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Statistics 100B **Instructor: Nicolas Christou**

The F distribution

Definition:
Let $U \sim \chi^2_{n_1}$ and $V \sim \chi^2_{n_2}$. If U and V are independent the ratio

$\frac{\frac{U}{n_1}}{\frac{V}{n_2}}$ follows the F distribution with numerator d.f. n_1 and denominator d.f. n_2.

We write $X \sim F_{n_1,n_2}$.
The probability density function of $X \sim F_{n_1,n_2}$ is:

$$f(x) = \frac{\Gamma(\frac{n_1+n_2}{2})}{\Gamma(\frac{n_1}{2})\Gamma(\frac{n_2}{2})} \left(\frac{n_1}{n_2}\right)^{\frac{n_1}{2}} x^{\frac{n_1}{2}-1} \left(1 + \frac{n_1}{n_2}x\right)^{-\frac{1}{2}(n_1+n_2)}, \quad 0 < x < \infty.$$

Proof:
Let $X \sim \frac{\frac{U}{n_1}}{\frac{V}{n_2}}$ and $W = V$. Solving for U we get $U = \frac{n_1}{n_2}XW$. Since U, V are independent the joint pdf of U and V is

$$f(u,v) = \frac{u^{\frac{n}{2}-1}e^{-\frac{u}{2}}}{\Gamma(\frac{n}{2})2^{\frac{n}{2}}} \times \frac{v^{\frac{n}{2}-1}e^{-\frac{v}{2}}}{\Gamma(\frac{n}{2})2^{\frac{n}{2}}}$$

The inverse of the Jacobian of the transformation is $\frac{n_1}{n_2}W$ (why?). Therefore the joint pdf of X and W is

$$f(x,w) = \frac{1}{\Gamma(\frac{n_1}{2})\Gamma(\frac{n_2}{2})2^{\frac{(n_1+n_2)}{2}}} \left(\frac{n_1 x w}{n_2}\right)^{\frac{n_1}{2}-1} w^{\frac{n_2}{2}-1} e^{-\frac{w}{2}\left(\frac{n_1 x}{n_2}+1\right)} \frac{n_1 w}{n_2},$$

or

$$f(x,w) = \frac{(\frac{n_1}{n_2})^{\frac{n_1}{2}} x^{\frac{n_1}{2}-1}}{\Gamma(\frac{n_1}{2})\Gamma(\frac{n_2}{2})2^{\frac{(n_1+n_2)}{2}}} w^{\frac{n_1+n_2}{2}-1} e^{-\frac{w}{2}\left(\frac{n_1 x}{n_2}+1\right)}$$

Finally to find the marginal pdf of X we integrate the joint $f(x,w)$ w.r.t. w:

$$f(x) = \int_0^\infty \frac{(\frac{n_1}{n_2})^{\frac{n_1}{2}} x^{\frac{n_1}{2}-1}}{\Gamma(\frac{n_1}{2})\Gamma(\frac{n_2}{2})2^{\frac{(n_1+n_2)}{2}}} w^{\frac{n_1+n_2}{2}-1} e^{-\frac{w}{2}\left(\frac{n_1 x}{n_2}+1\right)} dw$$

Change the variable of integration by using the transformation, $y = \frac{w}{2}\left(\frac{n_1 x}{n_2}+1\right)$. It follows that $dw = \frac{2}{\frac{n_1}{n_2}x+1}dy$ and $w = \frac{2y}{\frac{n_1}{n_2}x+1}$. Therefore,

$$f(x) = \int_0^\infty \frac{(\frac{n_1}{n_2})^{\frac{n_1}{2}} x^{\frac{n_1}{2}-1}}{\Gamma(\frac{n_1}{2})\Gamma(\frac{n_2}{2})2^{\frac{(n_1+n_2)}{2}}} \left(\frac{2y}{\frac{n_1}{n_2}x+1}\right)^{\frac{n_1+n_2}{2}-1} e^{-y} \left(\frac{2}{\frac{n_1}{n_2}x+1}\right) dy$$

This is simplified to

$$f(x) = \frac{\Gamma(\frac{n_1+n_2}{2})}{\Gamma(\frac{n_1}{2})\Gamma(\frac{n_2}{2})} \left(\frac{n_1}{n_2}\right)^{\frac{n_1}{2}} x^{\frac{n_1}{2}-1} \left(1 + \frac{n_1}{n_2}x\right)^{-\frac{1}{2}(n_1+n_2)}, \text{ why?}$$