QUIZ 6 SOLUTIONS

EXERCISE 1:

$$P(D_2 \mid H_3) = \frac{P(D_2 \cap H_3)}{P(H_3)} = \frac{P(D_2 \cap H_3)}{P(H_3 \cap D_1) + P(H_3 \cap D_2)}$$

$$= \frac{P(H_3 \mid D_2)\,P(D_2)}{P(H_3 \mid D_1)\,P(D_1) + P(H_3 \mid D_2)\,P(D_2)}$$

$$= \frac{1 \cdot \frac{1}{3}}{\frac{1}{2} \cdot \frac{1}{3} + 1 \cdot \frac{1}{3}} = \frac{2}{3}$$

AND $$P(D_1 \mid H_3) = \frac{\frac{1}{2} \cdot \frac{1}{3}}{\frac{1}{2} \cdot \frac{1}{3} + 1 \cdot \frac{1}{3}} = \frac{1}{3}.$$

EXERCISE 2:

$$X_i \sim \exp\left(\frac{1}{i\theta}\right) \rightarrow M_{X_i}(t) = (1 - i\theta t)^{-1}$$

LET $$\hat{\theta} = \sum_{i=1}^{n} \frac{X_i}{n i}$$

$$M_{\hat{\theta}}(t) = M_{\frac{X_1}{n}}(t) \cdot M_{\frac{X_2}{2n}}(t) \cdots M_{\frac{X_n}{n^2}}(t)$$

$$= M_{X_1}\left(\frac{t}{n}\right) \cdot M_{X_2}\left(\frac{t}{2n}\right) \cdots M_{X_n}\left(\frac{t}{n^2}\right)$$

$$= \left(1-\theta\frac{t}{n}\right)^{-1}\left(1-\frac{2\theta t}{2n}\right)^{-1}\cdots\left(1-\frac{n\theta t}{n^2}\right)^{-1}$$

$$= \left(1-\frac{\theta}{n}t\right)^{-n}$$

$$\therefore \quad \hat{\theta} \sim \Gamma\left(n, \frac{\theta}{n}\right)$$

(b). $$E\hat{\theta}^{-1} = \frac{\Gamma(n-1)\left(\frac{\theta}{n}\right)^{-1}}{\Gamma(n)} = \frac{n}{n-1}\cdot\frac{1}{\theta}.$$

(c). $$MSE\left(c\hat{\theta}^{-1}\right) = Var\left(c\hat{\theta}^{-1}\right) + B^2 = *$$

$$B = E\hat{\theta}^{-1} - \frac{1}{\theta} = \frac{1}{\theta(n-1)}.$$

$$* = c^2\left\{E\left(\hat{\theta}^{-2}\right) - \left(E\hat{\theta}^{-1}\right)^2\right\} + \frac{1}{\theta^2(n-1)^2}$$

$$= c^2\left[\frac{\Gamma(n-2)\left(\frac{\theta}{n}\right)^{-2}}{\Gamma(n)} - \left(\frac{n}{n-1}\frac{1}{\theta}\right)^2\right] + \frac{1}{\theta^2(n-1)^2}$$

THEN $\frac{d\,MSE}{dc} = 0$ AND SOLVE FOR C.

EXERCISE 3 :

(a). $E\hat{\theta} = \frac{1}{n} E \sum (X_i^2 - X_i)$ $\qquad EX_i = \lambda$

$= \frac{1}{n} \left(\sum E X_i^2 - \sum E X_i \right)$ $\qquad var(X_i) = \lambda$

$= \frac{1}{n} \left(n(\sigma^2 + \mu^2) - n\mu \right)$

$= \frac{1}{n} \left(n\lambda + n\lambda^2 - n\lambda \right) = \lambda^2.$

(b) $f(X_i) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2} X_i^2} = (2\pi\sigma^2)^{-\frac{1}{2}} e^{-\frac{1}{2\sigma^2} X_i^2}$

$\ln f(X_i) = -\frac{1}{2} \ln 2\pi\sigma^2 - \frac{1}{2\sigma^2} X_i^2$

$S = \frac{\partial \ln f(X_i)}{\partial \sigma^2} = -\frac{1}{2\sigma^2} + \frac{1}{2\sigma^4} X_i^2$

$var(S) = var\left(-\frac{1}{2\sigma^2} + \frac{1}{2\sigma^4} X_i^2 \right)$

$= var\left[\frac{1}{2\sigma^2} \frac{X_i^2}{\sigma^2} \right] = \frac{1}{4\sigma^4} 2 = \frac{1}{2\sigma^4}.$

USING SECOND DERIVATIVE:

$\frac{\partial^2 \ln f(x)}{\partial \sigma^2 (2)} = +\frac{1}{2\sigma^4} - \frac{1}{\sigma^6} X_i^2$

$$I(\theta) = -E\left[\frac{1}{2\sigma^4} - \frac{1}{\sigma^6} X_i^2 \right]$$

$$= -\left(\frac{1}{2\sigma^4} - \frac{E X_i^2}{\sigma^6} \right) = -\left(\frac{1}{2\sigma^4} - \frac{\sigma^2}{\sigma^6} \right)$$

$$= \frac{1}{2\sigma^4}$$

THEREFORE THE CRAMER-RAO LOWER BOUND IS $\frac{1}{nI(\theta)} = \frac{2\sigma^4}{n}$.

OUR ESTIMATOR IS $\hat{\sigma}^2 = \frac{\sum X_i^2}{n}$

$$E\hat{\sigma}^2 = E\left(\frac{\sum X_i^2}{n}\right) = \frac{n\sigma^2}{n} = \sigma^2.$$

AND

$$VAR(\hat{\sigma}^2) = VAR\left(\frac{\sum X_i^2}{n} \frac{\sigma^2}{\sigma^2}\right) = \frac{2\sigma^4}{n}.$$

$\therefore$ YES, $\hat{\sigma}^2$ IS EFFICIENT ESTIMATOR OF σ^2.

<u>NOTE</u>: HERE WE USE

$$\frac{\sum X_i^2}{\sigma^2} \sim \chi^2_n$$

AND THEREFORE $VAR\left(\frac{\sum X_i^2}{\sigma^2}\right) = 2n.$

(c). $X_1, \ldots, X_n$ iid $N(\mu, \sigma^2)$

$E\bar{X} = \mu$, $VAR(\bar{X}) = \frac{\sigma^2}{n}$

AS $n \to \infty$ $VAR(\bar{X}) \to 0$

$\therefore$ $\bar{X}$ IS CONSISTENT ESTIMATOR OF σ^2.

$X_1, \ldots, X_n$ ARE NOT INDEPENDENT:

$EX_i = \mu$ AND $E\bar{X} = \mu$.

BUT $VAR(\bar{X}) = VAR(\bar{X}) = VAR\left(\frac{\Sigma X_i}{n}\right)$

$= \frac{1}{n^2} VAR(\Sigma X_i) = \frac{1}{n^2}\left(n\sigma^2 + \binom{n}{2} 2\rho\sigma^2\right)$

$= \frac{1}{n^2}\left(n\sigma^2 + n(n-1)\rho\sigma^2\right)$

$= \frac{\sigma^2}{n}\left[1 + (n-1)\rho\right]$

AS $n \to \infty$ $VAR(\bar{X}) \to \rho\sigma^2$

$\therefore$ $\bar{X}$ IS NOT CONSISTENT ESTIMATOR OF μ.