Rejection Sampling

STAT221

When it is difficult to sample directly from a known density function $f(x)$, rejection sampling allows us to generate a sample from $f(x)$ by sampling from a different distribution $g(x)$.

Rejection Sampling

- Find a random variable Y with density g satisfying

$$\frac{f(t)}{g(t)} \leq c,$$

  for all t such that $f(t) > 0$.
- Generate a random y from the distribution with density g.
- Generate a random u from the $Uniform(0, 1)$ distribution.
- If

$$u < \frac{f(y)}{c\,g(y)},$$

  accept y and set $x = y$; otherwise reject y and repeat the sampling procedure.

Example: Beta distribution

The Beta distribution has the density

$$f(x; \alpha, \beta) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} \qquad \text{with } 0 < x < 1,\ \alpha > 0, \beta > 0$$

where $\Gamma(\cdot)$ is the Gamma function

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx,$$

with $\Gamma(1) = 1$ and, for all positive integers n, with the recursive property $\Gamma(n+1) = n\Gamma(n)$, which leads to

$$\Gamma(n) = 1 \times 2 \times 3 \times \cdots \times (n-1) = (n-1)!$$

In this exampe, we assume that $\alpha = 3$ and $\beta = 2$.

Instead of sampling directly from the beta distribution, we can generate a random variable Y from a $Uniform(0, 1)$ distribution with density $g(y)$.

In order to ensure that

$$\frac{f(t)}{g(t)} \leq c,$$

we have to find constant c based on the maximum value of $f(t)$. With $\alpha = 3$ and $\beta = 2$ we get

$$f(x) = \frac{4!}{2!1!} x^2(1-x) = 12(x^2 - x^3)$$

$$\frac{df}{dx} = 24x - 36x^2 = 0$$

$$x = \frac{2}{3} \qquad f\left(\frac{2}{3}\right) = \frac{16}{9} = c$$

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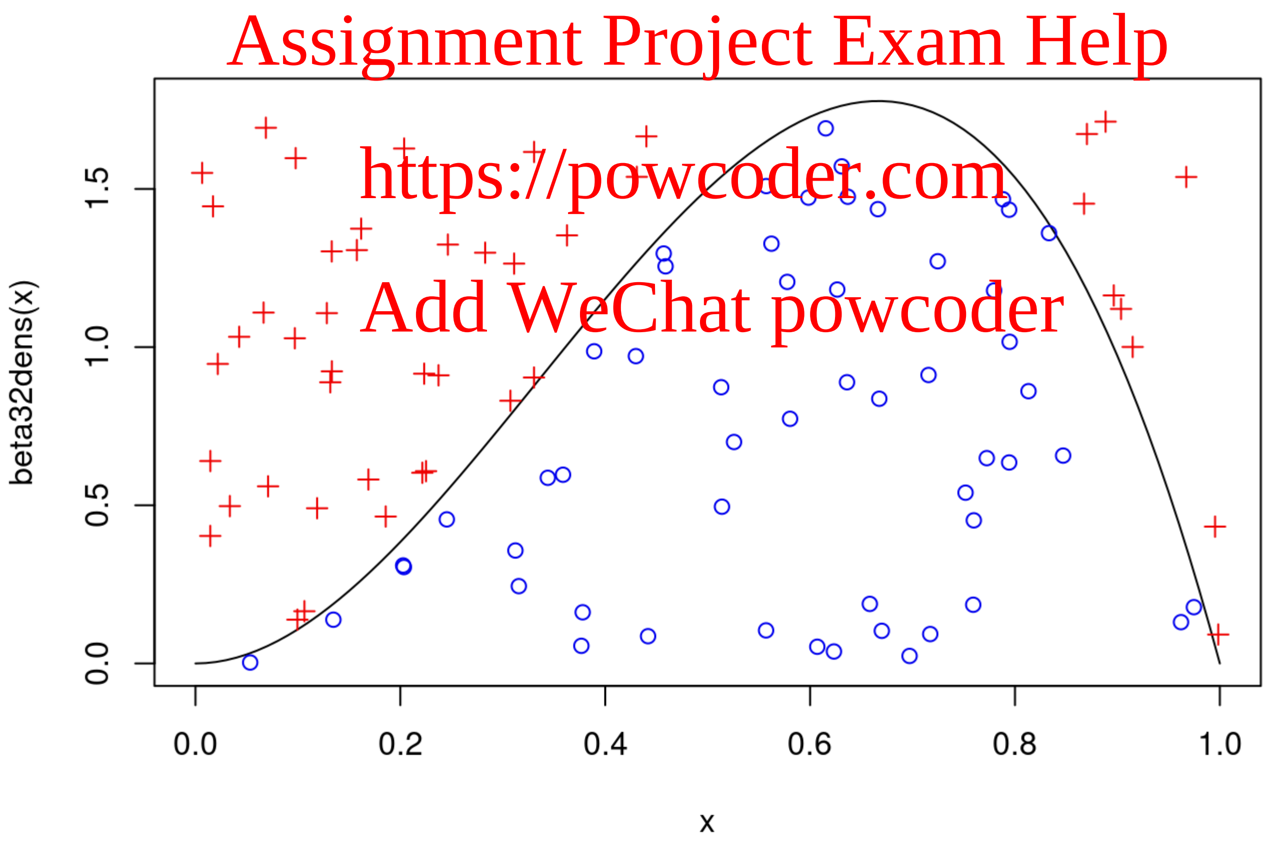
beta32dens <- function(x) 12 * x^2 * (1-x)
y <- runif(100)
u <- runif(100)
accept <- u < beta32dens(y)/(16/9)
x <- y[accept]

curve(beta32dens, from=0, to=1)
points(y[accept], u[accept]*16/9, col="blue2")
points(y[!accept], u[!accept]*16/9, col="red2", pch=3)
```



Acceptance Probability

The probability to accept y is

$$P(\text{accept} \mid Y) = P\left(U < \frac{f(Y)}{c\,g(Y)} \middle| \, Y\right) = \frac{f(Y)}{c\,g(Y)}$$

Then, the total probability of acceptance is

$$P(\text{accept}) = \sum_y P(\text{accept} \mid Y) P(Y = y) = \sum_y \frac{f(Y)}{c\, g(Y)}\, g(y) = \frac{1}{c}$$

The number of iterations until acceptance follows a geometric distribution with mean c; that is, on average it takes c iterations to accept a sample value of X. Hence, a smaller c will result in a more efficient sampler.

We can also show that the accepted sample has the same distribution as X using Bayes' theorem, i.e. for a discrete random variable with pmf $f(k)$:

$$P(k \mid \text{accepted}) = \frac{P(\text{accepted} \mid k) g(k)}{P(\text{accepted})} = \frac{\frac{f(k)}{c\, g(k)} g(k)}{\frac{1}{c}} = f(k)$$