STOCHASTIC METHODS IN FINANCE 2021–22
STAT0013
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Exercises 6 - Stochastic Calculus

1. Find the stochastic differential equation (SDE) satisfied by the square of a stock price that follows geometric Brownian motion. What is this process?

2. (a) Suppose that $g(t)$ is a *deterministic* differentiable function for $t > 0$, with $g(0) = 0$. Show that a solution to the ordinary differential equation

$$dx(t) = a\,x(t)\,dt + b\,x(t)\,dg(t)$$

   with boundary condition $x(0) = x_0 \neq 0$ is

$$x(t) = x_0 e^{at+bg(t)}.$$

   Hint: Write $dg(t)$ as $g'(t)dt$ and then use the variable separation technique and take integrals on both sides of the equation.

   (b) Show that the process $X_t = x\,e^{at+bW_t}$, where W_t is standard Brownian motion, satisfies the SDE

$$dX_t = \left(a + \tfrac{b^2}{2}\right)X_t\,dt + bX_t\,dW_t$$

   with initial condition $X_0 = x$.

3. Show that the Itô process $X_t = e^{W_t}e^{-t/2}$ (with W_t a standard Brownian motion) satisfies the stochastic differential equation

$$dX_t = X_t\,dW_t.$$

4. A zero-coupon government bond pays £100 at time T, and has price denoted by B_t. In the course so far we have assumed that the risk-free rate is constant and deterministic. In more advanced models, the risk-free rate can be modelled itself as a stochastic process. It has been

suggested that the short-term interest rate, r_t, will not be constant over time but will in fact follow the stochastic process

$$dr_t = a(b - r_t)dt + c\, r_t\, dz_t$$

where a, b, c are positive constants and z_t is a standard Brownian motion. Under this assumption, derive the SDE for the government bond price B_t for $t < T$.