Pricing of American Options.

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Outline

- The n-step Binomial model for a stock price is a discrete-time model with the following properties:

  - The asset (henceforth referred to as a stock) has value S at time 0.
  - The stock price changes only at discrete times $\delta t, 2\delta t, 3\delta t, \dots, n\delta t$.
  - At each time step $m\delta t$, with $m = 1, 2, \dots, n$, the stock can either move up, to a price:

$$u \times \text{(stock value at previous time step)}$$

    or it can move down to a price:

$$d \times \text{(stock value at previous time step)}$$

  - $d < 1 < e^{r\delta t} < u$, where r is the risk-free interest rate.
  - At each step, the probability of an up movement is p and of a down movement is $1 - p$.

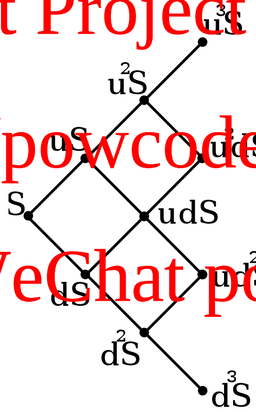
1. n-Step Binomial Model

- We can build up a ‘tree’ of possible stock prices.
- The tree is called a Binomial Tree, because the stock price will either move up or down at the end of each time period.

- Each node represents a possible future stock price. Note that u and d are the same at every node of the tree.

- We divide the time to expiration, T, into n steps, each of same duration $\delta t = T/n$.

- Example: We sketch below the Binomial tree for $n = 3$.



- The Binomial model becomes more realistic as we divide the periods to maturity into a larger number of sub-periods.

- As the number of time steps increases:
  - The length of each time period becomes smaller.
  - The number of possible stock values at maturity increases, thereby adding realism to the model.

- We introduce the following notation:
  - S_k^m is the k-th possible value of the stock price at time $m\delta t$, with $m \leq n$. Notice that we have $m+1$ possible prices:

$$S_k^m = u^k d^{m-k} S, \quad k = 0, 1, 2, \ldots, m.$$

  - Thus, k represents the number of upward steps among the m steps taken up to the instance $m\delta t$.
  - For example, at the 3^{rd} time-step, at time $3\delta t$, there are 4 possible stock prices: $S_0^3 = d^3 S$, $S_1^3 = ud^2 S$, $S_2^3 = u^2 dS$, $S_3^3 = u^3 S$.
  - At the final time-step $T = n\delta t$, there are $n+1$ possible values of the stock price.

- The following general strategy can be used for the valuation of European call options, under a n-step Binomial tree model, for any choice of steps $n \geq 1$:
  - Compute the risk-neutral probability $\hat{p}$ for every 1-step Binomial 'sub-tree' (part of the complete large tree).
  - Compute the option values at the terminal nodes – these are just the payoff function values.
  - Work backwards and compute the option values at each internal node using risk-neutral valuations.

- We will price an European call option that gives the right to buy a stock at strike price K, at time T. Recall that the call option payoff is $\max(S_T - K, 0)$.

1.1 Pricing of European Options

- We will price an European call option that gives the right to buy a stock at strike price K, at time T. Recall that the call option payoff is $\max(S_T - K, 0)$.

- Let f_k^m be the k^{th} possible value (equivalently, payoff) of the call option at time-step $m\delta t$, where $m \le n$ and $k = 0, 1, 2, \ldots, m$.

- At the final step we have a payoff $f_k^n = \max(S_k^n - K, 0)$ for the call option, with possible values:

$$S_k^n = u^k d^{n-k} S, \quad k = 0, 1, 2, \ldots, n.$$

- Let f be the price of the option at time 0.

- The aim is to find f.

- At each step the risk-neutral probability $\hat{p}$ – same at all steps – is:

$$\hat{p} = \frac{e^{r\delta t} - d}{u - d}, \quad \delta t = T/n$$

- Via risk-neutral valuation (recursion backward in time), we have:
  - At step $n-1$:

$$f_k^{n-1} = e^{-r\delta t}\left(\hat{p} f_{k+1}^n + (1-\hat{p}) f_k^n\right), \quad 0 \le k \le n-1$$

where, we recall that:

$$f_k^n = \max(S_k^n - K, 0), \quad S_T = S_k^n = u^k d^{n-k} S$$

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$$f_k^n = \max(S_k^n - K, 0), \quad S_T = S_k^n = u^k d^{n-k} S$$

for $0 \le k \le n$.

  - At step m $(m < n)$:

$$f_k^m = e^{-r\delta t}\left(\hat{p} f_{k+1}^{m+1} + (1-\hat{p}) f_k^{m+1}\right), \quad 0 \le k \le m$$

  - At step 0:

$$f = e^{-r\delta t}\left(\hat{p} f_1^1 + (1-\hat{p}) f_0^1\right).$$

- Example: Consider a 9 months European call option of strike price $\pounds 14$ on a stock with current price $\pounds 8$.

- There are 3 time steps and in each step the stock price either moves up by 100% with probability $2/5$ or moves down by 50% with probability $3/5$. The risk-free interest rate is 25%.

- Evaluate the option price.

- Example: Consider a 9 months European call option of strike price $\pounds 14$ on a stock with current price $\pounds 8$.

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- Evaluate the option price.

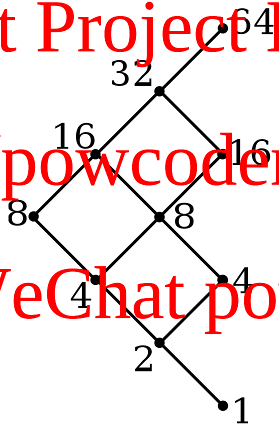
- We have $S = 8$, $u = 2$, $d = 1/2$, $n = 3$, $T = 0.75$, $\delta t = 0.25$, $K = 14$ and $r = 0.25$.

- Recall that for all $m \leq n$, we have $S_k^m = u^k d^{m-k} S, \;\; 0 \leq k \leq m$.

- We will next sketch the Binomial tree for this problem.

1.1 Pricing of European Options

- Example (Continued):



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- At step $n = 3$:
  - $f_k^3 = \max(S_k^3 - 14, 0), \;\; 0 \le k \le 3$.
  - That is: $f_0^3 = \max(1-14,0) = 0$, $f_1^3 = \max(4-14,0) = 0$, $f_2^3 = \max(16-14,0) = 2$ and $f_3^3 = \max(64-14,0) = 50$.

- We apply risk-neutral valuation (backward in time).
- At step 2

  - $f_k^2 = e^{-0.25\times 0.25}\left(\hat{p} f_{k+1}^3 + (1-\hat{p}) f_k^3\right), \;\; k = 0, 1, 2.$
  - That is: $f_0^2 = 0.939\left(0.376 f_1^3 + (1-0.376) f_0^3\right) = 0$, $f_1^2 = 0.939\left(0.376 f_2^3 + (1-0.376) f_1^3\right) = 0.706$ and $f_2^2 = 0.939\left(0.376 f_3^3 + (1-0.376) f_2^3\right) = 18.825$

- Example (Continued):
- At step 1:
  - $f_k^1 = e^{-0.25\times 0.25}\left(\hat{p} f_{k+1}^2 + (1-\hat{p}) f_k^2\right), \quad k = 0, 1.$
  - That is: $f_0^1 = 0.939\left(0.376 f_1^2 + (1-0.376) f_0^2\right) = 0.249$ and $f_1^1 = 0.939\left(0.376 f_2^2 + (1-0.376) f_1^2\right) = 7.059.$

- At step 0:
  - $f = e^{-0.25\times 0.25}\left(\hat{p} f_1^1 + (1-\hat{p}) f_0^1\right).$
  - That is: $f = 0.939\,(0.376 \times 7.059 + (1-0.376) \times 0.249) = 2.637$

- So far, we have taken the Binomial tree with u and d as given and proceeded to price derivatives.

- One way to set up the tree is to select values for u and d that match the stock return volatility as Cox, Ross and Rubinstein did:

$$u = e^{\sigma\sqrt{\delta t}} \approx 1 + \sigma\sqrt{\delta t}, \quad d = e^{-\sigma\sqrt{\delta t}} = \frac{1}{u} \approx 1 - \sigma\sqrt{\delta t}$$

- The volatility σ is a measure for the variation of the price of a financial instrument over time.

- In particular, $\sigma\sqrt{\delta t}$ is the volatility of the relative returns (equivalently, of the log-returns) over time-steps of δt years.

- Returns: The parameter of interest is the volatility σ.
  - Consider the stock prices S_i at a sequence of discrete times $i\delta$, $i = 0, 1, \ldots$, for some small δ.
  - E.g., when the S_i's correspond to daily observations – freely available on the web – we have $\delta = 1/365$.

  - In general, the S_i's can be easily observed from the stock market.
  - Let R_i be the corresponding log-returns, i.e., $R_i = \log(S_i/S_{i-1})$, and $\bar{R} = \frac{1}{n}\sum_{i=1}^{n} R_i$ be the mean log-return ($\bar{R}$ is typically very close to 0).

  - We can estimate the volatility σ using time series data as follows:

$$\hat{\sigma} = \sqrt{\frac{1}{\delta}\left[\frac{1}{n-1}\sum_{i=1}^{n}(R_i - \bar{R})^2\right]}.$$

- Implied Volatility: Alternatively, we can estimate the volatility σ by matching observed option prices.

- The two approaches (i.e., estimation of σ using observed returns or using the implied volatility) should generate similar estimates of σ, but they are different in practice.

- More on this 'Implied Volatility' approach for estimating σ when we discuss the Black-Scholes formula in the second half of the term.

- Recall that an American option can be exercised at any time prior to expiration time T.
- We need to determine the best time to exercise the option.
- This decision need not be subjective! It can be determined in a systematic way!

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- We need to determine the best time to exercise the option.
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- The American put option value P must be greater than or equal to the payoff function. If $P < \max(K - S_T, 0)$ then there is obvious arbitrage opportunity. We can buy stock for S and option for P and immediately (at time $t = 0$) exercise the option by selling stock for K. Since $K - (P + S_T) > 0$, there is an arbitrage opportunity.

2. Pricing of American Options

- The general pricing method for an American option is as follows.
- Compute the risk-neutral probability for every 1-step Binomial sub-tree (part of the complete large tree).
- Compute the option values at the terminal nodes using the payoff function.
- Work backwards and compute the option values at each internal node using risk-neutral valuation. Test if early exercise at each node is optimal. If it is, replace the value from the risk-neutral valuation with the payoff from early exercise.
- Continue with the nodes one step earlier.

- We denote by P_k^m the k^{th} possible value of a put option at time $m\delta t$.
- In the case of a European put option:

$$P_k^m = e^{-r\delta t}\left(\hat{p}P_{k+1}^{m+1} + (1-\hat{p})P_k^{m+1}\right)$$

for $0 \le k \le m$ and $\hat{p} = \frac{e^{r\delta t}-d}{u-d}$.

- In the case of an American put option:

$$P_k^m = \max\left\{ \max(K - S_k^m, 0),\ e^{-r\delta t}\left(\hat{p}P_{k+1}^{m+1} + (1-\hat{p})P_k^{m+1}\right) \right\}$$

where S_k^m is the k-th possible value of the stock price at time-step $m\delta t$.

- Final condition: $P_k^n = \max(K - S_k^n, 0),\ 0 \le k \le n$.

- Example: We assume that over each of the next 2 years a stock price either moves up by 20% or moves down by 20%. The risk-free interest rate is 5%.

- Find the price of a 2-step, 2-year American put option with a strike price of $\$52$ on a stock with current price $\$50$.

- Example: We assume that over each of the next 2 years a stock price either moves up by 20% or moves down by 20%. The risk-free interest rate is 5%.

- Find the price of a 2-step, 2-year American put option with a strike price of $\$52$ on a stock with current price $\$50$.

- In this case $u = 1.2$, $d = 0.8$, $r = 0.05$, $K = 52$, $Su = 60$, $Sd = 40$, $Su^2 = 72$, $Sud = 48$, $Sd^2 = 32$.

- Risk-neutral probability: $\hat{p} = \frac{e^{0.05} - 0.8}{1.2 - 0.8} = 0.6282$

- Example (Continued): At the terminal node, we compute:
  - $P_k^2 = \max(52 - S_k^2, 0)$, for $k = 0, 1, 2$.
  - That is: $P_0^2 = \max(52 - S_0^2, 0) = \max(52 - Sd^2, 0) = 20$, $P_1^2 = \max(52 - S_1^2, 0) = \max(52 - Sud, 0) = 4$ and $P_2^2 = \max(52 - S_2^2, 0) = \max(52 - Su^2, 0) = 0$.

- $e^{-0.05\times 1}\,(0.6282 \times 0 + 0.3718 \times 4) = 1.4147$
- $e^{-0.05\times 1}\,(0.6282 \times 4 + 0.3718 \times 20) = 9.4636.$

- Example (Continued):

- Payoff: $K - S_0^1 = 52 - 40 = 12 > 9.4636$. Early exercise is optimal! So $P_0^1 = \max(12, 9.4636) = 12$.

- Payoff: $K - S_1^1 = \max(52 - 60, 0) = 0 < 1.4147$. Early exercise is not optimal. So $P_1^1 = \max(0, 1.4147) = 1.4147$.

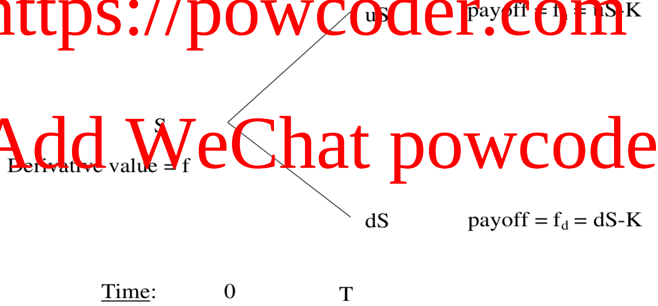
- $e^{-0.05\times 1}\,(0.6282 \times 1.4147 + 0.3718 \times 12) = 5.0894$.

- Payoff: $K - S_0 = 52 - 50 = 2 < 5.0894$. Early exercise is not optimal at the initial node. Therefore:

$$P_0 = \max(2, 5.0894) = 5.0894$$

- A forward contract with delivery price K is pictured in the diagram below. Its payoff f_T is given by:

$$f_T = \begin{cases} f_u = Su - K \\ f_d = Sd - K \end{cases}$$



3. Value of Forward Contract

- We can now determine the no-arbitrage price for the forward in the Binomial model by replicating the contract using an amount of stock and an amount of riskless investment (risk-free bonds).

- Consider the portfolio that consists of being long at 1 unit of stock, and being short at Ke^{-rT} units of the risk-free investment.

- Let S_T is the price of the stock at time T. At time T the value of the portfolio will be:

$$S_T - Ke^{-rT} \times e^{rT} = S_T - K$$

- This is exactly the payoff of the forward derivative at time T, so we have constructed a replicating portfolio for the forward.

- Thus, the price of the derivative at time 0 is the value of this portfolio at time 0, which is $S - Ke^{-rT}$. Note that this is the usual value for a forward contract we saw in a previous Meeting.

- For the contract to be fair at inception, this price must be 0, that is:

$$S - Ke^{-rT} = 0$$

which implies:

$$K = Se^{rT}.$$

- This finding is consistent with the result from a previous Meeting.

3. Value of Forward Contract

- We can now use the risk-neutral approach in the Binomial model and verify that the formula for the value f at time 0 gives the correct delivery price Se^{rT} (forward price).

- Under the usual risk-neutral equation, applied now for a different payoff function, the value of the contract at time 0 is given by:

$$f = e^{-rT}[\,\hat{p}\, f_u + (1-\hat{p}) f_d\,]$$

- This gives the following expression, once we plug in the appropriate values for f_u and f_d:

$$f = e^{-rT}[\hat{p}\,(Su-K) + (1-\hat{p})(Sd-K)]$$

which simplifies to:

$$f = e^{-rT}\,[\,\hat{p}\, Su + (1-\hat{p})Sd - K\,]$$

3. Value of Forward Contract

- Use now the formula for the risk-neutral probability $\hat{p}$:

$$\hat{p} = \frac{e^{rT} - d}{u - d}$$

- We obtain:

$$\begin{aligned} f &= e^{-rT}\left[\frac{e^{rT} - d}{u - d} \times Su + \frac{u - e^{rT}}{u - d} \times Sd - K\right] \\ &= e^{-rT}\left[\frac{e^{rT}Su - Sud + Sud - e^{rT}Sd}{u - d} - K\right] \\ &= e^{-rT}\left[\frac{e^{rT}S(u - d)}{u - d} - K\right] \\ &= S - Ke^{-rT} \end{aligned}$$

- This is the same value for a forward contract as obtained with the replicating portfolio approach.

- As with the replicating portfolio approach, the delivery price which gives zero initial value to the forward contract is:

$$K = Se^{rT}$$

- Therefore, we see that the Binomial model and the related risk-neutral equations give consistent results also for other derivatives (not only options).

- It is only the expression for the payoff function that changes.

Further Reading for the *Binomial Model*

- For further examples and explanations of the Binomial model and its applications, see the suggested reading below.

- John C. Hull (2003, 5th Edition). *Options Futures and Other Derivative Securities.* Section 9.1.

- Martin Baxter & Andrew Rennie (1996). *Financial Calculus*. Chapter 2.

Short Call Options

- There are two counter-parties in every option contract.
- The one party is taking a long position (i.e., buys the option).
- The other party is taking a short position (i.e, sells or 'writes' the option).
- The seller/writer of the option receives a premium up-front in exchange for potential liabilities later on.
- The profit or loss for the seller/writer of the option is the opposite of that for the buyer of the option.

Short Call Options

- The seller of a call (short call) might be obliged (depending on the movement of the stock) to deliver stock at the strike price.

- When the seller of the call already owns the underlying stock, the setting is referred to as 'writing a covered call'. If the stock movement favors the buyer, the seller is obliged to hand over the stock. If it favors the seller, the seller retains the shares.

- When the seller of the call does not own the underlying stock, the setting is referred to as 'writing a naked call'.