STOCHASTIC METHODS IN FINANCE 2021–22
STAT0013
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Exercises 9: Risk-neutral pricing

For all questions, assume that the risk-free rate is continuously compounded, is constant and is r.

1. A European digital call option[1] has an "all or nothing payoff" function, so that at expiration date the option pays off £1 if $S_T \geq K$, and nothing if $S_T < K$, where K is the strike price. Assume that the underlying stock follows geometric Brownian motion. Use the risk-neutral valuation approach to find the value of this digital call option with strike K.

2. Find the Delta of the European digital call option from question 1.

3. A stock follows a price process of geometric Brownian motion with volatility σ. A derivative based on this stock will provide a payoff at expiration time T of £H_1 if $S_T \leq K_1$ and £H_2 if $S_T \geq K_2$, where H_1, H_2, K_1 and K_2 are positive constants with $K_2 > K_1$. The payoff is zero if $K_1 < S_T < K_2$, and early exercise is not allowed. Use the risk-neutral valuation approach to find a formula for the price of the derivative.

4. Suppose you are given a vector of n independent, pseudo-random numbers generated from a uniform [0,1] distribution, i.e. $u_i \sim U[0,1]$ for $i = 1, 2, ..., n$. Determine an expression based on the Monte-Carlo simulation technique that uses these numbers to estimate the price of a European style derivative based on an underlying stock whose price process S_t follows the stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dB_t,$$

and whose payoff at expiration time T is S_T^2. Assume the risk-free rate is r and that the current stock price is S_0.

[1] Sometimes also called a binary call option

5. If a stock price in a risk-neutral world follows the SDE

$$dS_t = rS_t dt + \sigma S_t dB_t$$

then use Itô's lemma to show that the price process relative to the riskless bond, say R_t, (i.e., the process S_t/R_t), has zero drift in the risk-neutral world[2].

[2]Having zero drift is a property of a "fair game" process. A fair game is a stochastic process X_t that satisfies the result $E[X_T|X_t] = X_t$ for $T > t$, so that the expectation of any future value is the current value, given that we know the current value. If the process is an Ito process it will have zero drift. A "fair game" is also referred to as a martingale