x)}-1$ internal nodes.
- **Proof:** By induction on height of x.
  - **Base Case:** Height $h(x) = 0 \Rightarrow x$ is a leaf $\Rightarrow \text{bh}(x) = 0$. Subtree has $2^0-1 = 0$ nodes. $\surd$
  - **Induction Step:** Height $h(x) = h > 0$ and $bh(x) = b$.
    - Each child of x has height $h - 1$ and black-height either b (child is red) or $b - 1$ (child is **black**).
    - By ind. hyp., each child has $\geq 2^{\text{bh}(x)-1}-1$ internal nodes.
    - Subtree rooted at x has $\geq 2\,(2^{\text{bh}(x)-1}-1)+1$ $= 2^{\text{bh}(x)}-1$ internal nodes. (The $+1$ is for x itself.)

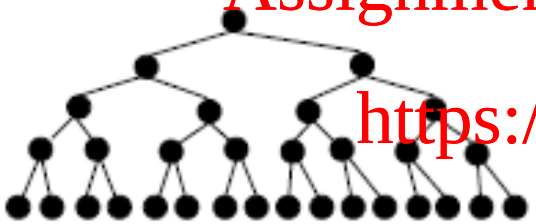
Bound on RB Tree Height

- Lemma: The subtree rooted at any node x has $\geq 2^{bh(x)}-1$ internal nodes.
- **Lemma 13.1:** A red-black tree with n internal nodes has height at most $2\lg(n+1)$.
- **Proof:**
  - By the above lemma, $n \geq 2^{bh} - 1$,
  - and since $bh \geq h/2$, we have $n \geq 2^{h/2} - 1$.
  - $\Rightarrow\ h \leq 2\lg(n + 1)$.

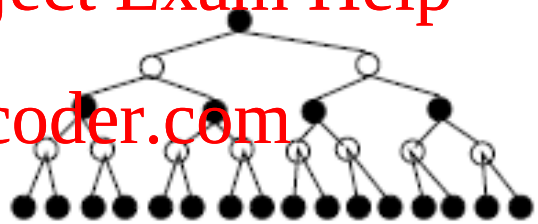
Examples of RB Trees based on full BST of 15 nodes



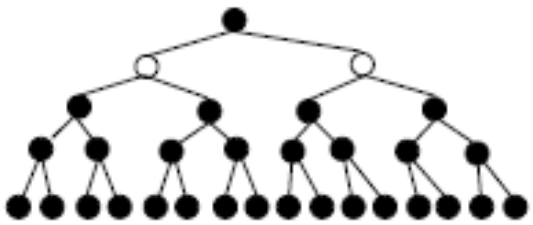
A Full Binary Search Tree of 15 Nodes



NULLs Shown as Nodes, all nodes black
Black height of tree = 4



NULLs Shown as Nodes, black height = 2



NULLs Shown as Nodes, black height = 3

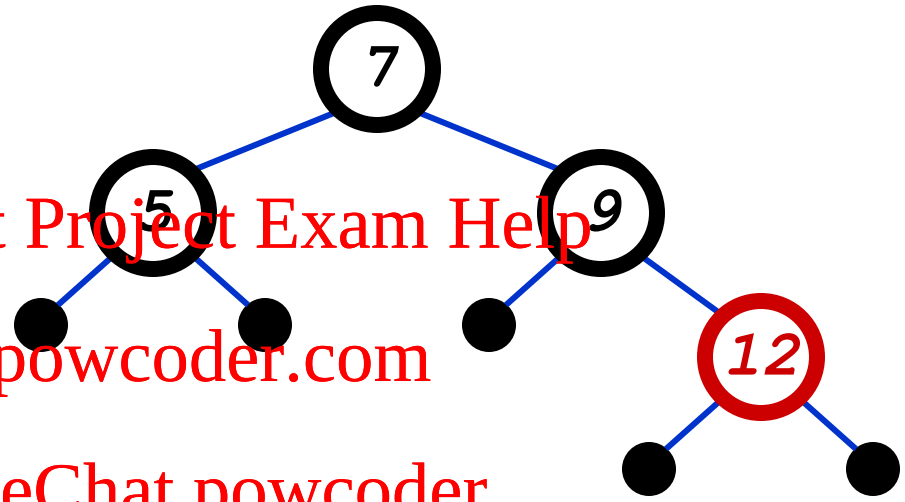
NULLs Shown as Nodes, black height = 3

Operations on RB Trees

- All operations can be performed in $O(\lg n)$ time.
- The query operations, which don't modify the tree, are performed in exactly the same way as they are in BSTs.
- These operations take $O(\lg n)$ time:
  - Minimum(), Maximum()
  - Successor(), Predecessor()
  - Search()
- Insert() and Delete():
  - Will also take $O(\lg n)$ time
- Insertion and Deletion are not straightforward. Why?

Red-Black Trees: An Example

- *Color this tree:*

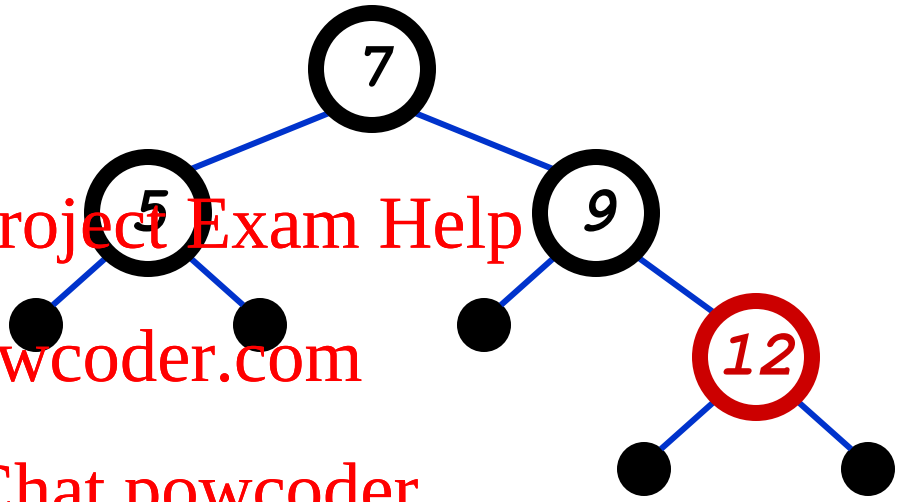


Red-black properties:

1. Every node is either red or black
2. The root is always black
3. Every leaf (NIL pointer) is black
4. If a node is red, both children are black
5. Every path from node to descendent leaf contains the same number of black nodes

Red-Black Trees: The Problem With Insertion

- Insert 8
  - *Where does it go?*



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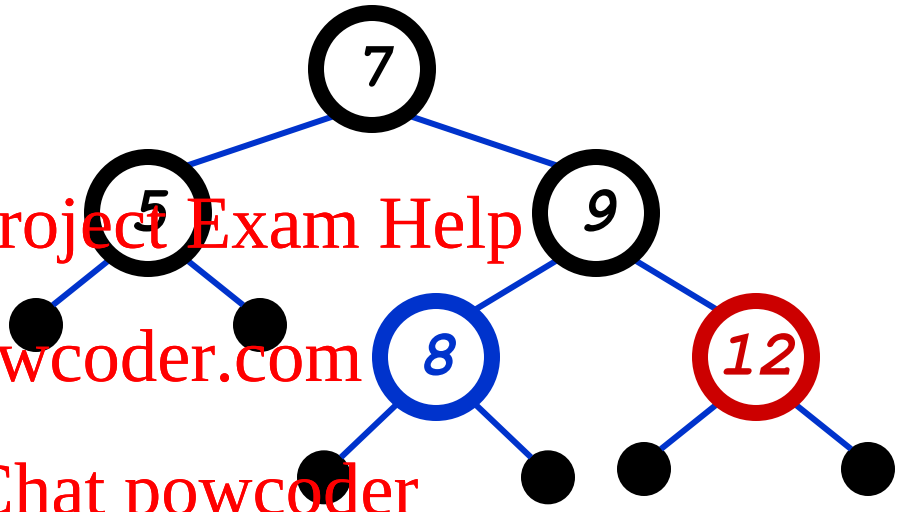
Red-Black Trees: The Problem With Insertion

- Insert 8
  - *Where does it go?*
  - *What color should it be?*



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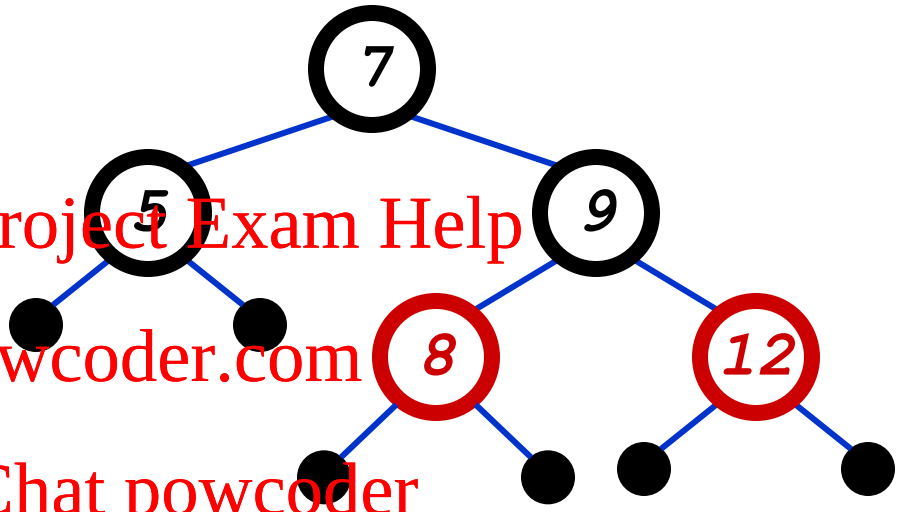

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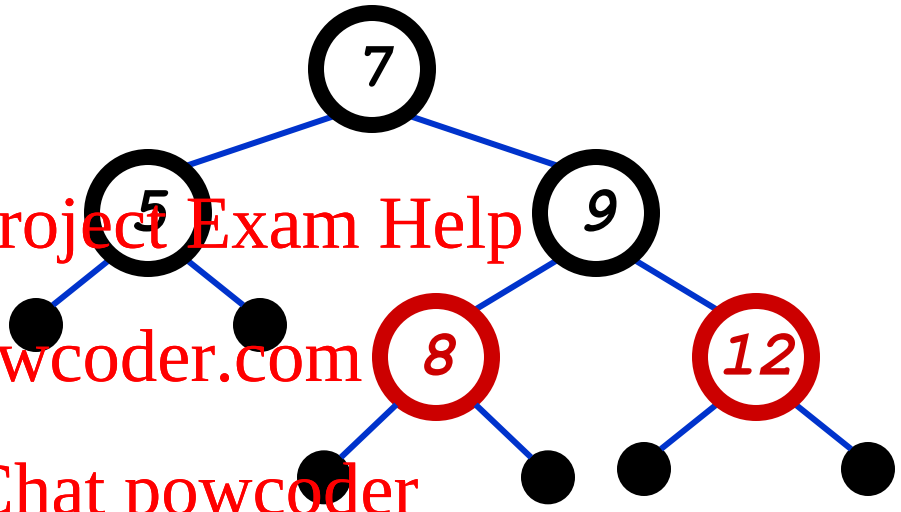
5. Every path from node to descendent leaf contains the same number of black nodes

Red-Black Trees: The Problem With Insertion

- Insert 11
  - *Where does it go?*



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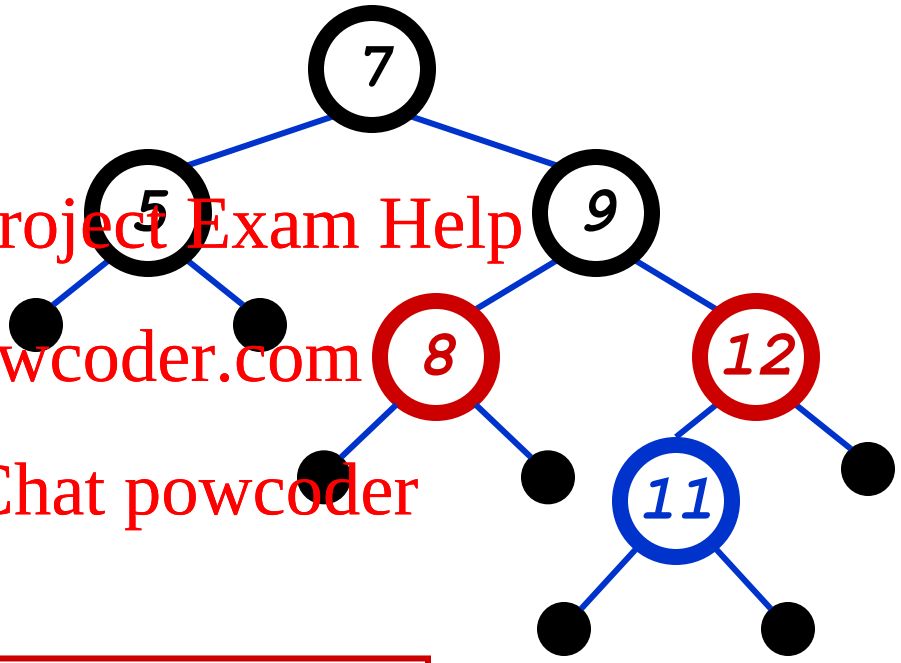
4. If a node is red, both children are black
5. Every path from node to descendent leaf contains the same number of black nodes

Red-Black Trees: The Problem With Insertion

- Insert 11
  - *Where does it go?*
  - *What color?*



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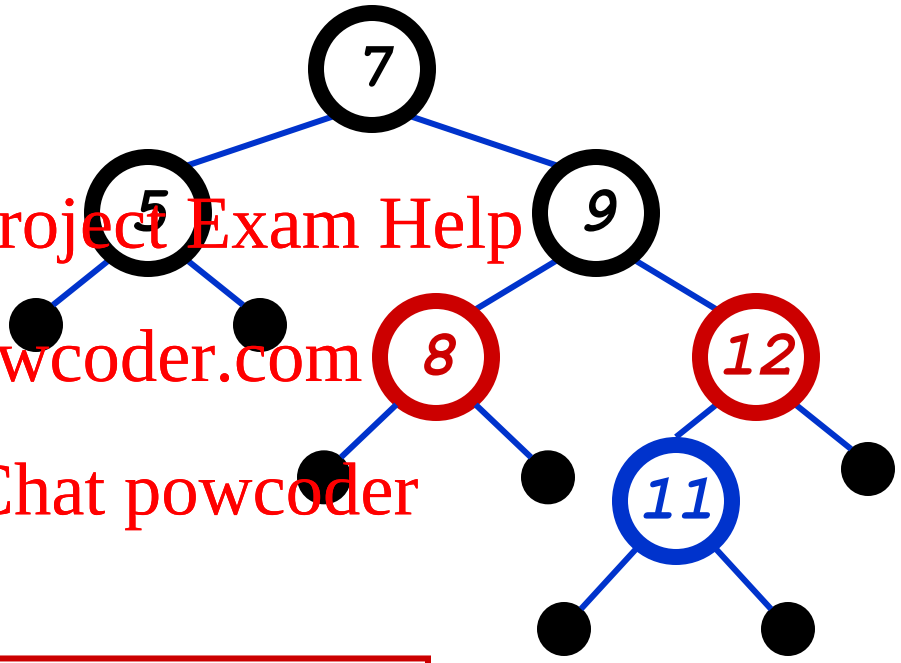
Red-Black Trees: The Problem With Insertion

- Insert 11
  - *Where does it go?*
  - *What color?*
    - Can't be red. (#4)



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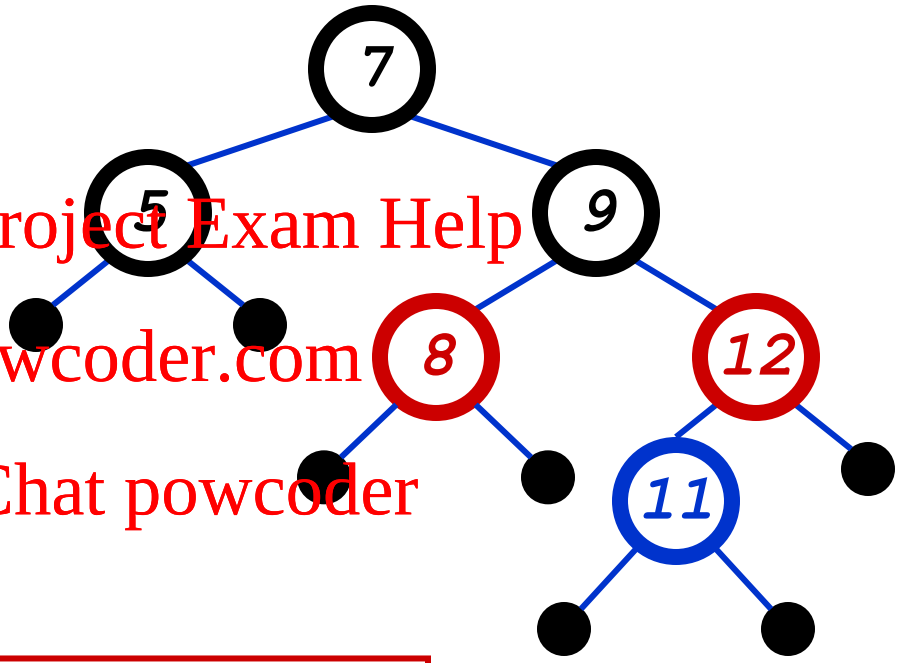
Red-Black Trees: The Problem With Insertion

- Insert 11
  - *Where does it go?*
  - *What color?*
    - Can't be red! (#4)
    - Can't be black! (#5)



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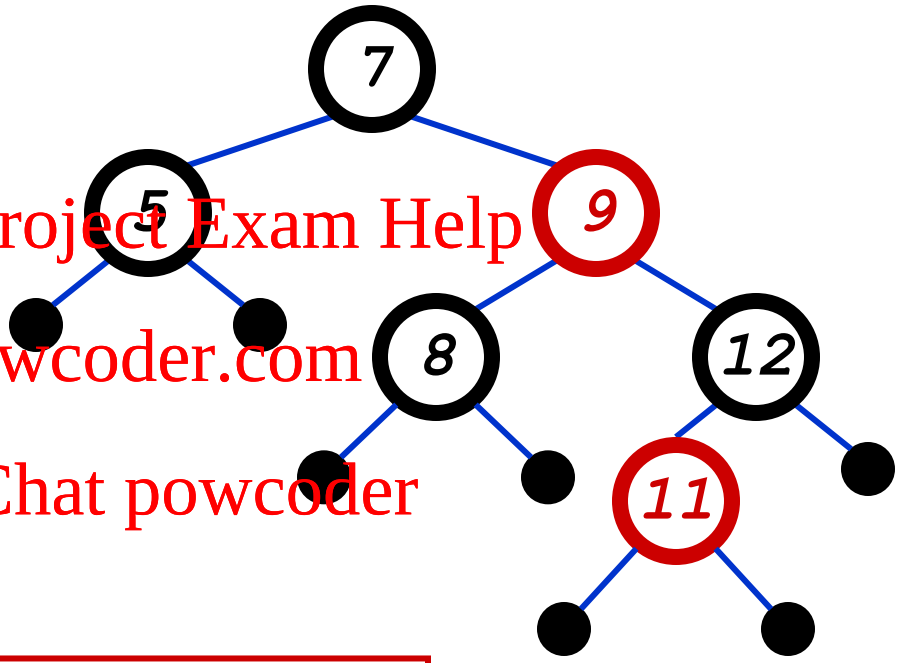
Red-Black Trees: The Problem With Insertion

- Insert 11
  - *Where does it go?*
  - *What color?*
    - Solution: recolor the tree



Red-black properties:

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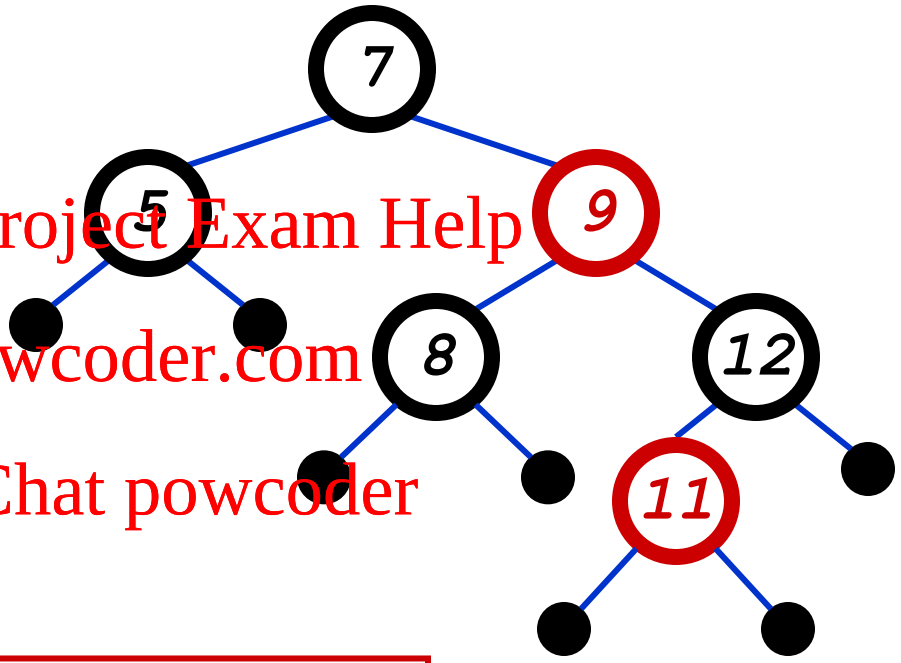
5. Every path from node to descendent leaf contains the same number of black nodes

Red-Black Trees: The Problem With Insertion

- Insert 10
  - *Where does it go?*



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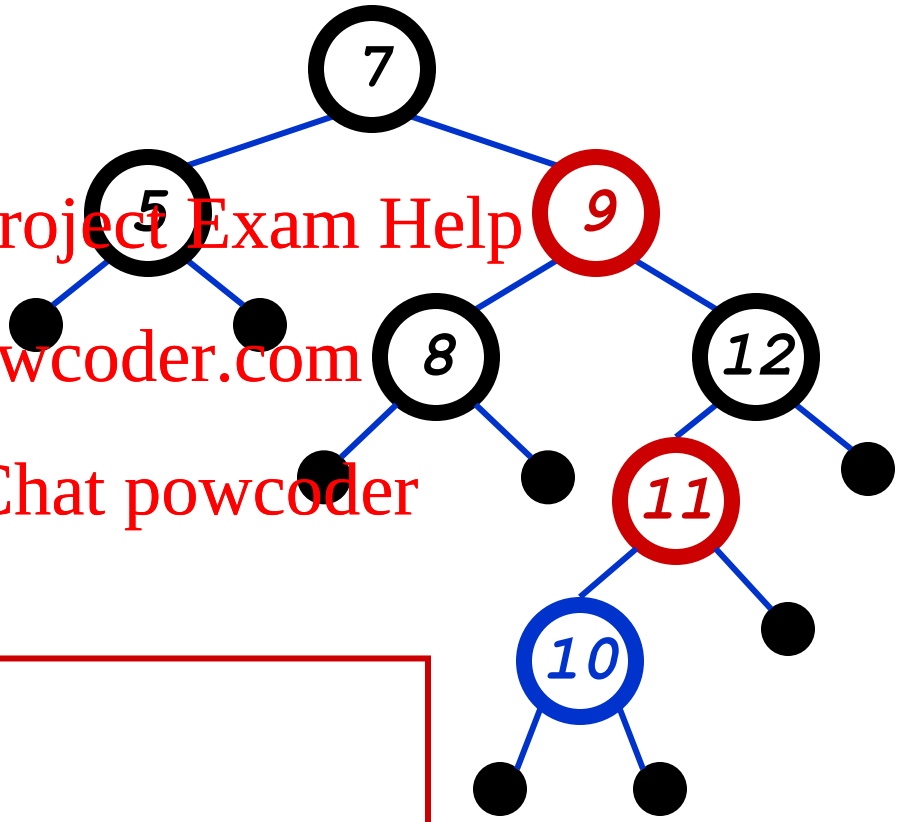
Red-Black Trees: The Problem With Insertion

- Insert 10
  - *Where does it go?*
  - *What color?*



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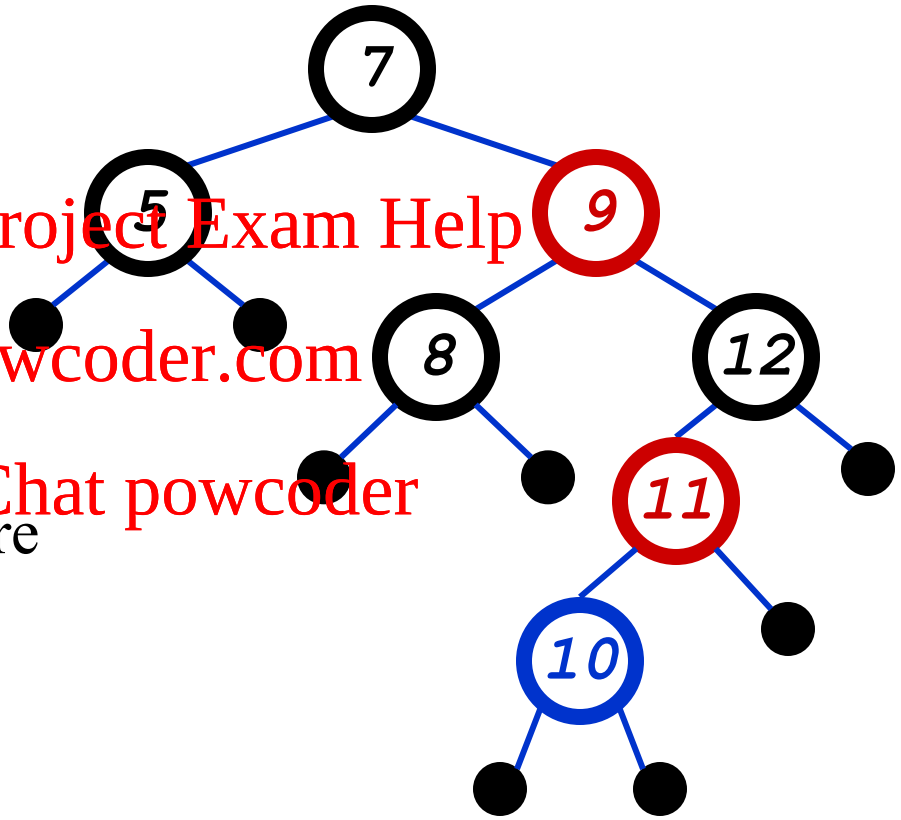
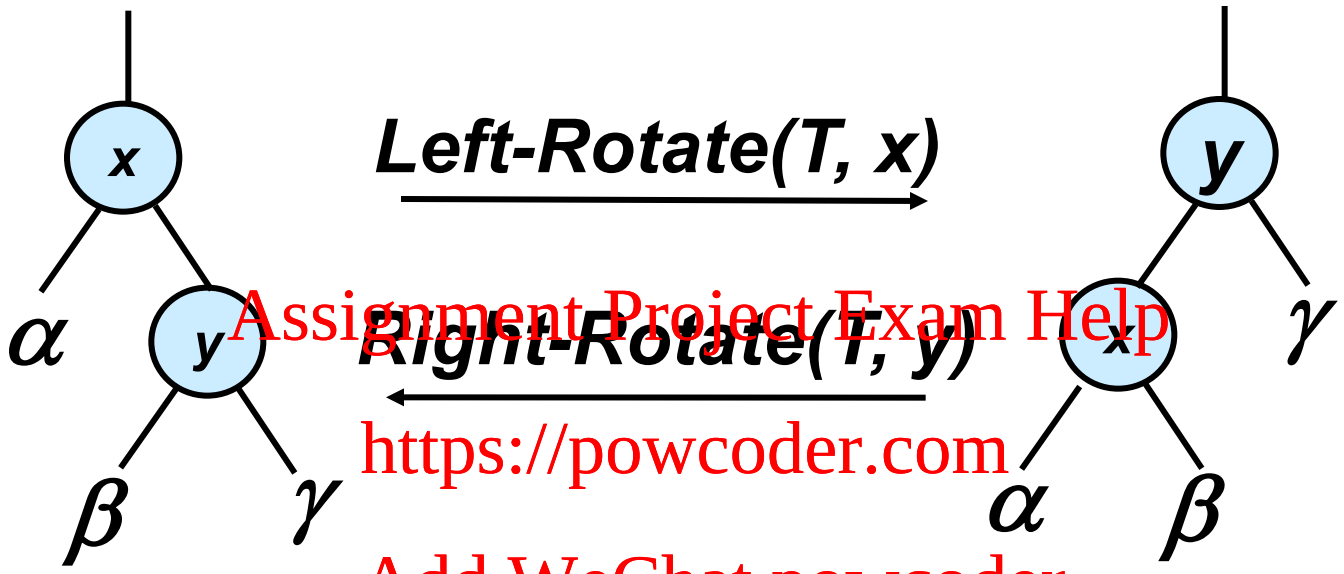
Red-Black Trees: The Problem With Insertion

- Insert 10
  - *Where does it go?*
  - *What color?*
    - Answer: no color! Tree is too imbalanced
    - Must change tree structure to allow recoloring
  - Goal: restructure tree in O(lg *n*) time

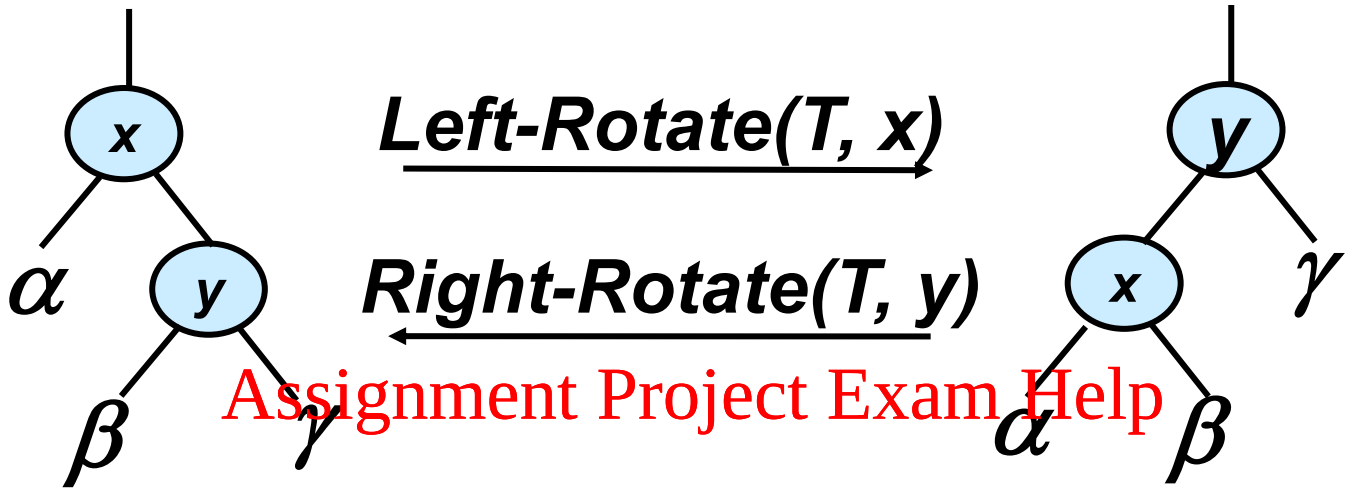


RB Trees: Rotation



- *Does rotation preserve inorder key ordering?*
- *What would the code for* **`rightRotate()`** *actually do?*

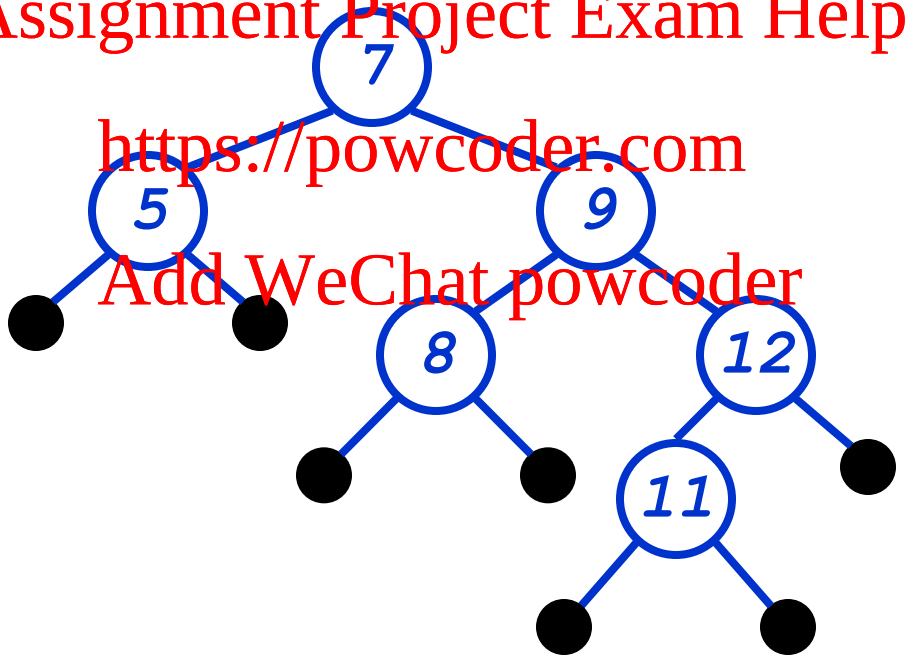
RB Trees: Rotation



- Answer: A lot of pointer manipulation
  - x keeps its left child
  - y keeps its right child
  - x's right child becomes y's left child
  - x's and y's parents change
- *What is the running time?*

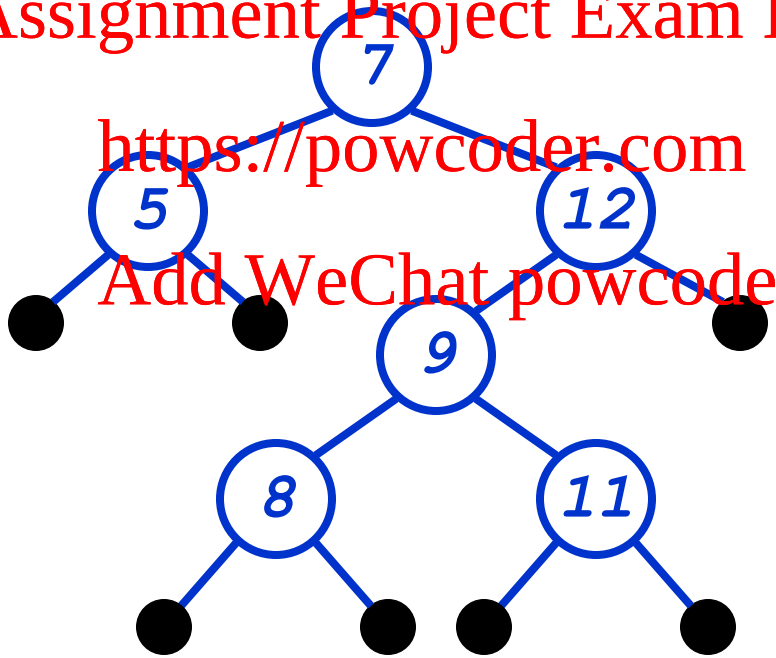
Rotation Example

- Rotate left about 9:



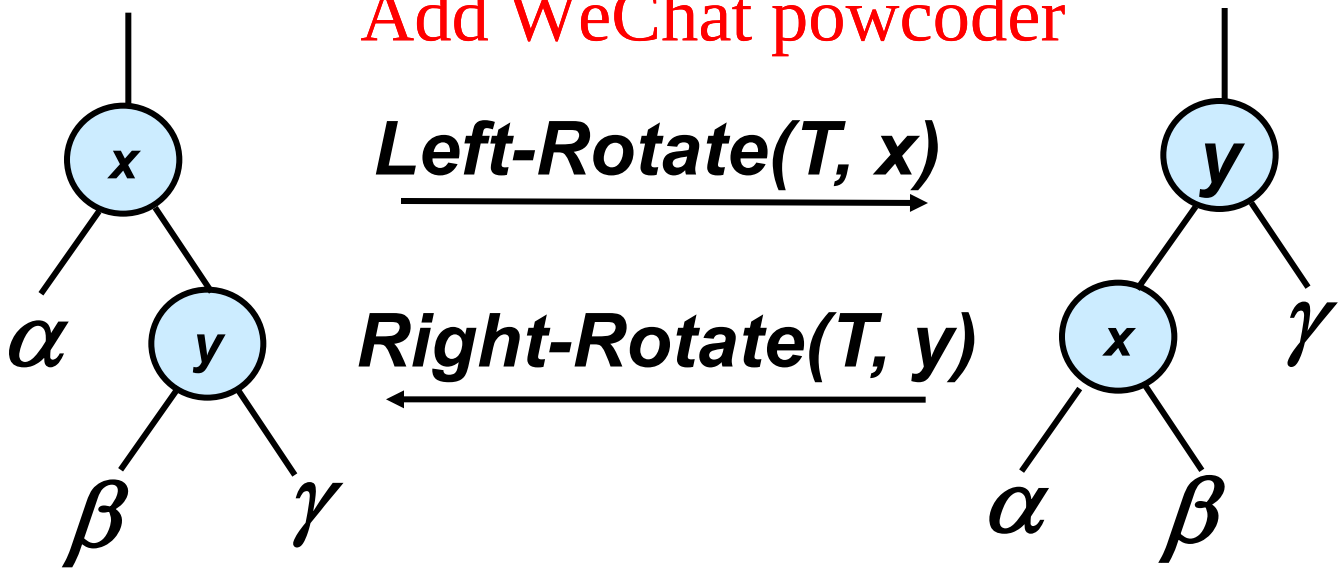
Rotation Example

- Rotate left about 9:



RB Trees: Rotation

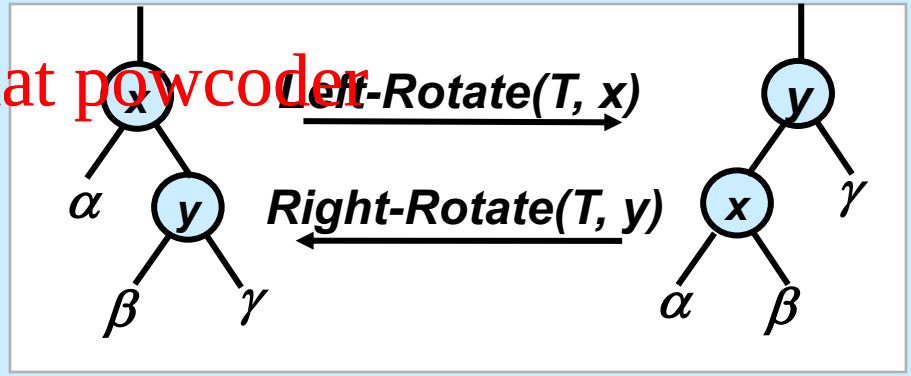
- Rotations are the basic tree-restructuring operation for almost all *balanced* search trees.
- Rotation takes a red-black-tree and a node,
- Changes pointers to change the local structure, and
- Won't violate the binary-search-tree property.
- Left rotation and right rotation are inverses.



Left Rotation – Pseudo-code

Left-Rotate (*T, x*)

```
1.  y = x.right          // Set y.
2.  x.right = y.left     //Turn y's left subtree into x's right subtree.
3.  if y.left ≠ T.nil
4.      then y.left.p = x
5.  y.p = x.p            // Link x's parent to y.
6.  if x.p == T.nil
7.      then T.root = y
8.  else if x == x.p.left
9.           then x.p.left = y
10. else x.p.right = y
11. y.left = x           // Put x on y's left.
12. x.p = y
```

Rotation

- The pseudo-code for Left-Rotate assumes that
  - $X.right \neq T.nil$, and
  - root's parent is $T.nil$.
- Left Rotation on x, makes x the left child of y, and the left subtree of y into the right subtree of x.
- Pseudocode for Right-Rotate is symmetric: exchange *left* and *right* everywhere.
- ***Time:*** $O(1)$ for both Left-Rotate and Right-Rotate, since a constant number of pointers are modified.

Insertion in RB Trees

- Insertion must preserve all red-black properties.
- Should an inserted node be colored Red? Black?
- Basic steps:
  - Use Tree-Insert from BST (slightly modified) to insert a node x into T.
    - Procedure **RB-Insert(x)**.
  - Color the node x red.
  - Fix the modified tree by re-coloring nodes and performing rotation to preserve RB tree property.
    - Procedure **RB-Insert-Fixup**.

Insertion

RB-Insert(T, z)

```
1.   y = T.nil
2.   x = T.root
3.   while x ≠ T.nil
4.         y = x
5.         if z.key < x.key
6.               x = x.left
7.         else x = x.right
8.   z.p = y
9.   if y == T.nil
10.        T.root = z
11.  else if z.key < y.key
12.        y.left = z
13.  else y.right = z
14.  z.left = T.nil
15.  z.right = T.nil
16.  z.color = RED
17.  RB-Insert-Fixup (T, z)
```

How does it differ from the Tree-Insert procedure of BSTs?

Line 14 to 17.

Which of the RB properties might be violated?

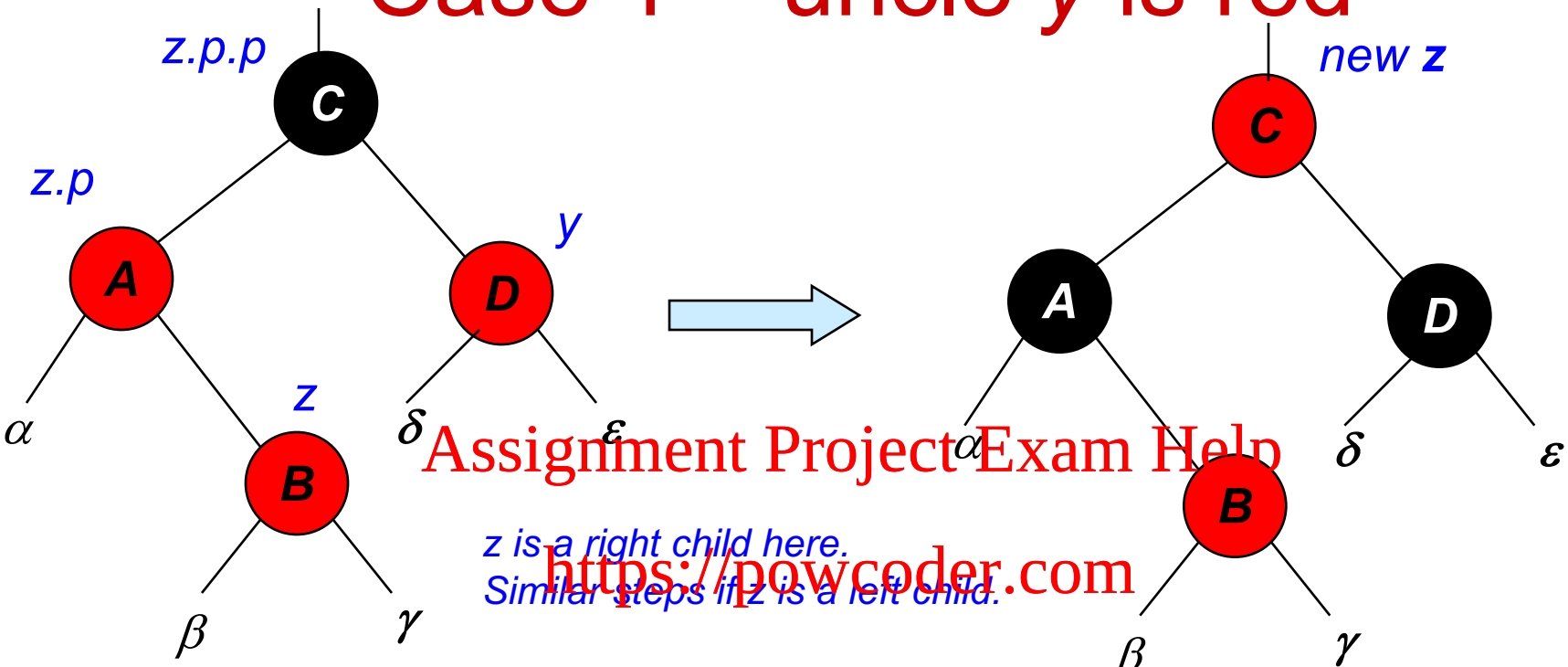
Fix the violations by calling RB-Insert-Fixup.

Insertion – Fixup

RB-Insert-Fixup (*T*, *z*)

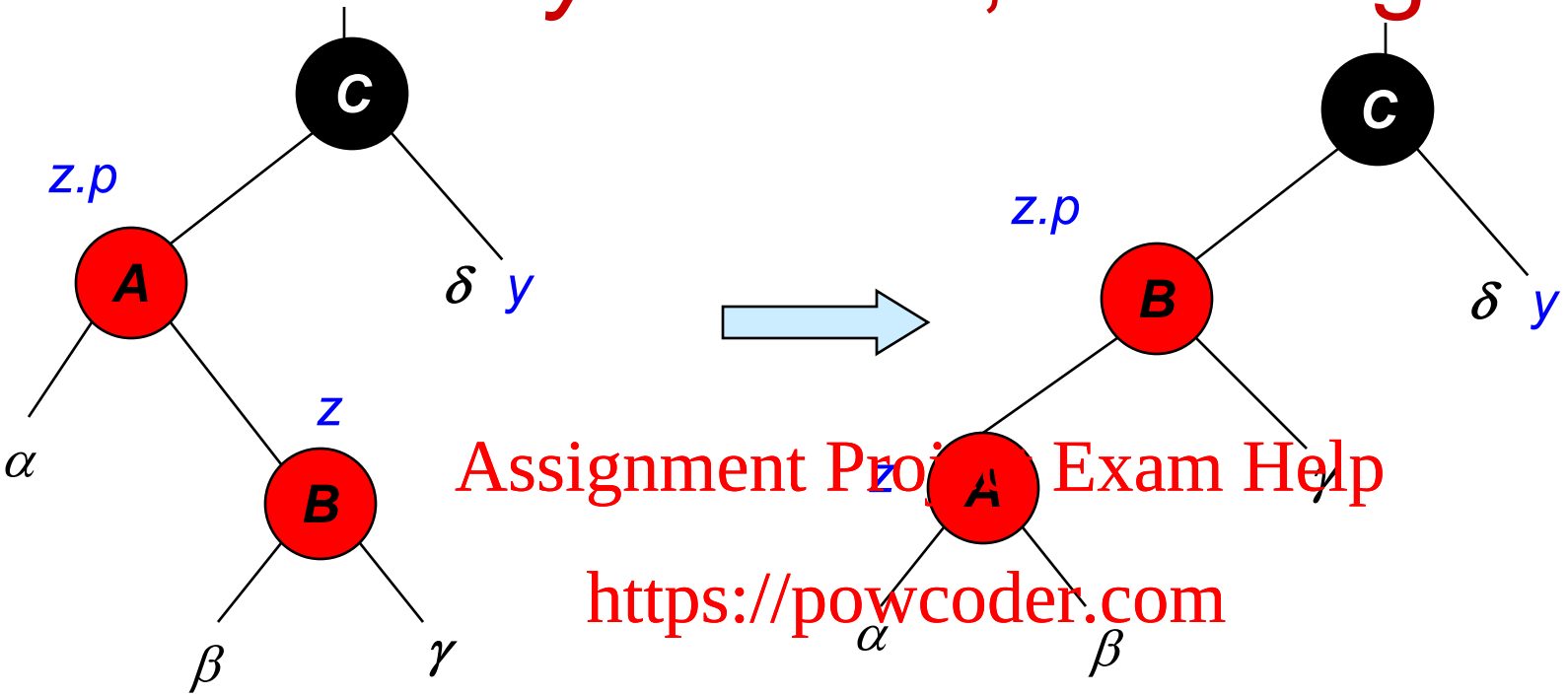
```
1.      while z.p.color == RED
2.          if  z.p == z.p.p.left
3.              y = z.p.p.right                    // y now is z's uncle
4.              if  y.color  == RED
5.                  z.p.color = BLACK              // Case 1
6.                  y.color = BLACK                // Case 1
7.                  z.p.p.color = RED              // Case 1
8.                  z  = z.p.p                     // Case 1
9.              else if  z == z.p.right            // y.color ≠ RED
10.                     z  = z.p                   // Case 2
11.                     LEFT-ROTATE(T, z)          // Case 2
12.             z.p.color  = BLACK                 // Case 3
13.             z.p.p.color  = RED                 // Case 3
14.             RIGHT-ROTATE(T, z.p.p)             // Case 3
15.         else (if z.p  == z.p.p.right)(same as 3-14
16.                 with "right" and "left" exchanged)
17.     T.root.color  = BLACK
```

Case 1 – uncle *y* is red



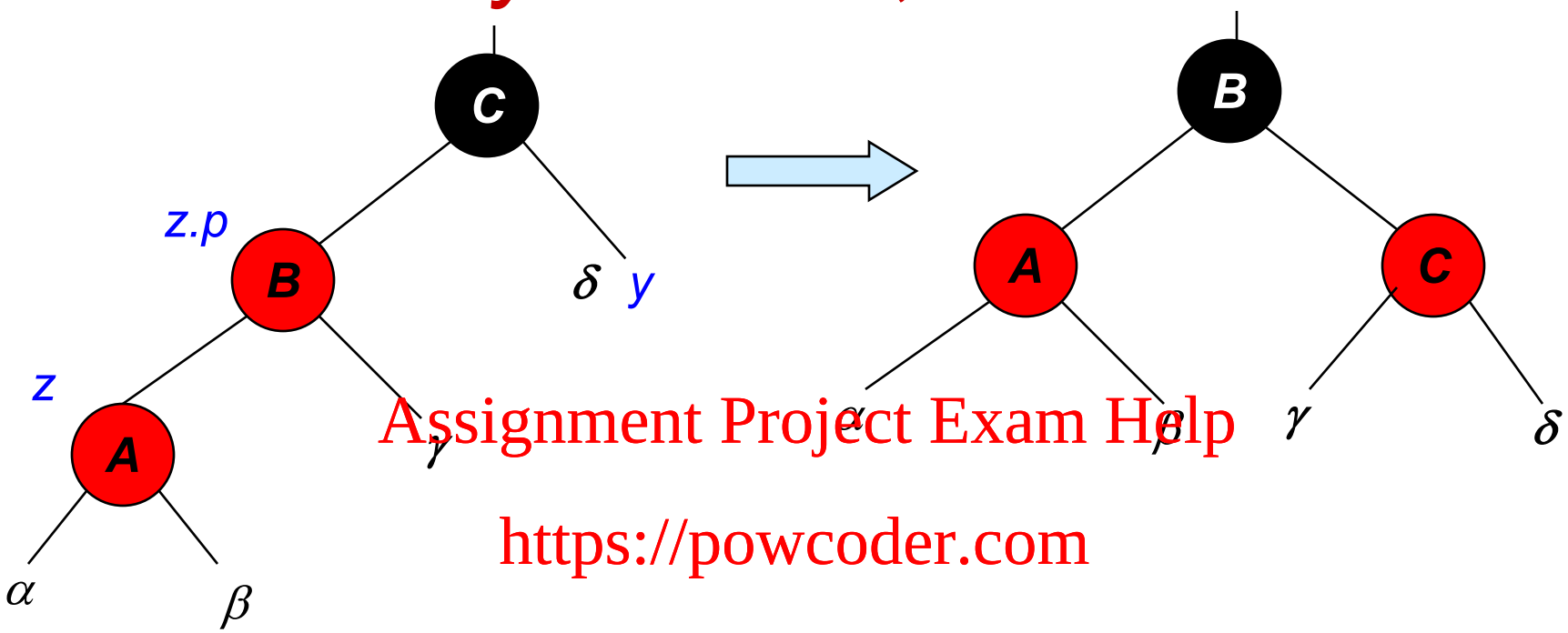
- *z.p.p* (*z*'s grandparent) must be black, since *z* and *z.p* are both red and there are no other violations of property 4.
- Make *z.p* and *y* black $\Rightarrow$ now *z* and *z.p* are not both red. But property 5 might now be violated.
- Make *z.p.p* red $\Rightarrow$ restores property 5.
- The next iteration has *z.p.p* as the new *z* (i.e., *z* moves up 2 levels).

Case 2 – y is black, z is a right child



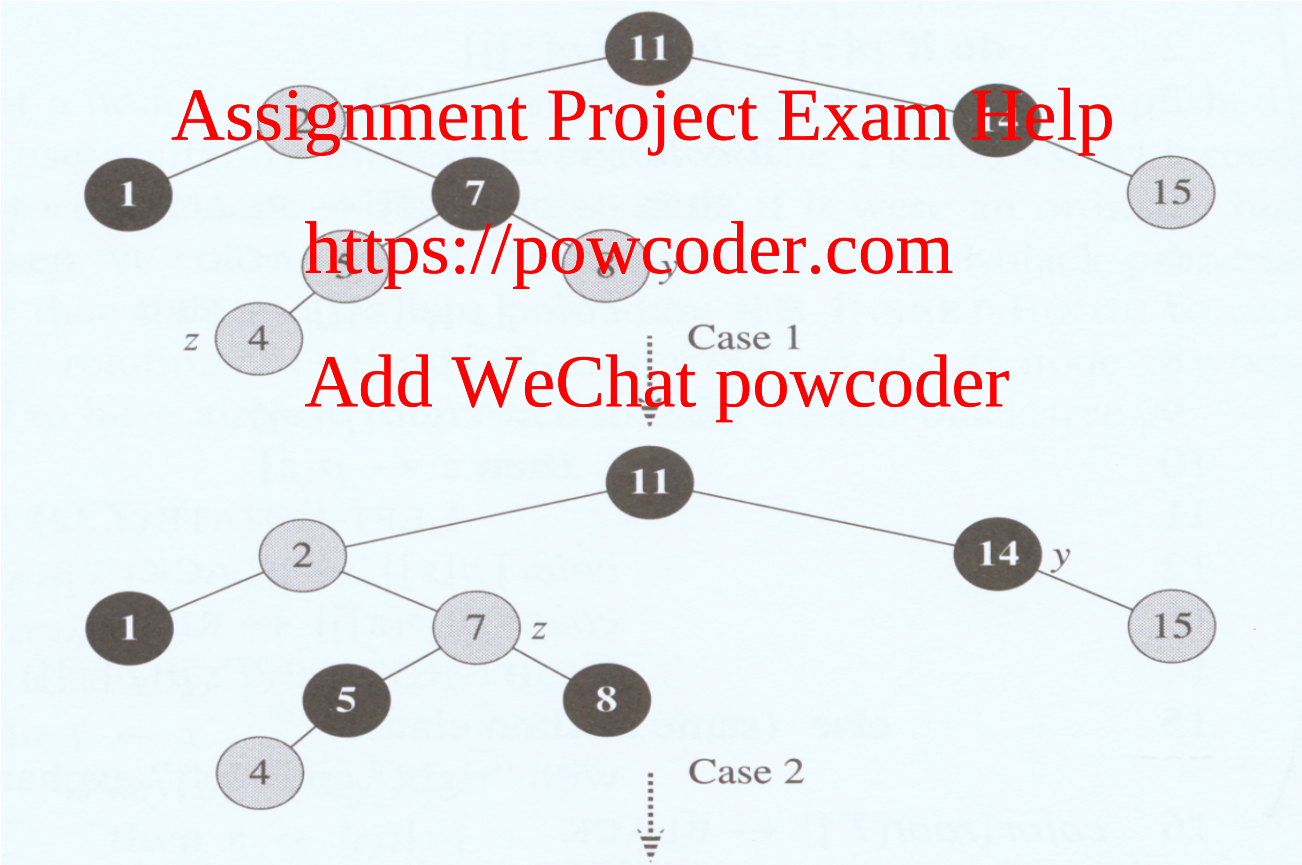
- Left rotate around $z.p$, $z.p$ and z switch roles $\Rightarrow$ now z is a left child, and both z and $z.p$ are red.
- Takes us immediately to case 3.

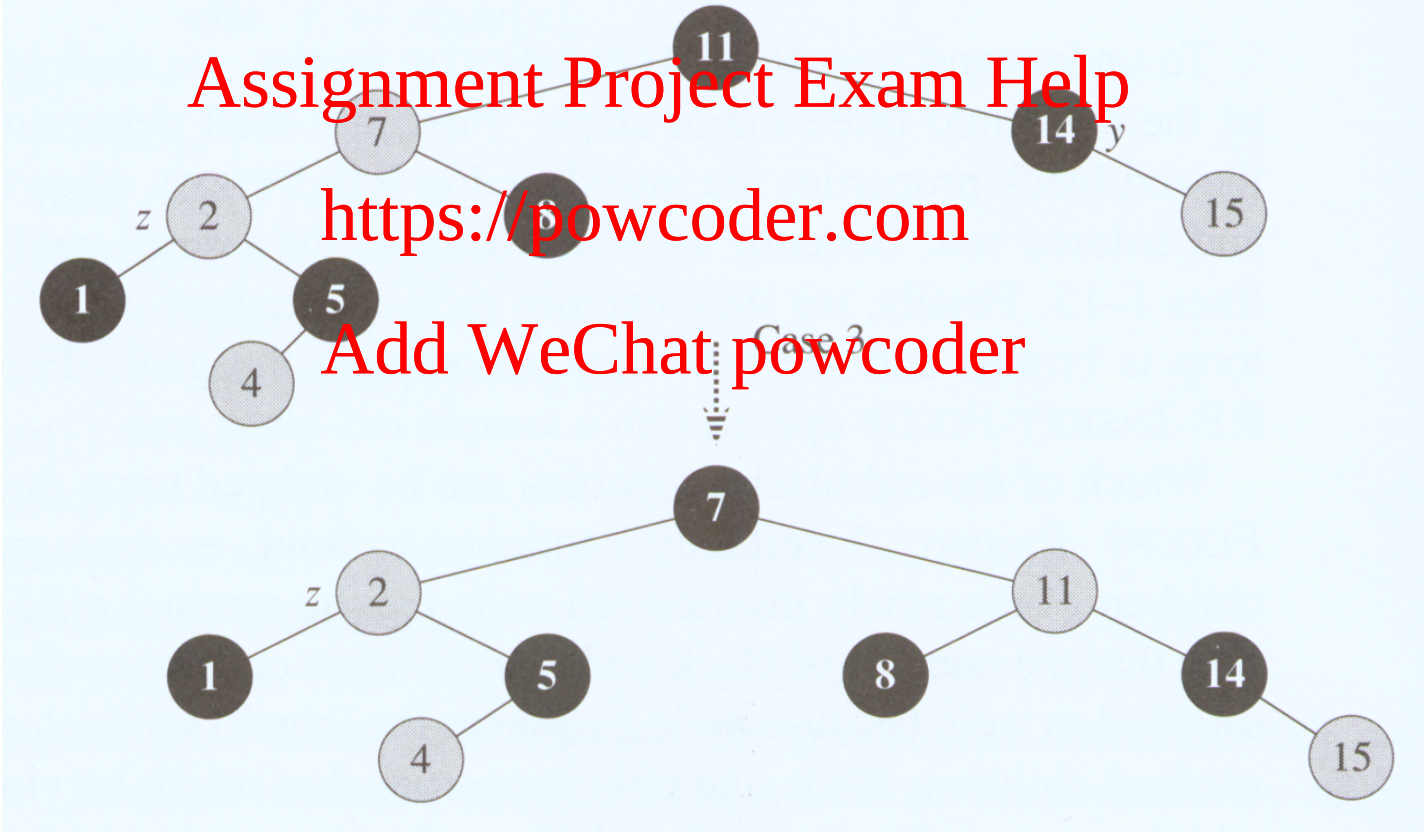
Case 3 – y is black, z is a left child



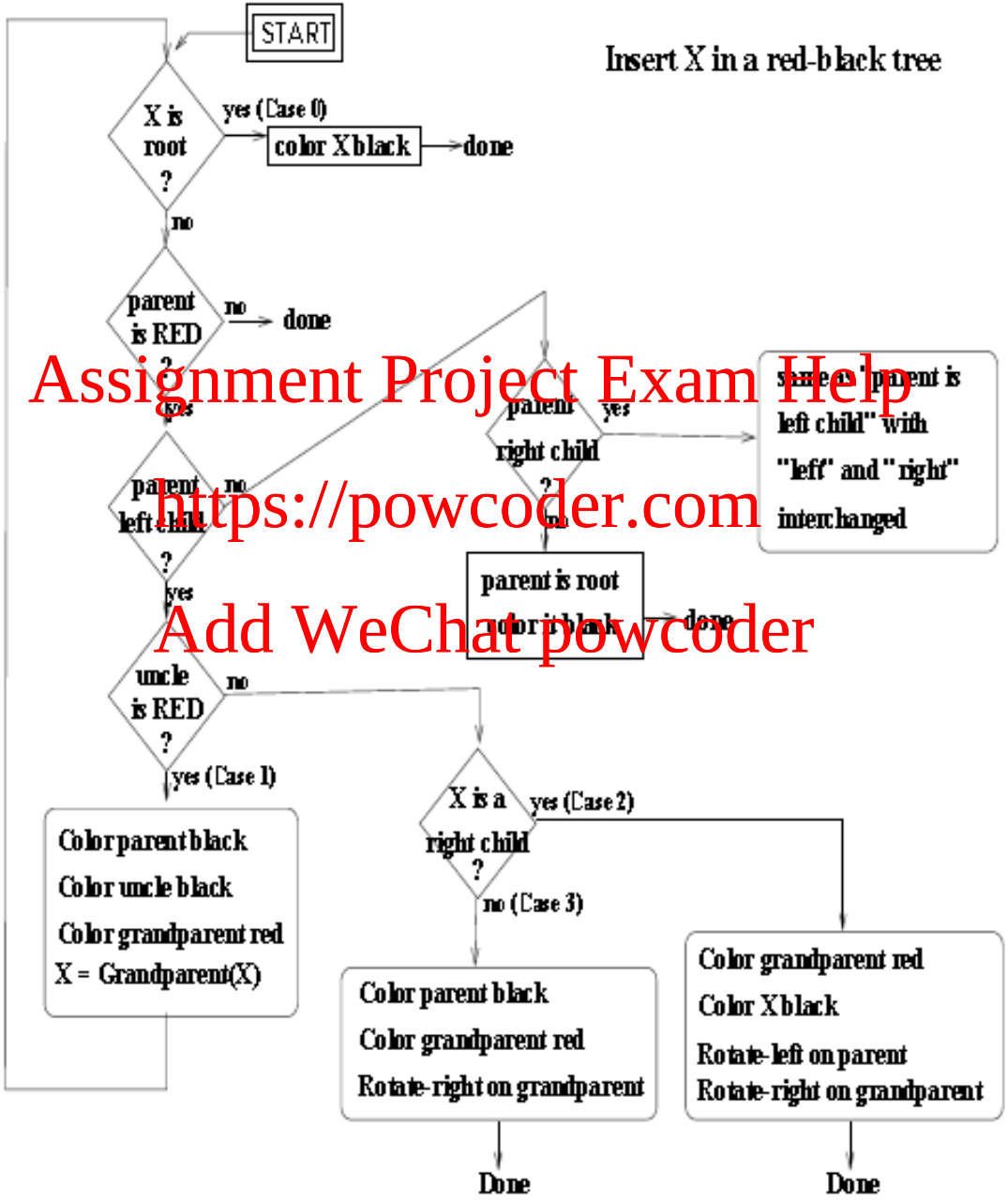
- Make $z.p$ black and $z.p.p$ red.
- Then right rotate on $z.p.p$. Ensures property 4 is maintained.
- No longer have 2 reds in a row.
- $z.p$ is now black $\Rightarrow$ no more iterations.

The operation of RB-INSERT-FIXUP





Flow Chart for Insertion in RB Tress



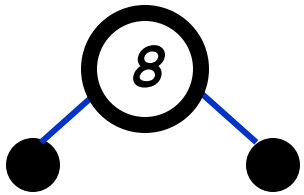
Algorithm Analysis

- $O(\lg n)$ time to get through RB-Insert up to the call of RB-Insert-Fixup.
- Within RB-Insert-Fixup:
  - Each iteration takes $O(1)$ time.
  - Each iteration but the last moves z up 2 levels.
  - $O(\lg n)$ levels $\Rightarrow O(\lg n)$ time.
  - Thus, insertion in a red-black tree takes ***O*(lg *n*)** time.
  - Note: there are at most 2 rotations overall.
    - It never performs more than two rotations, since the while loop terminates if case 2 or case 3 executed.

Example: RB-Tree Insertion

- Insert 8,6,9,11,4,5,12,7 to form a RBTree

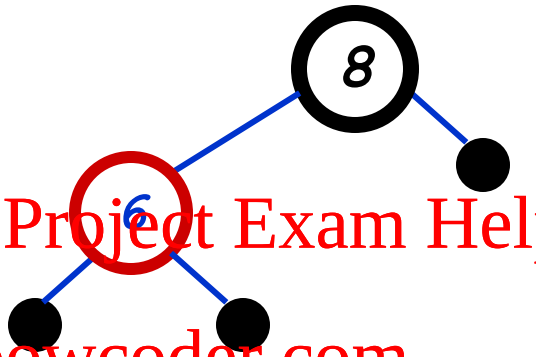
8,6,9,11,4,5,12,7



Red-black properties:

1. Every node is either red or black
2. The root is always black
3. Every leaf (NIL pointer) is black
4. If a node is red, both children are black
5. Every path from node to descendent leaf contains the same number of black nodes

8,6,9,11,4,5,12,7



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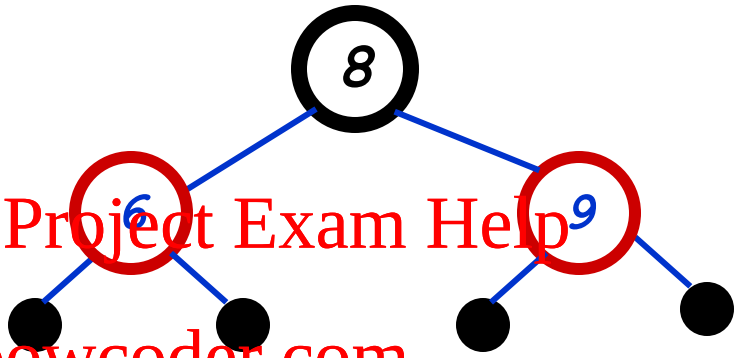

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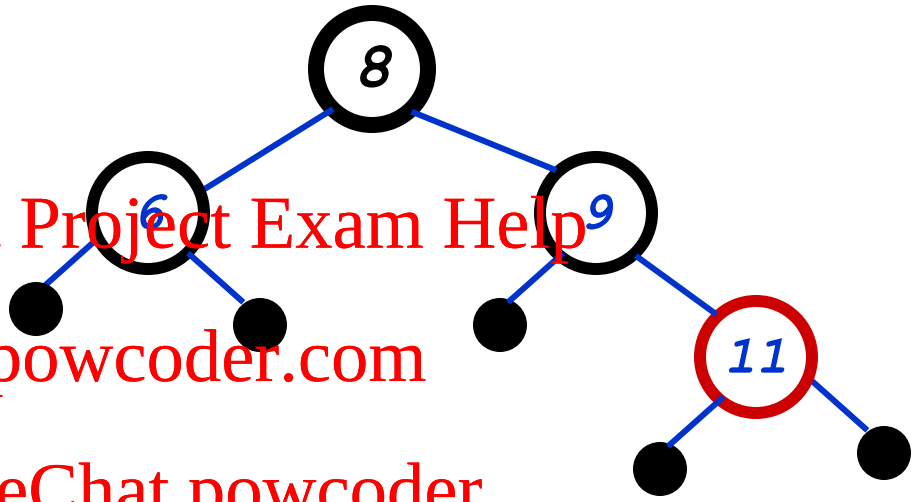

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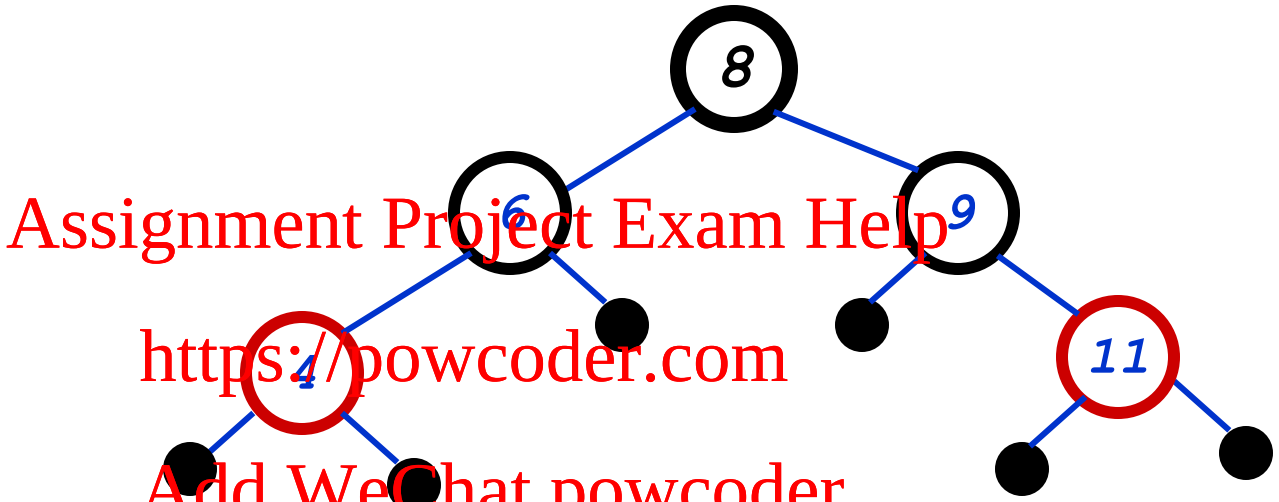

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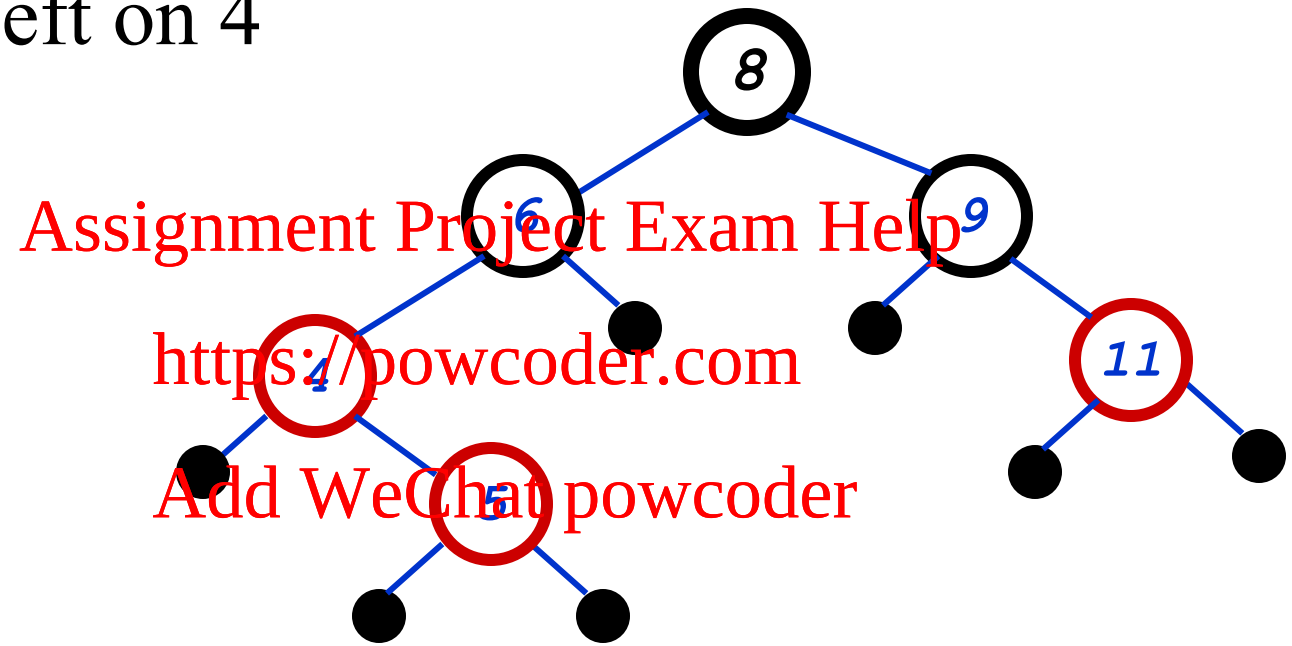
5. Every path from node to descendent leaf contains the same number of black nodes

8,6,9,11,4,5,12,7

- Rotate-left on 4



Red-black properties:

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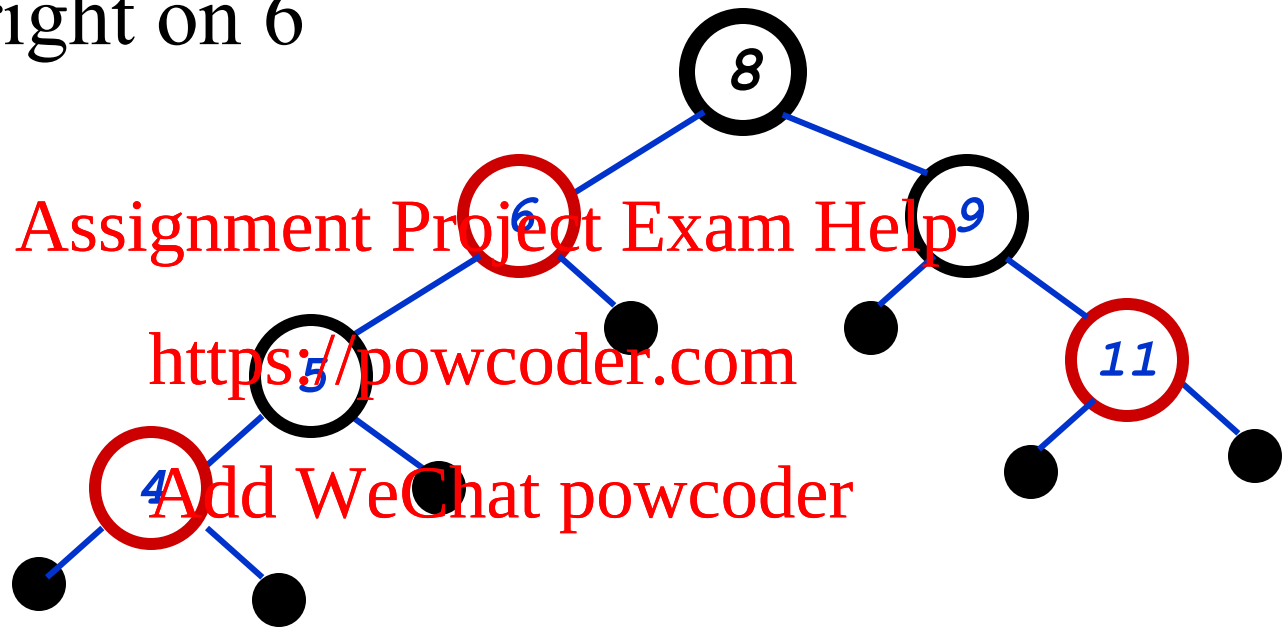
2. The root is always black
3. Every leaf (NIL pointer) is black
4. If a node is red, both children are black
5. Every path from node to descendent leaf contains the same number of black nodes

8,6,9,11,4,5,12,7

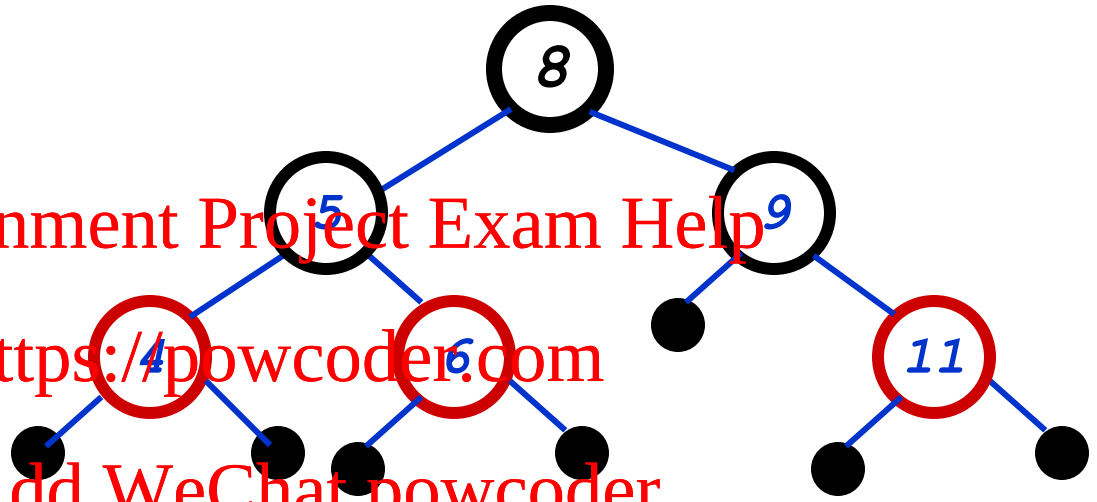
- Rotate-right on 6



Red-black properties:

1. Every node is either red or black
2. The root is always black
3. Every leaf (NIL pointer) is black
4. If a node is red, both children are black
5. Every path from node to descendent leaf contains the same number of black nodes

8,6,9,11,4,5,12,7



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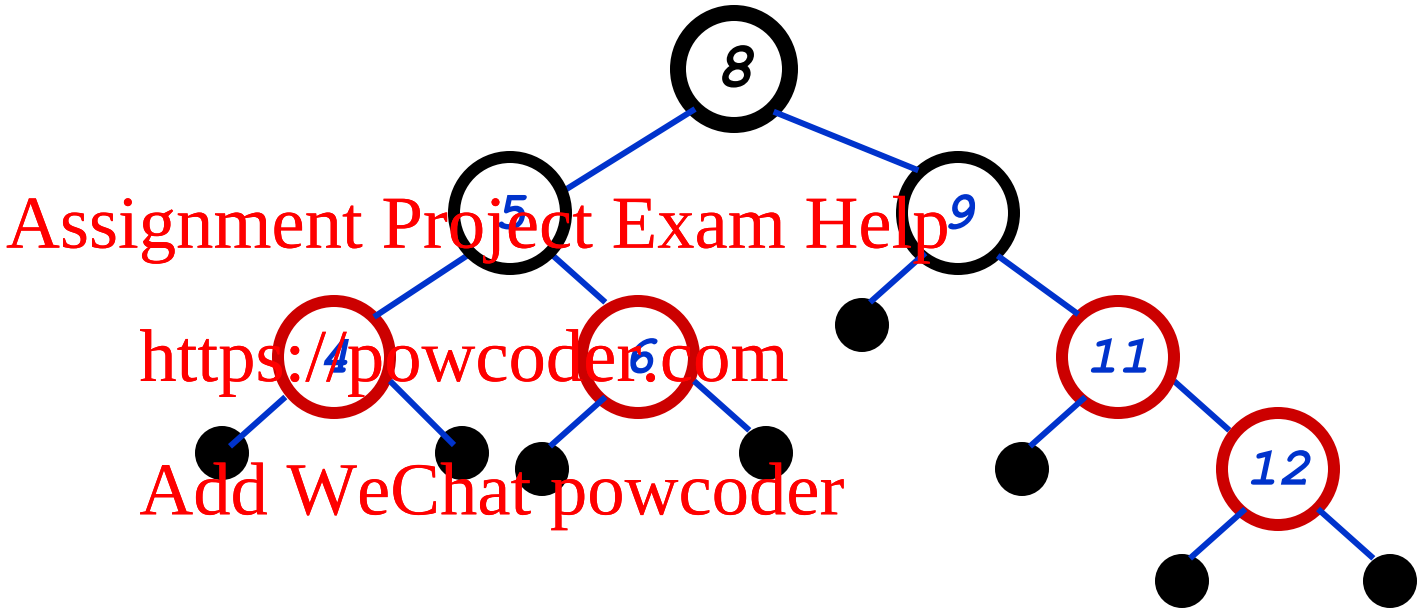
8,6,9,11,4,5,12,7



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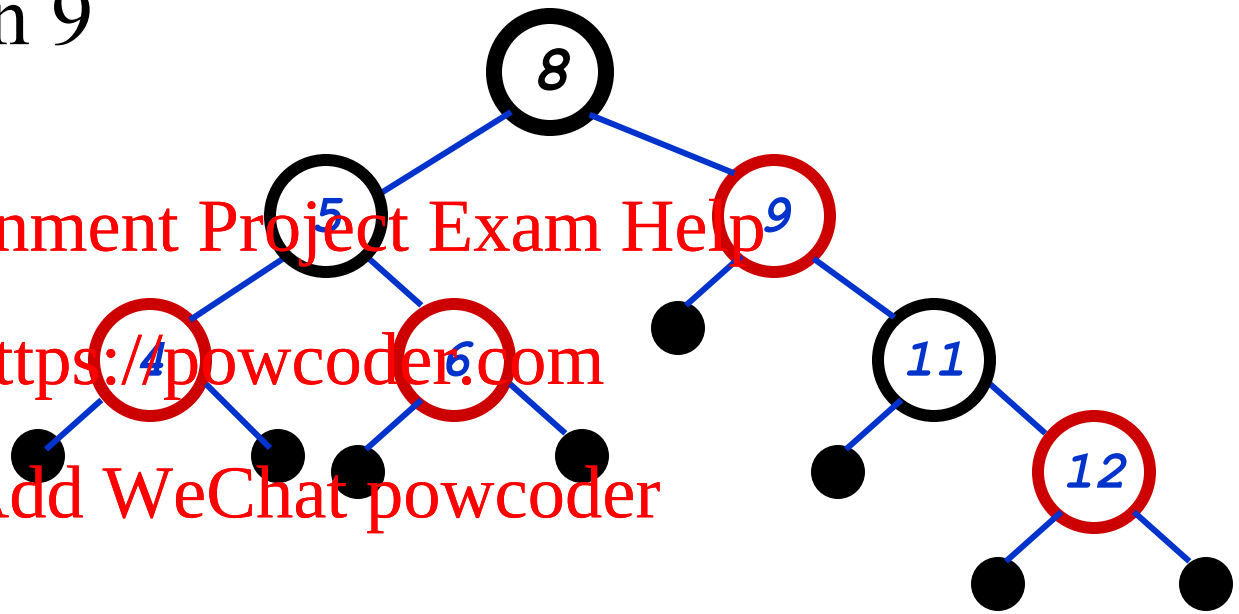
8,6,9,11,4,5,12,7

- Left-Rotate on 9



Red-black properties:

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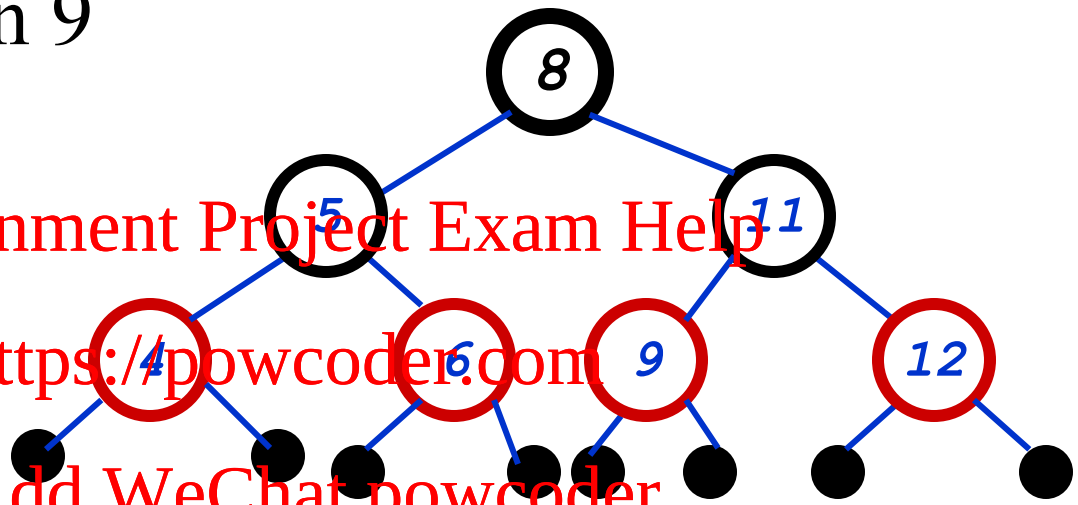
8,6,9,11,4,5,12,7

- Left-Rotate on 9



Red-black properties:

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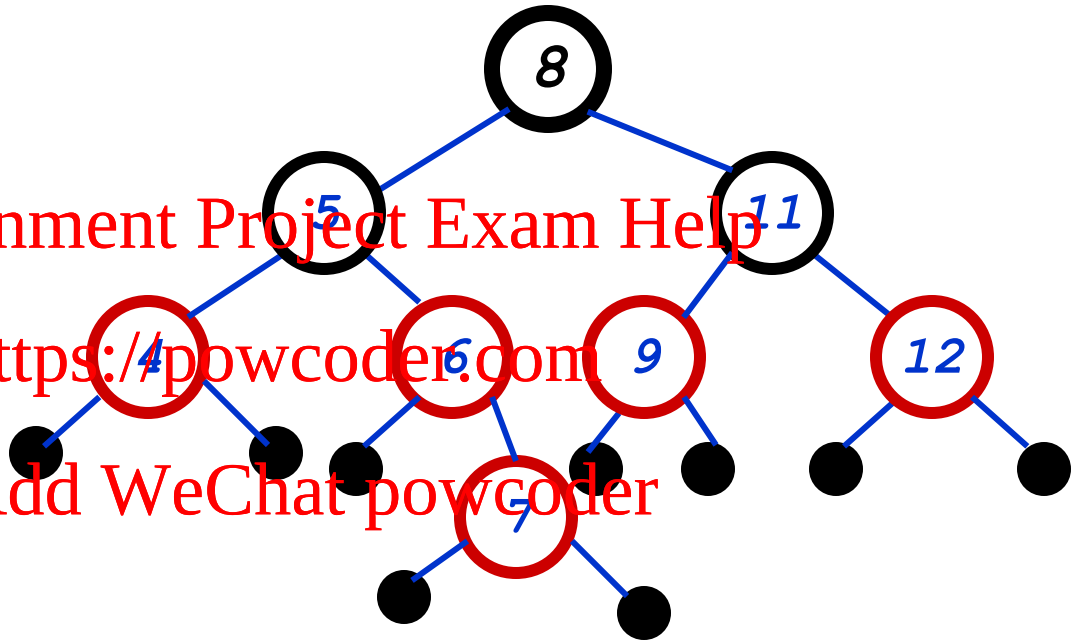
8,6,9,11,4,5,12,7



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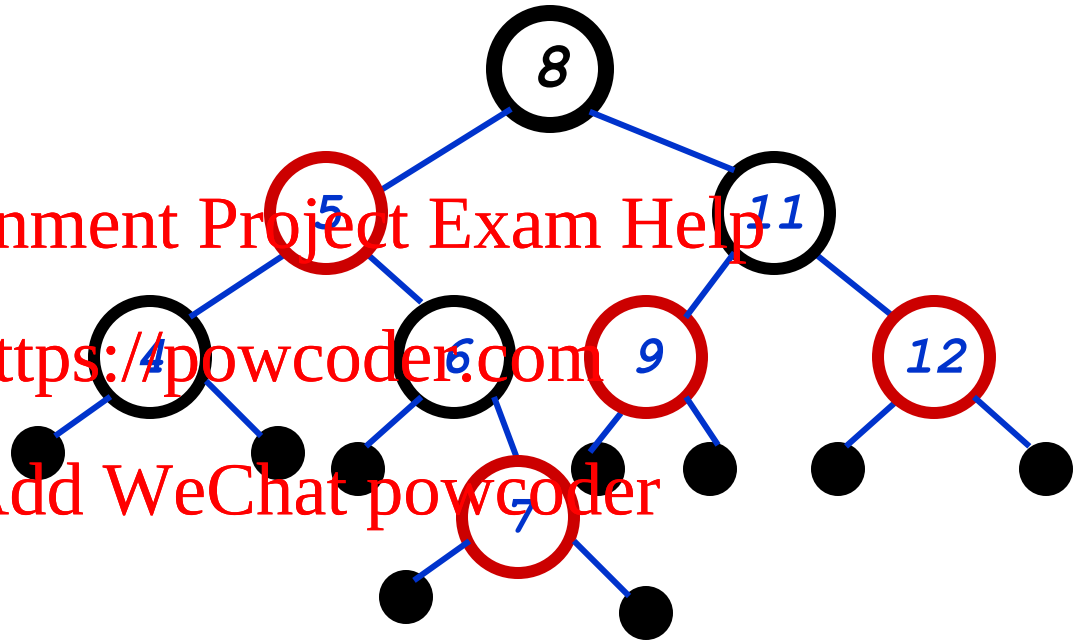
8,6,9,11,4,5,12,7

- Done



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Deletion

- Deletion, like insertion, should preserve all the RB properties.
- The properties that may be violated depends on the color of the deleted node.
  - Red – OK.
  - Black?
- Steps:
  - Do regular BST deletion.
  - Fix any violations of RB properties that may result.

Deletion - Transplant

Transplant(*T*, *u*, *v*)

```
/* Handle u is root of T */
1.  if u.p == NIL
2.        T.root = v
/* if u is a left child */
3.   else if u == u.p.left
4.        u.p.left = v
/* if u is a right child */
5.   else u.p.right = v
/* update v.p if v is non-NIL */
6.   if v ≠ NIL
7.        v.p = u.p
```

RB-Transplant(*T*, *u*, *v*)

```
/* Handle u is root of T */
1.  if u.p == T.nil
2.        T.root = v
/* if u is a left child */
3.   else if u == u.p.left
4.        u.p.left = v
/* if u is a right child */
5.   else u.p.right = v
6.   v.p = u.p
```

Deletion – BST v.s. RB Tree

```
Tree-Delete(T, z)
    /* (a) z has no left child */
1   if z.left == NIL
2.          Transplant(T, z, z.right)
    /* (b) z has a left child, but no right child */
3   else if z.right == NIL
4           Transplant(T, z, z.left)
    /* z has two children */
5    else y = Tree-Minimum(z.right)  /* find z's successor */
6       if y.p ≠ z /* (d) y lies within y's right subtree but is
        not the root of this subtree */
7               Transplant(T, y, y.right)
8               y.right  = z.right
9               y.right.p  = y
    /* (c) if y is z's right child */
10      Transplant(T, z, y)
11      y.left  =  z.left     /* replace y's left child by z's left
        child */
12      y.left.p = y
```

```
RB-Delete(T, z)
1.   y = z
2.   y-original–color = y.color
3.   if z.left == T.nil
4.          x = z.right
5.          RB-Transplant(T, z, z.right)
6.   else if z.right == T.nil
7.          x = z.left
8.          RB-Transplant(T, z, z.left)
9.   else y = Tree-Minimum(z.right)
10.         y-original–color = y.color
11.         x = y.right
12.         if y.p == z
13.                x.p = y
14.         else RB-Transplant(T, y, y.right)
15.                y.right  = z.right
16.                y.right.p  =y
17.         RB-Transplant(T, z, y)
18.         y.left  =  z.left
19.         y.left.p = y
20.         y.color = z.color
21.  if y-original-color == BLACK
22.         RB-Delete-Fixup(T, x)
```

RB-Deletion

```
RB-Delete(T, z)
1.  y = z
2.  y-original–color = y.color
3.  if z.left == T.nil
4.      x = z.right
5.      RB-Transplant(T, z, z.right)
6.  else if z.right == T.nil
7.      x = z.left
8.      RB-Transplant(T, z, z.left)
9.  else y = Tree-Minimum(z.right)   // z has two children
10.     y-original–color = y.color
11.     x = y.right
12.     if y.p == z
13.         x.p = y
14.     else RB-Transplant(T, y, y.right)
15.         y.right  = z.right
16.         y.right.p  =y
17.     RB-Transplant(T, z, y)
18.     y.left  =  z.left
19.     y.left.p = y
20.     y.color = z.color
21.  if y-original-color == BLACK
22.     RB-Delete-Fixup(T, x)
```

The node passed to the fixup routine is the lone child of the spliced up node, or the sentinel.

- 1~2: Maintain y either removed or moved.
- 3 ~ 8: if z has fewer than two children, z is removed.
- 9: z has two children, y points to z's successor. After deleting z, y will point to z's position
- 10: y may change color. keep y's color. We cannot remove or move y if it's black (line 21).
- 11: Keep track of x that moves into y's original position like BST.
- 12: if z's successor y is z's child
- 13: set x.p to y causes x.p to point to the original position of y's parent, even if x = T.nil.
- 14~20: Remove z and set y to z's position
- 21: if y is Black, we might have introduced one or more violations of properties. Use RB-Delete-Fixup to restore properties.

RB Properties Violation

- If y is black, we could have violations of red-black properties:
  - Prop. 1. OK.
  - Prop. 2. If y is the root and x is red, then the root has become red.
  - Prop. 3. OK.
  - Prop. 4. Violation if $y.p$ and x are both red.
  - Prop. 5. Any path containing y now has 1 fewer black node.

RB Properties Violation

- Prop. 5. Any path containing y now has 1 fewer black node.
  - Correct by giving x an "extra black."
  - Add 1 to count of black nodes on paths containing x.
  - Now property 5 is OK, but property 1 is not.
  - x is either ***doubly black*** (if $x.color$ = BLACK) or ***red & black*** (if $x.color$ = RED).
  - The attribute $x.color$ is still either RED or BLACK. No new values for *color* attribute.
  - In other words, the extra blackness on a node is by virtue of x pointing to the node.
- Remove the violations by calling RB-Delete-Fixup.

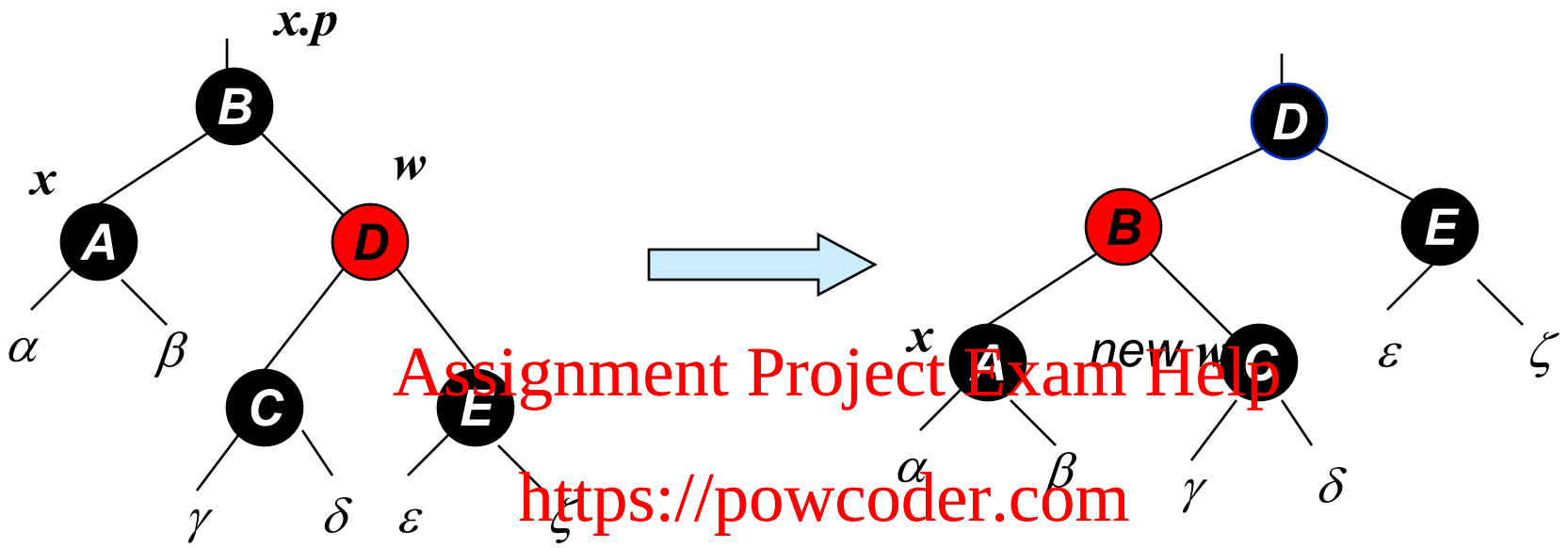
RB-Delete-Fixup(*T, x*) // to restores properties 1, 2, 4.

```
1.  while x ≠ T.root and x.color == BLACK
2.      if x == x.p.left
3.          w = x.p.right
4.          if w.color == RED
5.              w.color = BLACK                    // Case 1
6.              x.p.color = RED                    // Case 1
7.              LEFT-ROTATE(T, x.p)                // Case 1
8.              w = x.p.right                      // Case 1
9.          if w.left.color == BLACK and w.right.color == BLACK   // x is still x.p.left
10.             w.color = RED                      // Case 2
11.             x = x.p                            // Case 2
12.         else if w.right.color == BLACK
13.             w.left.color = BLACK               // Case 3
14.             w.color = RED                      // Case 3
15.             RIGHT-ROTATE(T,w)                  // Case 3
16.             w = x.p.right                      // Case 3
17.         w.color = x.p.color                    // Case 4
18.         x.p.color = BLACK                      // Case 4
19.         w.right.color = BLACK                  // Case 4
20.         LEFT-ROTATE(T, x.p)                    // Case 4
21.         x = T.root                             // Case 4
22.     else (same as then clause with "right" and "left" exchanged)
23. x.color = BLACK
```

Deletion – Fixup

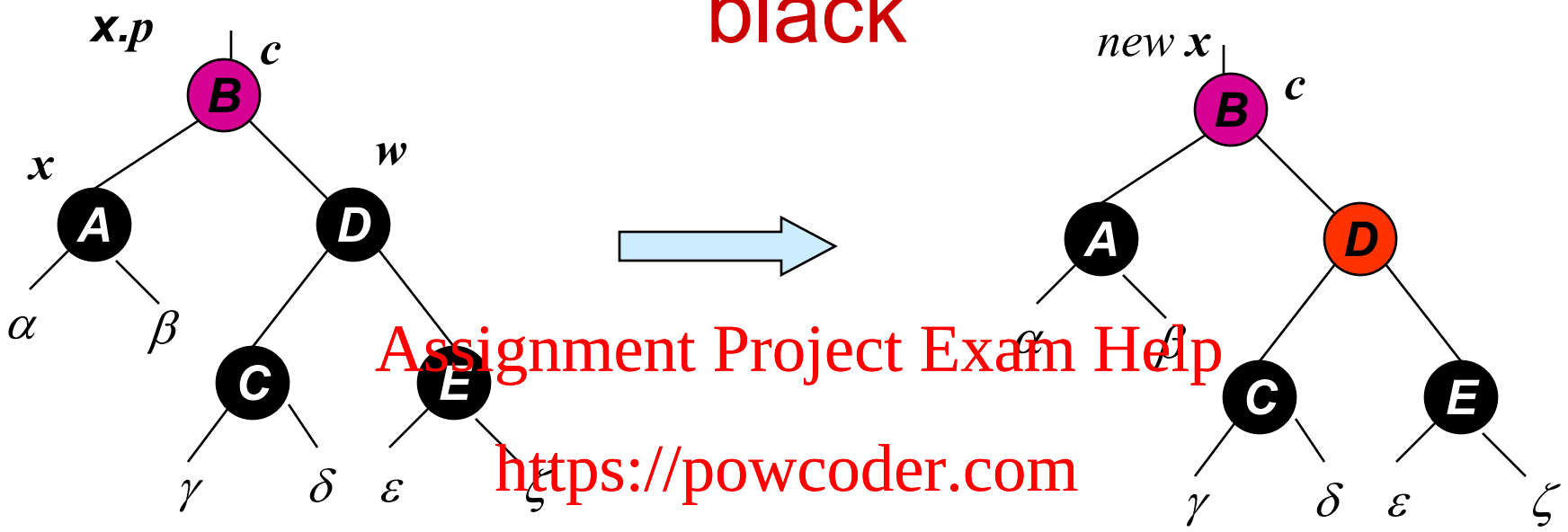
- ***Idea:*** Move the extra black up the tree until x points to a red-&-black node $\Rightarrow$ turn it into a black node (line 23),
- x points to the root $\Rightarrow$ just remove the extra black, or
- Having performed suitable rotations and recolorings and exit the loop.
- Within the **while** loop:
  - x always points to a nonroot doubly black node.
  - w is x's sibling.
  - w cannot be $T.nil$, since that would violate property 5 at $x.p$.
- 8 cases in all, 4 of which are symmetric to the other.

Case 1 – x's sibling *w* is red



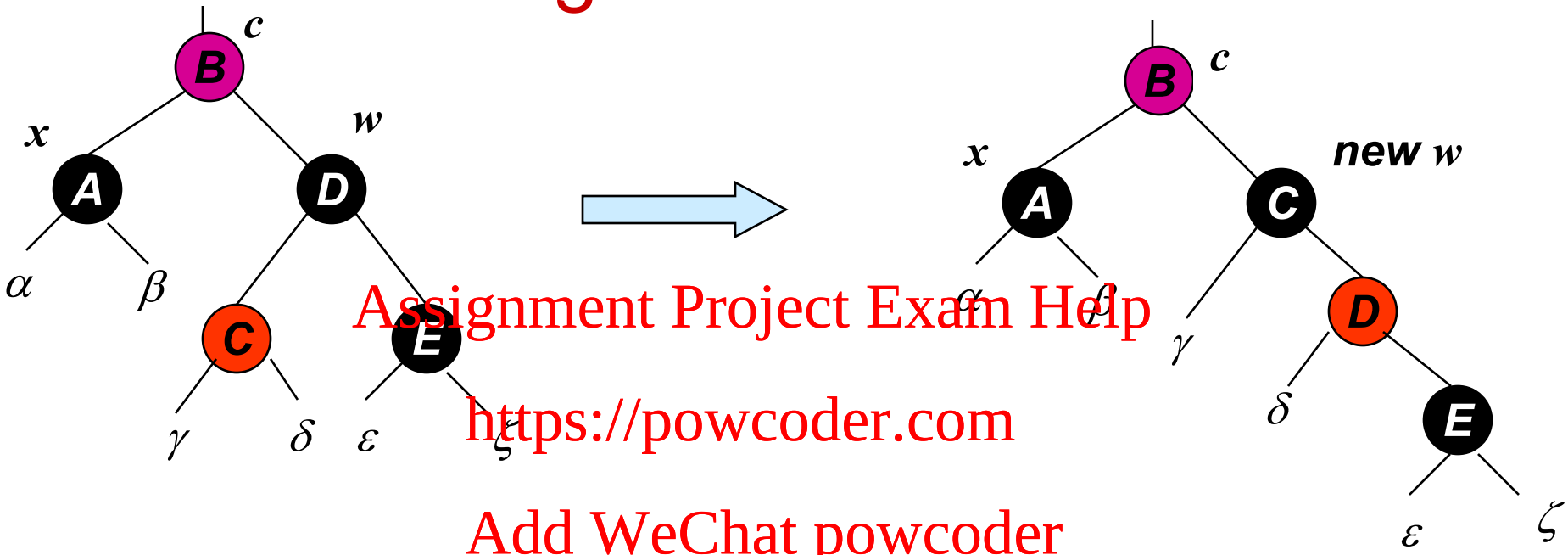
- *w* must have black children.
- Make *w* black and *x.p* red (because *w* is red *x.p* couldn't have been red).
- Then left rotate on *x.p*.
- New sibling of *x* was a child of *w* before rotation ⇒ must be black.
- Go immediately to case 2, 3, or 4.

Case 2 – w is black, both w's children are black



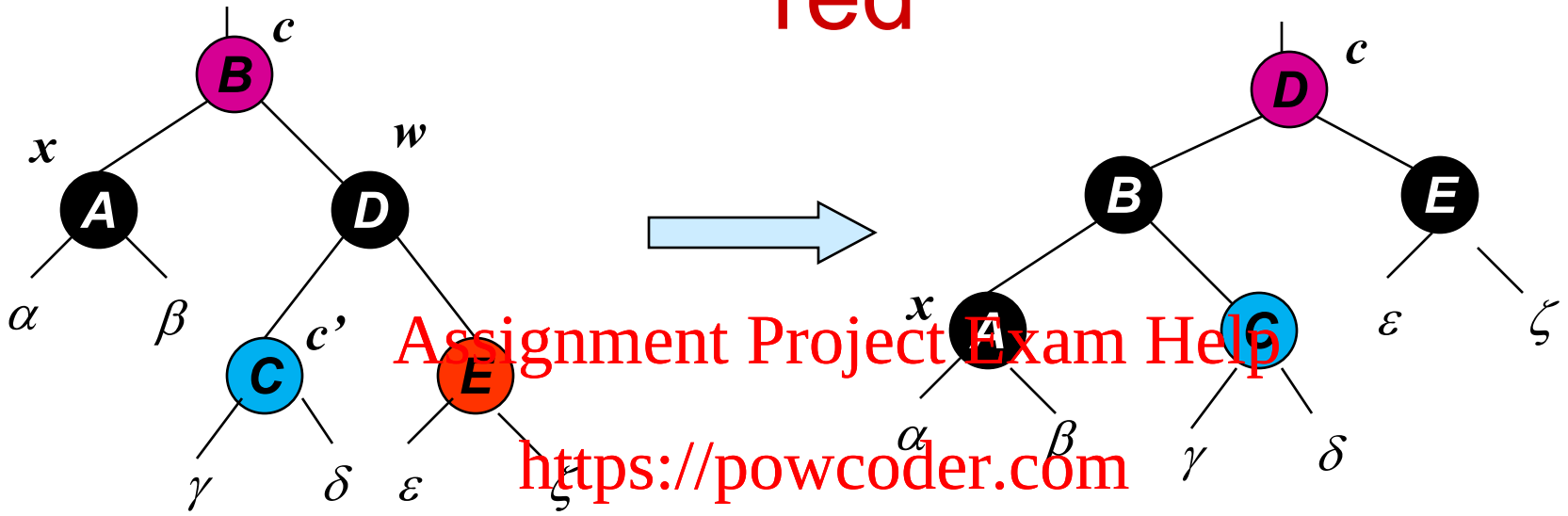
- Take 1 black off x ($\Rightarrow$ singly black) and off w ($\Rightarrow$ red).
- Move that black to $x.p$.
- Do the next iteration with $x.p$ as the new x.
- If entered this case from case 1, then $x.p$ was red $\Rightarrow$ new x is red & black $\Rightarrow$ color attribute of new x is RED $\Rightarrow$ loop terminates. Then new x is made black in the last line.

Case 3 – *w* is black, *w*'s left child is red, *w*'s right child is black



- Make *w* red and *w*'s left child black.
- Then right rotate on *w*.
- New sibling *w* of *x* is black with a red right child $\Rightarrow$ case 4.

Case 4 – w is black, w's right child is red



- Make w be $x.p$'s color (c).
- Make $x.p$ black and w's right child black.
- Then left rotate on $x.p$.
- Remove the extra black on x ($\Rightarrow$ x is now singly black) without violating any red-black properties.
- All done. Setting x to root causes the loop to terminate.

Analysis

- $O(\lg n)$ time to get through RB-Delete up to the call of RB-Delete-Fixup.
- Within RB-Delete-Fixup:
  - Case 2 is the only case in which more iterations occur.
    - x moves up 1 level.
    - Hence, $O(\lg n)$ iterations.
  - Each of cases 1, 3, and 4 has 1 rotation $\Rightarrow \leq 3$ rotations in all.
  - Hence, $O(\lg n)$ time.

RB Tree Visualization

- https://www.cs.usfca.edu/~galles/visualization/RedBlack.html