University of California, Los Angeles
Department of Statistics

Statistics 100B | **Instructor: Nicolas Christou**

Practice 1 - solutions

EXERCISE 1
Find the distribution of the random variable X for each of the following moment-generating functions:

a. $M_X(t) = \left[\frac{1}{3}e^t + \frac{2}{3}\right]^5$. X follows binomial with $n = 5$, $p = \frac{1}{3}$.

b. $M_X(t) = \frac{e^t}{2-e^t} = \frac{\frac{1}{2}e^t}{1-\frac{1}{2}e^t}$. X follows geometric with $p = \frac{1}{2}$.

c. $M_X(t) = e^{2(e^t-1)}$. X follows Poisson with $\lambda = 2$.

EXERCISE 2
Let $M_X(t) = \frac{1}{6}e^t + \frac{2}{6}e^{2t} + \frac{3}{6}e^{3t}$ be the moment-generating function of a random variable X.

a. To find $E(X)$ we need the first derivative of $M_X(t)$ w.r.t. t:
$M'_X(t) = \frac{1}{6}e^t + \frac{2}{6}2e^{2t} + \frac{3}{6}3e^{3t}$. The first moment is the previous derivastive evaluated at $t = 0$. The result is $E(X) = \frac{1}{6} + \frac{4}{6} + \frac{9}{6} = \frac{7}{3}$.

b. To find $var(X)$ we need the second moment. We need the second derivative of $M_X(t)$ w.r.t. t.
$M''_X(t) = \frac{1}{6}e^t + \frac{8}{6}2e^{2t} + \frac{27}{6}3e^{3t}$. And at $t = 0$ we obtain the second moment. It is equal to $EX^2 = \frac{1}{6} + \frac{8}{6} + \frac{27}{6} = 6$. The variance is equal to: $var(X) = EX^2 - (E(X))^2 = 6 - (\frac{7}{3})^2 = \frac{5}{9}$.

c. From the definition of moment-generating functions $M_X(t) = Ee^{tX}$ we see that X is discrete with possible values 1, 2, and 3, and corresponding probabilities $\frac{1}{6}$, $\frac{2}{6}$, and $\frac{3}{6}$.

EXERCISE 3
Let X follow the Poisson probability distribution with parameter λ. Its moment-generating function is $M_X(t) = e^{\lambda(e^t-1)}$.

a. The moment-generating function of $Z = \frac{X-\lambda}{\sqrt{\lambda}}$ is:

$$M_Z(t) = M_{\frac{X-\lambda}{\sqrt{\lambda}}}(t) = e^{-\frac{\lambda}{\sqrt{\lambda}}t} M_X\left(\frac{t}{\sqrt{\lambda}}\right) \Rightarrow M_Z(t) = e^{-\sqrt{\lambda}t} e^{\lambda(e^{\frac{t}{\sqrt{\lambda}}}-1)}.$$

b. Using the series expansion of

$$e^{\frac{t}{\sqrt{\lambda}}} = 1 + \frac{\frac{t}{\sqrt{\lambda}}}{1!} + \frac{\left(\frac{t}{\sqrt{\lambda}}\right)^2}{2!} + \frac{\left(\frac{t}{\sqrt{\lambda}}\right)^3}{3!} + \cdots$$

we get:

$$M_Z(t) = e^{-\sqrt{\lambda}t-\lambda+\lambda[1+\frac{t}{\sqrt{\lambda}}+\frac{t^2}{2\lambda}+\frac{t^3}{3!(\sqrt{\lambda})^3}+\cdots]} = e^{-\sqrt{\lambda}t-\lambda+\lambda+\sqrt{\lambda}t+\frac{t^2}{2}+\frac{\lambda t^3}{3!(\sqrt{\lambda})^3}+\cdots}$$

Therefore,

$$\lim_{\lambda\to\infty} M_Z(t) = e^{\frac{1}{2}t^2}.$$

In other words, as $\lambda \to \infty$, the ratio $Z = \frac{X-\lambda}{\sqrt{\lambda}}$ converges to the standard normal distribution.

EXERCISE 4
Here we use the normal approximation to Poisson. It is given that X follows the Poisson distribution with $\lambda = 100$. We know that for large λ the ratio $Z = \frac{X-\lambda}{\sqrt{\lambda}}$ follows the standard normal distribution. Therefore:

$$P(X \leq 110) = P\left(Z < \frac{110.5 - 100}{\sqrt{100}}\right) = P(Z < 1.05) = 0.8531.$$

The exact probability is

$$P(X \leq 110) = \sum_{x=0}^{110} \frac{e^{-100}100^x}{x!} = \cdots = 0.8529.$$

The approximation is not bad!

EXERCISE 5
We know that the moment generating function of a normal random variable is $e^{\mu t+\frac{1}{2}\sigma^2t^2}$. Because the sample is i.i.d the moment generating function of $T = \sum_{i=1}^n X_i$ is:
$M_T(t) = Ee^{Tt} = Ee^{(X_1+\cdots+X_n)t} = Ee^{X_1t}\cdots Ee^{X_nt} \Rightarrow M_T(t) = e^{n\mu t+\frac{1}{2}n\sigma^2t^2}$. This is the moment generating function of a normal random variable with mean $n\mu$ and variance $n\sigma^2$. Therefore $T \sim N(n\mu, \sigma\sqrt{n})$.

EXERCISE 6

The two sample are independent with $X \sim N(\mu_1, \sigma_1)$ and $Y \sim N(\mu_2, \sigma_2)$.

a. $E(\bar{X} - \bar{Y}) = E(\bar{X}) - E(\bar{Y}) = \mu_1 - \mu_2$.

b. $Var(\bar{X} - \bar{Y}) = Var(\bar{X}) + Var(\bar{Y}) - 2Cov(\bar{X}, \bar{Y}) = \frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{m}$. The covariance is zero because the two samples are independent.

c. $M_{\bar{X}-\bar{Y}} = Ee^{(\bar{X}-\bar{Y})t} = Ee^{\bar{X}t}Ee^{-\bar{Y}t} = M_{X_1}(\frac{t}{m}) \cdots M_{X_n}(\frac{t}{m})M_{Y_1}(-\frac{t}{n}) \cdots M_{Y_n}(-\frac{t}{n}) =$
$e^{m(\mu_1 \frac{t}{m} + \frac{1}{2}\sigma_1^2 \frac{t^2}{m^2})} e^{-n(\mu_2 \frac{t}{n} - \frac{1}{2}\sigma_2^2 \frac{t^2}{n^2})} = e^{t(\mu_1 - \mu_2) + \frac{1}{2}t^2(\frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n})}$. This is the moment generating function of a normal random variable with mean $\mu_1 - \mu_2$ and variance $\frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n}$.

d. It is given that $n = m$ and $\sigma_1^2 = 2, \sigma_2^2 = 2.5$. We want $P(-1 < \bar{X} - \bar{Y} - (\mu_1 - \mu_2) < 1) = 0.95$.
Since $\bar{X} - \bar{Y} \sim N(\mu_1 - \mu_2, \sqrt{\frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n}})$ we get:

$$P(\frac{-1}{\sqrt{\frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n}}} < Z < \frac{1}{\sqrt{\frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n}}}) = 0.95 \Rightarrow P(\frac{-1}{\sqrt{\frac{2}{n} + \frac{2.5}{n}}} < Z < \frac{1}{\sqrt{\frac{2}{n} + \frac{2.5}{n}}}) = 0.95$$

Therefore, $1.96 = \frac{1}{\sqrt{\frac{2}{n} + \frac{2.5}{n}}} \Rightarrow 1.96 = \frac{1}{\sqrt{\frac{4.5}{n}}} \Rightarrow n = 17.3 \approx 18$.

EXERCISE 7

We are given that X follows the normal distribution with $\mu = 0$ and $\sigma^2 = 1$. Therefore its probability density function (p.d.f.) is $f(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}, -\infty < x < \infty$. The random variable Y is said to follow the lognormal distribution if $Y = e^X$ because $log(y)$ follows the normal distribution. To find its p.d.f we start with its cumulative distribution function which by definition is:

$$F(y) = P(Y \leq y) = P(e^X \leq y) = P(X \leq log(y)) = F_x(log(y)) \Rightarrow f(y) = \frac{1}{y}f_x(log(y)) \Rightarrow \frac{1}{y}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(log(y))^2}.$$

EXERCISE 8

The radius of a circle, X, is an exponential random variable with parameter λ. Therefore its p.d.f. is $f(x) = \lambda e^{-\lambda x}$. Let Y be the area of the circle. Then $Y = \pi X^2$. We want to find the p.d.f. of Y. We start with the c.d.f. function: $F(y) = P(Y \leq y) = P(\pi X^2 \leq y) = P(X^2 \leq \frac{y}{\pi}) = P(-\sqrt{\frac{y}{\pi}} \leq X \leq +\sqrt{\frac{y}{\pi}}) = F_X(\sqrt{\frac{y}{\pi}}) - F_X(-\sqrt{\frac{y}{\pi}}) \Rightarrow f(y) = F(y)' = \frac{1}{2\sqrt{\pi y}}f_X(\sqrt{\frac{y}{\pi}}) - 0 \Rightarrow f(y) = \frac{1}{2\sqrt{\pi y}}\lambda e^{-\lambda\sqrt{\frac{y}{\pi}}}$.