

Basic Math for Computer Graphics

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Objectives

- Much of graphics is just translating math directly into code.
 - The cleaner the math, the cleaner the resulting code.
 - Also, clean codes result in better performance in many cases.

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- In this lecture, we review various tools from high school and college mathematics.

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- This chapter is not intended to be a rigorous treatment of the material; instead, intuition and geometric interpretation are emphasized.

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Sets and Mappings

- Sets

- a is a member of set S

$a \in S$
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- Cartesian product of two sets: given any two sets A and B ,

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 $A \times B = \{(a, b) | a \in A \text{ and } b \in B\}$

- * As a shorthand, we use the notation A^2 to denote $A \times A$.

Sets and Mappings

- Common sets of interest include:

- \mathbb{R} : the real numbers
- \mathbb{R}^+ : the non-negative real numbers (includes zero)
- \mathbb{R}^2 : the ordered pairs in the real 2D plane
- \mathbb{R}^n : the points in n -dimensional Cartesian space
- \mathbb{Z} : the integers

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Sets and Mappings

- Mappings (also called functions)

$$f : \mathbb{R} \mapsto \mathbb{Z}$$

- “There is a function called f that takes a real number as input and maps it to an integer.”

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- equivalent to the common programming notation :

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$$\text{integer } f(\text{real}) \quad \leftarrow \text{equivalent} \rightarrow \quad f : \mathbb{R} \mapsto \mathbb{Z}$$

“There is a function called f which has one real argument and returns an integer.”

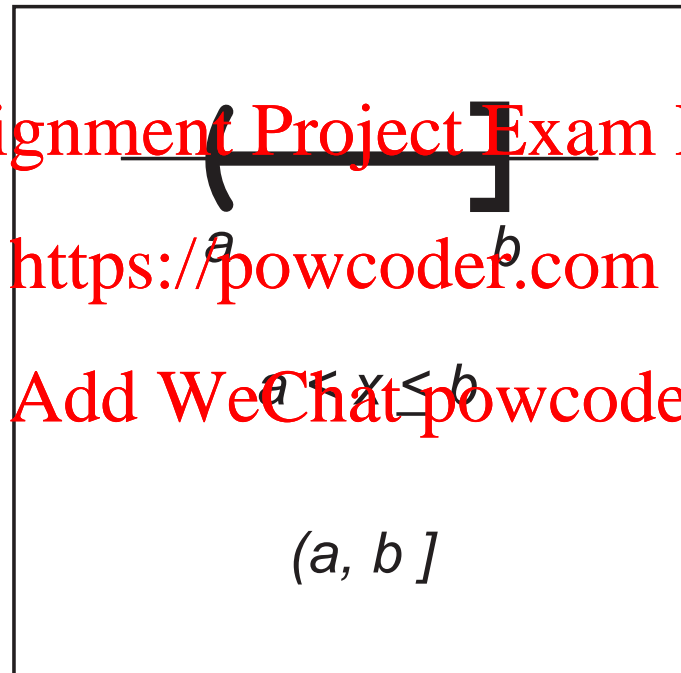
Intervals

- Three common ways to represent an interval

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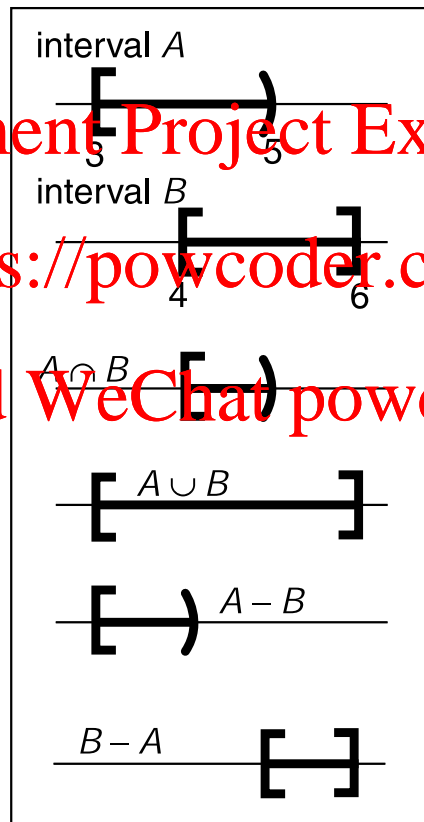
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a to b that includes b but not a

Intervals

- Interval operations on $[3, 5)$ and $[4, 6]$



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Logarithms

- Every logarithm has a base a .
“log base a ” of x is written

$$\log_a x$$

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- The exponent to which a must be raised to get x

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$$y = \log_a x \Leftrightarrow a^y = x$$

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Logarithms

- Several consequences:

- $a^{\log_a x} = x$

- $\log_a a^x = x \log_a a = x$

- $\log_a xy = \log_a x + \log_a y$

- $\log_a x/y = \log_a x - \log_a y$

- $\log_a x = \log_a b \log_b x$

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Logarithms

- Natural logarithm
 - The special number $e = 2.718\dots$
 - The logarithm with base e is called the natural logarithm

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$$\ln x \equiv \log_e x$$

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- * Note that the “ \equiv ” symbol can be read “is equivalent by definition.”

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Logarithms

- The derivatives of logarithms and exponents illuminate why the natural logarithm is “natural”:

$$\frac{d}{dx} \log_a x = \frac{1}{x \ln a}$$

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$$\frac{d}{dx} a^x = a^x \ln a$$

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The constant multipliers above are unity only for $a = e$.

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$$\frac{d}{dx} \ln x = \frac{1}{x \ln e} = \frac{1}{x}$$

$$\frac{d}{dx} e^x = e^x \ln e = e^x$$

Trigonometry

- The conversion between degrees and radians:

$$\text{degrees} = \frac{180}{\pi} \text{radians}$$

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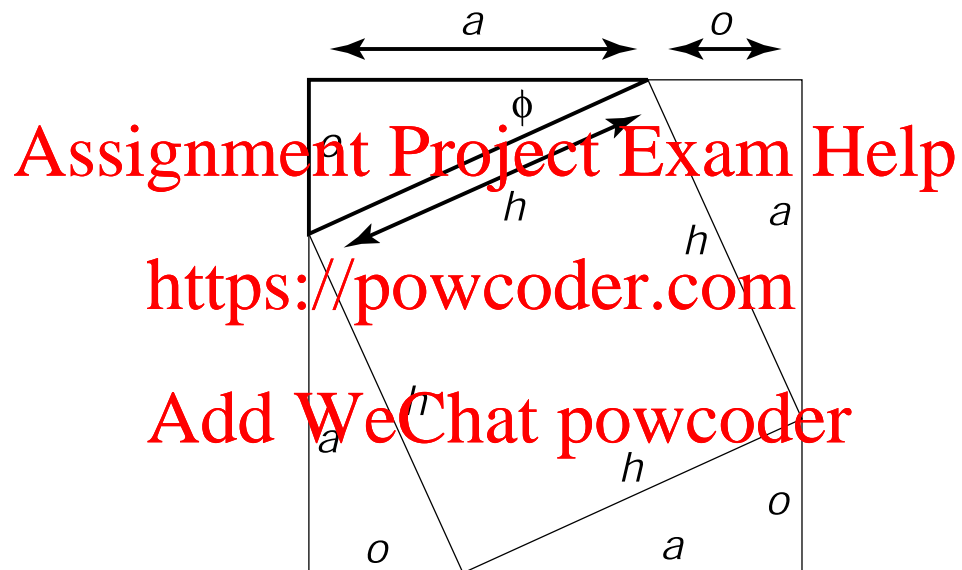
$$\text{radians} = \frac{\pi}{180} \text{degrees}$$

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Trigonometry

- Trigonometric functions
 - Pythagorean theorem: $a^2 + o^2 = h^2$



$$\sin \phi \equiv o/h$$

$$\cos \phi \equiv a/h$$

$$\tan \phi \equiv o/a$$

$$\csc \phi \equiv h/o$$

$$\sec \phi \equiv h/a$$

$$\cot \phi \equiv a/o$$

Trigonometry

- The functions are not invertible when considered with the domain \mathbb{R} . This problem can be avoided by restricting the range of standard inverse functions, and this is done in a standard way in almost all modern math libraries.

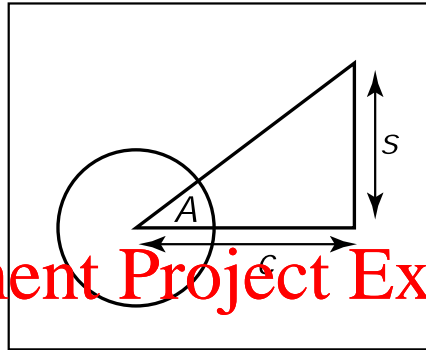
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- The domains and ranges are:

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$$\begin{aligned}\arcsin \text{ (asin)} &: [-1, 1] \mapsto [-\pi/2, \pi/2] \\ \arccos \text{ (acos)} &: [-1, 1] \mapsto [0, \pi] \\ \arctan \text{ (atan)} &: \mathbb{R} \mapsto [-\pi/2, \pi/2] \\ \arctan 2 \text{ (atan2)} &: \mathbb{R}^2 \mapsto [-\pi, \pi]\end{aligned}$$

Trigonometry



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- The $\text{atan2}(s, c)$ is often very useful in graphics: It takes an s value proportional to $\sin A$ and a c value that scales $\cos A$ by the same factor, and returns A .

Vector Spaces (in Algebraic Math)

- A *vector space* over a *field* F is a set V together with addition and multiplication that satisfy the eight axioms. Elements of V and F are called *vectors* and *scalars*.

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Axiom	Meaning
Associativity of addition	$u + (v + w) = (u + v) + w$
Commutativity of addition	$u + v = v + u$
Identity element of addition	There exists an element $0 \in V$ such that $v + 0 = v$ for all $v \in V$.
Inverse elements of addition	For every $v \in V$, there exists an element $v \in V$ such that $v + (-v) = 0$.
Distributivity of scalar multiplication with respect to vector addition	$a(u + v) = au + av$
Distributivity of scalar multiplication with respect to field addition	$(a + b)v = av + bv$
Compatibility of scalar multiplication with field multiplication	$a(bv) = (ab)v$
Identity element of scalar multiplication	$1v = v$, where 1 denotes the multiplicative identity in F .

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Vector Spaces (in Algebraic Math)

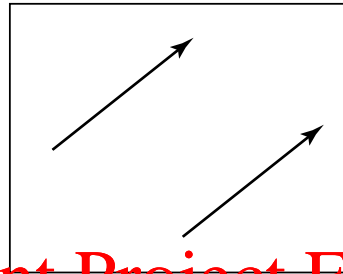
- A *vector space* over a *field* F is a set V together with addition and multiplication that satisfy the eight axioms. Elements of V and F are called *vectors* and *scalars*.
- Mathematical structures related to the concept of a field can be tracked as follows:
 - A *field* is a *ring* whose nonzero elements form a *abelian group* under multiplication.
 - A *ring* is an *abelian group* under addition and a *semigroup* under multiplication; addition is commutative, addition and multiplication are associative, multiplication distributes over addition, each element in the set has an additive inverse, and there exists an additive identity.
 - An *abelian group* (*commutative group*) is a group in which commutativity ($a \cdot b = b \cdot a$) is satisfied.
 - A *semigroup* is a set A in which $a \cdot b$ satisfies associativity for any two elements a and b and operator \cdot .
 - A *group* is a set A in which $a \cdot b$ satisfies closure, associativity, identity element, and inverse element for any two elements a and b and operator \cdot .

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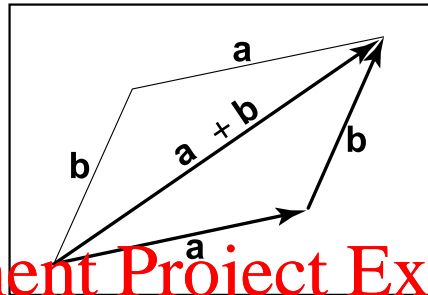
Vectors (Simply)



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- A quantity that encompasses a length and a direction.
- Represented by an arrow and not as coordinates or numbers
- Length: $\|\mathbf{a}\|$
- A unit vector: Any vector whose length is one.
- The zero vector: the vector of zero length. The direction of the zero vector is undefined.

Vectors (Simply)



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- Two vectors are added by arranging them head to tail. This can be done in either order.

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$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$$

- Vectors can be used to store an offset, also called a displacement, and a location (or position) that is a displacement from the origin.

Cartesian Coordinates of a Vector

- Vectors in a n -D vector space are said to be *linearly independent*, iff

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n = \mathbf{0}$$

has only the trivial solution ($a_1 = a_2 = \cdots = a_n = 0$).

- The vectors are thus referred to as basis vectors.

- For example, a 2D vector \mathbf{c} may be expressed as a combination of two basis vectors \mathbf{a} and \mathbf{b} :

$$\mathbf{c} = a_c\mathbf{a} + b_c\mathbf{b},$$

where a_c and b_c are the Cartesian coordinates of the vector \mathbf{c} with respect to the basis vectors $\{\mathbf{a}, \mathbf{b}\}$.

Cartesian Coordinates of a Vector

- Assuming vectors, \mathbf{x} and \mathbf{y} , are orthonormal,

$$\mathbf{a} = x_a \mathbf{x} + y_a \mathbf{y}$$

the length of \mathbf{a} is

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$$\|\mathbf{a}\| = \sqrt{x_a^2 + y_a^2}$$

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By convention we write the coordinates of \mathbf{a} either as an ordered pair (x_a, y_a) or a column matrix

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$$\mathbf{a} = \begin{bmatrix} x_a \\ y_a \end{bmatrix}$$

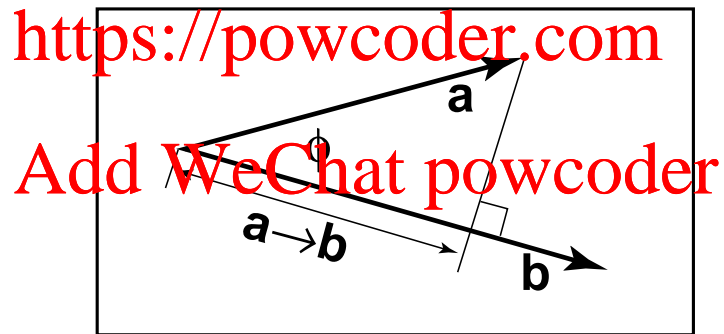
$$\mathbf{a}^\top = [x_a \quad y_a]$$

Dot Product

- The simplest way to multiply two vectors (also called the scalar product).

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos \phi$$

- The projection of one vector onto another: the length $\mathbf{a} \rightarrow \mathbf{b}$ of the projection of \mathbf{a} that is projected onto \mathbf{b}



$$\mathbf{a} \rightarrow \mathbf{b} = \|\mathbf{a}\| \cos \phi = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|}$$

Dot Product

- If 2D vectors **a** and **b** are expressed in Cartesian coordinates,

$$\begin{aligned}\mathbf{a} \cdot \mathbf{b} &= (x_a \mathbf{x} + y_a \mathbf{y}) \cdot (x_b \mathbf{x} + y_b \mathbf{y}) \\ &= x_a x_b (\mathbf{x} \cdot \mathbf{x}) + x_a y_b (\mathbf{x} \cdot \mathbf{y}) + x_b y_a (\mathbf{y} \cdot \mathbf{x}) + y_a y_b (\mathbf{y} \cdot \mathbf{y}) \\ &= x_a x_b + y_a y_b\end{aligned}$$

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- Similarly in 3D, <https://powcoder.com>

$$\mathbf{a} \cdot \mathbf{b} = x_a x_b + y_a y_b + z_a z_b$$

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Cross Product

$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\| \|\mathbf{b}\| \sin \phi$$

- By definition the unit vectors in the positive direction of the x -, y - and z -axes are given by

$$\mathbf{x} = (1, 0, 0),$$

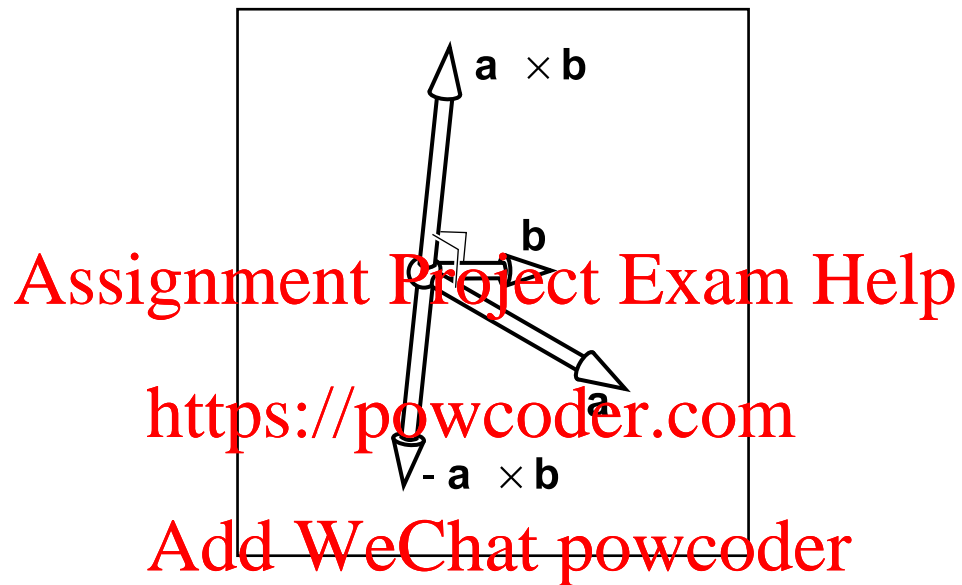
$$\mathbf{y} = (0, 1, 0),$$

$$\mathbf{z} = (0, 0, 1)$$

and we set as a convention that $\mathbf{x} \times \mathbf{y}$ must be in the plus or minus \mathbf{z} direction.

$$\mathbf{z} = \mathbf{x} \times \mathbf{y}$$

Cross Product



- The "right-hand rule":
Imagine placing the base of your right palm where \mathbf{a} and \mathbf{b} join at their tails, and pushing the arrow of \mathbf{a} toward \mathbf{b} . Your extended right thumb should point toward $\mathbf{a} \times \mathbf{b}$.

2D Implicit Curves

- Curve: A set of points that can be drawn on a piece of paper without lifting the pen.
- A common way to describe a curve is using an implicit equation.

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$$f(x, y) = 0$$

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$$f(x, y) = (x - x_c)^2 + (y - y_c)^2 - r^2$$

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- If $f(x, y) = 0$, the points where this equality hold are on the circle with center (x_c, y_c) and radius r .

2D Implicit Curves

- “implicit” equation: The points (x, y) on the curve cannot be immediately calculated from the equation, and instead must be determined by plugging (x, y) into f and finding out whether it is zero or by solving the equation.

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- The curve partitions space into regions where $f > 0$, $f < 0$ and $f = 0$.

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The 2D Gradient

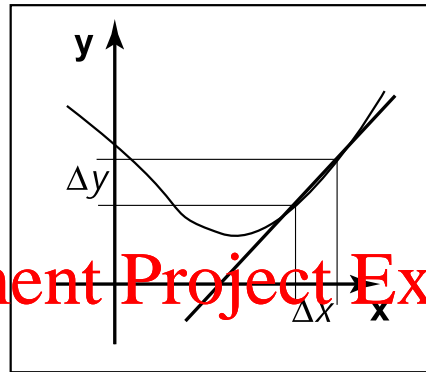
- If the function $f(x, y)$ is a height field with height $= f(x, y)$, the gradient vector points in the direction of maximum upslope, i.e., straight uphill. The gradient vector $\nabla f(x, y)$ is given by

$$\nabla f(x, y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)$$

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- The gradient vector evaluated at a point on the implicit curve $f(x, y) = 0$ is perpendicular to the tangent vector of the curve at that point. This perpendicular vector is usually called the *normal* vector to the curve.
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The 2D Gradient



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- The derivative of a 1D function measures the slope of the line tangent to the curve. Add WeChat powcoder
- If we hold y constant, we can define an analog of the derivative, called the partial derivative :

$$\frac{\partial f}{\partial x} \equiv \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

Implicit 2D Lines

- The familiar “slope-intercept” form of the line is

$$y = mx + b$$

This can be converted easily to implicit form

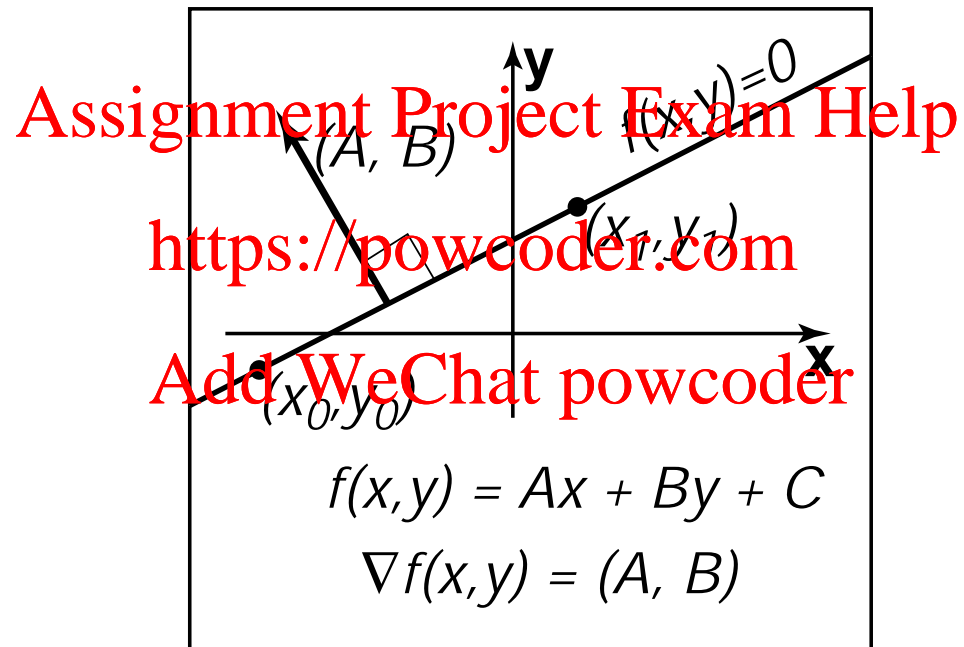
$$y - mx - b = 0$$

$$Ax + By + C = 0$$

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Implicit 2D Lines

- The gradient vector (A, B) is perpendicular to the implicit line $Ax + By + C = 0$



2D Parametric Curves

- A parametric curve: controlled by a *single* parameter, t ,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} g(t) \\ h(t) \end{bmatrix}$$

Vector form : **Assignment Project Exam Help**

$$\mathbf{p} = f(t)$$

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where f is a vector valued function $f : \mathbb{R} \mapsto \mathbb{R}^2$
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2D Parametric Lines

- A parametric line in 2D that passes through points $\mathbf{p}_0 = (x_0, y_0)$ and $\mathbf{p}_1 = (x_1, y_1)$ can be written

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_0 + t(x_1 - x_0) \\ y_0 + t(y_1 - y_0) \end{bmatrix}$$

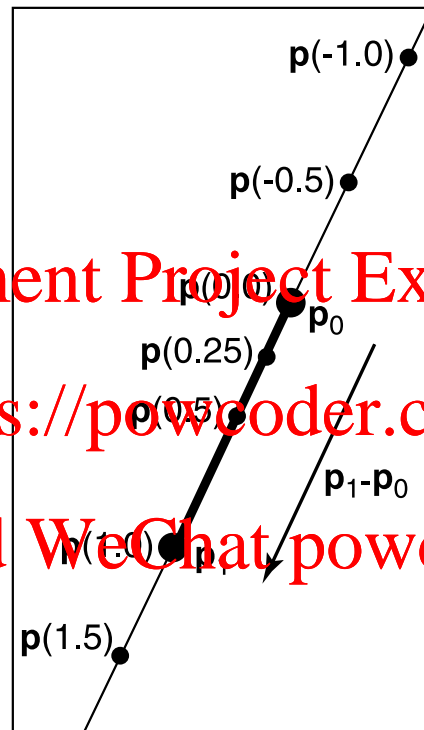
Because the formulas for x and y have such similar structure, we can use the vector form for $\mathbf{p} = (x, y)$.

$$\mathbf{p}(t) = \mathbf{p}_0 + t(\mathbf{p}_1 - \mathbf{p}_0)$$

Parametric lines can also be described as just a point \mathbf{o} and a vector \mathbf{d} :

$$\mathbf{p}(t) = \mathbf{o} + t(\mathbf{d})$$

2D Parametric Lines



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- A 2D parametric line through P_0 and P_1 . The line segment defined by $t \in [0, 1]$ is shown in bold.

Linear Interpolation

- Most common mathematical operation in graphics.
- Example of linear interpolation: Position to form line segments in 2D and 3D

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$$\mathbf{p} = (1 - t)\mathbf{a} + t\mathbf{b}$$

- interpolation: \mathbf{p} goes through \mathbf{a} and \mathbf{b} exactly at $t = 0$ and $t = 1$
- linear interpolation: the weighting terms t and $1 - t$ are linear polynomials of t

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Linear Interpolation

- Example of linear interpolation: A set of positions on the x -axis: x_0, x_1, \dots, x_n and for each x_i we have an associated height, y_i .
 - A continuous function $y = f(x)$ that interpolates these positions, so that f goes through every data point, i.e., $f(x_i) = y_i$.
 - For linear interpolation, the points (x_i, y_i) are connected by straight line segments.
 - It is natural to use parametric line equations for these segments. The parameter t is just the fractional distance between x_i and x_{i+1} :

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$$f(x) = y_i + \frac{x - x_i}{x_{i+1} - x_i}(y_{i+1} - y_i)$$

Linear Interpolation

- In the common form of linear interpolation, create a variable t that varies from 0 to 1 as we move from data A to data B . Intermediate values are just the function $(1 - t)A + tB$.

Assignment $t = \frac{x - x_i}{x_{i+1} - x_i}$ Project Exam Help

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2D Triangles

- With origin and basis vectors, any point \mathbf{p} can be written:

$$\mathbf{p} = (1 - \beta - \gamma)\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}$$

$$\alpha \equiv 1 - \beta - \gamma$$

$$\mathbf{p}(\alpha, \beta, \gamma) = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}$$

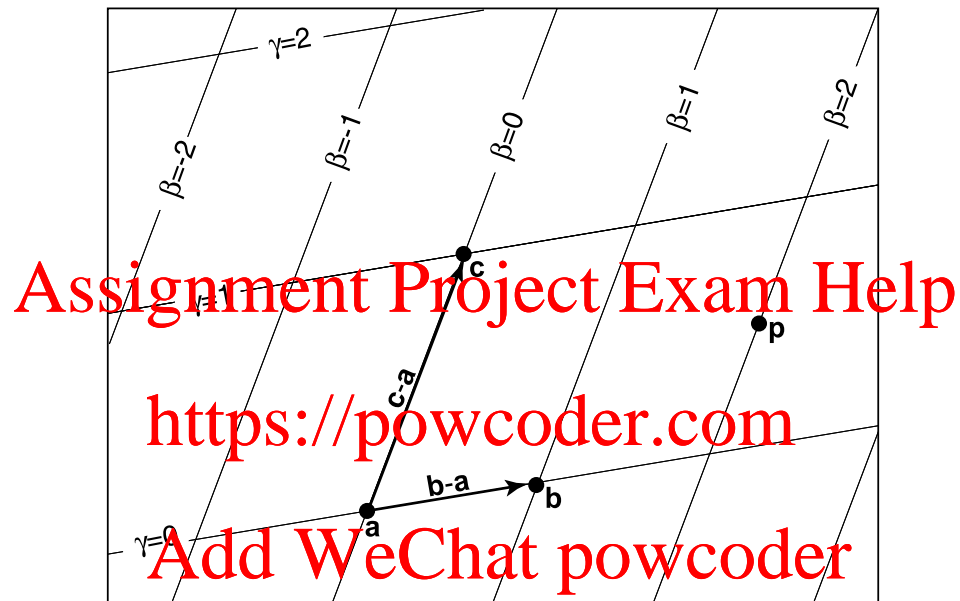
with the constraint that

$$\alpha + \beta + \gamma = 1$$

- A particularly nice feature of barycentric coordinates is that a point \mathbf{p} is inside the triangle formed by \mathbf{a} , \mathbf{b} , and \mathbf{c} if and only if

$$\begin{aligned} 0 &< \alpha < 1, \\ 0 &< \beta < 1, \\ 0 &< \gamma < 1. \end{aligned}$$

2D Triangles



- A 2D triangle (vertices a , b , c) can be used to set up a non-orthogonal coordinate system with origin a and basis vectors $(b-a)$ and $(c-a)$. A point is then represented by an ordered pair (β, γ) . For example, the point $p = (2.0, 0.5)$, i.e., $p = a + 2.0(b-a) + 0.5(c-a)$.

2D Triangles

- Defined by 2D points **a**, **b**, and **c**. Its area:

$$\begin{aligned} area &= \frac{1}{2} \|(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})\| \\ &= \frac{1}{2} \left\| \begin{pmatrix} x_b - x_a & y_b - y_a \\ x_c - x_a & y_c - y_a \end{pmatrix} \right\| \\ &= \frac{1}{2} ((x_b - x_a)(y_c - y_a) - (x_c - x_a)(y_b - y_a)) \\ &= \frac{1}{2} (x_a y_b + x_b y_c + x_c y_a - x_a y_c - x_b y_a - x_c y_b) \end{aligned}$$

3D Triangles

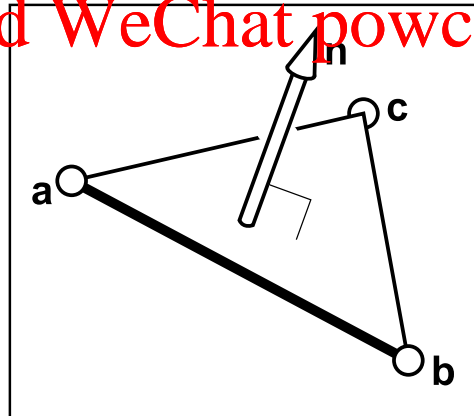
- Barycentric coordinates extend almost transparently to 3D. If we assume the points **a**, **b**, and **c** are 3D,

$$\mathbf{p} = (1 - \beta - \gamma)\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}$$

- The normal vector: taking the cross product of any two vectors.

$$\mathbf{n} = (\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})$$

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3D Triangles

This normal vector is not necessarily of unit length, and it obeys the right-hand rule of cross products.

- The area of the triangle. Taking the length of the cross product.

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$$area = \frac{1}{2} \|(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})\|$$
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Vector Operations in Matrix Form

- We can use matrix formalism to encode vector operations for vectors when using Cartesian coordinates; if we consider the result of the dot product a one by one matrix, it can be written:

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- If we take two 3D vectors we get:

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$$\begin{bmatrix} x_a & y_a & z_a \end{bmatrix} \begin{bmatrix} x_b \\ y_b \\ z_b \end{bmatrix} = [x_a x_b + y_a y_b + z_a z_b]$$

Matrices and Determinants

- The determinant in 2D is the area of the parallelogram formed by the vectors. We can use matrices to handle the mechanics of computing determinants.

$$\begin{bmatrix} a & A \\ b & B \end{bmatrix} = \begin{bmatrix} a & b \\ A & B \end{bmatrix} = aB - Ab$$

- For example, the determinant of a particular 3×3 matrix

$$\begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix} = 0 \begin{bmatrix} 4 & 5 \\ 7 & 8 \end{bmatrix} + 1 \begin{bmatrix} 3 & 5 \\ 6 & 8 \end{bmatrix} + 2 \begin{bmatrix} 3 & 4 \\ 6 & 7 \end{bmatrix}$$

$$= 0(32 - 35) + 1(30 - 24) + 2(21 - 24)$$

$$= 0$$