Lecture notes for STAT3006 / STATG017

Stochastic Methods in Finance 1

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Contents

STOCHASTIC METHODS IN FINANCE 1
2011–12

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Lectures: *Term 1, Tue 5–7pm, starting 4 October. No lecture in reading week.*

In Course Assessment: *In class assessment on* **Week 5; Tuesday 1 November 5pm**

Course guidance notes

This introductory note provides some guidance around the course.

Aims of course

To introduce mathematical and statistical concepts and tools used in the finance industry, in particular stochastic models and techniques used for derivative pricing.

Prerequisites

No prior knowledge of finance is assumed, though an interest in the subject is a distinct advantage.

A course covering probability and distribution theory, and a foundation in calculus and differential equations is required. For example you should be comfortable with ordinary differential equations (in particular be familiar with linear and exponential growth equations), Taylor expansions, partial differentiation and standard integration techniques such as integration by parts and substitution. You should also be comfortable with conditional probabilities, Markov processes, probability spaces, probability density functions and normal distributions.

Course content

1. Intro to financial products, markets and derivatives
2. Time value of money
3. Arbitrage pricing
4. The Binomial pricing model
5. Brownian motion and continuous time modelling of assets
6. Stochastic calculus
7. The Black-Scholes framework
8. Risk-neutral pricing

Lecture notes

The printed notes should be used as a guide to some of the key topics in the course, and are aimed at providing written copies of some of the working in the lectures so that students can concentrate on the subject without needing to copy too many lines of algebra. At the end of each set of printed lecture notes a selection of further reading is provided, which will give a guide to the sections of the course-related texts in which the lecture topic can be found, as well as a selection of more advanced reading for the interested student.

Note that the printed lecture notes do not contain everything that is outlined in the lectures, and so should not be used as a substitute for lecture attendance. The full set of course material consists of material presented in the lectures, combined with the handouts and the ideas and techniques developed through exercise sheets and the two workshops.

Exercise sheets

The exercise sheets are an integral part of the learning on the course. These should be attempted on a weekly basis. As well as providing opportunities for consolidating learning from the lectures, these exercises will also be used as a learning tool themselves to explore the concepts in the course. Therefore a student that does not complete all the exercises will not have achieved the full set of learning outcomes for the course.

These exercise sheets will not be marked and so do not need to be handed in.

The course builds quickly on material learnt previously. You should therefore plan each week to review your lecture notes, review the previous week's exercise solutions, as well as attempt the current week's exercises before the next lecture. It is strongly recommended that you attempt the exercises each week as subsequent lectures usually build on these exercises.

Workshops

The two workshops also form an integral part of the learning for the course. They will provide an opportunity to further develop your understanding of concepts introduced in the lectures, and also introduce some new techniques. They are usually timetabled for Friday afternoons - see the Statistics timetable for the specific dates and times for this course.

In course assessments

The in course assessment (ICA) will be a closed book classroom test during the lecture of the 5th week of term.

Office hours

Office hours for students requiring help with course material will be after each lecture, arranged by prior appointment only. Please indicate the area of the course and problem that you require help on when arranging this.

Reading List

The course is self sufficient, but this is a list of both related texts and some wider reading. There is now a vast range of books on mathematical finance and derivative pricing, so this is necessarily just a small selection with the aim of providing a good range of approaches taken.

Additional books are also referenced at the end of some of the lecture notes in the *reading list* sections where they are particularly relevant to the lecture topic.

Main Related Texts

- John C. Hull (2005) *Options Futures and Other Derivative Securities.* Now in the 8th edition, Prentice Hall.
- Martin Baxter & Andrew Rennie (1996) *Financial Calculus.* Cambridge University

Press.
- Paul Wilmott (2001) *Paul Wilmott Introduces Quantitative Finance* Wiley

Further introductory texts in the area

These books provides good introduction to the main topics covered in the course, each presenting the material with a different perspective.

- Salih Neftci (2000) *An Introduction to the Mathematics of Financial Derivatives.* Academic Press.
Excellent coverage of the mathematics needed for pricing derivatives, explained in an easy to follow way. Includes a good introductory presentation of stochastic calculus.

- Alison Etheridge (2002), *A course in Financial Calculus.* Cambridge University Press
A nice introduction to the principals underlying derivative pricing, with the emphasis on laying foundations of understanding. Takes a similar approach to the Baxter and Rennie book, but at a slightly simpler level.

- Wilmott, Howison, and Dewynne (1995), *The Mathematics of Financial Derivatives.* Cambridge University Press

This book presents derivative pricing from a physicist's perspective, and so focuses on solving partial differential equations, a less common perspective in more recent books. Worth a read for a good explanation of the pde.s relevant to finance, and particularly for those with a background in applied maths, but not essential for this course.

- Tomas Björk (1998) *Arbitrage Theory in Continuous Time.* Oxford University Press.
Good coverage of the key topics. Could be used as an excellent introductory text if you are comfortable with a more mathematical presentation. Provides some good intuition.

- Desmond Higham (2004), *An Introduction to Financial Option Valuation.* Cambridge University Press
A simple presentation of the basics of options pricing, this book also provides a guide to computational implementation of the results, with guides to Matlab code at the end of each chapter. Useful for those who want to work on their own to implement and extend the results we see in the course.

- Sean Dineen - *Probability theory in finance - a mathematical guide to the Black-Scholes formula.* American Mathematical Society, Graduate Series in Mathematics.

A good introduction to the the use of martingales in no-arbitrage pricing. Provides a more mathematically rigorous treatment of topics covered in this course, but also takes the time to introduce the motivation for and intuition of the results presented, and proceeds at a slow pace. Chatty style of writing.

More Advanced Texts

- Ricardo Rebonato (2004) *Volatility and Correlation*; Second Edition; Wiley.

- Steven Shreve (2004) - *Stochastic Calculus for Finance - 1 Discrete-Time models; 2 Continuous-Time models.* Springer.
A rigorous, mathematical approach, and goes into much more depth of the underlying probability theory results used in martingale pricing. Two volumes.

- Marek Musiela and Marek Rutkowski - *Martingale Methods in Financial Modelling* (Springer)
In-depth coverage of a wide range of models and products, using a rigorous, probability based approach.

- John Cochrane - *Asset Pricing* - Excellent book that presents general frameworks for the range of pricing techniques, with an aim of unifying asset pricing techniques and "clarifying, relating and simplifying the set of tools we have all learned in a hogdepodge manner". Hence goes beyond no-arbitrage derivative pricing covered in this course to include other types of pricing models (including CAPM etc), and on the way develops an awareness of the links between theoretical finance and macro-economics. Also includes interesting chapters on the theory of statistical estimation. Aimed at graduate level, so I suggest reading after completing some introductory courses in asset pricing.

- Stanley Pliska, *Introduction to Mathematical Finance: Discrete Time Models* (Blackwell Publishing)
More of an introductory text, but I have put in this section as it looks more widely at models beyond derivative pricing. Provides a rigorous study of use of risk-neutral probability measures, takes a mathematical approach, but is still accessible if you are comfortable with calculus and elementary probability theory, and prepared to put some work in.

- Glenn Shafer & Vladimir Vovk (2001) *Probability and Finance: It's Only a Game!* Wiley
Interesting book that presents a new framework for looking at probability and finance.

Tough going without a strong background in probability theory.

Background and references in finance

These books provide guides to financial concepts and jargon, and may be useful as references as the course progresses, or for students with no prior finance knowledge.

- Michael Brett (1995) *How to read the financial pages.* 4th edition, Century Business.

- Brian Butler, David Butler & Alan Isaacs (1997) *Dictionary of finance and banking.* Oxford University Press.

- John Downes, Jordan Goodman (2003) *Dictionary of Finance and Investment Terms* 6th edition, Barron's.

A selection of other finance related books

- Richard Lindsey and Barry Schachter (2007). *How I became a quant - insights from 25 of Wall Street's elite* (Wiley)

Twenty five different industry professionals contribute a chapter each, and therefore this book provides an eclectic mix of views on a range of areas and industries in which quantitative techniques are used. Also highlights some of the challenges of developing modelling techniques that are effective in a business environment, as well as a number of personal views on career steps and challenges. Worth bearing in mind that you don't have to empathise with all of the authors (or even any) to have a rewarding and successful career in finance - there is a huge range of different working cultures in industry.

An excellent read for those interested in using their stats, maths, economic or programming skills in the finance industry. An even better read for those of you who have these skills and are not currently interested. And probably essential reading if there is anyone who thinks they *know* they only want to research hybrid quantos stoch. vol. models for one of 4 select IBs/develop equity stat-arb trading algorithms on their favourite platform... etc.

- Roger Lowenstein (2000) - *When genius failed - the risk and fall of Long Term Capital Management*

Well told story of the potential financial crises resulting from hedge fund LTCM's positions in 1998, and the response of regulators and investment banks. A good case study in the importance of stress testing underlying model assumptions.

- Michael Lewis - *Liars Poker*
Classic book telling the story of the world of bond trading and investment banking in the eighties. Still relevant today.

- Andrew Ross Sorkin (2009) *Too big to fail - Inside the battle to save Wall Street*
Another well put together telling of a financial crisis, this time the story of the fall of Lehmans in 2008, based on interviews with many of the key players in banking and regulation at the time.

Chapter 1

Financial Markets and Products

In this lecture we provide a basic introduction to the financial markets that will provide the background to the problems in finance that we will study in the course. We also look at the types of risks that financial institutions operating in these financial markets are exposed to.

1.1 Financial markets

The financial markets are institutions and procedures that facilitate transactions in all types of financial securities (claim on future income).

We can have organised *security* or *stock exchanges*, or *over-the-counter markets* (for very specific requests).

Why do we have financial markets?

- In order to transfer efficiently funds from economic units who have them to people who can use them
- To reallocate risk and to manage risk

1.2 Equities

- *Equity* (or *stock* or *share*) is the ownership of a small piece of a company (claim on the earnings and on the assets of the company).
- The *shareholders* are the people who own the company, and have a say in the running of the business by the directors.
- Most companies give out lump sums every 6 months or a year, which are called *dividends*.

- The shares of large companies are traded in *regulated stock exchanges*.

Later in the course what we will want to study (and model) is the *stock price*. The price of a share is determined by *the market*, and depends on the *demand* and *supply* of shares in the market.

1.3 Fixed income (FI)

FI securities are financial contracts between two counterparties where a *fixed* exchange of cash flows is agreed (which depends on the interest rate).

Interest rate is the cost of borrowing or the price paid for the rental of funds and is usually expressed as % per year.

Broadly, there are two types of interest:

- *fixed* interest rate: locked for a certain period of time

- *floating* interest rate: changes from time to time

Bond: a debt security that promises to make payments periodically (are issued by government, companies, local authorities, etc.)

- *zero-coupon* bond: pays only a known fixed amount (the *principal*) at some given date in the future (the *maturity* date).

- *coupon-bearing* bond: similar to zero-coupon bond, except that it also pays smaller quantities (the *coupons*) at specific intervals up to and including the maturity date.

Example. A bond that pays 1,000 GBP in 3 years with a coupon of 10% per year paid semi-annually. Income from the bond:

time from now	£
6 months	50
12 months	50
18 months	50
24 months	50
30 months	50
36 months	1,050

⇒ In the US:

- *Treasury bills:* maturity ≤ 1 year (normally zero-coupon)
- *Notes:* maturity between 2 and 10 years (coupon-bearing bonds)
- *Bonds:* maturity ≥ 10 years (long bond = 30 years)

⇒ In the UK: bonds that are issued by the government are called *gilts*.

An important question is: what is the value of the bond? For instance, if a T-bill pays 1,000 USD in one year, how much should I pay for it now? (linked to the time value of money)

Issuance of bonds and equities are the two main sources of funding for companies. In the event of liquidation (or bankruptcy) of a company the debt holders (i.e. bond holders) are paid out of the remaining assets of the company before the equity holders. Only when all debt holders have received all promised cash flows will equity holders start to receive any compensation, if there is anything left by then. On the other hand, if a company increases its value considerably, then bond holders will still only receive the fixed payments agreed in the bond contract, whereas the value of the equities does not have a ceiling. This means that equities are generally more risky than bonds, in the sense that there is more variance around their expected returns.

1.4 Currencies

Different countries use different currencies. If you want to exchange one currency for another you have to use the *spot* exchange rate.

Example 1,000 GBP are worth 1,480 USD if the spot exchange rate GBP/USD is 1.48. This is important if you have to consider the exchange rate in transactions.

The spot exchange rate may be regulated by the government of the country (in order to control the growth and the investment of foreign capital) or it can fluctuate freely (determined by the market).

A currency is *strong* when its value is rising relative to other currencies (it appreciates) and *weak* when it is falling (depreciates).

1.5 Commodities

These are raw products such as precious oil, metals, food products, etc. Some commodities have organised exchanges for their trading. Prices fluctuate according to demand and supply.

1.6 Indices

An index is a weighted sum of a collection of assets, which is designed to represent the whole market. These are often held by investors to gain exposure to investment in a whole market.

Example FT-SE 100 is designed to represent the *equity* market in the UK.

1.7 Further reading

–Paul Wilmott *Introduces quantitative finance* – sections 1.1 - 1.5
–Robert Kolb and Ricardo Rodriguez (1996) *Financial Markets*, Blackwell
–Michael Brett (1995) *How to read the financial pages.* 4th edition, Century Business.

Chapter 2

Time value of money

One of the most important concepts in finance is that £1 today is worth more than £1 received in the future. This is referred to as the time value of money. One way of looking at this this concept is that £1 today can generate income without any risk through *interest*.

Note: To simplify things in this section we make the assumption that the interest rate is constant and the same for all maturity dates [1].

2.1 Compound interest and present value

The *compound interest* is the interest that occurs when the interest paid on an investment during the first period is added to the principal during the second period. In other words, interest is paid on the interest received in the previous period.

Example. We invest £100 for one year at 6%. The future value after one year (FV_1) equals the present value PV (the principal) plus the interest:

$$FV_1 = PV + PV * 0.06 = PV(1 + 0.06)$$

Similarly, the future value after two years (FV_2) equals the value after one year FV_1 plus the interest computed on the future value itself:

$$FV_2 = FV_1 + FV_1 * 0.06 = FV_1(1 + 0.06) = PV(1 + 0.06)^2$$

In general we obtain:

$$FV_n = PV(1 + 0.06)^n$$

[1] In reality the risk-free interest rate depends on the length of the time period over which the money is held. For more details of this see for example Hull chapter 4.

If we denote the interest rate by r, the value of the investment after n years is:

$$FV_n = PV(1+r)^n$$

2.2 Compound interest with non-annual payments

Let us study the previous example, but under the assumption that the interest rate is paid every 6 months, so that 6 / 2 = 3% is paid every 6 months.

- 0 months $\Rightarrow$ 100
- 6 months $\Rightarrow 100 + 100 * \left(\frac{0.006}{2}\right) = 100 * \left(1 + \frac{0.006}{2}\right)$
- 12 months $\Rightarrow 100 * \left(1 + \frac{0.006}{2}\right) * \left(1 + \frac{0.006}{2}\right) = 100 * \left(1 + \frac{0.006}{2}\right)^2$

  Note that: $100 * \left(1 + \frac{0.006}{2}\right)^2 > 100 * (1 + 0.006)$
- 18 months $\Rightarrow 100 * \left(1 + \frac{0.006}{2}\right)^3$
- 24 months $\Rightarrow 100 * \left(1 + \frac{0.006}{2}\right)^4$

So if compounding occurs m times during a year, then the future value after n years is:

$$FV_n = PV\left(1 + \frac{r}{m}\right)^{n \cdot m}$$

where nm is the total number of interest payments.

Example. Let us consider what is the value of £100 after one year compounded at 15% if we compound in different ways:

(a) annually ($m = 1$): £115.00

(b) semi-annually ($m = 2$): £115.56

(c) quarterly ($m = 4$): £115.87

(d) monthly ($m = 12$): £116.08

(e) weekly ($m = 52$): £116.16

(f) daily ($m = 365$): £116.18

There seems to be a limit as $m \to \infty$. If $n = 1$:

$$FV_1 = PV\left(1 + \frac{r}{m}\right)^m$$

$$\lim_{m\to\infty}\left(1 + \frac{r}{m}\right)^m = e^r$$

Therefore we obtain:

$$FV_1 = PVe^r$$

When we use the limit as $m \to \infty$ we say that we use *continuous compounding*.

Example. Find the FV of £100 after 1 year and 3 years continuously compounded at 10%.

$FV_1 = 100 \cdot e^{0.10\times 1}$

$FV_3 = 100 \cdot e^{0.10\times 3}$

This happens because

$$\lim_{m\to\infty}\left(1 + \frac{r}{m}\right)^{m\cdot n} = (e^r)^n = e^{rn}$$

and therefore

$$FV_n = PVe^{rn}$$

Similarly, if we want to compute the value, say, after 6 or 18 months, we convert the months into years (0.5 and 1.5 respectively) and then compute:

$FV_{6\text{months}} = 100 \cdot e^{0.10\times 0.5}$

$FV_{18\text{months}} = 100 \cdot e^{0.10\times 1.5}$

Let us consider now the following problem: we can either compound continuously at rate r_c or compound m times a year at rate r_m. What value should r_c take with respect to r_m in order to obtain the same FV after n years (and vice-versa)?

Let us start from the usual formulas:

- If we use continuous compounding we have: $FV_n = PVe^{r_c n}$
- If we use m period compounding we have: $FV_n = PV\left(1 + \frac{r_m}{m}\right)^{n\cdot m}$

From here, if we want the two future values to be the same, we have to equate them and solve:

$$e^{r_c n} = \left(1 + \frac{r_m}{m}\right)^{n\cdot m}$$

Therefore we obtain:

- $r_m = m\left(e^{r_c/m} - 1\right)$
- $r_c = m \log\left(1 + \frac{r_m}{m}\right)$

2.3 Present value

Using the formulae in the previous sections, we can calculate the *present value* of a future cash amount. This is essentially how much should be invested today to achieve a certain amount in the future *if all interest payments are re-invested at the same interest rate.*

- For $m = 1$ we have: $FV_n = PV(1+r)^n \Rightarrow PV = FV_n(1+r)^{-n}$
- For general m: $FV_n = PV\left(1 + \frac{r}{m}\right)^{nm} \Rightarrow PV = FV_n\left(1 + \frac{r}{m}\right)^{-nm}$
- For continuous compounding: $FV_n = PVe^{rn} \Rightarrow PV = FV_n e^{-rn}$

2.4 Government Bond valuation

The value of a bond is the *present value* of its expected *cash flow*. The best way to understand how this works is by looking at some examples.

Example. A T-bill [2] pays \$1,000 after one year and the interest rate is 6% with continuous compounding. What is the value of the bond today?

$$\text{Value} = PV(\$1,000) = \$1,000e^{-0.06\times 1}$$

If the interest rate *increases*, the value of the bond goes *down*.

Example. A coupon-bearing, default-free bond pays 6% coupons semi-annually and gives back a \$1,000 principal after 3 years. The rate that we use for discounting is 5% with continuous compounding. What is the value of the bond?

Future cash flow:

[2] A T-bill is issued by the US Government and therefore assumed riskless i.e. that there will be no default and so the promised payments are guaranteed.

6 months	\$30
12 months	\$30
18 months	\$30
24 months	\$30
30 months	\$30
36 months	\$1,030

How do we compute the value? We said that it equals the present value of its expected cash flow, so we have to compute the present values of the cash flows above and sum them:

Present values:

6 months	$\$30 \cdot e^{-0.05\times 0.5}$
12 months	$\$30 \cdot e^{-0.05\times 1}$
18 months	$\$30 \cdot e^{-0.05\times 1.5}$
24 months	$\$30 \cdot e^{-0.05\times 2}$
30 months	$\$30 \cdot e^{-0.05\times 2.5}$
36 months	$\$1,030 \cdot e^{-0.05\times 3}$
Total	$\cdots$

Example. What is the value of a bond that pays \$1,000 annually forever, when the discount rate is 10% and is applied using annual compounding? (*perpetual bond*)

- 1^{st} payment after 1 year: $PV = \frac{1,000}{1+r}$
- 2^{nd} payment after 2 years: $PV = \frac{1,000}{(1+r)^2}$
- $\ldots$

Once again, the value of the bond is the sum of these present values, that is:

$$\begin{aligned}
\text{Value} &= \sum_{i=1}^{\infty} \frac{1,000}{(1+r)^i} \\
&= 1,000 \left(\frac{1}{1-\frac{1}{1+r}} - 1 \right) \\
&= 1,000 \left(\frac{1+r}{r} - 1 \right) \\
&= 1,000 \left(1 + \frac{1}{r} - 1 \right) \\
&= \frac{1,000}{r}
\end{aligned}$$

In this case, therefore, the value of the bond is $\$\frac{1,000}{0.1} = \$10,000$.

2.5 Further reading

Paul Wilmott ***Introduces quantitative finance*** – section 1.6 covers ground similar to this lecture, and also provides a differential equation based presentation of continuously compounded interest rates, which will be useful for later lectures in the course.

Chapter 3

Introduction to Derivatives

A *derivative* is an instrument whose value depends on the values of other more basic *underlying* securities.

Some examples are:

- *Forward* contracts
- *Future* contracts

where you have the *obligation* to buy or sell a security at some time in the future. We also have:

- *Options*, i.e. the *option* to buy or sell something in the future, and
- *Swaps*, which involve the exchange of future cash flows.

Where can we buy a derivative?

(a) Organised exchanges (standard products)

(b) Over the counter (from financial institutions, can be non-standard products)

Why are derivatives useful?

(a) We can use them to protect ourselves from future uncertainty (*hedge* the risk)

(b) We can use them to *speculate* on the future direction of the market

(c) Or to change the nature of an asset or liability, often its risk profile (e.g. to swap a fixed rate loan with a floating rate loan) or its tax status

(d) Or to change the nature of an investment without selling one portfolio and buying another

Who uses derivatives?

(a) *Hedgers* (aim to reduce risk)

(b) *Speculators* ("gamble" on the future direction of the markets)

(c) *Arbitrageurs* (aim to make money without any risk)

3.1 Forwards

A *forward contract* is an agreement to sell or buy an asset at a certain time in the future for a certain price (*delivery price*).

If the buyer knows that he will need to buy the asset in the future, then entering into a forward contract provides the benefit of eliminating the uncertainty of the price the buyer will have to pay in the future. Thus the buyer has protection against rises in the price of the asset. Similarly, if a seller has an asset that he knows he will want to sell in the future, then entering into a forward contract guarantees the amount he will receive for it, eliminating uncertainty in the price received for the asset.

The main features of this contract are:

- It can be contrasted with a *spot* contract (buy or sell immediately)
- It is an *over-the-counter* product
- Usually no money changes hands until maturity
- If you agree to buy, you have a *long* position
- If you agree to sell, you have a *short* position

Example. A *forward* contract can be used, for instance, to hedge foreign currency risk. Suppose an American trader expects to receive £1m in 6 months and wants to hedge against exchange rate movements. He enters into an agreement to sell dollars and buy pounds after 6 months at a rate 1.60. He is long in pounds and short in dollars. His net payoff per forward contract is shown in Figure 3.1.

If after 6 months the rate is 1.7, then without the forward contract he would need \$1.7m to buy £1m, but he can actually buy them with only \$1.6m, due to his forward contract, $\Rightarrow$ he makes a profit compared to his situation without the forward contract.

On the other hand, if the rate falls to 1.5, then without the forward contract he would need only \$1.5m to buy £1m, but under the forward contract he *has to* buy it

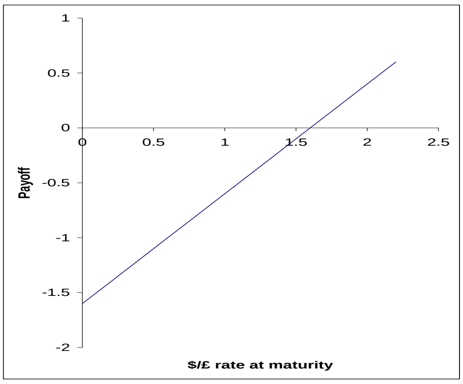


Figure 3.1: Payoff for foreign exchange forward, long position

at 1.6 $\Rightarrow$ he has a loss compared to his situation without the forward contract. The crucial point is that entering into the forward contract means that he has fixed the dollar amount he will receive for the £1m in 6 months time. He has eliminated the uncertainty - we say he has hedged his position.

Say, instead, that a trader has chosen to be long in dollars and short in pounds. His net payoff for this position is shown in Figure 2.

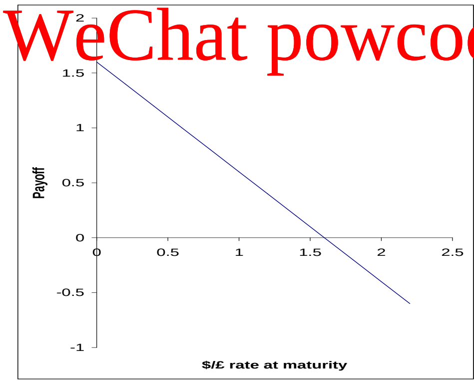


Figure 3.2: Payoff for foreign exchange forward, short position

So forward contracts are designed to neutralise risk by fixing the price that the hedger will pay or receive for the underlying asset.

3.2 Futures

A *future* contract is similar to a forward contract but has some differences:

(a) It is traded on an exchange

(b) Everything about the contract and the *underlying asset* is well specified

(c) The value of the contract is calculated daily and any profit or losses are adjusted in an account that you have with a *broker* (margin account)

(d) Closing out a futures position means entering into offsetting (trade)

(e) Most contracts are closed out before maturity

3.3 Options

- A *call* option is an option (but *not an obligation*) to buy a certain asset by a certain date for a certain price (*strike* price).

- A *put* option is an option to sell a certain asset by a certain date for a certain price.

- An *American* option can be exercised *at any time* up to expiration date.
- A *European* option can be exercised only on the expiration date itself.

There are two sides to every option contract:

(a) The party that has bought the option (long position)

(b) The party that has sold (or written) the option (short position)

3.4 Option Payoffs

A European call option will be exercised if the asset price is above the strike price at the option expiration date, and then the buyer of the option will receive a payoff worth $S_T - X$, where X is the strike price and S_T is the value of the asset at the option expiration date. If the asset price is below the strike price at the option expiration date, then the option will not be exercised, and will be worthless. Therefore the European call option has a payoff function $\max[S_T - X, 0]$. Similarly, the payoff function for a put option is $\max[X - S_T, 0]$.

The payoff from a derivative can be represented in a payoff diagram. Payoff diagrams are useful tools for understanding options and combinations of options. The payoffs from options are shown in the following diagrams, where X is the strike price and S_T is the asset price at time T (expiration date).

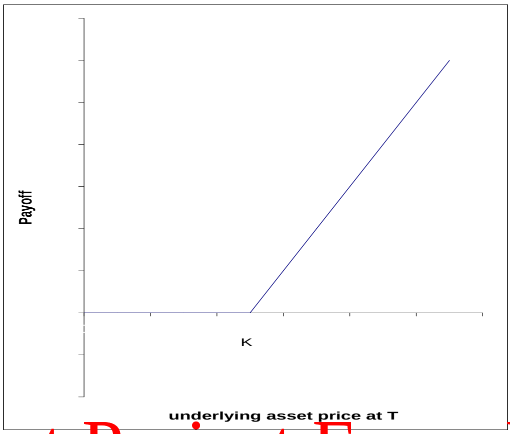


Figure 3.3: Payoff for long position in European call, strike price K

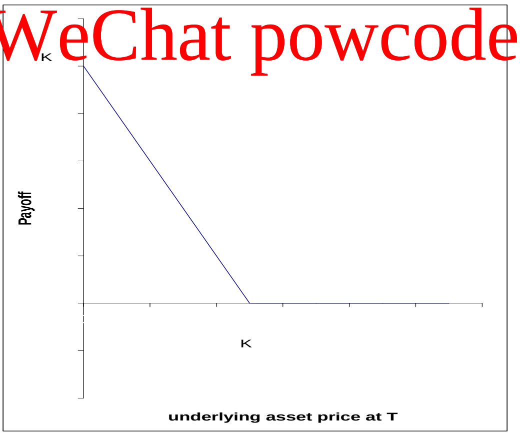


Figure 3.4: Payoff for long position in European put, strike price K

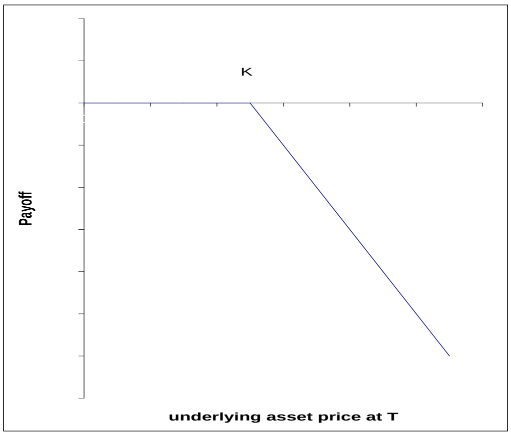


Figure 3.5: Payoff for short position in European call, strike price K

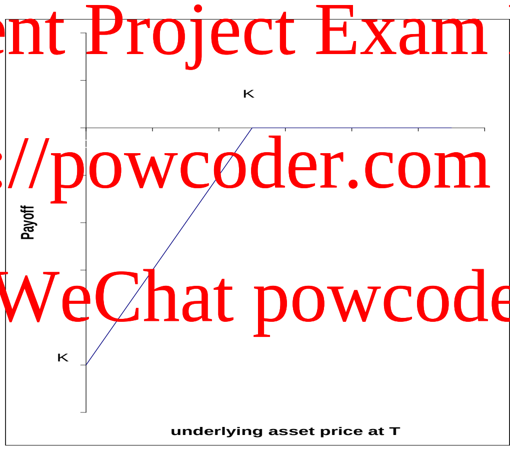


Figure 3.6: Payoff for short position in European put, strike price K

Derivatives can be used for a number of reasons, in particular, options can be used to;

(a) **Hedge:** We can use them to protect ourselves from future uncertainty (*hedge* the risk)

(b) **Speculation:** We can use them to *speculate* on the future direction of the market. As we shall see, options provide leaverage for speculation.

3.5 Hedging

We have seen in an earlier lecture how forward contracts can be used for hedging. Other types of hedging can be based on *options* contracts, where the objective is not to fix a price, but to provide insurance. Consider, for instance, the following example.

Example. An investor owns 1,000 shares at £102 each. He wants to hedge the risk of a fall in share price and buys six-month European put options with a strike price of £100 for £4 each. Therefore he pays £4,000 to buy the options, which guarantee that the shares can be sold for at least £100 each in six months.

If the share price goes below £100, he can exercise the options and obtain a payoff of £100,000, with a profit of £96,000. If the share price stays above £100, he keeps (or sells) the shares. Now the value of the holding is above £100,000 (profit above £96,000). Whatever happens, his total asset value, originally £102,000, can not fall below £96,000.

3.6 Speculation

An example of speculation using *forward* contracts could be the following.

Example. An agent believes that the \$/£ exchange rate will increase, and wants to exploit this change in the rate. He takes the risk and enters a three-month long forward contract, where he agrees to buy £100,000 at a \$/£ rate of 1.65. If the rate actually goes up, say to 1.7, he can buy for \$1.65 an asset worth \$1.7, and so he makes a profit of $(1.7 - 1.65) \times 100,000 = \$5,000$.

The way speculator use *options* is more complex, and is illustrated in the following example.

Example. Suppose a speculator believes a certain stock price will increase, and therefore wants to gain by buying now £6,400 worth of stock. Suppose the current price is £64 and that a three-month call option with £68 strike price is selling for £5. Two strategies are possible: buy 100 shares or buy 1,280 options.

If the price goes up to £75, the first strategy gives a profit of $100 \times (75 - 64) = \pounds 1,100$. The second strategy is more profitable: he can exercise the options and receive a payoff of $1,280 \times (75 - 68) = \pounds 8,940$; subtracting the cost paid for the options we have a profit of $8,940 - 6,400 = \pounds 2,540$.

However, if the price goes down to £55, then the first strategy gives a loss of $100 \times (64 - 55) = \pounds 900$, when the second strategy gives a loss of £6,400.

This example illustrates what is sometimes known as *leaveraging*. For a given amount invested (£6,400 in the above example), the use of options allows greater exposure to the movements of the underlying asset than investing in the asset alone.

3.7 Combining derivatives

The standard call and put options are termed "*plain vanilla*" derivatives. Financial institutions can design derivatives which are sold to customers or are combined with bond and stock issues in order to make them more attractive to the situation and needs of the customer. These derivatives are loosely called "*exotic options*".

The standard vanilla options can be combined to provide a wide range of risk profiles that may suit a number of customer requirements. Some simple examples are given here.

(a) *Straddle*: 1 call and 1 put at the same strike price

(b) *Strangle*: 1 call at strike price X_2 and one put at strike price $X_1 < X_2$ (used, for instance, if there is a lot of volatility in the stock)

(c) *Spread*: two options of the same type (i.e. two calls or two puts), one long and one short. We can have *bull* or *bear* spreads, and these can be obtained from both calls and puts.

BULL SPREAD from calls

BULL SPREAD from puts

The holder will benefit if the stock price increases.

BEAR SPREAD from calls

BEAR SPREAD from puts

The holder will benefit if the stock price decreases.

Chapter 4

Arbitrage and the pricing of forward contracts

4.1 Arbitrage

Inherent in most investments is a level of risk. You do not know for sure the return you are going to make on your investment, or sometimes even if you are going to make your initial investment back. The exception we have discussed earlier is an investment in Government bonds, the so-called risk-free investment. We assume that there will be no default on these bonds, hence they are "risk free".

Arbitrage is an important concept in finance. Arbitrage is a situation where by combining two or more financial products we make an investment that is *guaranteed* to yield a profit with no investment or cost, with absolute certainty. In other words, a successful arbitrage involves making money without any risk or investment.

We have discussed how a central question in finance is the valuation or pricing of financial products. A key concept in pricing or valuing products is the assumption that there are NO arbitrage opportunities in the financial markets.

This essentially relies on the fact that if there were arbitrage opportunities, then immediately someone would capitalize on these, thus affecting the market prices through demand and supply and eliminating the opportunity. In this course, and indeed in the majority of theoretical finance, it is assumed that this happens instantaneously. In other words, we assume that there are NO arbitrage opportunities in the markets.

4.2 Example - Arbitrage opportunities in a forward contract

Assume that a forward contract on a non-dividend paying stock matures in 3 months. i.e. the contract involves delivery of the stock in 3 months time. The stock price is

£100 now, the three-month risk-free interest is 12% p.a. Suppose the forward contract price is £105, i.e. the stock will be delivered for payment of £105 in 3 months time.

Questions to consider:

- Is this the correct price for the forward contract today?
- What do we mean by "correct price?"
- Can I make money without any risk?

Consider the following trading strategy:

- *Time 0*:
  - Borrow £100 at 12%
  - Buy a share at £100
  - Sell a forward at £105
- *3 months*:
  - Get £105 for the share
  - Pay back £103 (including £3 interest payment at the risk free rate of 12%)
  - $\Rightarrow$ profit £2

This has yielded a *riskless* profit of £2 with no initial investment. In other words, an arbitrage opportunity.

What about the case where the forward price is £102? Consider an investor who has a portfolio with one share of the stock. Can he make money without any risk?

- *Time 0*:
  - Sell the share at £100
  - Invest the £100 at 12%
  - Buy a forward at £102
- *3 months*:
  - Get £103
  - Pay £102 for the share
  - $\Rightarrow$ profit £1

This has yielded a *riskless* profit of £1 with no initial investment, so that an arbitrage opportunity exists if £102 is the forward price. Clearly neither of the two forward prices considered, £105 or £102, are sustainable in a market without arbitrage opportunities. They cannot be the correct price for the forward contract in such a market.

In fact, whenever the forward price in this example is higher or lower that £103 there is an *arbitrage opportunity*.

What happens in the market? The first strategy is profitable when the forward price is greater than £103. This will lead to an increased demand for short forward contracts, and therefore the three-month forward price of the stock will fall. On the other hand, the second strategy is profitable when the price is smaller than £103, therefore we will see an increase in the demand for long forward contracts and in turn an increase in the three-month forward price of the stock. These activities of traders will cause the three-month forward price to be exactly £103.

Therefore, from now on the basic assumption in derivative pricing is that

THERE ARE NO ARBITRAGE OPPORTUNITIES IN THE FINANCIAL MARKETS

Question: How did we calculate the figure £103 in this example?

4.3 Pricing forward contracts for securities that provides no income

The principal in the above example can be extended to price forward contracts in general.

Assumptions: **1)** No transaction costs **2)** The market participants can borrow or lend money at the same continously compounded risk-free rate r.

We also further assume that the underlying provide no income. Extensions to the argument are needed if the underlying provides an income, such as a share yielding dividends. We do not consider this here.

The example in the previous section indicates that the *delivery price* of a forward contract should be the Future Value (FV) of the underlying security price. If S is the spot price and F is the forward price, then

$$F = Se^{rT}$$

where T is the time to maturity and r is the corresponding risk-free interest rate. If $F < Se^{rT}$ or $F > Se^{rT}$, then there are arbitrage opportunities. See exercises for a proof of this.

Example. If the stock price is £40 with no dividends, and the interest rate is 5%, then a forward contract after 3 months should have delivery price equal to

$$F = 40e^{0.05\times 0.25} = 40.50$$

This is the correct price so that the *initial value* of the contract is *zero*, and the contract is therefore a fair one.

4.4 Value of a forward contract

As we have seen, the no-arbitrage principle requires that the value of a forward contract at the time it is first entered into is zero, i.e. the delivery price equals the forward price. The value of the contract, however, can change afterwards and become positive or negative, because the "fair" forward price, if recalculated, can change as the price of the underlying changes over time (while the delivery price of the contract remains the same).

At any time between the beginning of the contract and the delivery date, the value f of a long forward contract with delivery price K can be found by considering the following two portfolios:

- *Port. A:* One long forward contract + cash Ke^{-rT}
- *Port. B:* One security

Here T now denotes the *remaining* time until delivery, and r the associated risk-free interest rate. We consider only *non-wasting* securities, which can be kept indefinitely with no storage costs. Thus the argument is not directly applicable to *e.g.* commodity futures.

After time T, Ke^{-rT} will become K and I will use this money to buy one unit of the security. Therefore at time T the two portfolios have the same value (independent of the security price), which means that they *must* have the same value today (otherwise there is arbitrage).

- The value of portfolio A today is: $f + Ke^{-rT}$, where f is the value of the forward contract.
- The value of portfolio B today is S.

Therefore, since the value of the two portfolios is the same we have $f + Ke^{-rT} = S$, and so the value of the contract is:

$$f = S - Ke^{-rT}$$

This value is zero if and only if $K = Se^{rT}$.

Notice that $S = Fe^{-rT}$, where F is the current forward price, i.e. the delivery price that would apply if the contract were entered in today. Therefore, we can rewrite the above as $f = (F - K)e^{-rT}$. This equation shows that we can value a long forward contract on an asset by assuming that the cash value of the asset at the maturity of the forward contract is the forward price F. In fact, under this assumption, the contract will give a payoff of $F - K$, which is worth $(F - K)e^{-rT}$ today.

Example. Let us consider a six-month forward contract on a one-year T-bill with principal of \$1,000. The delivery price is \$950, and the six-month interest rate is 6% (with continuous compounding). The current bond price is \$930. Then the value of the contract is:

$$f = S - Ke^{-rT} = 930 - 950\ e^{-0.06\times\frac{6}{12}} = 8.08$$

Example. Consider now a forward contract on a non-dividend paying stock that matures in six months. The spot price is £1 and the risk-free interest rate is 10%. Therefore the forward price is $F = Se^{rT} = 1 \cdot e^{0.1\times\frac{6}{12}} = 1.05127$, and the value f is zero.

After three months the spot price is £1.05, and the interest rate remains the same. What is the value of the contract now? We obtain $f = 1.05 - 1.05127e^{-0.1\times\frac{3}{12}} = 0.02469$.

4.5 Forward contracts on a security that provides a known cash income

Let us consider now a (non-wasting) security that provides known cash incomes of c_i at time t_i for a number of time points in the future.

The spot price S already reflects the value of all future known incomes, and in particular up until T. However these incomes will not be received if we enter a forward contract with maturity at T as they will go to the holder of the security at the times t_i before T. Therefore, to compute the future price we have first to subtract from the spot price the present value of the missed future incomes, $I = PV(\text{income until time } T)$. $S - I$ is how much I would be prepared to pay now for this security if I would receive no income until time T. The forward *price* is then:

$$F = (S - I)e^{rT}$$

Work through the argument of Section 4.2 to show that this must be the no-arbitrage price in this known-dividend case.

Example. Consider a ten-month forward contract on a stock with spot price \$50. The interest rate is 8% (with continuous compounding) per annum. We assume that dividends of \$0.75 per share are expected after 3, 6 and 9 months. What is the forward price?

The present value of future incomes is given by:

$$I = 0.75\left(e^{-0.08\times\frac{3}{12}} + e^{-0.08\times\frac{6}{12}} + e^{-0.08\times\frac{9}{12}}\right) = 2.16$$

Therefore the price of the contract is $F = (50 - 2.16)e^{0.08\times\frac{10}{12}} = 51.14$.

The *value* of the contract is

$$f = S - I - Ke^{-rT}$$

for a delivery price of K.

4.6 Known dividend yield

Consider now the case where the underlying asset provides a known dividend yield which is paid continuously at an annual rate q. Then the forward *price* is

$$F = \left[Se^{-qT}\right]e^{rT} = Se^{(r-q)T}$$

Example. Let us consider a forward contract with maturity at 18 months, where the underlying asset provides a continuous dividend yield at 5%. The interest rate is 8% and the spot price is £1.20. The price of the contract is therefore $F = 1.20e^{(0.08-0.05)\times\frac{18}{12}} = 1.25523$.

If the delivery price is K, then the *value* of the contract is

$$f = Se^{-qT} - Ke^{-rT}$$

Note. The formula given above can be explained by a simple no-arbitrage argument. Consider a strategy as follows: at *time 0*

- Buy spot e^{-qT} of the asset at price S per unit and reinvest income from the asset in the asset. You spend Se^{-qT}.
- Short one forward contract on one unit of the asset

Since the holding of the asset grows at rate q, at *time* T I have $e^{-qT}\times e^{qT} = 1$ unit of the asset, and I sell it for F (forward price).

The present value of the cash inflow Fe^{-rT} must equal what I spend to enter the strategy, otherwise there is arbitrage, and therefore $Fe^{-rT} = Se^{-qT}$ i.e. our formula

above.
(What would an arbitrageur do if $F < Se^{(r-q)T}$? And what if $F > Se^{(r-q)T}$?)

4.7 Forward foreign exchange contracts

We consider now the case where we want to buy or sell foreign currencies in the future. For example, if we want to buy \$1m after six months, then we can buy a forward contract for \$1m at the appropriate exchange rate. Observe that the holder of the foreign currency earns interest at the risk-free rate prevailing in the foreign country. Denote that by r_f; if r is the domestic rate, then

$$F = Se^{(r-r_f)T}$$

Note that this equation is identical to the one in the previous section with q replaced by r_f. This is because a foreign currency can be regarded as an investment asset paying a known dividend yield, which in this case is the risk-free rate of interest in the foreign currency.

If the delivery price is K, then the value of a foreign exchange contract is given by:

$$f = Se^{-r_f T} - Ke^{-rT}$$

Example. The six-month interest rate in the US and the UK are 5% and 6% respectively. The current exchange rate is \$1.6/£1. The forward rate for a six-month contract is then:

$$F = 1.6e^{(0.05-0.06)\times\frac{6}{12}} = 1.592$$

Chapter 5

Pricing Options under the Binomial Model

We have seen in the last lecture that there is a fair contract price for forward contracts that does not allow arbitrage. Any other price will allow arbitrage. We now look at how we can determine this price in for other more general derivatives.

5.1 Modelling the uncertainty of the underlying asset price

In order to model the value of a *variable* that changes over time we will develop models based on *stochastic processes*.

We can use *discrete time*, where the variable changes only at certain fixed points in time, or we can use *continuous time*, where the variable changes at *any* time.

Also, the variable can be *continuous* (can take any value within a range), or it can be *discrete* (takes only certain values).

We will start now with *discrete time and discrete variables*.

In order to introduce the basic logic behind *option pricing* we start from an extremely simple model, the *one-step binomial tree*.

5.2 A simple example

Assume that the price of a stock is currently £20, and after three months it will, with equal probabilities, either be £22 or £18 (discrete time, discrete variable). We want to find the value of a European call option with strike price £21.

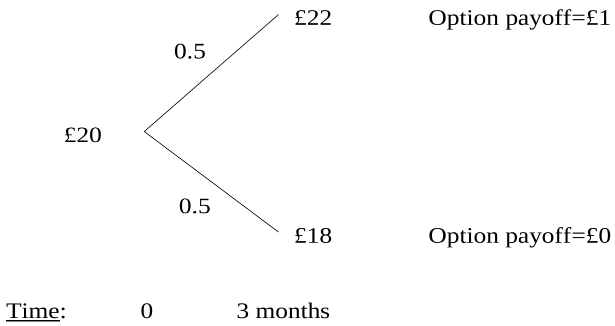
The payoff of the call option will be as follows:

		TIME	
		Discrete	Continuous
VARIABLE	Discrete	X	
	Continuous		

- If after three months the stock price goes up to £22, the option will be worth max(22-21,0) = £1

- If after three months the stock price goes down to £18, the option will be worth max(18-21,0)=0

We can see this as a game where with 50% probability you get £1 and with 50% probability you get £0.



The classical decision theory says that the game is "fair" when the expected payoff of the game is £0. This means that in this case:

$$E[\text{Payoff at T} - \text{Call option value}] = 0$$

and therefore we should compute the option value as

$$\begin{aligned} \text{Call option value} &= E[\text{Payoff at T}] \\ &= 0.5 \times 1 + 0.5 \times 0 = 0.5 \end{aligned}$$

In this case we would conclude that a "fair" price for the option would be £0.5.

But is this the "correct" price? We have learnt that the "correct" price is such that there are no arbitrage opportunities in the market, so we can check if a price of £0.5 allows for arbitrage.

5.2.1 Arbitrage opportunity

Suppose that the 3-month risk-free continuous interest rate is 4.4% p.a. Consider for instance the following portfolio:

- Short (sold) 0.25 shares
- Long (bought) 1 call option
- An investment in the riskless bond of £4.45 (at the risk-free rate)

If the option is sold for £0.5, then I could adopt the following strategy (setting up four portfolios like the one above):

- *Time 0*
  - Buy 4 options (-£2)
  - Sell 1 share at £20
  - Invest the £17.80 at 4.4%

  This provides a net gain of 20 - 17.80 - 2 = £0.20.

- *Time 3 months - two possible situations*
  - <u>Stock price is £22</u>
    * Get 4 × £1 = £4 payoff from options
    * Get $17.80 \times e^{0.044 \times 3/12} =$ £18 from cash risk-free investment
    * Buy back share for £22
    * $\Rightarrow$ Profit = 4 + 18 - 22 = 0
  - <u>Stock price is £18</u>
    * Get nothing from options
    * Get £18 from cash risk-free investment

    - Buy back share for £18
    - $\Rightarrow$ Profit = 18 - 18 = 0

Notice that, regardless of what happens in the future (whether the stock goes up or down), my position after 3 months is neutral, and I cannot lose any money. The initial profit of £0.20 at time zero means that regardless of the stock price in the future, I can make this profit of £0.20 today, with no chance of a loss in the future. This is an arbitrage opportunity. Hence we conclude that £0.50 cannot be the "correct" price for the option.

5.2.2 No-arbitrage pricing

So how can we obtain a price for the option such that no arbitrage opportunities can arise? Let us consider again the riskless and share assets in the portfolio we just introduced, but taking the reverse positions (i.e. long the stock and short the riskless asset); the value of the risk-free investment combined with the stock position, after three months, will be:

- If the stock price goes up to £22: $0.25 \times 22 - 4.45e^{0.044 \times 3/12} = 1$
- If the stock price goes down to £18: $0.25 \times 18 - 4.45e^{0.044 \times 3/12} = 0$

Compare this with the payoff of the option in each case. We can see that the value of the portfolio always equals the value of the call option: it is therefore called a *replicating portfolio* - it replicates the value of the option in all future states of the world (which in the case of the binomial model, is the the two possible stock movements, up and down). In order to avoid arbitrage, the value of this portfolio must equal the value of the derivative at all times (prove this to yourself).

The value of this portfolio at time zero is

$$PV = -4.45 + 20 \cdot 0.25 = 0.55$$

and so the no-arbitrage price for the option at time zero must also be £0.55.

Note that, unlike the earlier argument, this approach does not depend on the probabilities of the different possible outcomes; instead it depends on the risk-free interest rate.

We have found that if we can find a replicating strategy for a derivative, we are then able to price it. However the replicating portfolio we set up was no co-incidence. The amounts of stock and riskless bond were deliberately chosen to make the portfolio replicating. We now turn to the question of how we can determine what the amounts should be for a general derivative pricing problem.

5.3 One-step binomial tree

To look now at a general framework for the pricing of options we need to introduce some notation.

- Let S be the price of the stock at time 0, and let assume that at time T the stock can either go up to $u \cdot S$ or down to $d \cdot S$ ($u > 1$ and $d < 1$); $u - 1$ and $1 - d$ equal, respectively, the proportional increase or decrease in the stock price.
- We want to price a European call option that gives the right to buy at time T the stock at a strike price of X.
- Let us call f_u and f_d the values of the option at T when the price goes up or down, respectively, and let f be the value of the option at time 0 (i.e. its price). We have: $f_u = \max\{Su - X, 0\}$ and $f_d = \max\{Sd - X, 0\}$.
- Let p be the probability of the stock price going up at T (*market probability*).

All the above is summarised in the diagram in Figure 1.

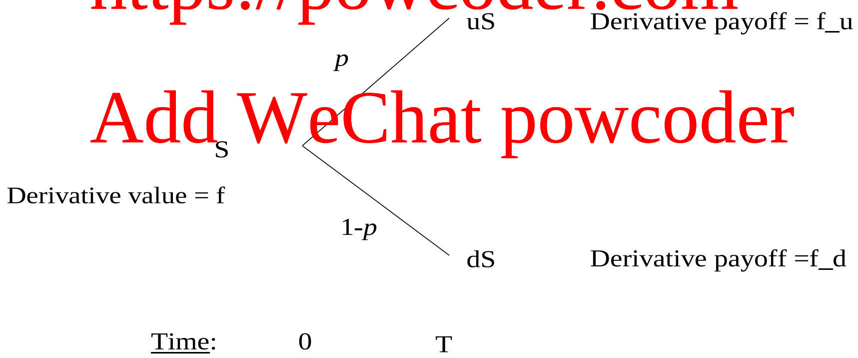


Figure 5.1: One-step binomial tree

This diagram will help us derive a no-arbitrage price for the call option. There are two ways of doing this: the first is similar to the example we saw earlier, i.e. it involves the construction of a *replicating portfolio* i.e. a portfolio of stock and cash that takes up the same values as the option at time T on both branches, while the second is based on the construction of a *riskless portfolio*, which we shall look at further in the next section.

5.4 A replicating portfolio

Let us consider a portfolio of x worth of riskless zero-coupon bonds and y stocks. A *replicating* portfolio has the property that its value tracks the target value (in this case the derivative value) exactly over time. It is constructed to have the same terminal value as the derivative. By no-arbitrage, it therefore has the same value as the derivative at all times prior to maturity, including time 0.

According to our notation, f is the value of the option at time 0, while f_u and f_d are the option values at T if the stock price has gone up or down, respectively (see the diagram above). How can we construct our replicating portfolio? Let us look at the value of the portfolio:

- At 0: $\Pi_0 = x + yS$
- At T:
$$\Pi_T = \begin{cases} \Pi_u = xe^{rT} + ySu & \text{if} S_T = Su \\ \Pi_d = xe^{rT} + ySd & \text{if} S_T = Sd \end{cases}$$

For the portfolio to be replicating, we must have $\Pi_u = f_u$ and $\Pi_d = f_d$, i.e. choose x and y such that

$$\begin{aligned} xe^{rT} + ySu &= f_u \\ xe^{rT} + ySd &= f_d \end{aligned}$$

The solution is given by

$$x = \frac{uf_d - df_u}{u-d}e^{-rT} \qquad y = \frac{f_u - f_d}{Su - Sd}$$

With this choice, the portfolio has the same value at T as the option. By no-arbitrage, the values at time zero must be the same, and we obtain:

$$f = x + yS = \frac{uf_d - df_u}{u-d}e^{-rT} + \frac{f_u - f_d}{Su - Sd}S$$

as the price of the derivative today. This can be rearranged as follows:

$$\begin{aligned} f &= e^{-rT}\left\{\frac{e^{rT} - d}{u-d}f_u + \frac{u - e^{rT}}{u-d}f_d\right\} \\ &= e^{-rT}\left\{\hat{p}f_u + (1-\hat{p})f_d\right\} \end{aligned}$$

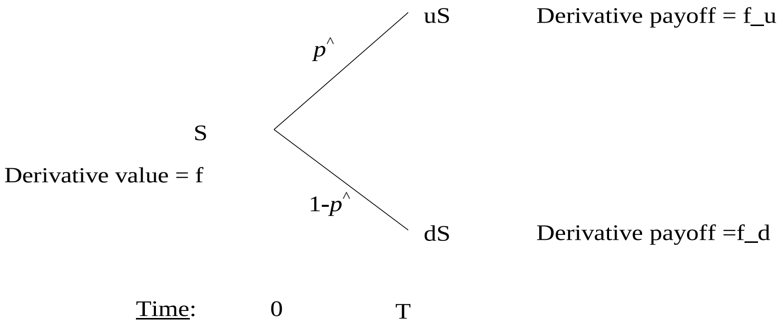
where we have written

$$\hat{p} := \frac{e^{rT} - d}{u-d}$$

Notice that $d < e^{rT} < u$, otherwise there would be arbitrage opportunities, and therefore $\hat{p}$ is a probability, since the above implies $0 < \hat{p} < 1$.

5.5 Risk-neutral valuation

Notice that the formula for f has a very nice interpretation: if we consider the following diagram:



we see that $f = e^{-rT}\hat{E}[f_T]$, where

$$f_T = \begin{cases} f_u & \text{if} S_T = Su \\ f_d & \text{if} S_T = Sd \end{cases}$$

and $\hat{E}$ denotes the expectation with respect to $\hat{p}$, i.e. $\hat{E}[f_T] = \hat{p}f_u + (1-\hat{p})f_d$. Therefore f is the *discounted, expected payoff* under the probability $\hat{p}$, which does not depend on the market probability p.

What is the interpretation of $\hat{p}$? If we calculate the expected stock price at time T with respect to this probability we obtain

$$\begin{aligned} \hat{E}[S_T] &= \hat{p}Su + (1-\hat{p})Sd \\ &= \hat{p}S(u-d) + Sd \\ &= \frac{e^{rT} - d}{u-d} \cdot S(u-d) + Sd = e^{rT}S \end{aligned}$$

Therefore

$$\hat{E}[S_T] = e^{rT}S$$

which is the forward price.

In a risk-neutral world, all individuals are indifferent to risk and require no compensation for risk, and the expected return on all securities is the same (the risk-free rate). The result above implies that setting the probability of the stock price going up to Su equal to $\hat{p}$ is equivalent to assuming that the return on the stock equals the risk-free rate. Therefore, $\hat{p}$ is the probability of an "up"-step in a risk-neutral world, and the option value is the discounted expected payoff in that risk-neutral world. This method of deriving option value is called *risk-neutral valuation*.

Example. Let us apply this to our numerical example: as before, $r = 0.044$, $S = 20$, $u = 1.1$, $d = 0.9$ and $T = 0.25$. We computed earlier $\hat{p} = 0.5553$.

The expected payoff form the option is then $0.5553 \times 1 + 0.4447 \times 0 = 0.5553$, and if we discount back to time 0 we obtain the call option value: $0.5553e^{-0.044\times0.25} = 0.55$ as before.

Risk-neutral valuation is also related to what is sometimes called the change in measure approach to option valuation. This refers to the fact that a probability is a measure. When we use $\hat{p}$ to calculate our discounted expected payoffs rather than the actual market probabilities p, we are "changing the probability measure".

We will explore this technique further in a continuous time framework later in the course.

5.6 Appendix: A riskless portfolio

5.6.1 Example revisited

Consider again the example outlined in section 5.2 above.

We will now look at the pricing problem in a slightly different way. Consider the following portfolio:

- Long (bought) 0.25 shares
- Short (sold) 1 call option

If the option is sold for £0.5 (which we already know is a price that offers arbitrage opportunities), then I could adopt the following strategy (selling four portfolios like the one above):

- *Time 0*
  - Buy 4 options (-£2)
  - Sell 1 share at £20
  - Invest the £18 at 4.4%

- *Time 3 months* Note that this time this portfolio costs nothing to set up and gains nothing. After three months:
  - Stock price is £22
    - Get £4 from options
    - Get £18.20 from investment
    - Buy back share for £22
    - $\Rightarrow$ Profit = 4 + 18.20 - 22 = 0.20
  - Stock price is £18
    - Get nothing from options
    - Get £18.5482 from investment
    - Buy back share for £18
    - $\Rightarrow$ Profit = 18.20 - 18 =0.20

Regardless of the stock price I make a profit of £0.20 - so we have again shown (using a slightly different approach) that a price of £0.50 for the option allows an arbitrage opportunity.

Notice that this differs from the replicating portfolio approach in that it costs nothing to set up, but guarantees a profit in the future. The replicating portfolio made a profit at set-up at time 0, but was guaranteed to produce a neutral position in the future.

5.6.2 No-arbitrage pricing

As with the replicating approach above, we can use this construction to find a price for the option such that no arbitrage opportunities can arise. Let us consider again the portfolio we just introduced; its value after three months will be:

- If the stock price is £22: $0.25 \times 22 - 1 = 4.5$
- If the stock price is £18: $0.25 \times 18 - 0 = 4.5$

We can see that the value of the portfolio does not change with the value of the stock: it is therefore called a *riskless portfolio*. In order to avoid arbitrage, its value at time 0 should be simply the present value of £4.5 computed using the risk-free discount rate (4.4%), since it is a riskless portfolio. Hence:

$$PV = 4.5 \cdot e^{-0.044 \times \frac{3}{12}} = 4.45$$

We know that the value of the portfolio at time 0 is 0.25 times the value of the share at time 0 minus the value of the call option f (I am in a short position), and this equals the PV above. Therefore:

$$0.25 \times 20 - f = 4.45$$

and so we obtain:

$$f = 5 - 4.45 = 0.55$$

the no-arbitrage price for the option is £0.55 as we found previously. We know need to look at how to establish a riskless portfolio in a general derivative pricing situation.

5.6.3 Riskless portfolio - general result

Consider the notation introduced in section 5.3. In order to construct a riskless portfolio we can start with Δ shares long and 1 call option short. We will then find what Δ needs to be. The value of this portfolio is as follows:

- At 0: $\Pi_0 = \Delta S - f$
- At T:

$$\Pi_T = \begin{cases} \Delta Su - f_u & \text{if} S_T = Su \\ \Delta Sd - f_d & \text{if} S_T = Sd \end{cases}$$

The portfolio is riskless iff the value at T is independent of the price of the stock, i.e. iff $\Delta Su - f_u = \Delta Sd - f_d$, which gives:

$$\Delta = \frac{f_u - f_d}{Su - Sd}$$

With this Δ the portfolio is riskless and its value at T equals $\Delta Su - f_u$.

In order to avoid arbitrage, the value of the portfolio at time 0 must equal the value at T discounted at the risk-free rate r:

$$\Pi_0 = \Pi_T e^{-rT}$$

This is equivalent to:

$$\Delta S - f = [\Delta Su - f_u]e^{-rT}$$

which gives the following expression for the value of the option at time 0:

$$f = e^{-rT}[\hat{p} f_u + (1 - \hat{p}) f_d]$$

where

$$\hat{p} = \frac{e^{rT} - d}{u - d}$$

(Check!)

This is exactly the same result as before under the replicating portfolio pricing approach.

5.7 Further reading

Martin Baxter & Andrew Rennie, *Financial Calculus.* – Chapter 1 and 2

Marek Capinski & Tomasz Zastawniak, *Mathematics for Finance: An Introduction to Financial Engineering* (Springer) – Chapter 1, 8.1-8.2.

John C. Hull, *Options Futures and Other Derivative Securities* – Section 9.1 (in the 5th edition)

Chapter 6

Applications of the Binomial Model

We are going to look now at some applications of the binomial model for the pricing of derivatives. We will show that the model can be used not only for option pricing, but also for other derivatives, giving the example of a forward contract. We then examine the case of a European put option.

We will then move on to the application of the binomial model for the evaluation of two-step and three-step trees, and look at the pricing of American options, addressing the issue of optimal early exercise.

6.1 The value of a forward contract

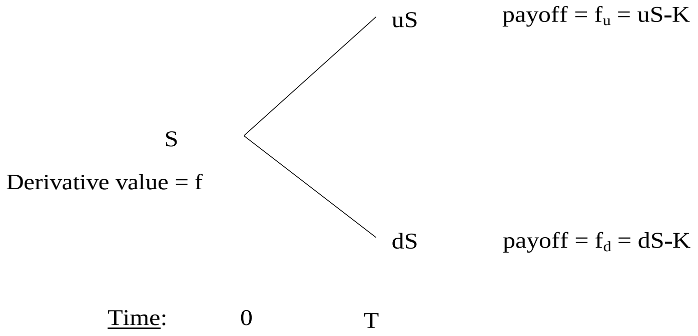
The binomial model can be used to compute the value of any derivative (not only options), provided that we can express its features using a binomial tree. In this section we look at an example considering a forward contract, but a similar analysis can be undertaken for other derivatives.

A forward contract with delivery price K is pictured in the diagram, and can be thought of as the payoff f_T given by

$$f_T = \begin{cases} f_u & = Su - K \\ f_d & = Sd - K \end{cases}$$

6.1.1 A replicating strategy

We can now determine the no-arbitrage price for the forward in the binomial model by *replicating* the forward contract using an amount of stock and an amount of the riskless investment (investment in risk-free bonds). Consider the portfolio that consists of long one unit of stock, and short Ke^{-rT} units of the riskless investment (i.e. you have sold the riskless, zero-coupon government bonds to "borrow" money, and so will have to pay the risk-free rate of interest).



At time T this will be worth

$$S_T - Ke^{-rT} \times e^{rT} = S_T - K$$

regardless of the future price of the stock (we have written S_T for the price of the stock at time T).

This is exactly the payoff of the forward derivative at time T, and so we have found a replicating portfolio for the forward. Therefore the price of the derivative at time zero is the value of this portfolio at this time, which is $S_0 - Ke^{-rT}$. Note that this is the usual value for a forward contract we saw in a previous lecture. For this to be a fair contract at construction, we need this price to be zero, and so we need

$$S_0 - Ke^{-rT} = 0$$

which implies

$$K = Se^{rT},$$

which is consistent with our previous result.

6.1.2 Risk-neutral valuation

We can now use the risk-neutral approach in the binomial model and verify that the formula for the value f at time zero gives the correct delivery price Se^{rT} (forward price).

We have seen that the value of the contract at zero is given by

$$f = e^{-rT}[\hat{p}\ f_u + (1 - \hat{p}) f_d]$$

This, in our case, gives the following expression once we substitute in the appropriate values for f_u and f_d:

$$f = e^{-rT}[\hat{p}\ (Su - K) + (1 - \hat{p})(Sd - K)]$$

which simplifies to

$$f = e^{-rT}[\hat{p}\ Su + (1 - \hat{p})Sd - K]$$

Replace now the formula for the risk-neutral probability $\hat{p}$

$$\hat{p} = \frac{e^{rT} - d}{u - d}$$

and obtain

$$\begin{aligned} f &= e^{-rT}\left[\frac{e^{rT} - d}{u - d} \times Su + \frac{u - e^{rT}}{u - d} \times Sd - K\right] \\ &= e^{-rT}\left[\frac{e^{rT}Su - Sud + Sud - e^{rT}Sd}{u - d} - K\right] \\ &= e^{-rT}\left[\frac{e^{rT}S(u - d)}{u - d} - K\right] \\ &= S - Ke^{-rT} \end{aligned}$$

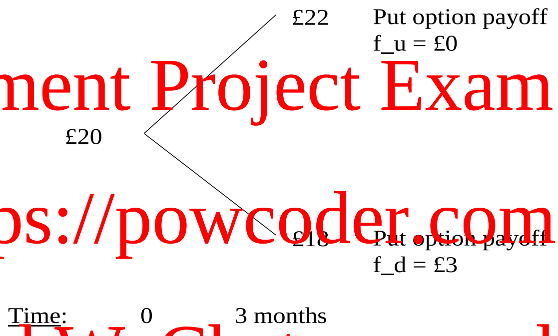
Again this is the usual value for a forward contract, and therefore we see that the binomial model gives consistent results also for other derivatives (not only options). Once again, we see that the only delivery price which gives zero initial value to the forward contract is $K = Se^{rT}$.

6.2 A European put option

We consider again the example presented in the last lecture, but this time we look at a put option. Once computed the price of the put option, we will compare it with the call price obtained previously, and check whether a result called put-call parity is satisfied.

Assume that the price of a stock is currently £20, and after three months it will either be £22 or £18 (discrete time, discrete variable). We want to find the value of a European put option with strike price £21.

If after three months the stock price is £22, the put option will be worth £0, otherwise it will be worth £3 (see diagram below).



Therefore $u = 1.1$, $d = 0.9$, $r = 0.044$, $T = 0.25$, $f_u = 0$ and $f_d = 3$. The risk-neutral probability of an up-step is

$$\hat{p} = \frac{e^{rT} - d}{u - d} = \frac{e^{0.044 \times 0.25} - 0.9}{1.1 - 0.9} = 0.5553$$

as before.

The price of the option is given by the usual formula

$$f = e^{-rT}[\hat{p}\ f_u + (1 - \hat{p}) f_d]$$

which gives in this case

$$f = e^{-0.044 \times 0.25}[0.5553 \times 0 + 0.4447 \times 3] = 1.32$$

So the price of a put option is £1.334, while the call option was priced £0.55 (from previous lecture).

We can know look at the put-call parity relationship for the price of put and call options, given by the following:

$$\text{Call} - \text{Put} = S - Ke^{-rT}$$

with the usual notation (K is the strike price). In our case, $S = 20$, $K = 21$ and r and T are the same as above. It follows that

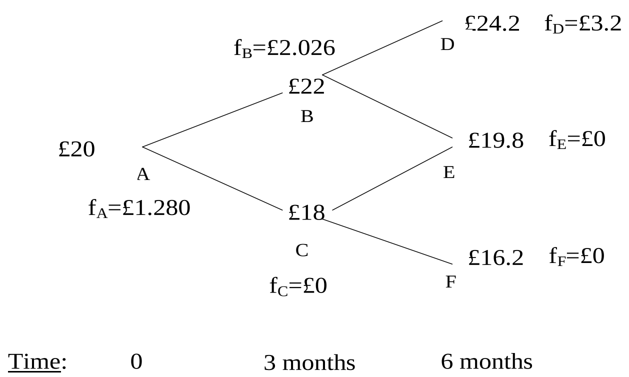
$$S - Ke^{-rT} = 20 - 21e^{-0.044\times 0.25} = -0.77$$

This equals the difference between the call and put price that we found, since $0.55 - 1.32 = -0.77$, and therefore the put-call parity result is satisfied.

6.3 Two-step binomial trees

We can now extend the discussion to two-step binomial trees where the asset price will change twice before maturity.

Example. Consider, for instance, a case similar to the previous one, but where the time to maturity is now six months, and the prices change twice, at three and at six months. The up- and down-steps are again of 10% at both times ($u = 1.1$ and $d = 0.9$). The stock price at zero and the strike price remain the same of the past example, i.e. $S = 20$, $X = 21$, but we now use $r = 12\%$ for the risk-free rate. We want to find the price of a European call option at £21.



To do so, we need to compute the value of the option at each node (denote the nodes with the letters A - F as in the diagram). The values at maturity are easy to compute at each of the final nodes. To obtain the values at three months we need to compute the risk-neutral probability $\hat{p}$ and then apply the one-step formula. From there we can then obtain the initial value applying again the one-step formula with the three-months values just obtained. Notice that the risk-neutral probability is the same for all steps, since we have constant u and d, and are given by

$$\hat{p} = \frac{e^{0.12\times 0.25} - 0.9}{1.1 - 0.9} = 0.6523 \quad \rightarrow 1 - \hat{p} = 0.3477$$

Let us compute the values. We know that at the final nodes the option is worth

- $f_D = 3.2$
- $f_E = f_F = 0$

We can obtain the values after three months as:

- $f_B = e^{-0.12\times 0.25}[0.6523 \times 3.2 + 0.3477 \times 0] = 2.0257$
- $f_C = 0$

It follows that the price of the option is

$$f = f_A = e^{-0.12\times 0.25}[0.6523 \times 2.0257 + 0.3477 \times 0] = 1.2823$$

6.4 General method for n-step trees

The following general approach can be used for the valuation of European options for two-step, three-step, ..., n-step binomial trees:

(a) Compute the risk-neutral probability for every one-step binomial tree (in the large tree)

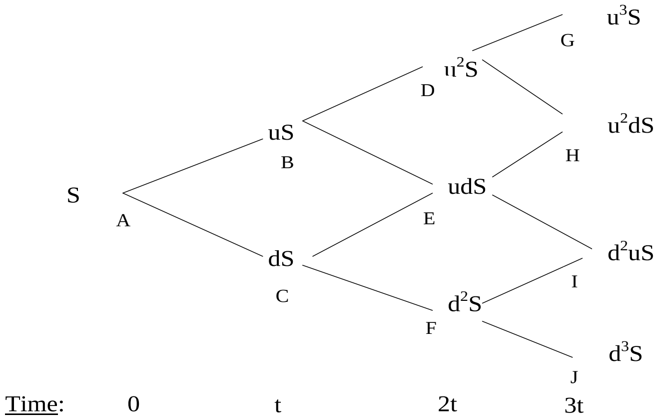
(b) Compute the option values at the terminal nodes (the payoff function)

(c) Work backwards and compute the option values at each intermediate node using risk-neutral valuation

Example. We give here an example of the general method using a three-step three for a European option, where each step is long t, and the up and down jumps are related as follows:

$$u = \frac{1}{d}$$

The three-step tree is shown in the diagram.



The method above gives:

(a) Compute $\hat{p} = \frac{e^{rt}-d}{u-d}$

(b) Find f_G, f_H, f_I and f_J

(c) Then:

- $f_D = e^{-rt}[\hat{p}\ f_G + (1-\hat{p})f_H]$
- $f_E = e^{-rt}[\hat{p}\ f_H + (1-\hat{p})f_I]$
- $f_F = e^{-rt}[\hat{p}\ f_I + (1-\hat{p})f_J]$

and:

- $f_B = e^{-rt}[\hat{p}\ f_D + (1-\hat{p})f_E]$
- $f_C = e^{-rt}[\hat{p}\ f_E + (1-\hat{p})f_F]$

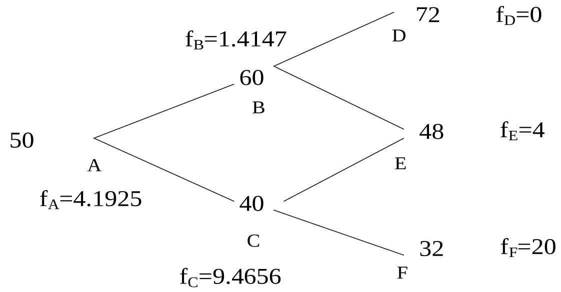
which give a value for the option equal to

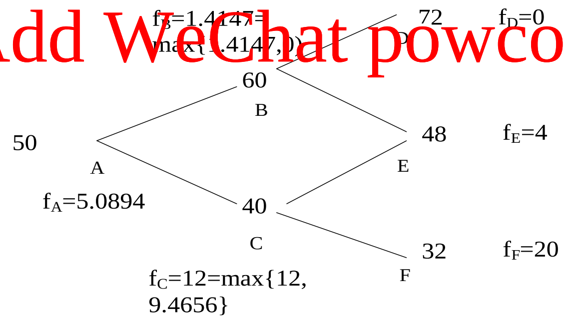
$$f = f_A = e^{-rt}[\hat{p}\ f_B + (1-\hat{p})f_C]$$

6.5 Pricing of American options

Let us start with an example of an American put option. Consider the following two-step tree with $u = 1.2$, $d = 0.8$, $r = 0.05$, and each step is one-year long. The initial price of the stock is £50, and the strike price is £52.

If you use risk-neutral valuation for European put option with strike price of £52 you obtain the values on the left tree. Let's see if the values are correct for an American option.





At the terminal nodes D, E and F you can exercise in both cases (European or American option), so the values are the same. At node B if you exercise (American) you get nothing, and therefore the value for an American option is the same as the European option value (1.4147). At node C, if you exercise you get £12, and the value if you do not exercise is 9.42636. Therefore you should exercise and the value for the American option at C is £12.

Now in order to find the value at A you use risk-neutral valuation

$$f = f_A = e^{-0.05\times 1}[\hat{p}\ 1.4147 + (1-\hat{p})12] = 5.0894$$

So the general valuation method for an American option is

(a) Compute the risk-neutral probability for every one-step binomial tree (in the large tree)

(b) Compute the option values at the terminal nodes using the payoff function

(c) Work backwards and compute the option values at each intermediate node using risk-neutral valuation. Test if early exercise at each node is optimal. If it is, replace the value from the risk-neutral valuation with the payoff from early exercise.

(d) Continue with the nodes one step earlier

Chapter 7

Calculus refreshers

These notes are intended to give you reminders of various important classical calculus results that we will use. The **result (10.3)** and the solution to **equation (7.4)** are particularly relevant. For more details see an introductory calculus book.

7.1 Taylor series

A function $f(x)$ can be expanded as a Taylor series around the point x_0 as follows

$$\begin{aligned} f(x) &= f(x_0) + f'(x)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 + \ldots \\ &= \sum_{i=0}^{\infty} \frac{f^{(i)}(x_0)(x - x_0)^i}{i!} \end{aligned}$$

where $f^{(i)}$ is the i^{th} derivative of f, and we need to assume that the derivatives at x_0 all exist. The special case of this expansion in which $x_0 = 0$ is sometimes called Maclaurin's series.

If $f()$ is a function of two variables, say $f(x, y)$, then a Taylor series expansion can be made about a point (x_0, y_0) as follows

$$\begin{aligned} f(x, y) = \; & f(x_0, y_0) + \frac{\partial f(x_0, y_0)}{\partial x}(x - x_0) + \frac{\partial f(x_0, y_0)}{\partial y}(y - y_0) + \\ & \frac{1}{2!}\frac{\partial^2 f(x_0, y_0)}{\partial x^2}(x - x_0)^2 + \frac{1}{2!}\frac{\partial^2 f(x_0, y_0)}{\partial y^2}(y - y_0)^2 + \\ & \frac{1}{2!}2\frac{\partial^2 f(x_0, y_0)}{\partial x \partial y}(x - x_0)(y - y_0) + \ldots \end{aligned} \tag{7.1}$$

This can be generalised to the case where f is a function of n variables.

7.2 Chain rule differentiation

The classical chain rule for differentiation says that for differentiable functions f and g we have

$$[f(g(s))]' = f'(g(s))\, g'(s)$$

or equivalently

$$\frac{d}{ds}f(g(s)) = \frac{df}{dg}(g(s))\,\frac{dg}{ds}(s)$$

or also equivalently

$$\begin{aligned} f(g(t)) - f(g(0)) &= \int_0^t f'(g(s))\, g'(s)ds \\ &= \int_0^t f'(g(s))dg(s) \end{aligned}$$

where we have written $h'(t)$ for the ordinary derivative of a function h at t.

7.3 Partial differentiation

For f a function of x and y, with x and y both functions of a single independent variable t, the extension of the chain rule gives us

$$\frac{df}{dt} = \frac{\partial f(x,y)}{\partial x}\frac{dx}{dt} + \frac{\partial f(x,y)}{\partial y}\frac{dy}{dt}$$

We can also write this as

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \tag{7.2}$$

when y and x depend on one or more other variables.

For our course it will be useful to consider this result heuristically as an application of the Taylor series. If we take the limit as x_0 tends to x and y_0 tends to y and we write $dx \approx x - x_0$, $dy \approx y - y_0$ and $df(x,y) \approx f(x,y) - f(x_0,y_0)$, then we can express result (7.1) as

$$\begin{aligned} df = {} & \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \\ & \frac{1}{2!}\left(\frac{\partial^2 f}{\partial x^2}dx^2 + \frac{\partial^2 f}{\partial y^2}dy^2 + 2\frac{\partial^2 f}{\partial x \partial y}dxdy\right) + \ldots \end{aligned} \tag{7.3}$$

In ordinary calculus the second order (and higher order) terms in (10.3) tend to zero so that we obtain result (7.2).

7.4 Linear ordinary differential equations

A vast topic - here we only present one simple ordinary differential equation (ODE) to remind the reader of one basic technique, and of the important example of exponential growth. The ODE

$$\frac{dx}{dt} = rx, \tag{7.4}$$

(also sometimes written as $dx = rx\ dt$), where r is a constant, can be solved using the variable separable technique, or in other words by writing the ODE as the integral equation

$$\int \frac{1}{rx} dx = \int 1\ dt$$

Integrating this then gives

$$\begin{aligned} \tfrac{1}{r} ln(x) &= t + k_1 \\ \Rightarrow \quad ln(x) &= rt + rk_1 \\ \Rightarrow \quad x &= e^{rt+rk_1} \\ \Rightarrow \quad x &= Ke^{rt} \end{aligned} \tag{7.5}$$

where k_1 and K are constants. This is the equation for exponential growth at rate r.

To obtain particular solutions, differential equations are usually solved together with **boundary conditions**. In the above simple example, we can determine the value of the arbitrary constant K with the added information that say

$$x(0) = x_0. \tag{7.6}$$

Then substituting this into (7.5) we can find K as

$$x(0) = x_0 = Ke^{r \times 0} = K$$

so that the particular solution to the ODE (7.4) together with the boundary condition (7.6) is

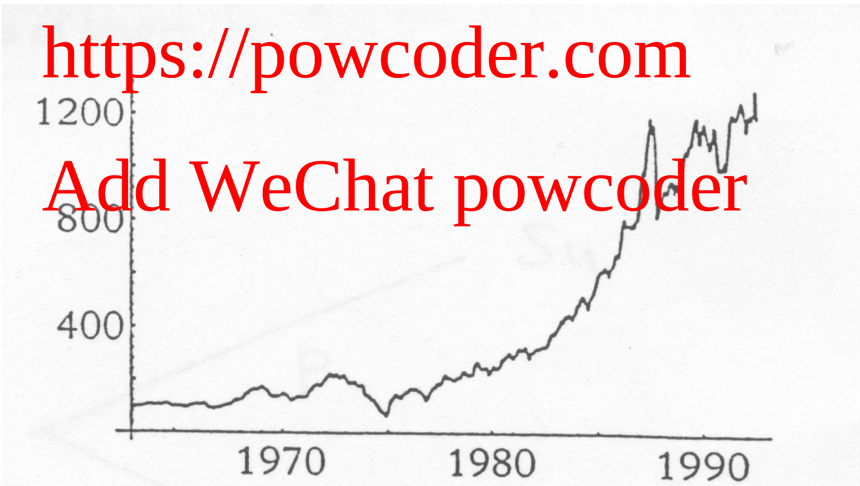
$$x = x_0 e^{rt}.$$

When we move into the world of partial differential equations an initial boundary condition can specify the value of the function across a curve. However boundary conditions do not always have to be initial conditions, they can, for example, be terminal conditions, or other more complicated types of boundary conditions.

Chapter 8

Continuous-time stochastic processes for stock prices

We now explore further suitable stochastic processes to describe the behaviour of stock prices. A typical example of stock price movements is given in the graph below, which shows the price of the UK FTA index from 1963-1992.



The binomial model we saw in the past lectures is one approach to modelling the movement of real prices. In order to implement it on realistic time scales we will need to have a large number of time steps and hence use a large number of nodes. Although there are sometimes benefits to the discrete time approach to modelling the stock prices, solving for derivative prices can be computationally expensive as we are often not able to obtain explicit solutions over many nodes. One alternative approach that can lead to the derivation of some explicit theoretical results for option pricing is to consider a continuous time limit.

A continuous time limit of the binomial model can lead us to a model based on the important stochastic process *Brownian motion*. This increases our flexibility for asset price modelling and allows the use of mathematical techniques such as stochastic calculus.

First of all we look again briefly at a special case of the binomial model, and how it can be extended over time to model the stock prices. We will then consider the appropriate limits to use to move into continuous time.

8.1 Random walk

We will start now from this simple discrete-time, discrete-value (discrete-state space) stochastic process that has natural similarities with the binomial model we just looked at.

Here $\{x_t\}$ is a random variable that begins at a known value x_0, and at time $t = i$ jumps by a random variable ϵ_i from x_{i-1}, for $i = 1, 2, 3, \ldots$ So for example $x_1 = x_0 + \epsilon_1$, and $x_2 = x_1 + \epsilon_2$. Assume that ϵ_i are independent, identically distributed random variables with

$$\begin{aligned} \epsilon_i &= -\delta \text{ with probability } \tfrac{1}{2} \\ \epsilon_i &= +\delta \text{ with probability } \tfrac{1}{2} \end{aligned}$$

for $i = 1, 2, \ldots$, so that x_t takes a jump at each time step of size δ, either up or down, each with probability $\frac{1}{2}$. The jumps are independent, and therefore we can describe the dynamics of x_t with the following equation

$$x_t = x_{t-1} + \epsilon_t = x_0 + \sum_{i=1}^{t} \epsilon_i$$

where ϵ_t is a random variable that takes only values δ or $-\delta$ with equal probability $\frac{1}{2}$.

Therefore the random variable $x_t = x_0 + \sum_{i=1}^{t} \epsilon_i$ is the position after t steps of a random walker who started from x_0, where each step is equally likely to be up or down by δ. The random process $\{x_t, t = 0, 1, \ldots, N\}$ is the path followed by the walker over time.

8.2 Other processes and Markov property

Because the probability of an up or down jump is $\frac{1}{2}$, at time $t = 0$ the expected value of x_t is x_0 for all t. One way to generalise this process is by changing the probabilities for an up- or down-jump to be greater than 0.5 (*random walk with drift*).

Another way to generalise this process is to let the size of the jump at each time t be a continuous random variable. For example, we could let the size of the jump be normally distributed with mean zero and variance σ^2. In this case we will call x_t a discrete-time, continuous-state stochastic process. Another example of a discrete-time continuous-state process is the *first-order autoregressive process*, abbreviated as $AR(1)$. It is given by the equation

$$x_t = \delta + \rho x_{t-1} + \zeta_t$$

where δ and ρ are constants, with $-1 < \rho < 1$, and ζ_t is normally distributed with zero mean.

Both the random walk (with discrete and continuous states, with or without drift) and the AR(1) process satisfy the *Markov property*, and are therefore called *Markov processes*. This property is that the probability distribution for x_{t+1} depends only on x_t, and not additionally on what happened before time t. For example, in the case of the simple random walk, if $x_t = 6$ and $\delta = 1$, then x_{t+1} can equal 5 or 7, each with probability $\frac{1}{2}$. The values before x_t are irrelevant once we know x_t.

The random walk (with discrete and continuous states, with or without drift) process satisfies the *Markov property*, and are therefore called *Markov processes*. This property is that the probability distribution for x_{t+1} depends only on x_t, and not additionally on what happened before time t. For example, in the case of the simple random walk, if $x_t = 6$ and $\delta = 1$, then x_{t+1} can equal 5 or 7, each with probability $\frac{1}{2}$. The values before x_t are irrelevant once we know x_t.

8.3 Taking limits of the random walk

If our N steps take time T to be carried out, each one takes $\Delta t = T/N$ time units. As we move from discrete to continuous time, both the time between jumps, Δt, and the size of the jumps, δ, will need to tend to zero. However the rate at which each tends to zero, and hence the relationship between the time gap Δt and the jump size δ, will also be important. We will need to fix this relationship in some way when we take limits.

We might initially think that as we take limits we should set $\delta = k\Delta t$ for some positive constant k, so that the ratio of $\delta/\Delta t$ is always constant. However it turns out that this choice will not work and we will need to find another approach (see appendix).

Remember that we are going to take limits as both $\Delta t \to 0$ and $\delta \to 0$, whereas t is fixed as the length of the overall interval we are looking at. It turns out that for the variance of the process to "behave well" and remain finite, it is the ratio $\delta^2/\Delta t$ that we need to keep constant while $\Delta t \to 0$ and $\delta \to 0$, rather than the ratio of $\delta/\Delta t$. If we call this constant

$$\sigma^2 := \delta^2/\Delta t,$$

then we can see that the variance of an incremental change in the process over time t is $\sigma^2 t$, and hence is proportional to t, the size of the time length we are looking at. The standard deviation is therefore proportional to the square root of t.

This property, that the variance of the incremental change in our process is proportional to the length of time we are considering, is one of the key properties of the continuous time stochastic process that we will use, Brownian motion.

8.4 Brownian motion (Wiener process)

Brownian motion or a *Wiener process* is a continuous-time stochastic process with the following important properties[1]:

- The Brownian motion process has *independent increments*. This means that the probability distribution for the change in the process over any time interval is independent of any other (non-overlapping) time interval (a property that is also true for the discrete time random walk).

- It is a *Markov process*. As explained earlier, this means that the probability distribution for all future values of the process depends only on its current value, and is unaffected by past values of the process or by any other current information. As a result, the current value of the process is all one needs to make a best forecast of its future value. This property follows from the independent time interval property.

- Changes in the process over a finite interval of time are *normally distributed*, with a variance that increases linearly with the time interval.

The Markov property is particularly important in the context of modelling stock markets, since it implies that only current information is useful for forecasting the future path of the process. Stock prices are often modelled as Markov processes, on the grounds that public information is quickly incorporated in the current price of the stock, so that the past pattern of the prices has no forecasting value. This is called the *weak form of market efficiency*. If it did not hold, investors could in principle "beat the market" through technical analysis, by using the past pattern of prices to forecast the future. The fact that a Wiener process has independent increments means that we can think of it as a continuous-time version of a random walk.

[1] In 1827, the botanist Robert Brown first observed and described the motion of small particles suspended in a liquid, resulting from the apparent successive and random impacts of neighbouring particles; hence the term Brownian motion. In 1905, Albert Einstein proposed a mathematical theory of Brownian motion, which was developed further and made more rigorous by Norbert Wiener in 1923.

8.5 Definition of Brownian motion

Brownian motion can be formally defined by the following properties. The process z_t is Brownian motion if and only if

1. All non-overlapping increments of the process are independent (so that $z_{s_2} - z_{s_1}$ is independent of $z_{t_2} - z_{t_1}$ for all $0 \leq s_1 < s_2 \leq t_1 < t_2$).

2. For $0 \leq s < t$ the increment $z_t - z_s$ is normally distributed with mean 0 and variance $t - s$.

3. z_t is continuous and $z_0 = 0$.

We can think of the second property as saying that a process $z = \{z_t\}_{t>0}$ following Brownian motion has increments that can be expressed as

$$\Delta z = z_{t+\Delta t} - z_t = \sqrt{\Delta t} \cdot \epsilon_t$$

where $\epsilon_t \sim N(0,1)$.

Let us examine what the conditions above imply for the change of z over some finite interval of time T. If we want to study the difference $z_T - z_0$, we can use N intervals with each interval step being $\Delta t = T/N$, and obtain

$$\begin{aligned} z_T - z_0 &= z_T - z_{T-T/N} + z_{T-T/N} - z_{T-2T/N} + \cdots + z_{T/N} - z_0 \\ &= \sum_{i=1}^{N} \epsilon_i \sqrt{\Delta t} \end{aligned}$$

where the $\epsilon_i \sim N(0,1)$ and they are independent. Each "small" difference has mean 0 and variance equal to the interval length Δt, which for N intervals will be T/N. It follows that $z_T - z_0$ is normally distributed with

$$E[z_T - z_0] = 0$$

and

$$\text{Var}[z_T - z_0] = N \times \frac{T}{N} = T$$

i.e.

$$(z_T - z_0) \sim N(0, T)$$

An example of a sample path of the Brownian motion process is shown below.

Example. Suppose z_t follows a Brownian motion process where $z_1 = 25$ and $\Delta t = 1$ year. What are the distributions of z_2 and z_6? From above, we know that $(z_{T_2} - z_{T_1}) \sim N(0, (T_2 - T_1))$. Therefore for $T_2 = 2$ and $T_1 = 1$ we have

$$(z_2 - 25) \sim N(0,1) \quad \Rightarrow z_1 \sim N(25,1)$$

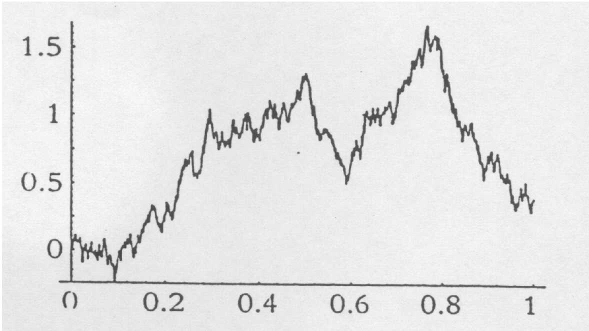


Figure 8.1: Example of a Brownian motion sample path

Similarly, when $T = 6$ we have $z_6 \sim N(25, 5)$.

Notice that in the case where the process starts at zero, i.e. $z_0 = 0$, the generic variable z_t follows the normal distribution $N(0, t)$.

Example. If $Z \sim N(0, 1)$, then the process $x_t = \sqrt{t}Z$ is continuous and is marginally distributed as a $N(0, t)$. Is x_t a Brownian motion process?

The answer is no, since the increments do not respect the conditions for a Wiener process. In fact, we have $x_{t+\Delta t} - x_t = \left(\sqrt{t+\Delta t} - \sqrt{t}\right) Z$. It follows that the increment follows a normal distribution with zero mean and variance given by

$$\left(\sqrt{t+\Delta t} - \sqrt{t}\right)^2 \times 1 = \Delta t - 2t\left[\sqrt{1+\frac{\Delta t}{t}} - 1\right]$$

which is not Δt.

Moreover, the increments are not independent. For instance, if we consider the two increments $x_{t+\Delta t} - x_t$ and $x_t - x_0$ we have that their correlation is given by the following (where we use the fact that $x_0 = 0$ and Z^2 is a χ^2_1):

$$\begin{aligned}
E(x_{t+\Delta t} - x_t)(x_t - x_0) &= E(x_{t+\Delta t} - x_t)x_t \\
&= \sqrt{t+\Delta t}\sqrt{t}E(Z^2) - E(x_t^2) \\
&= t\left[\sqrt{1+\frac{\Delta t}{t}} - 1\right] \neq 0
\end{aligned}$$

It follows that x_t is not a Brownian motion process.

Example. If z_t and w_t are two independent Brownian motion processes starting at zero, and ρ is a constant between -1 and 1, then the process $x_t = \rho z_t + \sqrt{1-\rho^2}w_t$ is continuous and has marginal distributions $N(0, t)$. Is x_t a Brownian motion?

Here an increment is given by

$$x_{t+\Delta t} - x_t = \rho(z_{t+\Delta t} - z_t) + \sqrt{1-\rho^2}(w_{t+\Delta t} - w_t)$$

We know from the properties of Brownian motion that both the increments of z_t and of w_t appearing above follow a $N(0, \Delta t)$ distribution. It follows that the increment in x_t is the sum of a $N(0, \rho^2 \Delta t)$ and a $N(0, (1-\rho^2)\Delta t)$, i.e. a $N(0, \Delta t)$, which is consistent with the Brownian motion process properties.

Moreover, the increment in x_t shown above will be independent of any other increment in the process x_t over a non-overlapping time interval, since this is true for both $z_{t+\Delta t} - z_t$ and $w_{t+\Delta t} - w_t$. It follows that x_t is indeed a Brownian motion.

8.6 Generalised Brownian Motion process

If z is a Brownian motion (or equivalently Wiener) process , we have seen that $E[\Delta z] = 0$ *(drift)* and $\mathrm{Var}[\Delta z] = \Delta t$ *(variance)*.

We can construct a general class of processes $x = \{x_t\}$ such that for "small" time periods Δt

$$\Delta x = x_{t+\Delta t} - x_t = a\Delta t + b\Delta z$$

where a and b are constants. Then $\Delta x \sim N(a\Delta t, b^2\Delta t)$ since Δz is normally distributed and:

$$E[\Delta x] = a\Delta t \quad \text{(new drift)}$$

and

$$\mathrm{Var}[\Delta x] = b^2\Delta t \quad \text{(new variance)}$$

With arguments similar to those in Section 1, we obtain that the behaviour of the change over a time interval T is given by

$$(x_T - x_0) \sim N(aT, b^2T)$$

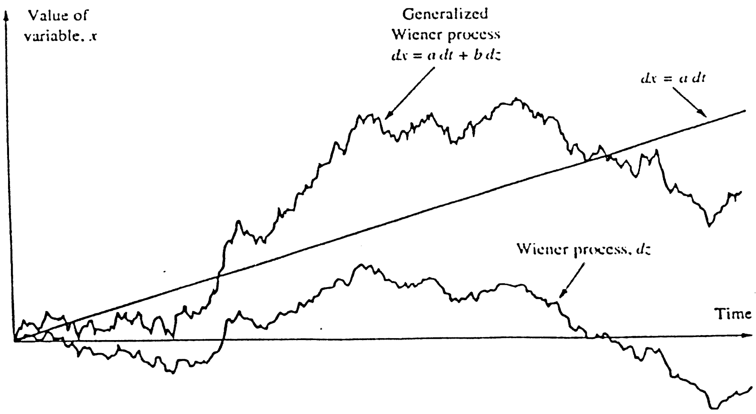
Example. Consider a generalised Wiener process with $x_0 = 50$, $a = 20$ and $b^2 = 900$. At the end of six months ($T = 0.5$) we have $(x_{0.5} - x_0) \sim N(0.5a, 0.5b^2)$, and therefore the value of the variable after six months has distribution $N(x_0 + 0.5a, 0.5b^2)$, i.e. a $N(60, 450)$.

We say that x satisfies the stochastic differential equation

$$dx = adt + bdz$$

where a is the *drift rate*, b^2 is the *variance rate* and dz is the "error" term. If $b = 0$ (no variability) then $dx = adt$, i.e. $x = x_0 + at$: the process grows linearly with time. When $b \neq 0$, the term $b\Delta z$ adds variability around the line $x = x_0 + at$.

Sample paths for a generalised Brownian motion process and Brownian motion are shown below.



8.7 Itô process

A further type of stochastic processes can be defined where the drift and variance rate are not constant anymore. A random process $x = \{x_t\}$ is an *Itô process* if for any t and very small Δt

$$\Delta x = x_{t+\Delta t} - x_t = a(x,t)\Delta t + b(x,t)\Delta z$$

where $a(x,t)$ is the *drift rate* and $[b(x,t)]^2$ is the *variance rate*. We say that x satisfies the following stochastic differential equation

$$dx = a(x,t)dt + b(x,t)dz$$

Itô processes are sometimes called diffusion processes, as they can be used to model the diffusion of gas particles.

8.8 A process for stock prices: the geometric Brownian motion

We are looking for an appropriate process to model stock prices. One model we may consider using for non-dividend paying stock prices is the generalised Brownian motion process

$$dS = \mu dt + \sigma dz$$

where μ is the drift of the stock price and the σ, is the square root of the variance rate. The model implies that in a period of Δt

$$\Delta S = S_{t+\Delta t} - S_t = \mu \Delta t + \sigma \epsilon \sqrt{\Delta t}$$

where $\epsilon \sim N(0,1)$ as usual.

Observe that the expected increase in the stock price in this time period is $\mu \Delta t$, which is independent of the stock price itself. This does not seem appropriate because growth in a stock price is usually related to the size of the stock price itself. It is the return from a stock that we are interested in, which is measured in terms of a percentage change in price.

This model will also allow the possibility of negative values for the stock price S which is clearly not appropriate.

A more appropriate model is to look at the percentage change as following generalised Brownian motion, i.e.

$$\frac{\Delta S}{S} = \frac{S_{t+\Delta t} - S_t}{S_t} = \mu \Delta t + \sigma \Delta z$$

This gives a model for the actual stock price S as

$$\Delta S = \mu S_t \Delta t + \sigma S_t \Delta z$$

i.e. $\Delta S / S \sim N(\mu \Delta t, \sigma^2 \Delta t)$. We call this process a *geometric Brownian motion*, and we say that it satisfies the following stochastic differential equation

$$dS = \mu S dt + \sigma S dz \qquad (*)$$

If $\sigma = 0$ (no variance) then S is a risk-free asset and $dS = \mu S dt$, which is equivalent to $dS/dt = \mu S$, i.e. $S' = \mu S$. The solution to this differential equation is

$$S_T = S_0 e^{\mu T}$$

For $\sigma = 0$ the price grows at a continuously compounded rate of μ per unit.

The geometric Brownian motion has two parameters: the rate of return μ and the volatility σ (risk).

It turns out that when $\sigma \neq 0$ the process that solves the *stochastic* differential equation $(*)$ is given by

$$S_t \;=\; S_0 \exp\left\{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma z\right\} \tag{8.1}$$

i.e. $(*)$ corresponds to the exponential of a generalised Brownian motion. We shall look into this further in lectures to come. Following the formula above, the geometric Brownian motion is also called *exponential Brownian motion*.

Example. Consider a stock that pays no dividends, has a volatility of 30% per annum and provides an expected return of 15% per annum. Then $\mu = 0.15$ and $\sigma = 0.30$:

$$\frac{dS}{S} = 0.15dt + 0.30dz$$

This means that as an approximation we can write

$$\frac{\Delta S}{S} = 0.15\Delta t + 0.30\epsilon\sqrt{\Delta t}$$

If the price is now £100 what is the distribution of the price after one week?

One week is 1/52 years, i.e. $\Delta t = 0.0192$; $S_t = 100$. We obtain:

$$\frac{S_{t+(1\,\text{week})} - 100}{100} = 0.15 \times 0.0192 + 0.30 \times \sqrt{0.0192} \cdot \epsilon$$

where $\epsilon \sim N(0,1)$. It follows:

$$S_{t+(1\,\text{week})} = 100.288 + 4.16\epsilon$$

Therefore our approximation gives:

$$S_{t+(1\,\text{week})} \sim N(100.288, 4.16^2)$$

An approximate 95% probability interval is then given by

$$100.288 - 1.96 \times 4.16 \leq S_{t+(1\,\text{week})} \leq 100.288 + 1.96 \times 4.16,$$

which gives

$$92.14 \leq S_{t+(1\,\text{week})} \leq 108.44.$$

With the more precise lognormal formula for a 95% probability interval, derived from (8.1) above,

$$Se^{(\mu - \frac{\sigma^2}{2})T - 1.96\sigma\sqrt{T}} \leq S_T \leq Se^{(\mu - \frac{\sigma^2}{2})T + 1.96\sigma\sqrt{T}}$$

we obtain

$$92.17 \leq S_{t+(1\,\text{week})} \leq 108.49.$$

They are almost identical, since Δt here is very small (0.0192). If Δt is larger the first procedure is no longer accurate, and we need to use the lognormal approach.

8.9 Appendix: Brownian Motion as a limit of a discrete time random walk

In later lectures we will be working with *Brownian motion* (also called a *Wiener process*), which is a continuous-time stochastic process. Here we provide an outline of how to move from a version of the discrete-time binomial model (the random walk) to continuous-time Brownian motion, by taking appropriate limits.

Consider a process x_n that follows the random walk outlined in section 8.1. After n steps the process can be expressed as

$$x_n = x_0 + \sum_{i=1}^{t} \epsilon_i.$$

We are interested in the incremental change $x_N - x_0$ (i.e. the distribution of the increase or decrease in the process after N time steps) when N is large.

Suppose now that the time between the jumps up or down of the random walk is small and is Δt, so that jumps take place at times $\Delta t, 2\Delta t, 3\Delta t \dots$ The number of jumps up to a time point t will be $n = t/\Delta t$. As we move from discrete to continuous time, both the time between jumps, Δt, and the size of the jumps, δ, will need to tend to zero. However the rate at which each tends to zero, and hence the relationship between the time gap Δt and the jump size δ, will also be important. We will need to fix this relationship in some way when we take limits.

We might initially think that as we take limits we should set $\delta = k\Delta t$ for some positive constant k, so that the ratio of $\delta/\Delta t$ is always constant. However as we shall see, this choice will not work and we will need to find another approach.

As we know that n, the number of jumps between time 0 and time t, is going to become large in the limit, we can start by using the central limit theorem. One version of the central limit theorem tells us that if $X_1, X_2, \dots, X_n$ are a sequence of independent identically distributed random variables with finite means μ and finite non-zero variances σ^2, and $S_n = \sum_{i=1}^{n} X_i$, then

$$\frac{S_n - n\mu}{\sigma\sqrt{n}} \to \mathrm{N}(0,1)$$

in distribution as $n \to \infty$. In our case the mean of each jump ϵ_i is zero ($\mu = \frac{1}{2}\delta - \frac{1}{2}\delta = 0$), and we can calculate the variance as

$$\begin{aligned} \mathrm{E}[\epsilon_i^2] - (\mathrm{E}[\epsilon_i])^2 &= \delta^2(\tfrac{1}{2} + \tfrac{1}{2}) - 0 \\ &= \delta^2. \end{aligned}$$

Therefore, the variance of $\sum_{i=1}^{n} \epsilon_i$ is $n\delta^2$ as all covariances are zero by independence of the ϵ_i. So for large n, we have that approximately

$$\sum_{i=1}^{n} \epsilon_i \sim \mathrm{N}(0, n\delta^2).$$

Substituting $n = t/\Delta t$ as the number of jumps in our time interval for given Δt, the variance of the incremental change in the process, $x_t - x_0$, is

$$\frac{\delta^2}{\Delta t} t.$$

Remember that we are going to take limits as both $\Delta t \to 0$ and $\delta \to 0$, whereas t is fixed as the length of the overall interval we are looking at. We can see from this variance result that, in order for the variance of the process to "behave well" and remain finite, it is the ratio $\delta^2/\Delta t$ that we need to keep constant while $\Delta t \to 0$ and $\delta \to 0$, rather than the ratio of $\delta/\Delta t$. If we call this constant

$$\sigma^2 := \delta^2/\Delta t,$$

then we can see that the variance of an incremental change in the process over time t is $\sigma^2 t$, and hence is proportional to t, the size of the time length we are looking at. The standard deviation is therefore proportional to the square root of t.

This property, that the variance of the incremental change in our process is proportional to the length of time we are considering, is one of the key properties of the continuous time stochastic process that we will use, Brownian motion.

Chapter 9

Introduction to stochastic calculus and Itô's lemma

In the last few lectures we have tried to model the behaviour of stock prices using some specific stochastic processes. But what about the behaviour of the price of a derivative? We expect that in general the price of a derivative will be a function of both *time* and the *price of the underlying asset*. If we specify a particular process for the price of the underlying asset, can we obtain the dynamic of the price of the derivative? The answer is yes, and this can be achieved using *stochastic calculus*.

9.1 Ordinary and stochastic calculus

In previous lectures we have seen that the various stochastic processes we analysed satisfied certain stochastic differential equations (SDE). The most general form of SDE was of the type

$$dx_t = a(x_t, t)dt + b(x_t, t)dz_t$$

where $\{z_t\}$ is a standard Brownian motion, and a and b are some known (non-random) functions of x_t and time. In fact, when we write an SDE we are really just using convenient notation to describe an integral equation. If we add an initial condition, say that the process starts at a given point x_0, then the SDE above can be interpreted as a version of the integral equation

$$x_t = x_0 + \int_0^t a(x_s, s)ds + \int_0^t b(x_s, s)dz_s \tag{9.1}$$

Example. In the case of constant $a(x_t, t) = a$ and $b(x_t, t) = b$ we obtain the SDE of the generalised Brownian motion

$$dx_t = adt + bdz_t$$

and the associated integral equation simplifies to

$$x_t = x_0 + at + bz_t$$

We are of course comfortable with the integral with respect to t (or equivalently with respect to s in (9.1)), but we need to be careful with what it really means to talk about the second integral in (9.1), which is with respect to Brownian motion (z_t).

We have seen in the lectures on Brownian motion that in the case of no volatility ($b = 0$) the process is just a deterministic function equal to the straight line $x_t = x_0 + at$, while in the case $b \neq 0$ we add noise around that line according to the Brownian motion z_t.

But what about the general case of variable $a(x_t, t)$ and $b(x_t, t)$? We can try and interpret our integral equation as follows: the ds-integral is an ordinary Riemann integral, and the dz_s-integral perhaps could be interpreted as a Riemann-Stieltjes integral for each trajectory. Unfortunately, this is not possible since one can show that the trajectories are of locally unbounded variation[1], and therefore the stochastic dz_s-integral cannot be defined in the traditional way.

To make the characteristics of the trajectories of a Brownian motion even clearer, let us take a closer look at some graphs. First, consider a deterministic smooth (differentiable) function with a quite jagged trajectory which could be considered "similar" to a Brownian motion trajectory. As we zoom in, we see that the function becomes smoother and straighter, until eventually it becomes a straight line.

Differentiable functions, however strange their global behaviour, are built from straight line segments, and classic calculus is the formal acknowledgement of this.

Let us now take a look at a Brownian motion. As we zoom in we cannot obtain a straight line. The process replicates itself (self-similarity property) even if rescaling (zooming in).

Unfortunately this implies that the trajectory is not differentiable at any t, and therefore the tools of traditional calculus cannot be applied. What we need now is *stochastic calculus*.

We are going to look here at *stochastic differentials* and in particular at Itô's formula for the differential of a function of a stochastic process. Underlying the subject is the careful definition of the integral with respect to Brownian motion in (9.1). We will not look at this in detail in this introductory course, but instead will concentrate on some intuition and some tools needed for manipulating stochastic differentials.

[1]The *variation* of a process X_t over an interval $[0, T]$ is defined as $\sup_\tau \sum_{i=1}^n |X_{t_i} - X_{t_{i-1}}|$ where $\tau : 0 = t + 0 < t_1 < ... < t_n = T$, so that the supremum is taken over all possible partitions of $[0, T]$.

9.2 Itô's formula

Itô's formula is a fundamental result of stochastic calculus, and gives us an explicit understanding of the behaviour of functions of stochastic processes. The formula is also known as *Itô's lemma*, and states the following:

Itô's formula. *Consider a random variable x that follows an Itô process*

$$dx = a(x,t)dt + b(x,t)dz$$

where z is a Wiener process, $a(x,t)$ is the drift rate and $[b(x,t)]^2$ is the variance rate (non-constant). Consider also a continuous function $G = G(x,t)$ twice differentiable in x and once differentiable in t. Then G is itself a random process and satisfies the following stochastic differential equation:

$$dG = \left[\frac{\partial G}{\partial x}a(x,t) + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2(x,t)\right]dt + \frac{\partial G}{\partial x}b(x,t)dz$$

Therefore G also follows an Itô process, with drift rate of

$$\frac{\partial G}{\partial x}a(x,t) + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2(x,t)$$

and variance rate of

$$\frac{\partial^2 G}{\partial^2 x}b^2(x,t)$$

The rigorous proof of the formula is outside the scope of this course.

However, when we are dealing with a continuous and differentiable function of two variables x and t, in ordinary calculus we usually express the differential as

$$dG = \frac{\partial G}{\partial x}dx + \frac{\partial G}{\partial t}dt$$

This reflects the fact that we are making a Taylor expansion

$$dG = \frac{\partial G}{\partial x}dx + \frac{\partial G}{\partial t}dt + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}(dx)^2 + \frac{1}{2}\frac{\partial^2 G}{\partial t^2}(dt)^2 + \frac{\partial^2 G}{\partial x \partial t}dxdt + \cdots$$

and then neglecting the terms of second or higher order. In our case, however, we have to be careful because the variable x has special properties, since it satisfies the SDE above. Therefore, after we make the Taylor expansion, we have to look carefully at the second order terms to see if we can actually neglect them. It turns out that whilst the terms in $(dt)^2$ and in $dxdt$ are indeed of higher order and hence can be dropped, the term in $(dx)^2$ is actually of order dt and so cannot be dropped (see Appendix for more explanation of why this is the case). Hence if we go back now to our Taylor expansion, we see that we cannot neglect anymore all the terms that looked "second order", since the term in $(dx)^2$ is actually of order dt. The terms in $(dt)^2$ and in $dxdt$, instead, are of higher order, and can be dropped. Therefore we are left with

$$dG = \frac{\partial G}{\partial x}dx + \frac{\partial G}{\partial t}dt + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2(x,t)dt \qquad (*)$$

If we substitute in the definition of dx according to the SDE of the process we obtain Itô's formula.

Note. Sometimes Itô's result is reported directly as above, i.e. stating the following:

Itô's formula (2). *If x is an Itô process and $G(x,t)$ is $C^{2,1}$, then dG is given by (*).*

This is absolutely equivalent to the formula we gave earlier (just substitute in dx from the SDE).

9.3 Examples

We are going to apply now Itô's formula to derive the SDE followed by some specific processes. The first two were given without proof in the last handout, and are derived here using Itô's formula. The other examples are more general applications. In all these examples we always start from an Itô process

$$dx = a(x,t)dt + b(x,t)dz$$

and then specify $a(x,t)$ and $b(x,t)$ in order to obtain the processes of interest.

9.3.1 Derivation of the SDE of a generalised Brownian motion

Let us start from a standard Brownian motion. In this case our process x is simply z, so the SDE will obviously be $dx = dz$, which is an Itô process with $a(x,t) = 0$ and $b(x,t) = 1$. Let us now consider a transformation of the process with a shift, a change of scale and the addition of a trend term:

$$G(x,t) = x_0 + \alpha t + \beta x$$

where α and β are constants. We know that G follows a generalised Brownian motion, since the differences are given by

$$\Delta G = \alpha \Delta t + \beta \Delta x$$

which by definition is a generalised Brownian motion, since x in this case is a standard Brownian motion.

To apply Itô's formula we compute:

$$\begin{aligned}
\frac{\partial G}{\partial x} &= \beta \\
\frac{\partial G}{\partial t} &= \alpha \\
\frac{\partial^2 G}{\partial x^2} &= 0
\end{aligned}$$

It follows that

$$dG = \alpha dt + \beta dz$$

is the SDE of a generalised Brownian motion with drift α and volatility β.

9.3.2 Solution of the SDE of a geometric (exponential) Brownian motion

In the last handout we said that the solution of the SDE of the geometric Brownian motion was the exponential of a generalised Brownian motion with certain parameters. Now we have the show to prove it: we start from a generalised Brownian motion satisfying the following SDE

$$dx = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma dz$$

In this case we have an Itô process with $a(x,t) = \mu - \frac{\sigma^2}{2}$ and $b(x,t) = \sigma$.

From the previous example we know that the solution of the SDE is given by

$$x = \left(\mu - \frac{\sigma^2}{2}\right) t + \sigma z$$

Let us consider now the exponential transformation

$$G(x,t) = e^x$$

We see that G is only function of x, so all the derivatives in t will be zero. Compute now:

$$\begin{aligned}
\frac{\partial G}{\partial x} &= e^x = G \\
\frac{\partial^2 G}{\partial x^2} &= e^x = G
\end{aligned}$$

Then we have

$$dG = \left[G\left(\mu - \frac{\sigma^2}{2}\right) + \frac{1}{2}G\sigma^2\right] dt + G\sigma dz$$

which simplifies to

$$dG = \mu G dt + \sigma G dz$$

i.e. the SDE of the geometric Brownian motion.

So we have proved that the solution of the SDE of a geometric Brownian motion is

$$G = \exp\left\{\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma z\right\}$$

i.e. the exponential of a generalised Brownian motion.

9.3.3 Logarithm of stock prices

In the last handout we established that an appropriate model for stock prices is the geometric Brownian motion. From the previous example, then, we can write

$$S = S_0 \ \exp\left\{\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma z\right\}$$

and therefore we have that the logarithm of stock prices follows a generalised (not geometric!) Brownian motion. In fact:

$$Y = \log S = Y_0 + \left(\mu - \frac{\sigma^2}{2}\right) t + \sigma z$$

where $Y_0 = \log S_0$, i.e. a generalised Brownian motion with drift rate $\mu - \frac{\sigma^2}{2}$ and variance rate σ^2.

This result can also be obtained starting from the SDE for S

$$dS = \mu S dt + \sigma S dz$$

and applying Itô's formula to the function $G(S,t) = \log S$.

9.3.4 Generic transformation of stock prices

We can derive a general formula valid for *any* function of stock prices. Consider the usual geometric Brownian motion for S as above. We have

$$\begin{aligned} a(S,t) &= \mu S \\ b(S,t) &= \sigma S \end{aligned}$$

Applying Itô's formula, we obtain that a generic transformation of S and time, $G = G(S,t)$, follows a process with SDE

$$dG = \left[\frac{\partial G}{\partial S}\mu S + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial S^2}\sigma^2 S^2\right] dt + \frac{\partial G}{\partial S}\sigma S dz$$

9.3.5 Forward price

A direct application of the previous formula can be the determination of the SDE satisfied by the forward price.

Consider a forward contract on a non-dividend paying stock. The forward price is $F = Se^{rT}$, where T is the time to maturity. We can express this as a function of current time t by setting $T = t_m - t$, where t_m is the maturity date. We obtain

$$F(S,t) = Se^{r(t_m - t)}$$

our function of interest. The process of stock prices is as usual a geometric Brownian motion. We have

$$\begin{aligned} \frac{\partial F}{\partial S} &= e^{r(t_m - t)} \\ \frac{\partial F}{\partial t} &= -rSe^{r(t_m - t)} \\ \frac{\partial^2 F}{\partial S^2} &= 0 \end{aligned}$$

From the formula derived in the previous example we have

$$dF = \left[e^{r(t_m - t)}\mu S - rSe^{r(t_m - t)}\right] dt + e^{r(t_m - t)}\sigma S dz$$

Substituting $F = Se^{r(t_m - t)}$ this becomes

$$dF = (\mu - r)F dt + \sigma F dz$$

i.e. F is a geometric Brownian motion too, with expected growth rate $\mu - r$ and volatility σ. The growth rate of F is the excess return of S over the risk-free rate.

9.4 Appendix: second order effects in the limit

Here we look at the limit of dx^2. Remember that x follows an Itô process, and hence small changes in x are given by

$$\Delta x = a(x,t)\Delta t + b(x,t)\sqrt{\Delta t}\epsilon$$

where we used the Brownian motion property $\Delta z = \sqrt{\Delta t}\epsilon$. Dropping the arguments of functions a and b, the differences squared are then given by

$$(\Delta x)^2 = a^2(\Delta t)^2 + b^2 \Delta t \epsilon^2 + 2ab(\Delta t)^{3/2}\epsilon$$

The first and last terms are of order higher than Δt, but the central term has the same order as Δt; we also know that $\epsilon^2 \sim \chi_1^2$ and therefore

$$\begin{aligned} E[\Delta t \epsilon^2] &= \Delta t \\ \mathrm{Var}[\Delta t \epsilon^2] &= 2(\Delta t)^2 \end{aligned}$$

As $\Delta t \to 0$, the variance of $\Delta t \epsilon^2$ will tend to zero faster than its expectation, so we can treat the term as non-stochastic and equal to its expected value Δt. It follows that $(\Delta x)^2$ will tend to $b^2 dt$ as Δt tends to zero, i.e.

$$(dx)^2 \approx b^2 dt$$

9.5 Further reading

Wilmott (ch. 7) and Hull (Section 10A in 4th edition) provide a heuristic discussion of Ito's lemma similar to the one presented in these lectures, while Neftci (ch. 9 & 10) and Bjork (ch. 3) go into slightly more detail.

See for example *Stochastic differential equations: An introduction with applications* by Bernt Oksendal (*Springer*) for a more advanced treatment of the Itô integral and of stochastic differential equations.

Chapter 10

The Black-Scholes model

Earlier in the course we have seen some examples of the pricing of options using the binomial tree model. In that case the time was discrete, i.e. the price of the stock underlying the option could only change at some specific moments, and the stock price was discrete too, being able only to take two different values in the next time period. The model we present here allows the stock price to change at any time, and to take any value according to a continuous distribution. It is therefore a *continuous time continuous variable* model.

We are going to outline the Black-Scholes method for pricing derivatives. In particular we are going to analyse the pricing of European call and put options on a non-dividend-paying stock.

10.1 Lognormal property of stock prices

We assume as usual that the stock price follows a geometric Brownian motion:

$$dS = \mu S dt + \sigma S dz$$

where μ is the drift and σ is the volatility. As we have seen in earlier lectures, the logarithm of stock prices follows a generalised Brownian motion with

$$d(\log S) = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma dz$$

Therefore we know that the differences in log-prices have a Normal distribution as follows:

$$(\log S_T - \log S_0) \sim N\left[\left(\mu - \frac{\sigma^2}{2}\right) T,\ \sigma^2 T\right]$$

so that when S_0 is given we obtain:

$$\log S_T \sim N\left[\log S_0 + \left(\mu - \frac{\sigma^2}{2}\right) T,\ \sigma^2 T\right]$$

i.e. the logarithm of the stock prices has a Normal distribution. If a variable X is such that $\log X \sim$ Normal then we say that X has *Lognormal distribution*.

It follows that S_T has a Lognormal distribution, and the moments are given by

$$\begin{aligned} E(S_T) &= S_0 e^{\mu T} \\ \mathrm{Var}(S_T) &= S_0^2 e^{2\mu T}\left[e^{\sigma^2 T} - 1\right] \end{aligned}$$

10.2 The Black-Scholes-Merton differential equation

We now look at how we can determine methods for calculating derivative prices for the model we have introduced for the stock price process.

Earlier in the course we looked at simple discrete time, descrete state space models for the stock price, and we saw that there were a number of techniques that we could apply to price derivatives, all based on the assumption of no-arbitrage.

In this section we start to look at these techniques again, and apply them to our continuous time model for the stock price process.

10.2.1 Assumptions of the Black-Scholes model

Assumptions:

1) The stock price process, S_t, follows a geometric Brownian motion
2) We can have any long or short position in the stock
3) No transaction costs
4) The stock pays no dividends (note: the model can be easily adapted to include dividends)
5) No arbitrage
6) Trading is continuous in time
7) There is a risk-free interest rate that is constant and the same for all maturities (this assumption can also be relaxed relatively easily to allow for varying but still deterministic rate)

Denote the option value process by f; it is a function of the stock price S and of time t[1], and note that the above assumption number 1) means that we assume, as before, that

$$dS = \mu S dt + \sigma S dz \tag{10.1}$$

In our Black-Scholes world we also have a riskless asset with price process B_t say, i.e. B_t a riskless zero coupon bond that is worth 1 at time zero, with

$$dB = rBdt, \tag{10.2}$$

as we have seen in previous lectures.

We present here a heuristic outline of the derivation of a partial differential equation This will provide a flavour of the important concepts underlying the approach.

We are going to If we can find a portfolio that replicates the derivative then we can use the absence of arbitrage to argue that this value of the derivative will be the price process for this replicating portfolio. However, in order for us to be able to demonstrate that any other price process allows arbitrage, and hence that this must be the price of the derivative if there is no arbitrage, we need an additional one more property for the replicating portfolio.

Consider a replicating portfolio consisting of ϕ_t amount of the underlying stock, and ψ_t of the riskless asset. At time t this portfolio will be worth

$$\Pi_t = \phi_t S_t + \psi_t B_t \tag{10.3}$$

where B_t is the value of the riskless asset.

Recall our replicating strategy for the binomial model. We argued that if we could find a portfolio strategy that was always worth the same as the derivative at the maturity of the derivative in the future, then using no arbitrage we could argue that the value of this portfolio today must be the value of the derivative. We therefore need $\Pi_T = f(S_T, T) = Payoff(S_T)$ where $Payoff(S_T)$ is the payoff function for the particular derivative at maturity.

The difference in our continuous time set up is that we now need a dynamic portfolio strategy to replicate the derivative, as we have possible changes in the stock price over continuous time. However, to apply our no-arbitrage argument throughout the lifetime of the derivative, we also need the replicating portfolio strategy to be *self-financing*.

Self-financing portfolios

The replication argument we are going to use relies on the trading strategy we will use to replicate the derivative not having an injection or withdrawal of money at any

[1]The fact that the derivative is a function of the stock and time is an assumption that we make. It can be shown to be the case, but for this course we simply assume it.

point prior to maturity - the no-arbitrage argument will break down if at any point funds are added to the portfolio.

This means that for example any change in the amount of stock in the portfolio must be funded entirely by changes in the amount of bond, and vice-versa.

We say that the portfolio must be **self-financing**. In other words, whatever the cost of setting up the portfolio, it must not use additional, exogenous money to subsidise it at any point, nor have money taken out of it. Mathematically this requires

$$dV_t = \phi_t dS(t) + \psi dB(t), \tag{10.4}$$

so that changes in the portfolio value are driven by changes to the stick and bond prices only. Changes in the value of the portfolio are explainable in term of changes in the value of the tradable constituent assets alone.

Any replicating portfolio we use to price must satisfy this result.

This means that changes in the value of the portfolio are solely due to changes in the value of the two assets, so that we do not add or take out capital from the portfolio after the initial construction. A self financing strategy neither requires nor generates funds between time 0 and time T. This requirements results in the equation

$$\Pi_t = \int_0^t \phi_s dS_s + \int_0^t \psi_s dB_s$$

which gives the differential equation for the value of the replicating portfolio:

$$d\Pi_t = \phi_t dS + \psi_t dB$$

Substituting in for the dynamic processes for the stock and riskless asset price processes from (10.1) and (10.2), we can now write

$$\begin{aligned} d\Pi_t &= \phi_t dS + \psi_t dB \\ &= (\phi_t \mu S + \psi_t r B_t)dt + \phi_t \sigma S dZ \end{aligned} \tag{10.5}$$

We can also apply Itô's formula to the derivative value process, as it is a function of the stock price and time. For Π to replicate the derivative it must also be a function of the stock price and time, so that Itô's result gives

$$d\Pi = \left[\frac{\partial \Pi}{\partial S}\mu S + \frac{\partial \Pi}{\partial t} + \frac{1}{2}\frac{\partial^2 \Pi}{\partial S^2}\sigma^2 S^2\right] dt + \frac{\partial \Pi}{\partial S}\sigma S dz \tag{10.6}$$

where the first term is deterministic and the second is random.

Combining (10.6) and (10.5) we have

For the portfolio to be both replicating and self-financing we need to equate both the stochastic terms and the deterministic terms in the dynamics of these two equation.

These two conditions are analogous to our simultaneous equations that we needed to solve in the binomial model.

In other words, comparing (10.6) and (10.5) we find for the portfolio to be replicating we need

$$\phi_t \sigma S = \frac{\partial \Pi}{\partial S}\sigma S \tag{10.7}$$

$$(\phi_t \mu S + r\psi_t B_t) = \frac{\partial \Pi}{\partial S}\mu S + \frac{\partial \Pi}{\partial t} + \frac{1}{2}\frac{\partial^2 \Pi}{\partial S^2}\sigma^2 S^2 \tag{10.8}$$

Now recall that our aim is to find parameters ϕ_t and ψ_t to make this portfolio replicating. We can already see what ϕ_t must be - we should chose

$$\phi_t = \frac{\partial \Pi}{\partial S}$$

for (10.7) to hold.

Eliminating ψ_t from (10.8) by using a re-arrangement of (10.3) ($\psi_t B_t = \Pi_t - \phi_t S_t$), we also have that

$$(\phi_t \mu S + r(\Pi_t - \phi_t S_t)) = \frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 \tag{10.9}$$

Substituting our result for $\phi_t = \frac{\partial f}{\partial S}$ into this we have

$$(\frac{\partial \Pi}{\partial S}\mu S + r(\Pi_t - \frac{\partial \Pi}{\partial S}S_t)) = \frac{\partial \Pi}{\partial S}\mu S + \frac{\partial \Pi}{\partial t} + \frac{1}{2}\frac{\partial^2 \Pi}{\partial S^2}\sigma^2 S^2$$

which can be simplified as

$$r\Pi_t = r\frac{\partial \Pi}{\partial S}S_t + \frac{\partial \Pi}{\partial t} + \frac{1}{2}\frac{\partial^2 \Pi}{\partial S^2}\sigma^2 S^2$$

which is called the Black-Scholes-Merton partial differential equation

The differential equation (10.11) is an example of a class of partial differential equations called parabolic.

Exercise: Convince yourself of the need for the replicating portfolio to be self-financing by re-visiting the proof of this result, and showing where the argument breaks down if the portfolio is not self-financing.

10.2.2 Boundary conditions

In order to solve this differential equation and obtain f we need to impose some *boundary conditions* which will depend on which derivative (f) we want to price. This is common to all differential equations. For example, say that we look at a function $g(x, y)$ such that

$$\frac{\partial g}{\partial x} + \frac{\partial g}{\partial y} = 0$$

A solution is given by $a(x - y) + b$, and without boundary conditions we cannot find specific values for the constants a and b. If we impose values for the function g at specific points for x and y, say $g(0, 0) = 0$ and $g(1, 0) = 2$, we obtain $a = 2$ and $b = 0$. Similarly, we need to find boundary conditions for our derivatives prices to solve together with the partial differential equation (10.11). See the handout "Calculus Refresher" for more details.

The boundary conditions are determined by the properties of the particular derivative we are trying to value, usually depending on their payoff function at the exercise date. For a *European call option* with strike price X, the boundary condition is given by the payoff function at the exercise date

$$f = \max(S - X, 0) \quad \text{at time T}$$

or equivalently

$$f_T = \max(S_T - X, 0).$$

Similarly for a *European put option* we use the boundary condition

$$f = \max(X - S, 0) \quad \text{at time T}$$

10.3 Black-Scholes formulas for the pricing of vanilla options

We will not solve the partial differential equation (10.11) directly in this course, as this requires techniques from a course in partial differential equations. Here we just give the solution, but we will determine this solution from the PDE through an indirect risk-neutral pricing method in later lectures.

Using the above boundary conditions, the solutions of the Black-Scholes differential equation are given by the following formulas for the pricing of vanilla call and put options.

1. **European call option** with strike price X and time to expiry T.

$$\text{Call} = S_0 N(d_1) - X e^{-rT} N(d_2) \tag{10.10}$$

   where $N(.)$ denotes the cumulative distribution function of a standard Normal, and

$$d_1 = \frac{\log(\frac{S_0}{X}) + \left(r + \frac{\sigma^2}{2}\right) T}{\sigma \sqrt{T}}$$

$$d_2 = \frac{\log(\frac{S_0}{X}) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

2. **European put option** with strike price X and time to expiry T.

$$\text{Put} = Xe^{-rT}N(-d_2) - S_0N(-d_1)$$

   where the notation is the same as above.

Notice that both formulas *do not depend on* μ.

Put-call parity. Recall from workshop 1 the important result we determined between the price of a European call and a European put on the same underlying, with equal strike prices and maturity times. We can verify that these Black-Scholes pricing formulas satisfy this *put-call parity* result, i.e.

$$\text{Call} - \text{Put} = S_0 - Xe^{-rT}$$

From the expressions above we have:

$$\begin{aligned}\left[S_0N(d_1) - Xe^{-rT}N(d_2)\right] &- \left[Xe^{-rT}N(-d_2) - S_0N(-d_1)\right] = \\ &= S_0[N(d_1) + N(-d_1)] - Xe^{-rT}[N(d_2) + N(-d_2)] \\ &= S_0 - Xe^{-rT}\end{aligned}$$

where the last result follows from the symmetry of the Normal distribution (i.e. $N(z) = 1 - N(-z)$).

Note - American options. When there are no dividends, it can be shown that early exercise of an *American call option* is never optimal, and therefore the price is the same as a European call option, and we can use the formula above. Unfortunately, there is no analytic formula for the pricing of American *put* options, so in this case we need to use numerical methods.

Example. The price of a non-dividend-paying stock is now £42, the risk-free interest rate is 10% per annum, and the volatility is 20% per annum. What are the prices of a European call and a European put option expiring in 6 months and with strike price of £40?

We have $S_0 = 42$, $X = 40$, $r = 0.1$, $\sigma = 0.2$ and $T = 0.5$ years. We first compute the constants d_1 and d_2 and obtain

$$\begin{aligned} d_1 &= \frac{\log(\frac{42}{40}) + \left(0.1 + \frac{(0.2)^2}{2}\right)0.5}{0.2\sqrt{0.5}} = 0.7693 \\ d_2 &= d_1 - 0.2\sqrt{0.5} = 0.6278 \end{aligned}$$

Therefore the prices of the options are given by

$$\begin{aligned}\text{Call} &= 42N(0.7693) - 40e^{-0.1\times 0.5}N(0.6278) = 4.76 \\ \text{Put} &= 40e^{-0.1\times 0.5}N(-0.6278) - 42N(-0.7693) = 0.81\end{aligned}$$

10.4 BSM PDE using a riskless portfolio approach

Here we demonstrate how a riskless portfolio argument can be used to give the B-S-M pde. The approach is similar to the replicating approach used above. We first use Itô's formula for the derivative price as a function of the stock price and time. This gives

$$df = \left[\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right]dt + \frac{\partial f}{\partial S}\sigma S dz$$

where the first term is deterministic and the second is random.

We can attempt to construct a riskless portfolio (i.e. get rid of the random term) by first constructing a portfolio Π consisting of:

$$\begin{aligned} -1 &: \text{ derivative} \\ \Delta &: \text{ share}\end{aligned}$$

This is the same as our approach under the binomial model. At any point in time the value of Π is:

$$\Pi = -f + \Delta S$$

and therefore the value of the portfolio changes according to

$$d\Pi = -df + \Delta dS$$

But we know that df and ΔdS are given by:

$$\begin{aligned}\Delta dS &= \Delta\mu S dt + \Delta\sigma S dz \\ df &= \left[\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right]dt + \frac{\partial f}{\partial S}\sigma S dz\end{aligned}$$

so together this gives

$$d\Pi = \left[-\frac{\partial f}{\partial S}\mu S - \frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 + \Delta\mu S\right]dt + \left[-\frac{\partial f}{\partial S}\sigma S + \Delta\sigma S\right]dz$$

We can see that if we chose $\Delta = \frac{\partial f}{\partial S}$ then we can eliminate the random component of this expression, and we have

$$d\Pi = \left[-\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right]dt$$

which is a riskless portfolio because there is no stochastic driver in the dynamics of the portfolio price process, and hence no uncertainty about its future value. By definition, a riskless portfolio grows at the risk-free rate r, so it satisfies

$$d\Pi = r\Pi dt$$

so from above we have

$$r\Pi = -\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2$$

But from the definition of the portfolio $r\Pi = -rf + r\frac{\partial f}{\partial S}S$, and thus we obtain

$$rf = r\frac{\partial f}{\partial S}S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 \tag{10.11}$$

which is the *Black-Scholes-Merton differential equation.*

Compare this outlined approach with the arguments used for derivative pricing under the binomial model in previous lectures.

Note. The portfolio we are using is not permanently riskless, but riskless only in an infinitesimally short period of time. As S and t change, $\partial f/\partial S$ also changes. To keep the portfolio riskless, we have to change frequently the quantity of stock.

10.5 Simple extensions to the Black-Scholes model

We have assumed that the risk-free rate is always constant, and crucially it is deterministic (i.e. entirely predictable). As long as we keep it deterministic we can allow the risk-free rate to vary over time, as the function $r(t)$ say, and still follow through the same Black-Scholes arguments outlined above. In this case the risk-free rate parameter r in equation (10.11) should be replaced by

$$\frac{1}{T-t}\int_t^T r(u)du,$$

which can informally be seen as a form of averaging r over the remaining lifetime of the option.

10.6 Further reading

The risk-free hedging approach to deriving the Black-Scholes formula is discussed in many option pricing books, including Wilmott (ch. 8), Hull (ch. 11 of 4th edition). Bjork (ch. 7) outlines the replicating strategy derivation of the BSM PDE.

For more about solving PDEs both analytically and numerically see for example the Wilmott, Howison and Dewynne book, referenced in the course guidance notes.

See also Rebonato - Volatility and Correlation for discussion of techniques for modelling of volatility smiles.

See Bjork chapter 7 for further discussion of the replicating approach to showing the B-S-M PDE result.

Chapter 11

Hedging and the Greeks

11.1 The Greeks

A financial institution that sells a derivative to a customer is faced with the problem of managing the risk. This can be done by forming an appropriate *hedging portfolio* as we have seen above. In order to do this they need to use the partial derivatives of the option price with respect to different variables. Such partial derivatives play an important role and have been assigned specific names, according to Greek letters; for this reason they are known as *the Greek letters* or simply *the Greeks*. They are defined as follows:

- Delta: $\Delta = \frac{\partial f}{\partial S}$
- Theta: $\Theta = \frac{\partial f}{\partial t}$
- Gamma: $\Gamma = \frac{\partial^2 f}{\partial S^2}$
- Vega: $\mathcal{V} = \frac{\partial f}{\partial \sigma}$
- Rho: $\mathrm{P} = \frac{\partial f}{\partial r}$

As we discussed earlier in the course, one of the main uses of derivatives is for the transfer and management of risk. Here we take an initial look at the concepts of hedging in derivatives, and explore the use of the Greeks that we have defined.

11.2 Delta Hedging

The Δ of a derivative or a portfolio is the sensitivity of it value with respect to movement in the underlying. As we have seen it is the first derivative with respect to the underlying, Delta: $\Delta = \frac{\partial f}{\partial S}$

We have also seen in previous lectures that is plays an important role in determining the pde for the price of a derivative. The delta of the derivative was the amount of the stock that we needed to hold to hedge the derivative, i.e. to create a replicating portfolio.

Consider an agent who holds a portfolio consisting of an amount of an underlying (say a stock) and a derivative on the stock. The agent wishes to make the value of his portfolio, $P(S,t)$ say, immunized to small changes in the value of the stock, S. In other words he wants

$$\frac{\partial P}{\partial S} = 0,$$

and if this is the case we say the portfolio is *delta neutral*. One way to do this is to add an amount of a derivative to the portfolio to make the portfolio delta neutral. If the derivative has pricing function $F(S,t)$, and an amount x of the derivative is added, then the new portfolio, $V(S,t)$ say, is worth

$$V(S,t) = P(S,t) + xF(S,t).$$

To make this new portfolio delta neutral we need $\frac{\partial V}{\partial S} = 0$ which implies that

$$\frac{\partial V}{\partial S} = \frac{\partial P}{\partial S} + x\frac{\partial F}{\partial S} = 0$$

which gives

$$x = -\frac{\partial P}{\partial S} \Big/ \frac{\partial F}{\partial S}.$$

This is the amount of the derivative that needs to be added to a portfolio to make it delta neutral.

Consider the special case in which the original portfolio consists of just a short position in one derivative and no stock, so that $P(S,t) = -f(S,t)$ say. In other words we are an agent who has sold a derivative. Suppose we wish to hedge our position with the underlying asset by making it delta neutral. The asset to be added to the portfolio is the stock so that the price of the asset is $F(S,t) = S$, and hence $\frac{\partial F}{\partial S} = 1$. We then have that the amount of the stock that needs to be added to obtain a delta neutral position is

$$x = -\frac{\partial P}{\partial S} \Big/ \frac{\partial F}{\partial S} = -\frac{\partial f}{\partial S}.$$

In other words, the number of units of a stock needed to hedge one unit of a derivative is $\frac{\partial f}{\partial S}$, which is the **delta** of the derivative defined earlier, and also the amount used in the construction of the riskless portfolio during the Black-Scholes-Merton p.d.e. derivation.

For an agent who has sold an option, the Δ is the amount of stock that needs to be held to hedge his position. If the agent wants a perfectly hedged position he will need to constantly adjust this amount by re-balancing his portfolio, as delta $\frac{\partial f}{\partial S}$ itself is constantly changing as S changes.

It can be shown that the delta for a European call is

$$\Delta = \frac{\partial f}{\partial S} = N(d_1)$$

using the notation of the Black-Scholes formula (see exercise sheet). Deltas for call options are always positive, which means that a long (buy) call should be hedged with a short (sell) position in the underlying, and vice versa.

The delta of a European put can be similarly calculated as $N(d_1) - 1$, or simply derived from the delta of a call (again see exercise sheet). Deltas for put options are always negative, which means that a long put should be hedged with a long position in the underlying, and vice versa.

Delta is between 0 and +1 for calls and between 0 and -1 for puts. The delta for the underlying is always 1. A put option with a delta of 0.5 will drop 0.5 in price for each 1 rise in the underlying (i.e. increasingly out-of-the-money), a call option with the same delta will rise 0.5 instead (i.e. increasingly in-the-money).

Example

If, for example, the share price is 10 and the call option price is 1 and the delta of the call option is 0.5, an investor who has sold 12 call option contracts (options to buy 1,200 shares) can delta-hedge his/her position by buying 0.5 x 1,200 = 600 shares. A rise in share price will produce a loss of 0.5 x 1,200 = 600 on the call options but a gain of 600 on the shares.

The delta of the portfolio can be determined by adding up all his/her positions.The delta of the short option position is -0.5 x 1,200 = -600 and delta of the long share position is 0.5 x 1,200 = 600,thus his/her position has a delta of zero, this is referred as being delta neutral.

Unfortunately, delta-hedging only works for a short period of time during when delta of the option is fixed. The hedge will have to be readjusted periodically to reflect changes in delta, which could be affected by the share price, time to expiry, risk-free rate of return and volatility of the underlying. See exercises for an exploration of how how delta changes with the underlying share price and time to expiry.

11.3 Gamma and gamma neutral portfolios

In theory a derivative can be perfectly hedged by continuous re-balancing. This is the basis behind the no-arbitrage derivation of the option prices we have been looking at.

In a continuously re-balanced hedge the value of the stock and the money holdings in the bank will replicate the value of the derivative. This is analogous to the replicating portfolio approach to derivative pricing that we saw in the binomial model.

In practice of course, any hedging will need to be done at discrete time points. The following steps show how this can be done:

- Sell one unit of the derivative at time $t = 0$ at price $f(0, S)$.
- Find Δ, and buy Δ shares. Use the income from the derivative sale, and borrow from the bank at the risk free rate if necessary.
- Wait one time period (day, week, minute...). The stock price has moved, and so the old Δ is no longer correct.
- Calculate the new current Δ and adjust stock holdings, borrowing or lending with the bank if necessary.
- Repeat until time T, the maturity date.

In practice the discrete hedging suffers from a trade off between balancing more often to obtain a better hedge, but suffering higher transaction costs as a result.

Of course the reason why we have to re-balance the portfolio constantly is that delta $\frac{\partial f}{\partial S}$ itself is constantly changing as S changes. For this reason we may wish to know how quickly our portfolio will become unbalanced as S changes, in other words how quickly Δ changes as S changes. This measure of the sensitivity of a portfolio's Δ to S is given by $\Gamma = \frac{\partial \Delta}{\partial S} = \frac{\partial^2 P}{\partial S^2}$.

If Gamma is high then delta is very sensitive to the underlying price, and the portfolio has to be re-balanced frequently, whereas a low Gamma means delta only changes slowly and so to keep the portfolio delta-neutral adjustments to can be made less frequently, and so we can keep the hedge for a longer period.

Consider also the change in value of a portfolio of shares and options, P. Using a Taylor expansion and ignoring higher order terms we can write an approximation of

$$\delta\P = \Delta\delta S + \Theta\delta t + \frac{1}{2}\Gamma\delta S^2$$

so that for a delta neutral portfolio with $\Delta = 0$ we can see that for positive gamma, any changes in the underlying share price value *over a short time period* will result in an increase in the value of the portfolio. The reverse is also true.

It is therefore desirable to keep a portfolio Gamma neutral as well as Delta neutral, in other words, keep $\frac{\partial^2 P}{\partial S^2} = 0$. Delta neutrality will protect against small movements in the underlying between rebalancing, whereas gamma neutrality will also protect against large movements in the underlying price *between* delta-hedge rebalances.

To make the portfolio gamma neutral as well as delta neutral, we need to add to our portfolio an amount of two derivatives, so that it is now worth

$$V(S,t) = P(S,t) + x_F F + x_G G,$$

where F and G are the two derivatives to be added in amounts given by x_F and x_G. Differentiating once and then again, and setting both results to zero for a delta and gamma neutral portfolio, gives

$$\begin{aligned} \Delta_P + x_F\,\Delta_F + x_G\,\Delta_G &= 0 \\ \Gamma_P + x_F\Gamma_F + x_G\Gamma_G &= 0 \end{aligned}$$

which can be solved if we know Delta (Δ) and Gamma (Γ) for F, G and the current portfolio P.

Of course one of the derivatives may be the underling asset itself, which has a Delta of 1 and a Gamma of 0 (as $\frac{\partial S}{\partial S} = 1$ and $\frac{\partial 1}{\partial S} = 0$). Then the value of the new portfolio V is

$$V(S,t) = P(S,t) + x_F F + x_S S$$

which is delta and gamma neutral if

$$\begin{aligned} \Delta_P + x_F\Delta_F + x_S &= 0 \\ \Gamma_P + x_F\Gamma_F &= 0. \end{aligned}$$

This can be solved (check this) to give

$$\begin{aligned} x_F &= -\frac{\Gamma_P}{\Gamma_F} \\ x_S &= \frac{\Delta_F\Gamma_P}{\Gamma_F} - \Delta_P. \end{aligned}$$

11.4 Further reading

For a discussion of hedging and the Greeks see for example chapter 10 in Wilmott, chapter 8 in Bjork, or chapter 13 in Hull (4th edition).

Chapter 12

Volatility

We have seen that in the Black-Scholes model the option value is a function of the stock price S, of time t and time to expiry T, of the strike price X, and of volatility of the underlying asset σ, of the risk-free interest rate r, and of volatility of the underlying asset σ. The main challenge in applying the standard Black-Scholes model is to determine the most appropriate volatility to use. The volatility has a big impact on the price of the option, though it is a forward looking parameter in the sense that it is the volatility of the underlying over the future life of the derivative (until the derivative maturity date) that is important. Therefore, despite its importance on the derivative price, the volatility is in some sense the parameter that can not be directly observed.

In practice at least two methods can be used to estimate or gain a view on the volatility: we can either estimate σ from historical data for the movements of the price of the underlying asset, or we can use the volatility implied by the prices of similar derivatives in the market. The two methods are illustrated below.

12.1 Estimating volatility from historical data

Earlier we derived the distribution for the differences in log-prices. We observe data at fixed intervals (e.g. every week or every day); denote by S_i the stock price at the end of the i-th interval, and let τ be the length of the interval, in years (e.g. for daily observations $\tau = 1/252$, where we assume that there are 252 trading days per year). Then we know that

$$u_i = (\log S_i - \log S_{i-1}) = \log\left(\frac{S_i}{S_{i-1}}\right) \sim N\left[\left(\mu - \frac{\sigma^2}{2}\right)\tau,\ \sigma^2\tau\right]$$

Therefore, if we have n observations, we can compute

$$s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(u_i - \bar{u})^2}$$

where $\bar{u}$ is the sample average of the u_i's, and then obtain an estimate of the volatility as

$$\hat{\sigma} = \frac{s}{\sqrt{\tau}}$$

Note that the volatility in the Black-Scholes formula corresponds to the *future* volatility of the underlying asset, and not the historic volatility. Therefore the historical volatility may not always be the best indicator of the volatility appropriate for the option price.

12.2 Implied Volatility

Another way to estimate σ is to use the volatility that the market is currently using or implicitly assuming. This involves calculating the volatility implied by an option price observed in a market for tradable options. By looking at other tradable, *similar* derivatives to the one we are pricing (based on the same underlying), we can substitute their prices into the Black-Scholes pricing formulas, and solve for σ to get the *implied* volatility. In other words, the implied volatility is the volatility of the underlying which, when substituted into the Black-Scholes formula, gives the theoretical price equal to the market price the option is currently trading at. [1]

The Black-Scholes pricing formula is not directly solvable for σ but can be solved by a numerical root finding procedure. Implied volatilities can be used to monitor the market's opinion about the volatility of a particular stock. One use for them is to calculate implied volatilities from actively traded options on a certain stock and use them to calculate the price of a less actively traded option on the same stock.

Note that the prices of deep in-the-money and deep out-of-the-money[2] options can be relatively insensitive to volatility. Implied volatilities from these options therefore tend to be unreliable if used for options on the same underlying that are at the money. This is one example of a volatility smile.

12.3 Volatility smiles

In practice if you calculate the implied volatility using the Black-Scholes formula for many different strike prices and expiry dates on the same underlying stock, then we find that the implied volatility is not constant across these strikes and expiry dates. If you plot the graph of implied volatility against strike price the shape is of a smile

[1] When using this approach, care needs to be taken to consider the impact of possible volatility smiles - see the following section.

[2] We say that an option is deep in-the-money if exercising the option today would provide a high payoff (i.e. $S_t >> K$ for a call option), and similarly an option is deep out-of-the-money if the stock is a long way from giving the option a payoff positive, i.e. $S_t << K$

or a smirk (roughly). Of course, if the assumptions and theory of Black-Scholes all stood exactly then the graph should be a straight line representing constant volatility for the stock, regardless of the strike price of any option used for the calculation.

In equity options the implied volatility often decreases as the strike price increases (see Figure 1).

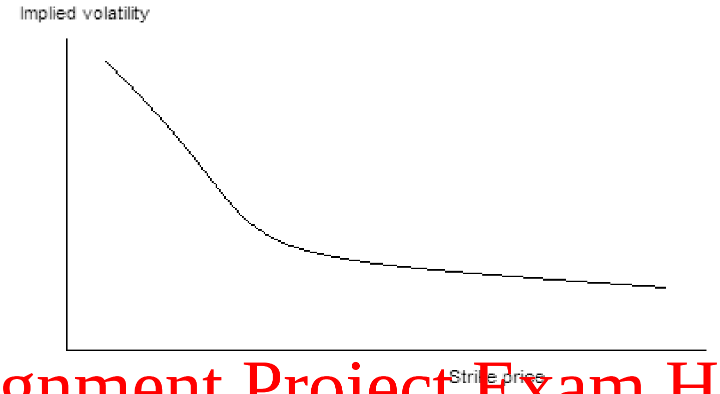


Figure 12.1: Volatility smile for equity options

In foreign currency options we often see that the volatility becomes higher as the option moves either in the money or out of the money (see Figure 2).

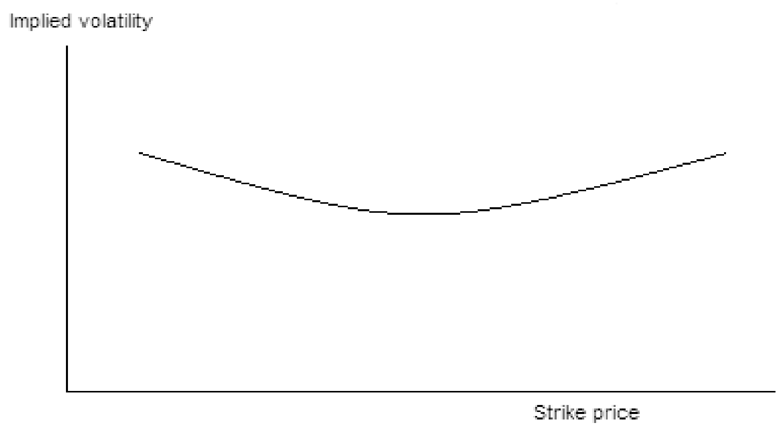


Figure 12.2: Volatility smile for foreign exchange options

There are many suggested reasons for volatility smiles. One of the most important involves a closer look at the assumptions about the movement of the underlying in the Black-Scholes model. As we have seen, the assumption that the price of the underlying follows geometric Brownian motion implies

- changes in asset price follow a lognormal distribution
- the volatility of the asset is constant
- The price of the asset changes smoothly with no jumps (which comes from the continuous property of Brownian motion).

If the movement of the prices of the underlying exhibit behaviour different from these assumptions, and in particular if extreme price movements occur more often than the lognormal distribution predicts, then we may observe a volatility smile, or other distortion from a straight line.

The smile effect clearly means that it can be dangerous to automatically use implied volatilities for underlying assets in pricing models for options on these assets which are not so readily traded as we would ideally like to do. Estimating and incorporating the potential impact of the volatility smiles is a key component of derivative pricing, though exploration of the many approaches for doing this is beyond the scope of this introductory course.

Volatility smiles and fat tailed distributions

Suppose that the movement of the underlying asset, or at least the market's view of it, is in fact different from this model. In particular, suppose that extreme price movements occur more often than the lognormal distribution predicts, so that the implied distribution has "fatter tails" than the model log normal distribution. Let's consider the effect this would have on the implied volatility graph.

Deep out-of-the-money (call) options will have a high strike prices relative to the current asset price, and will only pay out if there are relatively large movements in the underlying that take it above the strike price. If these large movements are more likely in the implied distribution than the model distribution due to fatter right-hand tails, then the call option price will be higher under the implied distribution. This is because the option is more likely to yield a positive payoff under the implied distribution. Thus, when this higher implied (market) price is put into the Black-Scholes formula, it will yield a higher implied volatility.

If the there are also fatter left-hand tails, then a similar argument concerning deep out-of-the-money *put* options for low strike prices will also give higher implied volatility (again, the larger probability of asset downward movements needed for a positive payoff under the implied distribution gives rise to higher put option prices and hence higher implied volatility). Put-call parity can be used to show that the same high volatility will be used to price a deep in-the-money call option with the same strike as the out-of-the-money put option.

We can therefore see that fatter tails in both the right and the left side of the distribution will lead to higher implied volatility for high and low strike prices when compared to implied volatility for options at-the-money. This will produce an upwards curve to the right and left of the implied volatility graph, hence the "smile" we see in Figure 2.

Possible causes of fat tailed distributions

As we have seen, if the movement of the underlying prices exhibits behaviour different from these assumptions, and in particular if extreme price movements occur more often than the lognormal distribution predicts, then we may observe a volatility smile, or other distortion from a straight line.

This means that the implied distribution of price movements is not in fact lognormal as assumed, but may be a distribution, for example, with "fatter tails". Why might these fatter tails occur?

Consider options for foreign exchange. Exchange rate prices may be subject not only to non-constant volatility, but also jumps in their movements, both up and down (maybe in response to central bank announcements). This is inconsistent with the continuity assumption for the movement of the underlying in the Black-Scholes model. The effect of both a nonconstant volatility and jumps is that extreme outcomes become more likely, i.e. a fatter tailed distribution in both the right and left hand side.

The case of equity options is usually slightly different. A look at Figure 1 shows that there is higher volatility at lower strike prices compared with at-the-money strike prices, but lower volatility at higher strike prices. This implies that the tails of the equity movement distribution are fatter on the left side, but not on the right side (follow through the above arguments to check this).

A possible explanation sometimes offered for this is that the volatility of an equity will increase as the equity value decreases because its financial leaverage will increase, making it a more risky investment[3] (do not worry if you are not familiar with this concept, this leaverage explanation will not be examined).

A further possible explanation that has been given for the volatility smile in equity options is what is referred to as "*crashophobia*". The argument is that traders are concerned about the possibility of further market crashes and price options accordingly.

[3] A company's leaverage refers to the relative amounts of debt and equity that are used to finance the firm. All other things equal, a firm is considered more risky if it contains more debt i.e. if it has a higher level of borrowing. If the equity value falls, then the amount of debt relative to equity becomes greater, which makes the firm more risky

12.4 Further reading

For a discussion of possible causes of the volatility smile and the impact of fat-tailed distributions, see Hull chapter 17 (in 4th edition).

More advanced models in which volatility is a *stochastic* parameter are described *Derivatives in financial markets with stochastic volatility* by Fouque, Papanicolaou and Sircar, published by *Cambridge University Press*.

See also Rebonato - Volatility and Correlation for discussion of techniques for modelling of volatility smiles.

Chapter 13

Risk-neutral pricing in continuous time

We saw in the discrete binomial model that we could use a convenient interpretation of our pricing formula in terms of “risk-neutral” probabilities. We now look at the analogous result in continuous time. See the Binomial Model lecture as background.

13.1 The risk-neutral process

The principle of risk-neutral pricing is that, for purposes of valuing a derivative, we can use a technique in which we proceed *as if* the expected stock price were increasing at the risk-free interest rate r (ignoring entirely its real-world rate of increase μ). We then use discounted expected derivative payoffs under this probability distribution. We have already seen this approach to derivative pricing in the *Binomial Model* lectures earlier in the course. As we saw under the Binomial model, the reason we can use this approach is essentially because it is possible to create a *replicating* portfolio for the derivative.

Suppose that a stock price S_t follows a geometric Brownian motion, with drift rate μ and variance rate σ^2 per unit time:

$$dS = \mu\, S\, dt + \sigma\, S\, dz. \tag{13.1}$$

In particular, this implies $\mathrm{E}(S_t) = S_0\, e^{\mu t}$.

In the artificial risk-neutral world, the stock price can be assumed to follow the process

$$dS = r\, S\, dt + \sigma\, S\, dz. \tag{13.2}$$

Then we have $\hat{\mathrm{E}}(S_t) = S_0\, e^{rt}$, where $\hat{\mathrm{E}}(\cdot)$ denotes expectation under the process (13.2), so that this does indeed imply that the expected stock price grows at the risk-free rate, as required. Note that we keep the volatility parameter σ unchanged when moving from the real world (13.1) to the risk-neutral world (13.2): this is analogous to keeping the up and down steps the same in the binomial model.

Note that this is the same argument we used in section 5.6, when we determined what the risk-neutral probabilities of a stock up movement must be in a risk-neutral world under the binomial model, to give a expected rate of return equal to the risk-free rate.

13.2 Risk-neutral pricing

Consider a derivative on the stock, and denote its value at time t, if the stock price is s, by $f(s,t)$. So its actual value at t will be $f_t = f(S_t, t)$, depending on the random quantity S_t and thus itself random. Under risk-neutral pricing, its value at time t should be the suitably discounted expectation, under the risk-neutral model (13.2), of its value at some later time t_m (in our applications t_m will be a maturity time, but that is not crucial). Thus, given that $S_t = s$, we should have

$$f_t = f(s,t) = e^{-rT}\,\hat{\mathrm{E}}(f_{t_m} \mid S_t = s) \tag{13.3}$$

where $T = t_m - t$. In cases where we know the function $f(\cdot, t_m)$ determining the value $f_{t_m} = f(S_{t_m}, t_m)$ of the derivative at t_m as a function of the stock price at that time, we can use (13.3) directly to calculate $f(s,t)$ for all $t \le t_m$. In section (**??**) below we give a proof that this formula does satisfy the Black-Scholes differential equation.

Moreover, it is clearly correct when $t = t_m$ i.e. at maturity. It follows that it must indeed be the correct derivative pricing formula. In particular, the initial value of the derivative is given by

$$f_0 = e^{-rT}\,\hat{\mathrm{E}}(f_T \mid S_0), \tag{13.4}$$

for any future time T.

13.2.1 Example: European call option

Consider a European call option with strike price X and exercise date T. At time T, this will have known value

$$f_T = \max\{S_T - X, 0\}. \tag{13.5}$$

We can thus apply (13.4) to calculate its initial value.

To evaluate $\hat{\mathrm{E}}(f_T \mid S_0) = \hat{\mathrm{E}}(\max\{S_T - X, 0\} \mid S_0)$, we first recall that the distribution of the future stock value S_T is log-normal. In fact, starting from current stock value S_0, and using the growth-rate r appropriate to the risk-neutral world, we have

$$\log S_T \sim \mathcal{N}\left(\log S_0 + (r - \tfrac{1}{2}\sigma^2)T,\, \sigma^2 T\right). \tag{13.6}$$

We can express this in terms of a standard normal variable U:

$$\log S_T = \log S_0 + (r - \tfrac{1}{2}\sigma^2)T + \sigma\sqrt{T}\,U. \tag{13.7}$$

In terms of U,

$$f_T = S_0\, e^{(r-\frac{1}{2}\sigma^2)T}\, e^{\sigma\sqrt{T}\,U} - X$$

if $\log S_T \geq \log X$, *viz* if

$$\begin{aligned} U &\geq \frac{\log\left(\frac{X}{S_0}\right) - (r - \frac{1}{2}\sigma^2)\,T}{\sigma\sqrt{T}} \\ &= d_2; \end{aligned}$$

otherwise $f_T = 0$.

We thus obtain

$$\begin{aligned} \hat{\mathrm{E}}(f_T) &= \int_{-d_2}^{\infty} \left\{S_0\, e^{(r-\frac{1}{2}\sigma^2)T}\, e^{\sigma\sqrt{T}\,u} - X\right\} \times \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2}\, du \\ &= S_0\, e^{rT} \int_{-d_2}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(u-\sigma\sqrt{T})^2}\, du - X\, N(d_2) \end{aligned} \tag{13.8}$$

where N denotes the standard normal distribution function:

$$N(x) := \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2}\, du.$$

To evaluate the remaining integral in (13.8), substitute $v = u + \sigma\sqrt{T}$. It becomes $\int_{-\infty}^{d_1} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}v^2}\, dv$, *i.e.* $N(d_1)$, where

$$\begin{aligned} d_1 &= d_2 + \sigma\sqrt{T} \\ &= \frac{\log\left(\frac{S_0}{X}\right) + (r + \frac{1}{2}\sigma^2)\,T}{\sigma\sqrt{T}}. \end{aligned}$$

Thus $\hat{\mathrm{E}}(f_T) = S_0\, e^{rT} N(d_1) - X N(d_2)$, and hence, on applying (13.4),

$$f_0 = S_0\, N(d_1) - X e^{-rT} N(d_2). \tag{13.9}$$

Equation (13.9) is the formula for pricing a European call option. A similar analysis yields the following pricing formula for a European put option, having $f_T = \max\{X - S_T, 0\}$:

$$f_0 = X e^{-rT} N(-d_2) - S_0\, N(-d_1).$$

See also workshop 2 for a similar look at this problem.

13.3 A useful log-normal distribution result

The following result is useful for pricing derivatives when the underlying asset follows a log normal distribution. We have effectively already used it to derive the Black-Scholes formula in the workshop 2 and example above.

If V follows a log normal distribution where the standard deviation of $ln(V)$ is $= s$, then we can write

$$E[max(V - X, 0)] \;\; = \;\; E[V]\, N(d_1) - X N(d_2). \tag{13.10}$$

where

$$d_1 = \frac{ln\,(E[V]/X) + s^2/2}{s}$$

and

$$d_2 = \frac{ln\,(E[V]/X) - s^2/2}{s}$$

Exercise: prove this result - this requires the same approach as the one used in Workshop 2 to determine the expectation integral in the Black-Scholes derivation, but with a very slight modification to the notation.

13.4 The risk-neutral Monte Carlo approach to derivative pricing

As we have discussed in previous lectures, the solution to the Black-Scholes-Merton partial differential equation with appropriate boundary conditions often has to be calculated using numerical methods such as finite difference techniques. It is, however, sometimes easier to price exotic options through Monte Carlo simulation in a risk-neutral framework.

This uses the risk-neutral valuation approach in continuous time that we have covered above. However, with some exotic options it may not be possible to calculate algebraic expressions for the expected payoffs under the risk-neutral probability distribution. We need to use computer based-simulation to calculate the risk-neutral expectations, and a technique called Monte Carlo simulation.

This involves using a computer to generate (pseudo) random numbers, and modelling realisations of the risk-neutral process from these. A summary of the steps for Monte Carlo based risk-neutral valuation is:

1) Simulate the risk-neutral random walk of the required time horizon up until the maturity date, starting at today's value of the asset.

2) For this realisation, calculate the derivative payoff.

3) Repeat many times.

4) Calculate the average payoff from all realisations.

5) Take the present value of the average for the option value (i.e. discount at the risk-free rate).

Note that for European style path independent options step 1 is straightforward as the jump to time T can be simulated in one go. For example under a Geometric Brownian motion model for the underlying asset price we know that the movement of the asset price from today to the maturity time will follow a log-normal distribution, which can be easily simulated.

For path dependent options the value of the asset price up to the maturity price may need to be tracked at small time intervals to produce accurate results. American options are very hard to use Monte Carlo simulation for, as at each time point there needs to be an assessment of whether early exercise should be made

13.4.1 Simulating Geometric Brownian motions

We now look more closely at step (1) above. We know from work above that the risk-neutral process for the stock price also follows a Geometric Brownian motion, with drift r. We can now also use the result we obtained earlier about the log normal distribution price movements under Geometric Brownian motion. Equation 13.7 above gave us

$$\log S_T = \log S_0 + (r - \tfrac{1}{2}\sigma^2)T + \sigma\sqrt{T}\,\epsilon.$$

where ϵ is a standard normal variable. If we can then simulate a standard normal variable then we are able to simulate the log of the stock price at time T, and hence by taking exponentials, the stock price itself.

13.4.2 Generating random variables

Many methods for (pseudo) random number generation provide samples from a uniform distribution on $[0, 1]$ (which we denote by U[0,1]). For the purpose of generating Monte Carlo simulations we then usually need to turn this random variable into a sample from the distribution we are interested in. For example in the above section we were interested in a sample from the normal distribution.

One useful result that can be used to generate samples from a range of probability distributions is as follows. If $u \sim$ U[0,1] is a sample from a uniform distribution and

$F_X(x)$ an invertible cumulative probability density function for the random variable of interest X then the transformation

$$y = F^{-1}(u)$$

will be a random sample from a distribution with density F_x.

So in the case of generating a sample from a standard normal distribution we can use $N^{-1}(u)$, where $N()$ is the cumulative distribution function of a standard normal.

13.5 Appendix - Risk neutral pricing and the Black-Scholes equation

Here we demonstrate that, under mild conditions, the formula 13.3 satisfies the Black-Scholes differential equation

$$rf = rs\frac{\partial f}{\partial s} + \frac{\partial f}{\partial t} + \tfrac{1}{2}\sigma^2 s^2 \frac{\partial^2 f}{\partial s^2},$$

or equivalently, in terms of $T = t_m - t$ (keeping t_m fixed),

$$rf = rs\frac{\partial f}{\partial s} - \frac{\partial f}{\partial T} + \tfrac{1}{2}\sigma^2 s^2 \frac{\partial^2 f}{\partial s^2}. \tag{13.11}$$

To simplify formulas, we introduce $Y = \log S_{t_m}$, and define $h(x) = f(e^x, t_m)$, so that $f(s, t_m) = h(\log s)$. Then $f_{t_m} = h(Y)$. In the risk-neutral world the distribution of Y (starting from an initial stock-price S_0) is given by 13.6 or 13.7, with $T = t_m$. The conditional distribution of Y given S_t $(t < t_m)$ is, similarly, obtained by replacing S_0 by S_t, and interpreting T as $t_m - t$, in the right-hand sides of 13.6 and 13.7. Thus, given $S_t = s$, we can represent

$$Y = \log s + (r - \tfrac{1}{2}\sigma^2)T - \sigma\sqrt{T}\,U \tag{13.12}$$

with $T = t_m - t$. Thus we can write 13.3 as

$$f(s,t) = e^{-rT}\,\mathrm{E}\{h(Y)\}, \tag{13.13}$$

where Y is given by 13.12, and the expectation is under the standard normal distribution for U. We now differentiate 13.13, assuming that we can take this operation through the expectation, and using, from 13.12,

$$\begin{aligned} \frac{dY}{ds} &= \frac{1}{s} \\ \frac{dY}{dT} &= (r - \tfrac{1}{2}\sigma^2) - \tfrac{1}{2}\,\sigma\,T^{-\frac{1}{2}}\,U. \end{aligned}$$

We obtain:

$$\frac{\partial f}{\partial s} = e^{-rT}\,\mathrm{E}\left\{\frac{1}{s}h'(Y)\right\} \tag{13.14}$$

$$\frac{\partial^2 f}{\partial s^2} = e^{-rT}\,\mathrm{E}\left\{-\frac{1}{s^2}\,h'(Y) + \frac{1}{s^2}\,h''(Y)\right\} \tag{13.15}$$

$$\frac{\partial f}{\partial T} = -rf + e^{-rT}\,\mathrm{E}\left[\left\{(r-\tfrac{1}{2}\sigma^2) - \tfrac{1}{2}\sigma\,T^{-\frac{1}{2}}\,U\right\}h'(Y)\right]. \tag{13.16}$$

Substituting these into the right-hand side of 13.11 and simplifying then gives

$$rf + \tfrac{1}{2}\,\sigma\,T^{-\frac{1}{2}}e^{-rT}\,\mathrm{E}\,\{\sigma\sqrt{T}h''(Y) + U\,h'(Y)\}\,. \tag{13.17}$$

To complete the analysis we need the following result.

Lemma:

Let k be a piecewise differentiable real function such that $k(u)\phi(u) \to 0$ as $|u| \to \infty$ (where $\phi(u) = (1/\sqrt{2\pi})\,\exp(-\frac{1}{2}\,u^2)$ denotes the standard normal density function), and let U have a standard normal distribution. Then

$$\mathrm{E}\{U\,k(U)\} = \mathrm{E}\{k'(U)\}. \tag{13.18}$$

Proof:

We can write

$$\begin{aligned}\mathrm{E}\{U\,k(U)\} &= \int_{-\infty}^{\infty} u\,k(u)\,\phi(u)\,du \\ &= -\int_{-\infty}^{\infty} k(u)\,\phi'(u)\,du,\end{aligned}$$

since $\phi'(u) = -u\,\phi(u)$. The result follows on integrating by parts. The same result will hold if there are isolated points at which k is undefined or non-differentiable, since we can split the range of integration into intervals inside each of which everything is well-behaved, and then combine. This extension is needed for dealing with functions such as 13.5, which is not differentiable at X.

Now take $k(u) = h'(y)$. We have $k'(u) = h''(y)(dy/du) = -\sigma\sqrt{T}\,h''(y)$. Thus, from 13.18,

$$\mathrm{E}\,\{\sigma\sqrt{T}h''(Y) + U\,h'(Y)\} = 0, \tag{13.19}$$

so that 13.17 becomes rf, verifying that the Black-Scholes differential equation 13.11 is satisfied.