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ST227 Survival Models-Part II Continuous Time Markov Chains

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1 Stochastic Processes

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1.1 Introduction

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- A **stochastic process** is a model for a time-dependent random phenomenon.

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- Thus, just as a single random variable describes a static random phenomenon, **a stochastic process is a collection of random variables, $Y(t) = Y_t$, one for each time t in some set J.**

- The **process is denoted** $\{Y_t : t \in J\}$. The set of values that the random variables Y_t can take is called the **state space** of the process, S.

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- The **first choice that one faces** when selecting a stochastic process to model a real-life situation **is that of the nature (discrete or continuous) of the time set J and of the state space S.**

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Example 1: Discrete state space with discrete time changes

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- A motor insurance company reviews the status of its customers yearly. Three levels of discount are possible $(0, 25\%, 40\%)$ depending on the accident record of the driver.

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- In this case the appropriate state space is $S = \{0, 25, 40\}$ and the time set is $J = \{0, 1, 2, ...\}$ where each interval represents a year.

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Example 2: Discrete state space with continuous time changes

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- A life insurance company classifies its policyholders as healthy, ill or dead.

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- Hence the state space $S = \{h, i, d\}$.

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- As for the time set, it is natural to take $J = [0, \infty)$ as illness or death can occur at any time. This problem is studied in some detail in what follows (Continuous Time Markov Chains).

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Example 3: Continuous state space

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- Claims of unpredictable amounts reach an automobile insurance company at unpredictable times; the company needs to forecast the **cumulative claim** over $[0, t]$ in order to assess the risk that it might not be able to meet its liabilities.

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- It is standard practice to use $[0, \infty)$ both for S and J in this problem.

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- It is important to be able to **conceptualise the nature of the state space of any process** which is to be analysed, **and to establish whether it is most usefully modelled using a discrete, a continuous, or a mixed time domain**.

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- Usually the **choice of state space will be clear from the nature of the process being studied** (as, for example, with the healthy-ill-dead model), **but whether a continuous or discrete time set is used** will often **depend on** the **specific aspects of the process** which are of interest, **and** upon **practical issues** like the time points for which data are available.

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1.1.1 The Markov property

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- In probability theory and statistics, the term **Markov property** refers to the **memoryless property of a stochastic process**. It is named after the Russian mathematician Andrey Markov.

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- A stochastic process $\{Y(t)\}_{t \geq 0}$ is a Markov process if the conditional probability distribution of future states depends only on the previous state.

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- For example, if $s \geq 0$ and we consider the states i and j, then we can say that $Y(t)$ has the **Markov Property** if $P(Y(t+s) = j | Y(t) = i)$ does not depend on any information before t.

- Hence, the **future development of $Y(t)$ can be predicted from its present state alone, without any reference to its history.**

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2 Continuous Time Markov Chains

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2.1 Introduction

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- Up till now our approach has been to specify a future lifetime in terms of a random variable T_x. In this chapter we look at things rather differently, and use a **Continuous Time Markov Chain (or a Markov model)** of transfers between states.

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- In the simplest case, a life can be alive or dead, and this gives a two-state model of mortality which is known as the **dead-or-alive model**. We often represent this in a diagram like:

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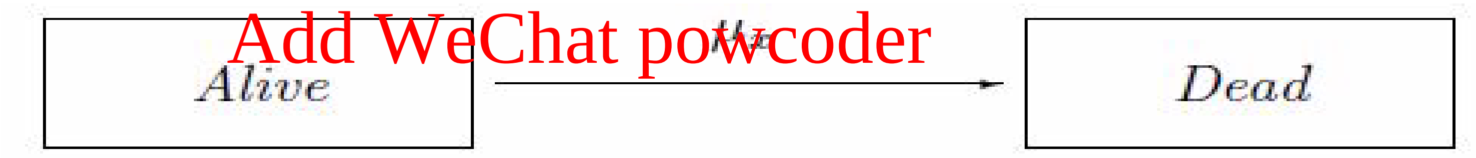

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- The probability that a life alive at any age x should be dead at some future age is determined by an age dependent force of mortality μ_{x+t}, $t \geq 0$, or **transition intensity**.

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- The **main advantages of the Markov model approach** to modelling mortality over the random variable approach are that:

  – it can be **generalised to multiple decrements or multiple state models**, e.g. the three state model $\{Well, Ill, Dead\}$, and

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– it **deals easily with censoring**, a common feature of mortality data, which we will talk about later on in the chapters ahead.

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- We make **two fundamental assumptions** in the 2-state model above.

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1. **(AS1) Markov assumption.** The probability that a life now aged x will be in either state $\{Alive, Dead\}$ at any future time $x+t$ depends only on the age x and the state currently occupied.

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2. **(AS2)** The probability ${}_{dt}q_{x+t}$ is given by

$$ {}_{dt}q_{x+t} = \mu_{x+t}dt + 0(dt), t \geq 0. \quad (1) $$

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- Informally, **assumption 1** says that the **future depends only on the present** and not on the past, while **assumption 2** says that the **probability of death in a short interval of time**, dt, **is approximately proportional to the force of mortality at that time.**

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- **Reminder:** The function $f(h)$ is said to be $o(h)$ or "little o of h" if

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$$\lim_{h \to 0} \frac{f(h)}{h} = 0. \tag{2}$$

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In other words, $f(h)$ is $o(h)$ if $f(h) \to 0$ faster than h.

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2.2 Computation of ${}_tp_x$ in the dead-or-alive model

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Our model is defined in terms of the transition intensities μ_x. How can we compute probabilities like ${}_tp_x$, ${}_tq_x$, etc?

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Lemma 1 *Assumptions 1 and 2 imply that*

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$$ {}_tp_x = \exp\left(-\int_0^t \mu_{x+s} ds\right) . \tag{3}$$

. This agrees with the well known result obtained with the future lifetime approach in Part 1.

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Proof: Let $s < t$. Notice first that ${}_sp_x$ is well defined by the Markov property, i.e. how the life got to age x is irrelevant. We consider the small interval of time *immediately after* $x + s$, and ask: how can a life aged x become a life aged $x + s + ds$? The diagram may help:

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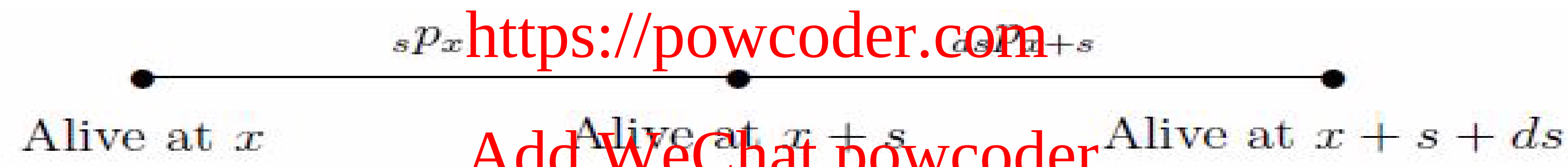

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Hence, the **probability that** (x) **survives** $s+ds$ **years** is equal to the **probability that they survive** s **years** times the **probability that, when at age** $x+s$**, they survive** ds **years**.

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$$\begin{aligned} {}_{s+ds}p_x &= {}_sp_x \times {}_{ds}p_{x+s} \\ &= {}_sp_x \times (1 - {}_{ds}\,q_{x+s}) \\ &= {}_sp_x \times (1 - \mu_{x+s}ds + 0(ds)) \quad \text{by Assumption 2.} \end{aligned} \tag{4}$$

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Now bring the term ${}_sp_x$ from the right to the left hand hand side of (4) we get

$${}_{s+ds}p_x -{}_s p_x = -{}_sp_x\mu_{x+s}ds + 0(ds). \tag{5}$$

Then divide both sides of (5) by ds and let $ds \to 0$. We find

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$$
\begin{aligned}
\frac{{}_{s+ds}p_x - {}_s p_x}{ds} &= -{}_s p_x \mu_{x+s} + \frac{0(ds)}{ds} \\
\Rightarrow \quad \frac{\partial}{\partial s}\, {}_s p_x &= -{}_s p_x \mu_{x+s} \qquad (6) \\
\Rightarrow \quad {}_t p_x &= \exp\left(-\int_0^t \mu_{x+s} ds\right)
\end{aligned}
$$

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on integrating (6) from 0 to t, this result will be proved in the class.

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2.3 Multi-state Markov models

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- The 2-state model of mortality can be extended to any number of states. Many insurance products, e.g. Permanent Health Insurance (PHI), can be modelled by a **multi-state model**. The set $S = \{Healthy, Ill, Dead\} = \{h, i, d\}$ is the **state space**. Here is a 3-state model for PHI:

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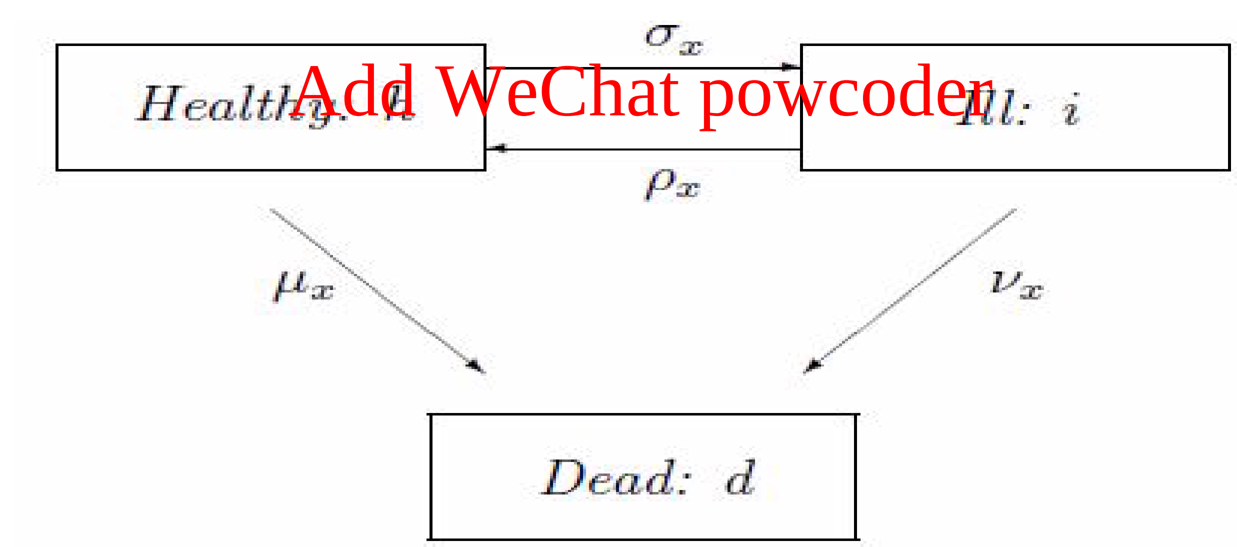

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- Let g and h be any two states. We extend the assumptions for the 2-state model to cope with a multi-state model. We do this in terms of the **transition probability** ${}_tp_x^{gh}$ (analogous to ${}_tp_x$) and the **force of transition (aka transition intensity)** μ_x^{gh} (analogous to μ_x).

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- We define the **transition probability**

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$$ {}_tp_x^{gh} = P_r(\text{In state } j \text{ at time } x+t \mid \text{In state } i \text{ at time } x) \tag{7} $$

for any two states i and j.

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- Also, for $z > 0$ define the **force of transition**

$$ \mu_x^{gh} = \lim_{z \longrightarrow 0^+} \frac{{}_zp_x^{gh}}{z}. \tag{8} $$

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2.4 Fundamental Assumptions for Multi-State models

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1. **(AS1) Markov assumption**. The probability that a life now aged x will be in a particular state at any future time $x+t$ depends only on the age x and the state currently occupied.

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2. **(AS2)** For any two **distinct states** g and h the transition probability ${}_{dt}p^{gh}_{x+t}$ is given by

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$$ {}_{dt}p^{gh}_{x+t} = \mu^{gh}_{x+t}dt + 0(dt),\ t \geq 0. \tag{9}$$

3. **(AS3)** The probability that **a life makes two or more transitions** in time dt is $o(dt)$. **Assumption 3 says, in effect, that only one transfer can take place at one time**.

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- What does ${}_tp_x^{gg}$ mean?

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- This is the probability that we are in state g at time $x+t$, given that we are in state g at time x.

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- This **does not imply** that we have been in state g for the whole of the time interval from x to $x+t$, for this we define the **occupation probability**.

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- **Occupation Probability:**

$$ {}_tp_x^{\overline{gg}} = P_r(\text{In state } g \text{ from } x \text{ to } x+t \mid \text{In state } g \text{ at time } x) \qquad (10)$$

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- Note that ${}_tp_x^{\overline{gg}} \leq {}_tp_x^{gg}$.

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- Because ${}_tp_x^{\overline{gg}}$ is the occupation probability, i.e. the individual never leaves state g between ages x and $x+t$.

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- The important distinction is that ${}_tp_x^{gg}$ includes the possibility that the individual leaves state g between ages x and $x+t$, provided they back in state g at age $x+t$. This result will be shown in the next class.

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- However ${}_tp_x^{\overline{gg}}$ will be equal to ${}_tp_x^{gg}$ in one common situation, namely when return to state g is impossible.

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- For example, in this **model of terminal illness**

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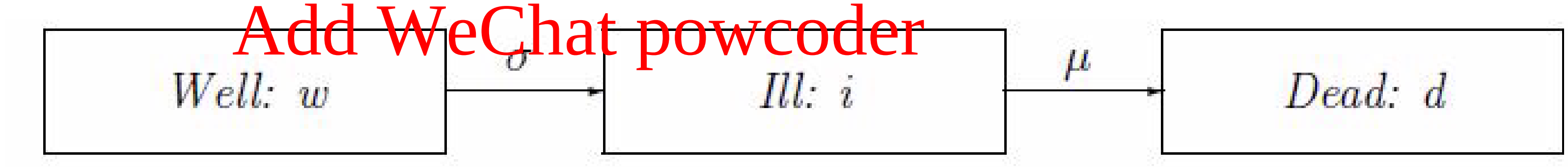

we have ${}_tp_x^{\overline{ww}} = {}_tp_x^{ww}$ since return to the well state is impossible.

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- In a similar fashion, we also have ${}_tp_x^{\overline{ii}} = {}_tp_x^{ii}$.

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2.5 Kolmogorov forward equations

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- What can we say about the **relationship between the transition intensities** μ_{x+t}^{gh}, $g \neq h$ **and the transition probabilities** ${}_tp_x^{gh}$?

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- We will look at **two examples** in detail **before giving the general result**.

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Example 1 Consider the 3-state model for working, retiring and dying.

- In this simple example we will assume **two constant transition intensities** μ (from working to dying) and σ (from working to retiring).

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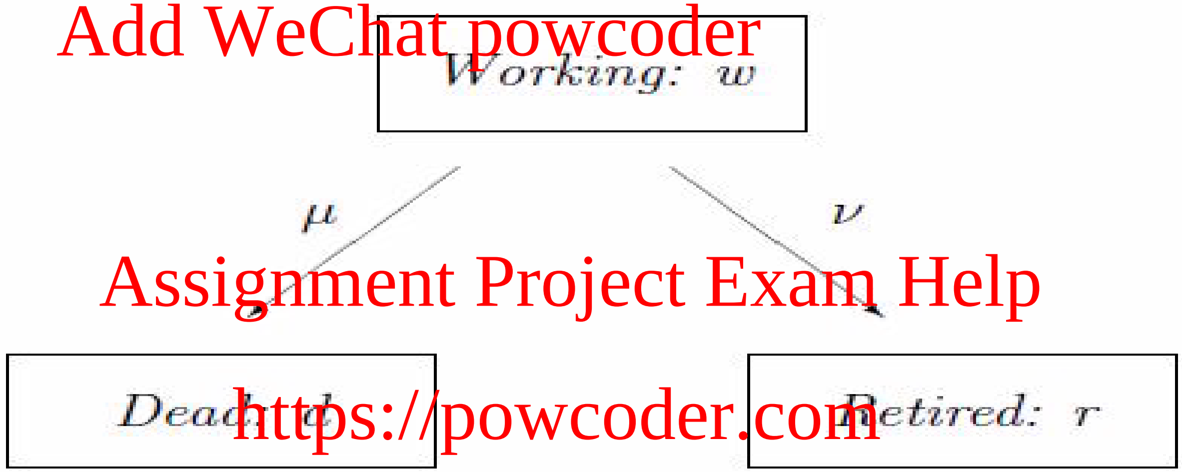

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- There are **three transition probabilities** which correspond to the events: (a) *Working to Dead*, (b) *Working to Retired and* (c) *Working to Working*.

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(a) *Working to Dead or* ${}_tp_x^{wd}$. We use a **standard method in all of these kind of problems**:

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- **Step 1:** We suppose we are in the destination state (here Dead) at time $x + t + dt$.

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- **Step 2:** We list the states we could be in at time $x + t$, i.e. just before time $x + t + dt$.

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- The diagram might help:

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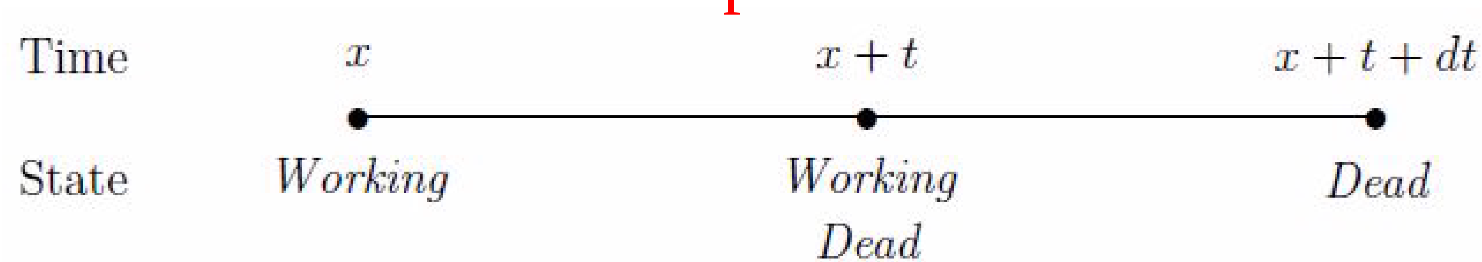

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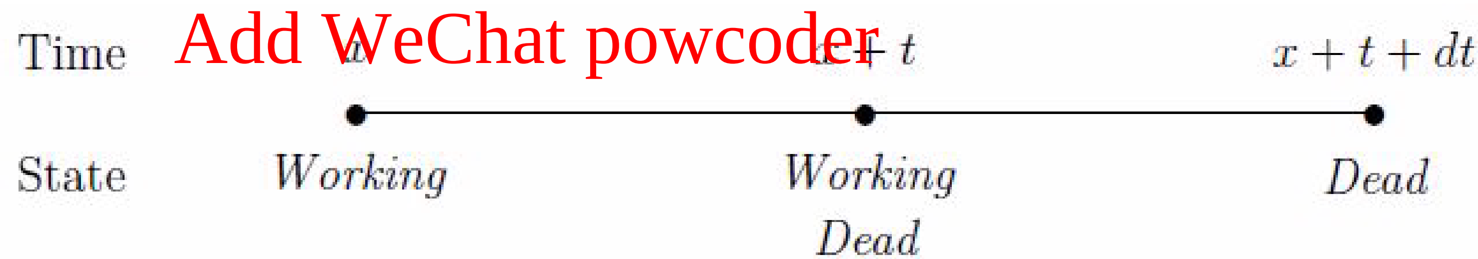

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- The **left end** represents the starting position at time x.

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- The **right end** represents the final position at time $x + t + dt$.

- The **middle position** lists the states that can be occupied at $x + t$ **immediately before** the final position at $x + t + dt$.

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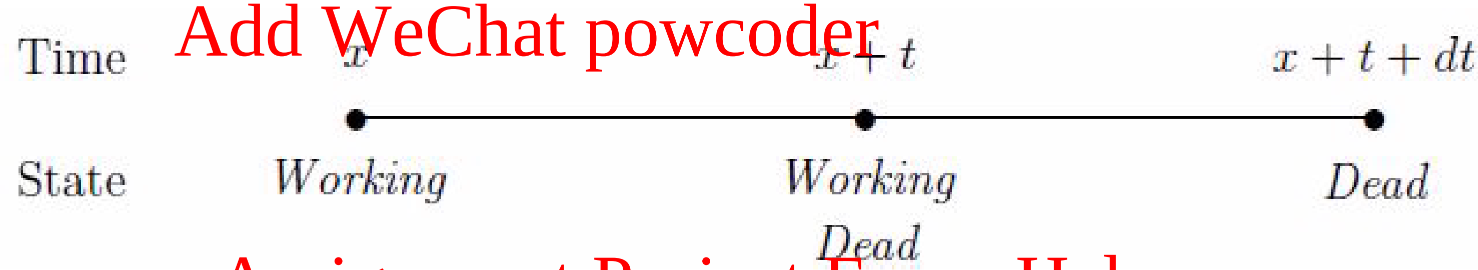

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- Thus, we have that

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$$
\begin{aligned}
{}_{t+dt}p_x^{wd} &= {}_tp_x^{wd} + {}_t\, p_x^{ww} \times {}_{dt}\, p_{x+t}^{wd} && (11)\\
&= {}_tp_x^{wd} + {}_t\, p_x^{ww} \times (\mu dt + 0(dt)). && (12)
\end{aligned}
$$

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Rearranging we get

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$$\frac{{}_{t+dt}p_x^{wd} - {}_tp_x^{wd}}{dt} = \mu \cdot {}_t\, p_x^{ww} + \frac{o(dt)}{dt} \tag{13}$$

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so taking the limit $dt \to 0$ we get

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$$\frac{\partial}{\partial t}\, {}_tp_x^{wd} = \mu \cdot {}_tp_x^{ww}. \tag{14}$$

This is the Kolmogorov forward equation for ${}_tp_x^{wd}$.

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(b) *Working to Retired or* ${}_tp_x^{wr}$ *.* We can apply exactly the same argument to ${}_tp_x^{wr}$ but it is better to use the **symmetry of the diagram**. This gives the Kolmogorov forward equation for ${}_tp_x^{wr}$ as

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$$\frac{\partial}{\partial t}\,{}_tp_x^{wr} = \nu \cdot {}_tp_x^{ww} . \qquad (15)$$

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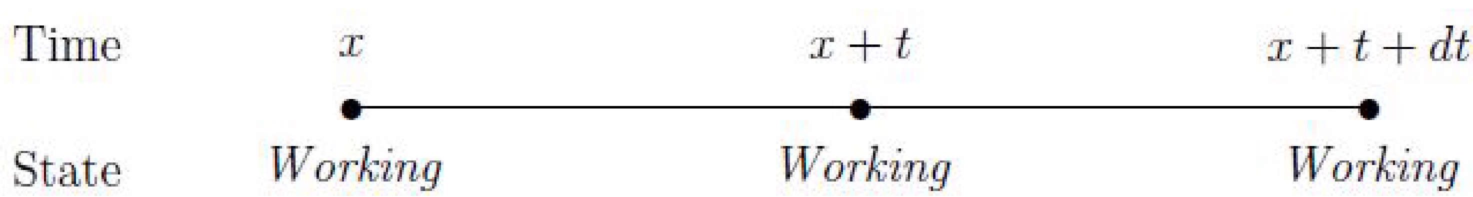
(c) *Working to Working.* First, notice that return to the *Working state* is impossible so ${}_tp_x^{ww} = {}_tp_x^{\overline{ww}}$. The diagram of possible routes is very simple:



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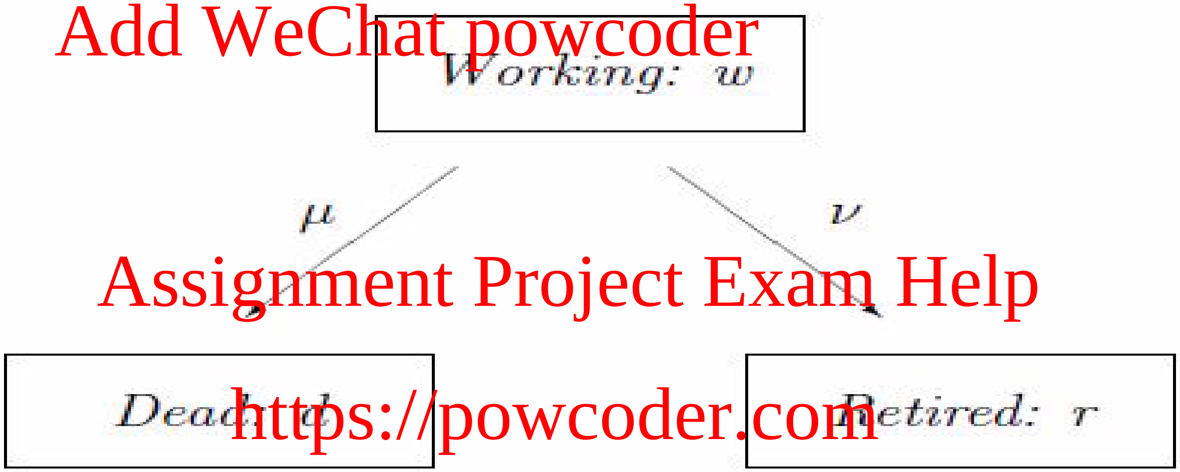

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The probability of transfer out of state *Working* in time dt is

$$
\begin{aligned}
{}_{dt}p^{wd}_{x+t} + {}_{dt}p^{wr}_{x+t} &= \mu dt + 0(dt) + \nu \cdot dt + 0(dt) && (16)\\
&= \mu \cdot dt + \nu \cdot dt + 0(dt). && (17)
\end{aligned}
$$

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Hence, the probability we **remain in** state *Working* for time dt is

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$$1 - \mu \cdot dt - \nu \cdot dt + 0(dt). \tag{18}$$

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Thus, since

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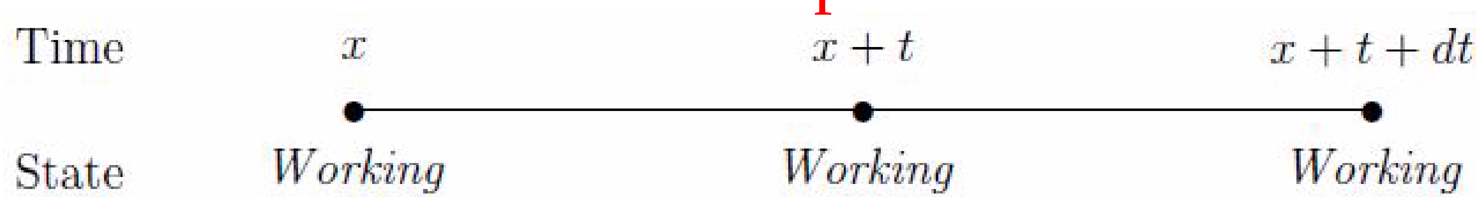

we get

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$$ {}_{t+dt}p_x^{\overline{ww}} = {}_tp_x^{\overline{ww}}(1 - \mu \cdot dt - \nu \cdot dt + 0(dt)). \tag{19} $$

Rearranging and letting $dt \to 0$, we find

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$$ \frac{\partial}{\partial t}\, {}_tp_x^{\overline{ww}} = -(\mu + v)\, {}_tp_x^{\overline{ww}} \tag{20} $$

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and putting ${}_tp_x^{\overline{ww}} = {}_tp_x^{ww}$ gives the Kolmogorov equation for ${}_tp_x^{ww}$:

$$ \frac{\partial}{\partial t}\, {}_tp_x^{ww} = -(\mu + v)\, {}_tp_x^{ww}. \tag{21} $$

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- We now have a system of three differential equations (14), (15) and (21) for the three unknown transition probabilities, ${}_tp_x^{ww}, {}_tp_x^{wr}$ and ${}_tp_x^{wd}$.

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- Note that

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$$ {}_tp_x^{ww} + {}_tp_x^{wd} + {}_tp_x^{wr} = 1 \tag{22} $$

since a life in state w at time x must be in some state at time $x + t$.

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Example 2 For our second example we return to the 3-state model for PHI.

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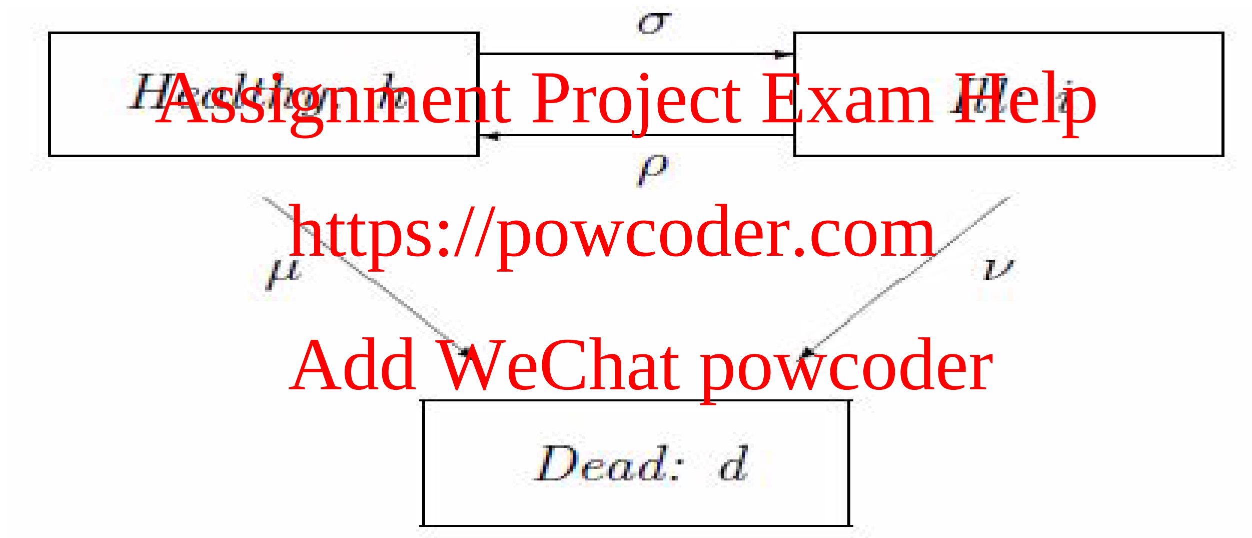

- We have assumed that the transition intensities μ, σ, ρ and ν are constant.

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- There are six transition probabilities ${}_tp_x^{hh}, {}_tp_x^{hi}, {}_tp_x^{hd}$, ${}_tp_x^{ii}, {}_tp_x^{ih}$ and ${}_tp_x^{id}$.

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- We look in detail at the derivation of the Kolmogorov equations for three of these, ${}_tp_x^{hh}$, ${}_tp_x^{hi}$ and ${}_tp_x^{hd}$. *(The remaining three equations can then be written down by using symmetry arguments.)*

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(a) *Healthy* to *Ill* or ${}_tp_x^{hi}$. The *Ill state* at time $x + t + dt$ can be reached from either the *Healthy* or the *Ill state* at time $x + t$. Our diagram is

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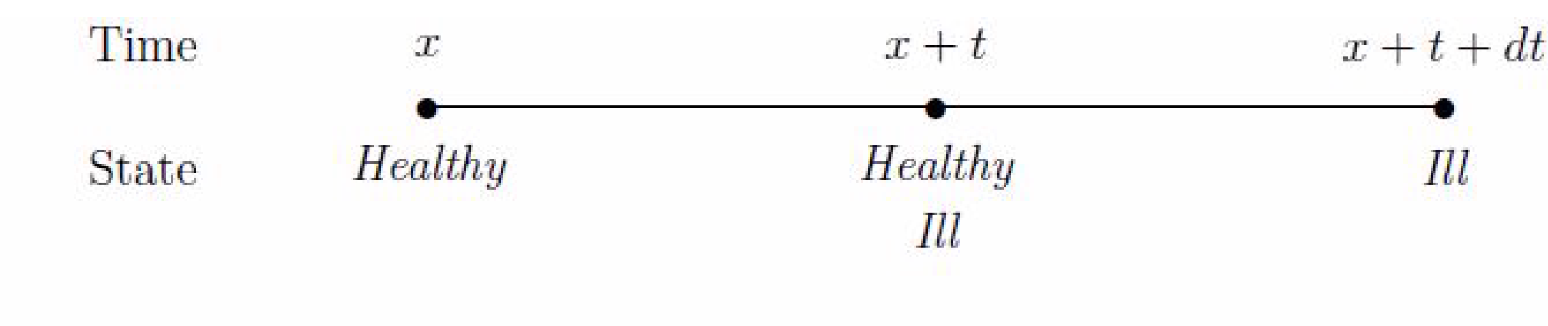

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Hence

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$$ {}_{t+dt}p_x^{hi} = {}_tp_x^{hh} \times (\sigma dt + 0(dt)) + {}_tp_x^{hi} \times (1 - \rho dt - \nu dt + 0(dt)) \quad (23) $$

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Rearranging we get

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$$ \frac{{}_{t+dt}p_x^{hi} - {}_t p_x^{hi}}{dt} = \sigma \cdot {}_t p_x^{hh} - (\rho + \nu)\,{}_tp_x^{hi} + \frac{0(dt)}{dt} \quad (24) $$

and taking the limit $dt \to 0$ gives the Kolmogorov forward equation for ${}_tp_x^{hi}$ (next slide).

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$$\frac{\partial}{\partial t}\,{}_tp_x^{hi} = \sigma\,{}_tp_x^{hh} - (\rho + \nu)\,{}_tp_x^{hi}. \tag{25}$$

(b) *Healthy* to *Healthy* or ${}_tp_x^{hh}$. The *Healthy state* at time $x + t + dt$ can be reached from either the $Healthy$ or the $Illstate$ at time $x + t$. Our diagram is

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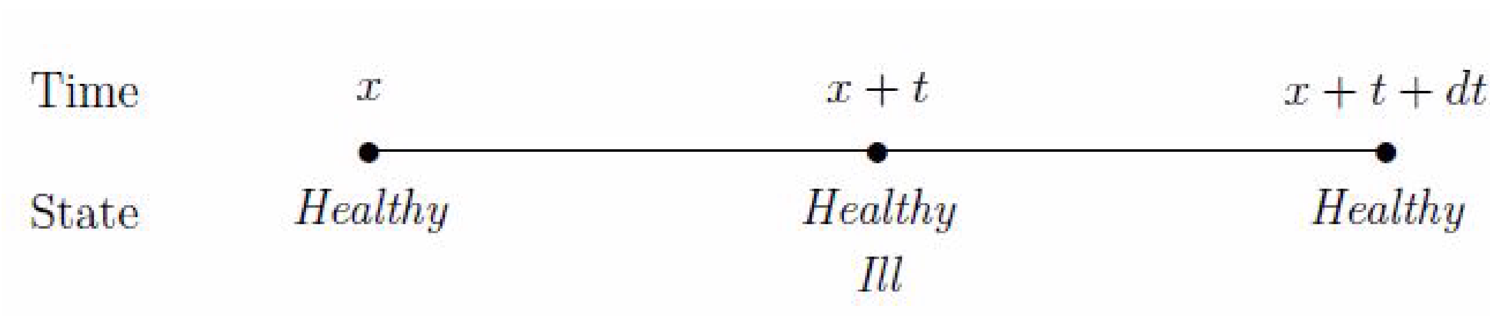

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Hence

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$$ {}_{t+dt}p_x^{hh} = {}_tp_x^{hh} \times (1 - \sigma dt - \mu dt + 0(dt)) + {}_tp_x^{hi} \times (\rho dt + 0(dt)) \quad (26) $$

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Rearranging we get

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$$ \frac{{}_{t+dt}p_x^{hh} - {}_tp_x^{hh}}{dt} = \rho \cdot {}_tp_x^{hi} - (\sigma + \mu){}_tp_x^{hh} + \frac{0(dt)}{dt} \quad (27) $$

and taking the limit $dt \to 0$ gives the Kolmogorov forward equation for ${}_tp_x^{hh}$ (next slide).

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$$\frac{\partial}{\partial t}\,{}_tp_x^{hh} = \rho \cdot {}_tp_x^{hi} - (\sigma + \mu)\,{}_tp_x^{hh} \tag{28}$$

(c) *Healthy* to *Dead* or ${}_tp_x^{hd}$. The *Dead state* at time $x + t + dt$ can be reached from either the *Healthy*, the *Ill* or the *Dead state* at time $x + t$. Our diagram is

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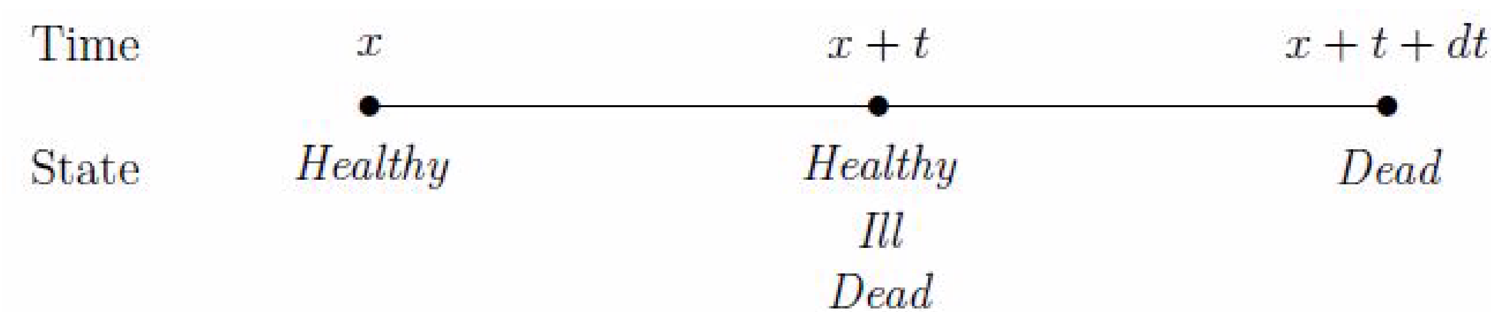

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Hence

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$$ {}_{t+dt}p_x^{hd} = {}_tp_x^{hh} \times (\mu dt + 0(dt)) + {}_tp_x^{hi} \times (\nu dt + 0(dt)) + {}_tp_x^{hd} \times 1 \quad (29) $$

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The Kolmogorov forward equation for p_x^{hd} follows by rearranging and taking the limit $dt \to 0$. We find

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$$ \frac{\partial}{\partial t} {}_tp_x^{hd} = \mu \cdot {}_tp_x^{hh} + \nu \cdot {}_tp_x^{hi}. \quad (30) $$

Comment: Note that

$$ {}_tp_x^{hd} = 1 - {}_tp_x^{hh} - {}_tp_x^{hi}. \quad (31) $$

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2.6 The general Kolmogorov equations

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- What can we say about the relationship between the transition intensities μ^{gh}_{x+t}, $g \neq h$ and the transition probabilities ${}_tp^{gh}_x$?

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- The **general Kolmogorov equations** generalise the previous two examples. We show that

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$$\frac{\partial}{\partial t}\,{}_tp^{gh}_x = \sum_{j \neq h} \left({}_tp^{gj}_x\,\mu^{jh}_{x+t} - {}_tp^{gh}_x\,\mu^{hj}_{x+t} \right), \quad g \neq h \tag{32}$$

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- We are interested in transfers from state g at time x to state h at time $x + t + dt$. So at time $x + t$ we are already in state h, or we have still to reach state h from some other state j.

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- Our diagram is

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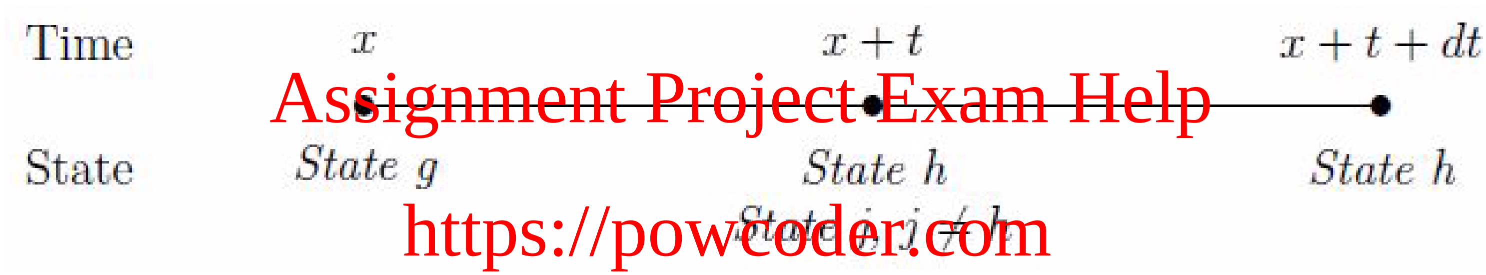

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- Hence we have that (next slide):

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$$
\begin{aligned}
{}_{t+dt}p_x^{gh} &= {}_tp_x^{gh} \times {}_{dt}p_{x+t}^{hh} + \sum_{j\neq h} {}_tp_x^{gj} \times {}_{dt}p_{x+t}^{jh} \qquad (33) \\
&= {}_tp_x^{gh} \times \left(1 - \sum_{j\neq h} \mu_{x+t}^{hj} dt + 0(dt)\right) + \sum_{j\neq h} {}_tp_x^{gj} \left(\mu_{x+t}^{hj} dt + 0(dt)\right)
\end{aligned}
$$

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- Rearranging we get

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$$
\frac{{}_{t+dt}p_x^{gh} -{}_t p_x^{gh}}{dt} = \sum_{j\neq h_t} (p_x^{gj} \mu_{x+t}^{jh} -{}_t p_x^{gh} \mu_{x+t}^{hj}) + \frac{0(dt)}{dt} \qquad (34)
$$

and the result follows on letting $dt \longrightarrow 0$.

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- We can also apply the same argument to finding the Kolmogorov equation for ${}_tp_x^{\overline{gg}}$, the probability that the state g is occupied continuously from time x to time $x + t$.

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- As in the previous examples, the resulting differential equation can be solved to give a closed form expression for ${}_tp_x^{\overline{gg}}$. The diagram

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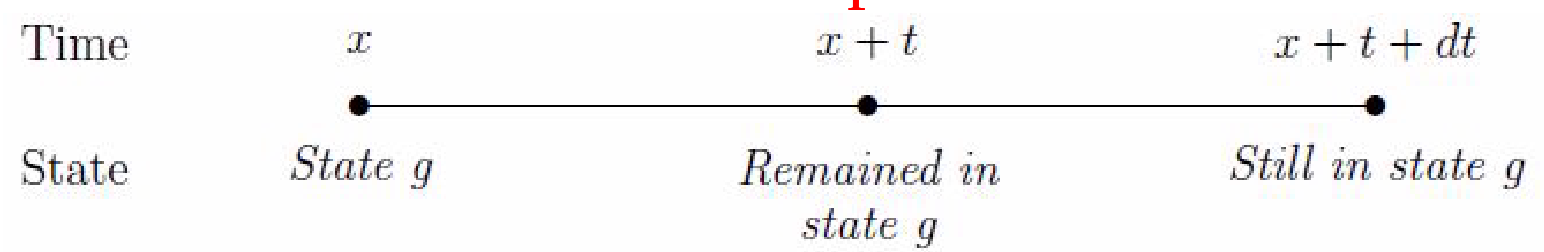

tells us that (next slide):

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$$
{}_{t+dt}p_x^{\overline{gg}} = {}_tp_x^{\overline{gg}} \times \left(1 - \sum_{j \neq g} \mu_{x+t}^{gj} dt + 0(dt)\right) \tag{35}
$$

- Rearranging we get

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$$
\frac{{}_{t+dt}p_x^{\overline{gg}} - {}_tp_x^{\overline{gg}}}{dt} = - {}_tp_x^{\overline{gg}} \sum_{j \neq g} \mu_{x+t}^{gj} + \frac{0(dt)}{dt}
$$

$$
\Rightarrow \frac{\partial}{\partial t} {}_tp_x^{\overline{gg}} = - {}_tp_x^{\overline{gg}} \sum_{j \neq g} \mu_{x+t}^{gj} \tag{36}
$$

on letting $dt \longrightarrow 0$.

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- Integrating (36) we find

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$$ {}_t p_x^{\overline{gg}} = \exp\left(-\int_0^t \sum_{j \neq g} \mu_{x+s}^{gj} ds \right) \tag{37} $$

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Comment: This formula generalises the well-known formula:

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$$ {}_t p_x = \exp\left(-\int_0^t \mu_{x+s} ds \right). \tag{38} $$