Homework 3

Exercise 1

THE JOINT PDF OF Y_1 AND Y_2 IS:

$$f(y_1, y_2) = f(y_1) \cdot f(y_2) = e^{-y_1} \cdot e^{-y_2} = e^{-(y_1+y_2)}$$

(INDEPENDENT)

LET $U = \frac{Y_1}{Y_1+Y_2}$ AND $V = Y_2$

IT FOLLOWS THAT $Y_2 = V$ AND $Y_1 = \frac{UV}{1-U}$

JACOBIAN:

$$J = \begin{vmatrix} \frac{\partial U}{\partial Y_1} & \frac{\partial U}{\partial Y_2} \\ \frac{\partial V}{\partial Y_1} & \frac{\partial V}{\partial Y_2} \end{vmatrix} = \begin{vmatrix} \frac{Y_2}{(Y_1+Y_2)^2} & \frac{-Y_1}{(Y_1+Y_2)^2} \\ 0 & 1 \end{vmatrix} = \frac{Y_2}{(Y_1+Y_2)^2} = \frac{(1-U)^2}{V}$$

THEREFORE THE JOINT PDF OF U AND V IS:

$$f(u,v) = e^{-\left[\frac{uv}{1-u} + v\right]} \frac{v}{(1-u)^2} = e^{-\frac{v}{1-u}} \frac{v}{(1-u)^2}$$

TO FIND PDF OF U INTEGRATE THE JOINT PDF W.R.T. V:

$$f(u) = \int_0^\infty e^{-\frac{v}{1-u}} \frac{v}{(1-u)^2} \, dv = 1.$$

EXERCISE 2 :

(a) $X \sim N(0,1)$, $Y \sim N(0,1)$ X, Y ARE INDEPENDENT

JOINT PDF OF X, Y:

$$f(x,y) = f(x) \cdot f(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}$$

$$= \frac{1}{2\pi} e^{-\frac{1}{2}x^2} \cdot e^{-\frac{1}{2}y^2}$$

$$\left.\begin{aligned} U &= X+Y \\ V &= X-Y \end{aligned}\right\} \rightarrow x = \frac{U+V}{2} \text{ AND } y = \frac{U-V}{2}$$

JACOBIAN :

$$J = \begin{vmatrix} \frac{\partial U}{\partial x} & \frac{\partial U}{\partial y} \\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2$$

JOINT PDF OF U, V.

$$f(u,v) = \frac{1}{2\pi} e^{-\frac{1}{2}\left(\frac{u+v}{2}\right)^2} \cdot e^{-\frac{1}{2}\left(\frac{u-v}{2}\right)^2} |-2|^{-1}$$

THIS IS SIMPLIFIED TO:

$$f(u,v) = \frac{1}{\sqrt{2}\sqrt{2\pi}} e^{-\frac{1}{2}\frac{u^2}{2}} \cdot \frac{1}{\sqrt{2}\sqrt{2\pi}} e^{-\frac{1}{2}\frac{v^2}{2}} \longrightarrow$$

EXERCISE 3:

$X \sim \text{beta}(\alpha, \beta)$, $Y \sim \text{beta}(\alpha+\beta, \gamma)$

$$f(x,y) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} \cdot \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha+\beta)\Gamma(\gamma)} x^{\alpha+\beta-1}(1-x)^{\gamma-1}$$

$$\left.\begin{array}{l} u = xy \\ v = x \end{array}\right\} \quad \begin{array}{l} y = \frac{u}{v} \\ x = v \end{array}$$

$$J = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} y & x \\ 1 & 0 \end{vmatrix} = -x$$

$$f(u,v) = \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} v^{\alpha-1}(1-v)^{\beta-1}\left(\frac{u}{v}\right)^{\alpha+\beta-1}\left(1-\frac{u}{v}\right)^{\gamma-1}\frac{1}{v}$$

Exercise 4

(a) CDF OF R:

$$P[R \le r] = 1 - P[R > r] = 1 - P\left(X = 0 \text{ PARTICLES IN SPHERE WITH VOLUME } \tfrac{4}{3}\pi r^3\right)$$

HERE $\lambda_{SPHERE} = \lambda \frac{4}{3}\pi r^3$

$$= 1 - \frac{\lambda_s^0 e^{-\lambda_s}}{0!} = 1 - e^{-\lambda \frac{4}{3}\pi r^3}$$

$$f(r) = 4\lambda\pi r^2 e^{-\lambda \frac{4}{3}\pi r^3}$$

(b) Let $U = R^3$

$$F_U(u) = P(U \le u) = P(R^3 \le u)$$

$$= P(R \le u^{1/3}) = F_R(u^{1/3})$$

$$f_U(u) = \frac{1}{3} u^{-\frac{2}{3}} 4\lambda\pi u^{2/3} e^{-\lambda \frac{4}{3}\pi u}$$

$$f_U(u) = \frac{4}{3}\lambda\pi e^{-\frac{4}{3}\lambda\pi u}$$

$$\therefore \quad U \sim \exp\left(\frac{4}{3}\lambda\pi\right)$$

And $$EU = \frac{3}{4\lambda\pi}$$

Exercise 5

$$X_1 \sim \chi^2_{r_1} \qquad X_2 \sim \chi^2_{r_2}$$

$$f(x_1,x_2) = f(x_1)f(x_2) = \frac{x_1^{\frac{r_1}{2}-1} e^{-x_1/2}}{\Gamma\left(\frac{r_1}{2}\right) 2^{r_1/2}} \cdot \frac{x_2^{\frac{r_2}{2}-1} e^{-x_2/2}}{\Gamma\left(\frac{r_2}{2}\right) 2^{r_2/2}}$$

$$J = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \end{vmatrix} = \begin{vmatrix} \frac{1}{x_2} & -\frac{x_1}{x_2^2} \\ 1 & 1 \end{vmatrix} = \frac{1}{x_2} + \frac{x_1}{x_2^2} = \frac{x_2+x_1}{x_2^2} = \frac{y_2}{x_2^2}$$

$$\left.\begin{aligned} y_1 &= \frac{x_1}{x_2} \\ y_2 &= x_1+x_2 \end{aligned}\right\} \qquad \begin{aligned} x_1 &= \frac{y_1 y_2}{1+y_1} \\ x_2 &= \frac{y_2}{1+y_1} \quad \frac{y_2}{2} \quad \frac{y_2}{2(1+y_1)} \end{aligned}$$

$$f(y_1,y_2) = \frac{\left(\frac{y_1 y_2}{1+y_1}\right)^{\frac{r_1}{2}-1} e^{-\frac{y_1 y_2}{2(1+y_1)}} \left(\frac{y_2}{\ldots}\right) e}{\Gamma\left(\frac{r_1}{2}\right) 2^{r_1/2}\, \Gamma\left(\frac{r_2}{2}\right) 2^{r_2/2}} \cdot \frac{y_2^2}{(1+y_1)^2 y_2} * \frac{\Gamma\left(\frac{r_1+r_2}{2}\right)}{\Gamma\left(\frac{r_1+r_2}{2}\right)}$$

$$= \frac{y_1^{\frac{r_1}{2}-1} (1+y_1)^{-\frac{r_1+r_2}{2}} \Gamma\left(\frac{r_1+r_2}{2}\right)}{\Gamma\left(\frac{r_1}{2}\right)\Gamma\left(\frac{r_2}{2}\right)} \cdot \frac{y_2^{\frac{r_1+r_2}{2}-1} e^{-y_2/2}}{\Gamma\left(\frac{r_1+r_2}{2}\right) 2^{r_1+r_2}}$$

$$= f(y_1) \cdot f(y_2)$$

$\therefore Y_1, Y_2$ ARE INDEPENDENT AND $Y_2 \sim \chi^2_{r_1+r_2}$

SINCE $\frac{X_1}{X_2}$ IND OF X_3 (GIVEN) AND $\frac{X_1}{X_2}$ IND OF X_1+X_2 $\Big\}$ $\frac{X_1/r_1}{X_2/r_2}$ IND OF $\frac{X_3/r_3}{X_1+X_2/(r_1+r_2)}$

Exercise 6

(a). $P(X_1, \ldots, X_{r-1} \mid X_r) = \dfrac{P(X_1, \ldots, X_r)}{P(X_r)}$

NOTE: $X_r \sim b(n, p_r)$

$$= \frac{\frac{n!}{x_1! x_2! \cdots x_r!} p_1^{x_1} p_2^{x_2} \cdots p_r^{x_r}}{\frac{n!}{(n-x_r)!\, x_r!} p_r^{x_r} (1-p_r)^{n-x_r}}$$

$$= \frac{(n-x_r)!}{x_1! x_2! \cdots x_{r-1}!} \left(\frac{p_1}{1-p_r}\right)^{x_1} \cdots \left(\frac{p_{r-1}}{1-p_r}\right)^{x_{r-1}}$$

$\therefore \sim$ MULTINOMIAL WITH $n - X_r$ # of TRIALS

AND PROBABILITIES $\dfrac{p_1}{1-p_r}, \dfrac{p_2}{1-p_r}, \ldots, \dfrac{p_{r-1}}{1-p_r}$

Exercise 7

$X_i \mid P_i \sim$ BERNOULLI (P_i), $P_i \sim \text{beta}(\alpha, \beta)$

(a) 1. $E X_i = E\left(E(X_i \mid P_i)\right) = E P_i = \dfrac{\alpha}{\alpha+\beta}$

$E \sum X_i = n \dfrac{\alpha}{\alpha+\beta}$

2. $\text{VAR}(X_i) = E\left(\text{VAR}(X_i \mid P_i)\right) + \text{VAR}\left(E(X_i \mid P_i)\right)$

$= E\left(P_i(1-P_i)\right) + \text{VAR}(P_i)$

$$= E p_i - E p_i^2 + \operatorname{VAR}(p_i)$$

$$= \frac{\alpha}{\alpha+\beta} - \left(\frac{\alpha^2}{(\alpha+\beta)^2} + \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} \right) + \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

$$= \frac{\alpha(\alpha+\beta)(\alpha+\beta+1) - \alpha^2(\alpha+\beta+1) - \alpha\beta + \alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} = \frac{\alpha\beta}{(\alpha+\beta)^2}$$

$$\therefore \operatorname{VAR}\left(\sum X_i\right) = n \frac{\alpha\beta}{(\alpha+\beta)^2}$$

(b) $X_i \mid p_i \sim b(n_i, p_i)$, $p_i \sim \text{beta}(\alpha, \beta)$

$$EY = \frac{\alpha}{\alpha+\beta} \sum n_i$$

$$\operatorname{VAR}(Y) = \sum \operatorname{VAR}(X_i)$$

WHERE $\operatorname{VAR}(X_i) = n_i \frac{\alpha\beta(\alpha+\beta+n_i)}{(\alpha+\beta)^2(\alpha+\beta+1)}$

Exercise 8

CDF of $Q = \prod X_i$

$$F_Q(q) = P(Q \le q)$$

$$= P\left(\prod X_i \le q\right)$$

$$= P\left(-\ln \prod X_i > -\ln q\right)$$

$$= P\left(-\ln X_1 - \dots - \ln X_n > -\ln q\right)$$

Now each $-\ln X_i \sim \exp(1)$ $\quad 1 - F_T(-\ln q)$

Then $T = -\sum \ln X_i \sim \Gamma(n, 1)$

Therefore, $F_Q(q) = 1 - F_T(-\ln q)$

$$f(q) = -f_T(-\ln q) \cdot \frac{-1}{q} = \frac{1}{q} \frac{(-\ln q)^{n-1} e^{-(-\ln q)}}{\Gamma(n)} = \frac{(-\ln q)^{n-1}}{\Gamma(n)}.$$