

HOMEWORK 4 SOLUTIONS

EXERCISE 1 :

$$\underline{w} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ y_1 \\ \vdots \\ y_n \end{pmatrix}$$

$$\underline{E} \underline{w} = \begin{pmatrix} \mu_1 \underline{1} \\ \mu_2 \underline{1} \end{pmatrix}$$

$$\text{var}(\underline{w}) = \begin{pmatrix} \sigma_1^2 \underline{I} & \sigma_{12} \underline{I} \\ \sigma_{12} \underline{I} & \sigma_2^2 \underline{I} \end{pmatrix}$$

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EXERCISE 2 :

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$$y_i = \varepsilon_i + c \varepsilon_{i-1}$$

$$y_1 = \varepsilon_1 + c \varepsilon_0$$

$$y_2 = \varepsilon_2 + c \varepsilon_1$$

$$y_3 = \varepsilon_3 + c \varepsilon_2$$

$$\vdots$$

$$y_n = \varepsilon_n + c \varepsilon_{n-1}$$

$$\left. \begin{matrix} y_1 = \varepsilon_1 + c \varepsilon_0 \\ y_2 = \varepsilon_2 + c \varepsilon_1 \\ y_3 = \varepsilon_3 + c \varepsilon_2 \\ \vdots \\ y_n = \varepsilon_n + c \varepsilon_{n-1} \end{matrix} \right\} \rightarrow \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} c & 1 & 0 & 0 & \dots & 0 \\ 0 & c & 1 & 0 & \dots & 0 \\ 0 & 0 & c & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & c & 1 \end{pmatrix} \begin{pmatrix} \varepsilon_0 \\ \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{pmatrix}$$

$$\underline{y} = \underline{A} \underline{\varepsilon} \quad \text{where} \quad \underline{\varepsilon} \sim N_{n+1} \left(\underline{0}, \sigma^2 \underline{I} \right)$$

$$\therefore \underline{y} \sim N_n \left(\underline{0}, \sigma^2 \underline{A} \underline{A}' \right)$$

Exercise 3

$$X_1 = Y_1 Y_3$$

$$X_2 = Y_2 Y_3$$

$$X_3 = Y_3 - Y_1 Y_3 - Y_2 Y_3 = Y_3 (1 - Y_1 - Y_2)$$

JOINT PDF OF X_1, X_2, X_3 is:

$$f(x_1, x_2, x_3) = f(x_1) \cdot f(x_2) \cdot f(x_3)$$

$$= \frac{x_1^{\alpha_1-1} e^{-x_1}}{\Gamma(\alpha_1)} \cdot \frac{x_2^{\alpha_2-1} e^{-x_2}}{\Gamma(\alpha_2)} \cdot \frac{x_3^{\alpha_3-1} e^{-x_3}}{\Gamma(\alpha_3)}$$

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JACOBIAN:

$$J = \begin{vmatrix} \frac{\partial Y_1}{\partial X_1} & \frac{\partial Y_1}{\partial X_2} & \frac{\partial Y_1}{\partial X_3} \\ \frac{\partial Y_2}{\partial X_1} & \frac{\partial Y_2}{\partial X_2} & \frac{\partial Y_2}{\partial X_3} \\ \frac{\partial Y_3}{\partial X_1} & \frac{\partial Y_3}{\partial X_2} & \frac{\partial Y_3}{\partial X_3} \end{vmatrix} = \begin{vmatrix} \frac{Y_2+X_2}{(X_1+X_2+X_3)^2} & \frac{X_1}{(X_1+X_2+X_3)^2} & -\frac{X_1}{(X_1+X_2+X_3)^2} \\ \frac{X_1+X_2}{(X_1+X_2+X_3)^2} & \frac{X_1+X_3}{(X_1+X_2+X_3)^2} & -\frac{X_2}{(X_1+X_2+X_3)^2} \\ 1 & 1 & 1 \end{vmatrix}$$

$$= \frac{1}{Y_3^2}$$

WE CAN ALSO COMPUTE THE JACOBIAN USING

$$J = \begin{vmatrix} \frac{\partial X_1}{\partial Y_1} & \frac{\partial X_1}{\partial Y_2} & \frac{\partial X_1}{\partial Y_3} \\ \frac{\partial X_2}{\partial Y_1} & \frac{\partial X_2}{\partial Y_2} & \frac{\partial X_2}{\partial Y_3} \\ \frac{\partial X_3}{\partial Y_1} & \frac{\partial X_3}{\partial Y_2} & \frac{\partial X_3}{\partial Y_3} \end{vmatrix} = \begin{vmatrix} Y_3 & 0 & Y_1 \\ 0 & Y_3 & Y_2 \\ -Y_3 & -Y_3 & 1-Y_1-Y_2 \end{vmatrix} = Y_3^2 \rightarrow$$

$$f(y_1, y_2, y_3) = \frac{(y_1 y_3)^{\alpha_1 - 1} e^{-y_1 y_3} (y_2 y_3)^{\alpha_2 - 1} e^{-y_2 y_3} (y_3 (1 - y_1 - y_2))^{\alpha_3 - 1} e^{-y_3 (1 - y_1 - y_2)} y_3^2}{\Gamma(\alpha_1) \Gamma(\alpha_2) \Gamma(\alpha_3)}$$

$$= \frac{y_1^{\alpha_1 - 1} y_2^{\alpha_2 - 1} (1 - y_1 - y_2)^{\alpha_3 - 1} y_3^{\alpha_1 + \alpha_2 + \alpha_3 - 1} e^{-y_3}}{\Gamma(\alpha_1) \Gamma(\alpha_2) \Gamma(\alpha_3)} = \star$$

NOTE: $\int_0^{\infty} \frac{y_3^{\alpha_1 + \alpha_2 + \alpha_3 - 1} e^{-y_3}}{\Gamma(\alpha_1 + \alpha_2 + \alpha_3)} dy_3 = 1$

$\hookrightarrow \frac{y_3^{\alpha_1 + \alpha_2 + \alpha_3 - 1} e^{-y_3}}{\Gamma(\alpha_1 + \alpha_2 + \alpha_3)}$

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$$\star = \frac{\Gamma(\alpha_1 + \alpha_2 + \alpha_3)}{\Gamma(\alpha_1) \Gamma(\alpha_2) \Gamma(\alpha_3)} y_1^{\alpha_1 - 1} y_2^{\alpha_2 - 1} (1 - y_1 - y_2)^{\alpha_3 - 1}$$

Exercise 4

$$M_{Q,Q}(t,s) = E e^{tQ_1 + sQ_2}$$

$$= E e^{t\bar{X} + s(X - \bar{X})} = E e^{t \frac{x_1 + \dots + x_n}{n} + s \left(x_1 - \frac{x_1 + \dots + x_n}{n} \right)}$$

$$= E e^{x_1 \left(\frac{t}{n} + s - \frac{s}{n} \right) + x_2 \left(\frac{t}{n} - \frac{s}{n} \right) + \dots + x_n \left(\frac{t}{n} - \frac{s}{n} \right)}$$

$$= E e^{x_1 \left(\frac{t}{n} + s - \frac{s}{n} \right)} \cdot \left\{ M_{X_i} \left(\frac{t}{n} - \frac{s}{n} \right) \right\}^{n-1}$$

where $M_X(t) = E e^{t\mu + \frac{1}{2}t^2\sigma^2}$

FOR X_1 REPLACE t WITH $\frac{t}{n} + s - \frac{s}{n}$

FOR THE OTHER $n-1$ X_i 's REPLACE t WITH $\frac{t}{n} - \frac{s}{n}$

SIMILARLY, $t\bar{X}$

$$M_{\bar{X}}(t) = E e^{t\bar{X}} = \left(M_{X_i} \left(\frac{t}{n} \right) \right)^n$$

$$M_{X_1 - \bar{X}}(s) = E e^{s(X_1 - \bar{X})} = E e^{x_1 \left(s - \frac{s}{n} \right) + x_2 \left(-\frac{s}{n} \right) + \dots + x_n \left(-\frac{s}{n} \right)}$$

$$= M_{X_1} \left(s - \frac{s}{n} \right) \cdot \left(M_{X_i} \left(-\frac{s}{n} \right) \right)^{n-1}$$

Exercise 5

$$\begin{aligned} \bar{Y} &= \frac{1}{n} \mathbf{1}' \mathbf{Y} \\ \begin{pmatrix} Y_1 - \bar{Y} \\ Y_2 - \bar{Y} \\ \vdots \\ Y_n - \bar{Y} \end{pmatrix} &= \left(\mathbf{I} - \frac{1}{n} \mathbf{1} \mathbf{1}' \right) \mathbf{Y} \end{aligned} \quad \left\{ \begin{pmatrix} \frac{1}{n} \mathbf{1}' \\ \mathbf{I} - \frac{1}{n} \mathbf{1} \mathbf{1}' \end{pmatrix} \mathbf{Y} = \underline{\mathbf{A}} \mathbf{Y} \right.$$

$$\mathbf{Y} \sim N_n \left(\mu \mathbf{1}, \sigma^2 \left((1-\rho) \mathbf{I} + \rho \mathbf{1} \mathbf{1}' \right) \right)$$

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$$\text{we } (\underline{\mathbf{A}} \mathbf{Y}) = \underline{\mathbf{A}} \text{ we } (\mathbf{Y}) \mathbf{A}'$$

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$$= \begin{pmatrix} \frac{1}{n} \mathbf{1}' \\ \mathbf{I} - \frac{1}{n} \mathbf{1} \mathbf{1}' \end{pmatrix} \sigma^2 \left((1-\rho) \mathbf{I} + \rho \mathbf{1} \mathbf{1}' \right) \begin{pmatrix} \frac{1}{n} \mathbf{1}, & \mathbf{I} - \frac{1}{n} \mathbf{1} \mathbf{1}' \end{pmatrix}$$

CHECK COVARIANCE:

$$\begin{aligned} \frac{1-\rho}{n} \mathbf{1}' + \frac{\rho}{n} \mathbf{1} \mathbf{1}' \mathbf{1}' - (1-\rho) \frac{1}{n} \mathbf{1}' \mathbf{1} \mathbf{1}' - \rho \frac{1}{n} \mathbf{1}' \mathbf{1} \mathbf{1}' \mathbf{1}' \\ = 0 \end{aligned}$$

Yes, \bar{Y} and $\sum (Y_i - \bar{Y})^2$ are independent

because \bar{Y} is independent of the vector of the deviations.