QUIZ 7 SOLUTIONS

EXERCISE 1

$$E\,\hat{\mu} = E\,\frac{\underline{1}'\underline{\Sigma}^{-1}\underline{y}}{\underline{1}'\underline{\Sigma}^{-1}\underline{1}} = \frac{\underline{1}'\underline{\Sigma}^{-1}\underline{1}\,\mu}{\underline{1}'\underline{\Sigma}^{-1}\underline{1}} = \mu .$$

$$\operatorname{var}(\hat{\mu}) = \frac{1}{(\underline{1}'\underline{\Sigma}^{-1}\underline{1})^2}\operatorname{var}(\underline{1}'\underline{\Sigma}^{-1}\underline{y})$$

$$= \frac{\underline{1}'\underline{\Sigma}^{-1}\underline{\Sigma}\,\underline{\Sigma}^{-1}\underline{1}}{(\underline{1}'\underline{\Sigma}^{-1}\underline{1})^2} = \frac{1}{\underline{1}'\underline{\Sigma}^{-1}\underline{1}}$$

WHERE, $\underline{\Sigma}^{-1} = \frac{1}{\sigma^2(1-\rho)}\left[\underline{I} - \frac{\rho}{1+(n-1)\rho}\underline{J}\right]$

THEREFORE

$$\underline{1}'\,\frac{1}{\sigma^2(1-\rho)}\left(\underline{I} - \frac{\rho}{1+(n-1)\rho}\underline{J}\right)\underline{1}$$

$$= \frac{1}{\sigma^2(1-\rho)}\left(n - \frac{\rho\, n^2}{1+(n-1)\rho}\right) = \frac{1}{\sigma^2(1-\rho)}\left(\frac{n+n(n-1)\rho-n^2\rho}{1+(n-1)\rho}\right)$$

$$= \frac{1}{\sigma^2(1+(n-1)\rho)}.$$

THEREFORE $\operatorname{var}(\hat{\mu}) = \sigma^2\left[1+(n-1)\rho\right] > 0$

$$\therefore\ \rho > -\frac{1}{n-1}$$

EXERCISE 2 $\therefore\ Y_i \sim N(i\theta,\ i\sigma)$

$$f(y_i) = \frac{1}{i\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{y_i - i\theta}{i\sigma}\right)^2}$$

$$L = (2\pi\sigma^2)^{-n/2} \prod i^{-1}\, e^{-\frac{1}{2}\sum\left(\frac{y_i - i\theta}{i\sigma}\right)^2}$$

$$\ln L = -\frac{n}{2}\ln 2\pi\sigma^2 + \ln \prod i^{-1} - \frac{1}{2\sigma^2}\sum\left(\frac{y_i - i\theta}{i}\right)^2$$

$$\frac{d\ln L}{d\theta} = +\frac{2}{2\sigma^2}\sum\frac{(y_i - i\theta)\,i}{i^2} = 0$$

$$\hat\theta = \sum\frac{y_i}{ni}$$

$\therefore$

AND $$\operatorname{var}(\hat\theta) = \frac{1}{n^2}\sum\frac{\operatorname{var}(y_i)}{i^2} = \frac{\sigma^2}{n}.$$

CRAMER-RAO LOWER BOUND:

$$\frac{d^2\ln L}{d\theta^2} = -\frac{n}{\sigma^2}$$

$$\frac{1}{-E\left(-\frac{n}{\sigma^2}\right)} = \frac{\sigma^2}{n}$$

$\therefore\ \hat\theta$ IS EFFICIENT ESTIMATOR OF θ.

EXERCISE 3 :

$$L = (2\pi\sigma^2)^{-n/2} e^{-\frac{1}{2\sigma^2}\sum (x_i-\mu)^2} \cdot (2\pi\sigma^2)^{-m/2} e^{-\frac{1}{2\sigma^2}\sum (y_i-\mu)^2}$$

$$\ln L = -\frac{n}{2}\ln 2\pi\sigma^2 - \frac{1}{2\sigma^2}\sum (x_i-\mu)^2$$

$$-\frac{m}{2}\ln 2\pi\sigma^2 - \frac{1}{2\sigma^2}\sum (y_i-\mu)^2$$

$$\frac{\partial \ln L}{\partial \mu} = +\frac{2}{2\sigma^2}\sum (x_i-\mu) + \frac{2}{2\sigma^2}\sum (y_i-\mu) = 0$$

$$\hat{\mu} = \frac{n\bar{x} + m\bar{y}}{n+m}.$$

$$\frac{\partial \ln L}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum (x_i-\mu)^2$$

$$-\frac{m}{2\sigma^2} + \frac{1}{2\sigma^4}\sum (y_i-\mu)^2 = 0$$

$$\hat{\sigma}^2 = \frac{\sum (x_i-\hat{\mu})^2 + \sum (y_i-\hat{\mu})^2}{n+m}.$$

EXERCISE 4:

$$y_i = \beta_1 X_i + \varepsilon_i$$

$$y_i \sim N(\beta_1 X_i, \sigma)$$

$$f(y_i) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(y_i - \beta_1 X_i)^2}$$

$$L = (2\pi\sigma^2)^{-n/2} e^{-\frac{1}{2\sigma^2}\sum(y_i - \beta_1 X_i)^2}$$

$$\ln L = -\frac{n}{2}\ln 2\pi\sigma^2 - \frac{1}{2\sigma^2}\sum(y_i - \beta_1 X_i)^2$$

$$\frac{d\ln L}{d\beta_1} = +\frac{2}{2\sigma^2}\sum(y_i - \beta_1 X_i) = 0$$

$$\hat{\beta}_1 = \frac{\sum x_i y_i}{\sum x_i^2}$$

$$E\,\hat{\beta}_1 = \frac{\sum x_i E y_i}{\sum x_i^2} = \frac{\beta_1 \sum x_i^2}{\sum x_i^2} = \beta_1.$$

$$\frac{d\ln L}{d\sigma^2} = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum(y_i - \beta_1 X_i)^2 = 0$$

$$\hat{\sigma}^2 = \frac{\sum(y_i - \hat{\beta}_1 X_i)^2}{n}.$$