**STOCHASTIC METHODS IN FINANCE 2021–22**
**STAT0013**
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# Exercises 10: Black-Scholes formula. Risk-Neutral pricing. SDEs.

The Black-Scholes formula for the price of a European call option under the standard assumptions, with strike price $K$ and time to expiry $T$, is



$$S_0 N(d_1) - Ke^{-rT} N(d_2),$$

where $N(\cdot)$ denotes the cumulative distribution function of a standard Normal, and



$$d_1 = \frac{\log(\frac{S_0}{K}) + \left(r + \frac{\sigma^2}{2}\right) T}{\sigma\sqrt{T}}$$



$$d_2 = \frac{\log(\frac{S_0}{K}) + \left(r - \frac{\sigma^2}{2}\right) T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

The Black-Scholes formula for the price of a European put option with strike price $K$ and time to expiry $T$, is

$$Ke^{-rT} N(-d_2) - S_0 N(-d_1),$$

where the notation is the same as above.

1. A financial institution sold for \$300,000 a European call option on 100,000 shares of a non-dividend paying stock. We assume that the stock price is $S_0 = 49$, the strike price is $K = 50$, the risk-free interest rate is $r = 5\%$ per annum, the stock price volatility is $\sigma = 20\%$ per annum and the time to maturity is 20 weeks. Assume that the assumptions of the Black-Scholes model hold.

   (a) Find the value of the European call option on the 100,000 shares.

   (b) Find the value of a European put option with same strike price and expiration date on the 100,000 shares.

   (c) Verify that the put-call parity holds in this case.

2. Assume that the current stock price is £110, the riskless interest rate is 3% per annum and the volatility is 7% per annum. Let $S_2$ be the stock price after two years. What is the value of a financial product that pays $S_2 - 120$ if $S_2 > 120$, $100 - S_2$ if $S_2 < 100$ and zero otherwise? Assume that all Black-Scholes assumptions hold (including no dividends).

3. For each of the following European-style derivatives with the given payoffs (in £), draw the payoff diagram at maturity and find the price of the derivative. In each case assume that the Black-Scholes assumptions hold, that all derivatives have maturity dates 1 year from now and that the underlying asset price process follows the stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dB_t,$$

   with parameters $\mu = 0.1$ and $\sigma = 0.4$. Assume also that the underlying asset is currently priced at £4 and that the risk-free rate is 0%.

   (a) A payoff of 0 if $S_T \leq 4$, and a payoff of $S_T - 4$ if $S_T \geq 4$.

   (b) A payoff of 0.2 if $S_T \leq 4$, a payoff of $S_T - 4$ if $4 \leq S_T \leq 5$, and a payoff of 1 if $S_T \geq 5$.

4. A stock price $S_t$ follows the usual model $dS_t = \mu S_t dt + \sigma S_t dB_t$ with expected return $\mu = 0.16$ and a volatility $\sigma = 0.35$. The current price is £38.

   (a) What is the probability that a European call option on the stock with an exercise price of £40 and a maturity date in six months will be exercised? Using the Black- Scholes formula, find the price of the call option if the risk-free interest rate is $\mu$.

(b) What is the probability that a European put option on the stock with the same exercise price and maturity will be exercised? Find the price of the put option, using put-call parity.

5. Assume that $\{B_t\}_{t\geq 0}$ is a standard Brownian motion. Let

$$X_t = B_{2t} - B_t$$

Is $X_t$ a Brownian motion? Explain briefly your answer.

6. Assume that $\{B_t\}_{t\geq 0}$ is a standard Brownian motion.

(a) Let $X_t = 2B_t/4$. Show that $X_t$ is a Brownian motion. Explain briefly your answer.



(b) Compute for $0 < s < t$ the covariance:



$$\mathrm{cov}(B_{3t} - 2B_{2t}, B_s)$$

(c) Recall that $\sinh(x) = \frac{e^x - e^{-x}}{2}$ , $x \in \mathbb{R}$. For $t > s$, compute:



$$E[e^{-t/2}\sinh(B_t)|B_s]$$

7. The CIR model for a security with price $X_t$ is determined via the following SDE:

$$dX_t = r(\theta - X_t)dt + \sigma\sqrt{X_t}dB_t$$

(a) What is the interpretation of the parameters $r$, $\theta$, $\sigma$?

(b) Find a function $f$ so that the process $Y_t = f(X_t)$ satisfies an SDE with unit diffusion coefficient.