STOCHASTIC METHODS IN FINANCE 2021–22
STAT0013
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Exercises 5 - Brownian motion

1. a) Before observing any of the path of a standard Brownian motion, B_t, (i.e., considering only the fact that $B_0 = 0$), what is the mean and variance of the Brownian motion at time T?

   b) Given that we now observe that $B_1 = k$, for a constant k, then what is the mean and variance of B_3?

2. Consider a process $\{S_t\}$ that follows the SDE:

$$dS_t = \mu dt + \sigma dB_t$$

   i.e. follows generalised Brownian motion. For the first three years, we have that $\mu = 2$ and $\sigma = 3$; for the next three years, $\mu = 3$ and $\sigma = 4$. If the initial value is $S_0 = 5$, what is the distribution of S_6?

3. A company's cash position, measured in millions of pounds, follows a generalised Brownian motion with a drift rate of 0.5 per quarter and a variance rate of 4 per quarter. How high does the company's initial cash position have to be for the company to have less than a 5% chance of a negative cash position by the end of one year?

4. Determine whether $Z_t := -W_t$ is a Brownian motion, if W_t is itself a Brownian motion.
   Hint: Are the increments independent? How are they distributed? Are the paths continuous? Is it true that $Z_0 = 0$?

5. The partial differential equation (PDE)

$$\frac{\partial f}{\partial t} = \tfrac{1}{2}\frac{\partial^2 f}{\partial x^2}$$

   is called the (forward) diffusion equation – or sometimes the heat equation – and is a model of the flow of heat (or diffusion of gas) in a continuous medium, in one dimension. Here t is time, x is distance and $f(x,t)$ is the temperature at time t and at a position given by x.

You are reminded that the density of a normal distribution with mean μ and variance σ^2 is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\},$$

for $-\infty < x < \infty$, $-\infty < \mu < \infty$, $\sigma > 0$. Using the appropriate mean and variance for a Brownian motion at time t, i.e. B_t, show that the pdf of B_t satisfies the diffusion equation.

6. Show that for a Brownian motion $\{W_t\}$ we have that

$$E\left[W_s W_t\right] = \min(s, t)$$