Multivariate normal - practice problems solutions

EXERCISE 1:

$$M_{\bar{X},\bar{Y}}(t_1,t_2) = E\, e^{t_1\bar{X} + t_2\bar{Y}}$$

$$= E\, e^{t_1 \frac{X_1+\cdots+X_n}{n} + t_2 \frac{Y_1+\cdots+Y_n}{n}}$$

$$= \left[E\, e^{\frac{t_1}{n}X_1 + \frac{t_2}{n}Y_1}\right] \cdots\cdots \left[E\, e^{\frac{t_1}{n}X_n + \frac{t_2}{n}Y_n}\right]$$

(BECAUSE (X_i, Y_i) $i=1,2,\dots,n$ ARE INDEPENDENT)

$$= \left\{\underbrace{M_{X_i,Y_i}\left(\frac{t_1}{n}, \frac{t_2}{n}\right)}\right\}^n$$

THIS IS THE JOINT MGF OF BIVARIATE NORMAL

$$= \left(e^{\frac{\underset{\sim}{t}'}{n}\underset{\sim}{\mu} + \frac{1}{2}\frac{\underset{\sim}{t}'}{n}\underset{\sim}{\Sigma}\frac{\underset{\sim}{t}}{n}}\right)^n \qquad \text{WHERE } \underset{\sim}{t} = \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}$$

$$= e^{\underset{\sim}{t}'\underset{\sim}{\mu} + \frac{1}{2}\underset{\sim}{t}'\frac{\underset{\sim}{\Sigma}}{n}\underset{\sim}{t}} \qquad \therefore \begin{pmatrix} \bar{X} \\ \bar{Y} \end{pmatrix} \sim N_2\left(\underset{\sim}{\mu}, \frac{\underset{\sim}{\Sigma}}{n}\right)$$

EXERCISE 2:

(a). $\underset{\sim}{X} \sim N(\underset{\sim}{0}, I)$

$$\left.\begin{aligned} Y_1 &= X_1 + \delta X_3 \\ Y_2 &= X_2 + \delta X_3 \end{aligned}\right\} \quad \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \delta \\ 0 & 1 & \delta \end{pmatrix} X = \underset{\sim}{A}\underset{\sim}{X}$$

$$\mathrm{VAR}(AX) = AA' = \begin{pmatrix} 1 & 0 & \delta \\ 0 & 1 & \delta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \delta & \delta \end{pmatrix} = \begin{pmatrix} 1+\delta^2 & \delta^2 \\ \delta^2 & \delta^2+1 \end{pmatrix}$$

$$\mathrm{COR}(Y_1, Y_2) = \frac{\delta^2}{\sqrt{\delta^2+1}\sqrt{1+\delta^2}} = \frac{1}{2} \qquad \frac{\delta^2}{1+\delta^2} = \frac{1}{2} \qquad 2\delta^2 = 1+\delta^2, \quad \delta^2 = 1, \quad \underline{\delta = 1}$$

(b).

$$\left.\begin{aligned} W &= X - \mu_1 \\ Q &= (Y - \mu_2) - \rho\frac{\sigma_2}{\sigma_1}(X - \mu_1) \end{aligned}\right\}$$

$$\mathrm{Var}(Q) = \sigma_2^2 + \rho^2\frac{\sigma_2^2}{\sigma_1^2}\sigma_1^2 - 2\rho\frac{\sigma_2}{\sigma_1}\rho\sigma_1\sigma_2 = \sigma_2^2(1-\rho^2).$$

$$\begin{pmatrix} W \\ Q \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ -\rho\frac{\sigma_2}{\sigma_1} & 1 \end{bmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} + \begin{bmatrix} -\mu_1 \\ +\rho\frac{\sigma_2}{\sigma_1}\mu_1 - \mu_2 \end{bmatrix} \quad 2$$

$$\sigma_2^2 + \rho^2\frac{\sigma_2^2}{\sigma_1^2}.$$

$$\begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ 0 & \sigma_2^2 - \rho^2\sigma_2^2 \end{bmatrix} \begin{bmatrix} 1 & -\rho\frac{\sigma_2}{\sigma_1} \\ & 1 \end{bmatrix}$$

CHECK VAR-COVAR MATRIX

$$\mathrm{VAR}\begin{pmatrix} W \\ Q \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ -\rho\frac{\sigma_2}{\sigma_1} & 1 \end{bmatrix} \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix} \begin{bmatrix} 1 & -\rho\frac{\sigma_2}{\sigma_1} \\ 0 & 1 \end{bmatrix}$$

$$= \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2(1-\rho^2) \end{pmatrix} \quad 3 \quad \therefore \text{INDEPENDENT}$$

$$\sigma_2^2\left(1 - \frac{\sigma_{12}^2}{\sigma_1^2\sigma_2^2}\right) = \sigma_2^2 - \frac{\sigma_{12}^2}{\sigma_1^2}.$$

EXERCISE 3:

(a) $$\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = \begin{pmatrix} \sigma_1 & 0 \\ \rho\sigma_2 & \sigma_2\sqrt{1-\rho^2} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} + \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$$

$$E\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$$

$$VAR\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = \begin{pmatrix} \sigma_1 & 0 \\ \rho\sigma_2 & \sigma_2\sqrt{1-\rho^2} \end{pmatrix} \begin{pmatrix} \sigma_1 & \rho\sigma_2 \\ 0 & \sigma_2\sqrt{1-\rho^2} \end{pmatrix}$$

$$= \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$$

$$\therefore \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \sim N_2\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}\right)$$

(b). $M_{x_1,x_2}(t_1,t_2) = E e^{t_1x_1+t_2x_2}$

$= e^{t_1\mu_1+t_2\mu_2+\frac{1}{2}t_1^2\sigma_1^2+\frac{1}{2}t_2^2\sigma_2^2+t_1t_2\sigma_1\sigma_2}$

$E y_1^3 = E e^{3x_1}$, so set $t_1=3$, $t_2=0$

$E y_1^3 = e^{3\mu_1+\frac{9\sigma_1^2}{2}}$

$\text{Cov}[y_1^3, y_2^3] = E y_1^3 y_2^3 - (E y_1^3)(E y_2^3)$

$= E e^{3x_1} e^{3x_2} - (E e^{3x_1})(E e^{3x_2})$

$= E e^{3x_1+3x_2} - (E e^{3x_1})(E e^{3x_2})$

set $t_1=3$, $t_2=3$ | set $t_1=3$, $t_2=0$ | set $t_1=0$, $t_2=3$

$= \cdots$

$= \cdots$