Quiz 8 solutions

(a). JOINT PMF OF X_i's SINCE THEY ARE INDEPENDENT IS:

$$P(x_1,\ldots,x_n) = \prod_{i=1}^{n} \frac{\lambda^{x_i} e^{-\lambda}}{x_i!} = \frac{\lambda^{\sum x_i} e^{-n\lambda}}{x_1!\,x_2!\cdots x_n!}$$

$$P(S) = \frac{(n\lambda)^N e^{-n\lambda}}{N!}, \text{ BECAUSE } \sum X_i \sim \text{Poisson}(n\lambda)$$

THEREFORE,

$$P[X_1=x_1,\ldots,X_n=x_n \mid S=N] = \frac{P[(X_1=x_1,\ldots,X_n=x_n)\cap(S=N)]}{P(S=N)} =$$

NOTE: $P[(X_1=x_1,\ldots,X_n=x_n)\cap(S=N)] = \frac{\lambda^N e^{-n\lambda}}{x_1!\cdots x_n!}$

$$= \frac{\dfrac{\lambda^N e^{-n\lambda}}{x_1!\cdots x_n!}}{\dfrac{(n\lambda)^N e^{-n\lambda}}{N!}} = \frac{N!}{n^N x_1!\cdots x_n!}$$

$$= \frac{N!}{x_1!\cdots x_n!}\left(\frac{1}{n}\right)^{x_1}\cdots\left(\frac{1}{n}\right)^{x_n} \quad \text{WHERE } \sum x_i = N.$$

(5). $E[X(X-1)] = \lambda^2$, $\quad VAR[X(X-1)] = 2\lambda^2 + 4\lambda^3$

LET $T_1 = \frac{1}{n} \sum X_i(X_i - 1)$

$$ET_1 = \frac{1}{n} \sum E[X_i(X_i-1)] = \frac{1}{n} n\lambda^2 = \lambda^2.$$

$$VAR(T_1) = \frac{1}{n^2} \sum VAR[X_i(X_i-1)]$$

$$= \frac{1}{n^2} n (2\lambda^2 + 4\lambda^3) = \frac{2\lambda^2 + 4\lambda^3}{n}.$$

TO VERIFY MEAN AND VARIANCE OF $X(X-1)$:

$$E[X(X-1)] = E[X^2 - X] = EX^2 - EX$$
$$= (\sigma^2 + \mu^2) - \mu = \lambda + \lambda^2 - \lambda = \lambda^2$$

AND

$$VAR[X(X-1)] = VAR(X^2) + VAR(X) - 2\,Cov(X^2, X)$$
$$= EX^4 - (EX^2)^2 + VAR(X) - 2[EX^3 - (EX^2)(EX)]$$

ETC.

(c). $T_2 = E[T_1 | S]$, WHERE $S = \sum X_i$

IT IS GIVEN THAT $E[X_i | S] = \frac{S}{n}$

$$\mathrm{VAR}[X_i | S] = S \frac{1}{n}\left(1 - \frac{1}{n}\right) \rightarrow E X_i^2 | S = \frac{S^2 + (n-1)S}{n^2}$$

THEREFORE,

$$T_2 = E[T_1 | S] = E\left[\frac{1}{n}\sum X_i (X_i - 1) \,\Big|\, S\right]$$

$$= E\left[\left(\frac{1}{n}\sum X_i^2 - \frac{1}{n}\sum X_i\right) \Big|\, S\right]$$

$$= E\left[\frac{1}{n}\sum X_i^2 \,\Big|\, S\right] - E\left[\frac{1}{n}\sum X_i \,\Big|\, S\right]$$

$$= \frac{1}{n} \frac{n[S^2 + (n-1)S]}{n^2} - \frac{1}{n}\frac{nS}{n} = \frac{S^2 + (n-1)S}{n^2} - \frac{S}{n}$$

$$= \frac{S(S-1)}{n^2}. \quad \text{BUT } S \sim \exp(n\lambda)$$

THEREFORE $ET_2 = E\frac{S(S-1)}{n^2} = \frac{ES^2 - ES}{n^2} = \frac{n\lambda + n^2\lambda^2 - n\lambda}{n^2} = \lambda^2$

YES.

<u>VARIANCE OF T_2</u>: FROM (b) $\mathrm{VAR}[X(X-1)] = 2\lambda^2 + 4\lambda^3$

HERE $\mathrm{VAR}(S(S-1)) = 2(n\lambda)^2 + 4(n\lambda)^3$, TO GET

$$\mathrm{VAR}(T_2) = \frac{2\lambda^2}{n^2} + \frac{4\lambda^3}{n}.$$

T_1 AND T_2 ARE UNBIASED ESTIMATORS OF λ^2.

WE FOUND IN (b) THAT

$$\text{var}(T_1) = \frac{2\lambda^2 + 4\lambda^3}{n}$$

AND IN (c) $\text{var}(T_2) = \frac{2\lambda^2}{n^2} + \frac{4\lambda^3}{n}$

THEREFORE THEY DO NOT ATTAIN THE CRAMÉR-RAO LOWER BOUND

T_2 HAS VARIANCE CLOSER TO THE CRAMÉR-RAO LOWER BOUND.

(d). $f(x) = \frac{\alpha x^{\alpha-1}}{\theta^{\alpha}}$, $\alpha > 0$, $\theta > 0$

$0 < x < \theta$

θ is known.

$$L = \frac{\alpha^n (x_1 x_2 \cdots x_n)^{\alpha-1}}{\theta^{n\alpha}} = \frac{\alpha^n}{\theta^{n\alpha}} \left(\prod x_i\right)^{\alpha-1}$$

LET $g(u, \theta) = \frac{\alpha^n}{\theta^{n\alpha}} \left(\prod x_i\right)^{\alpha-1}$

AND $h(x) = 1$

THEREFORE USING THE FACTORIZATION THEOREM WE CONCLUDE THAT

$U = \prod x_i$ IS A SUFFICIENT STATISTIC FOR α.