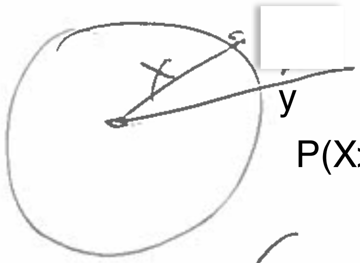
Question (a).



Q is poisson

P(X>x)=P(Q=0), therefore,

cdf of X: $P(X \le x) = 1 - P(Q=0 \text{ in circle with } \lambda c)$

$$= 1 - \frac{\lambda c^{0} e^{-\lambda c}}{0!} = 1 - e^{-\lambda \pi x^2}$$

pdf of X: $f(x) = 2\lambda\pi x\, e^{-\lambda\pi x^2}$

PDF of $\pi\lambda x^2$

Let $W = \pi\lambda x^2$

$$F_W(w) = P(W \le w) = P(\pi\lambda x^2 \le w) = P\left(x^2 \le \frac{w}{\pi\lambda}\right)$$

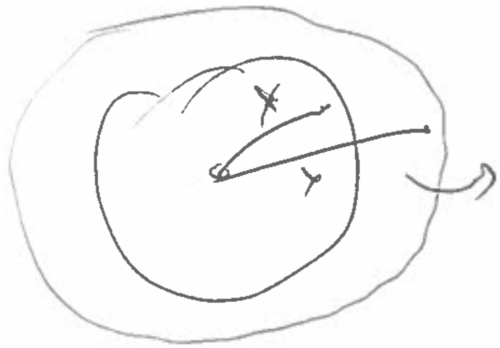
$$= P\left(X \le \sqrt{\frac{w}{\pi\lambda}}\right) = \frac{1}{\sqrt{\pi\lambda}}\,\frac{1}{2} w^{-1/2} f_X\left(\sqrt{\frac{w}{\pi\lambda}}\right)$$

$$f(w) = \frac{1}{\sqrt{\pi\lambda}}\,\frac{1}{2} w^{-1/2}\, 2\lambda\pi \sqrt{\frac{w}{\pi\lambda}}\; e^{-\lambda\pi \frac{w}{\pi\lambda}}$$

$$= e^{-w} \quad \therefore \quad w \sim \exp(1)$$

Question (b)

cdf of Y:



No event in ring

$$P(Y\le y) = 1 - P(Q=0 \text{ in ring with } \lambda_r = \lambda(\pi y^2 - \pi x^2))$$

$$= 1 - \frac{\lambda_r^0 e^{-\lambda_r}}{0!} = 1 - e^{-\lambda(\pi y^2 - \pi x^2)}$$

$$= 1 - e^{-\lambda\pi(y^2-x^2)}$$

pdf of Y:

$$f(y) = 2\lambda\pi y \, e^{-\lambda\pi(y^2-x^2)}$$

pdf of $\pi\lambda(y^2-x^2)$:

Let $Z = \pi\lambda(y^2-x^2)$

$$F_Z(z) = P(Z\le z) = P(\pi\lambda(y^2-x^2)\le z) = P\left(y\le\sqrt{\frac{z+\pi\lambda x^2}{\pi\lambda}}\right)$$

$$= P\left(y^2 \le \frac{z+\pi\lambda x^2}{\pi\lambda}\right)$$

$$f(z) = \frac{1}{\sqrt{\pi\lambda}}\,\frac{1}{2}(z+\pi\lambda x^2)^{-1/2}\, 2\lambda\pi\sqrt{\frac{z+\pi\lambda x^2}{\pi\lambda}}\cdot e^{-\lambda\pi\left(\frac{z+\pi\lambda x^2}{\pi\lambda}-x^2\right)} = e^{-z}$$

$\therefore Z\sim \exp(1)$

c. Suppose now we randomly select m points in this forest. Find the distribution of $2\lambda\pi\sum_{i=1}^{m} X_i^2$ and the distribution of $2\lambda\pi\sum_{i=1}^{m}(Y_i^2 - X_i^2)$.

SINCE $\lambda\pi X_i^2 \sim exp(1)$ AND $\lambda\pi(Y_i^2 - X_i^2) \sim exp(1)$

IT FOLLOWS THAT (using moment generating function properties)

$$2\lambda\pi\sum X_i^2 \sim \chi^2_{2m}$$

$$2\lambda\pi\sum(Y_i^2 - X_i^2) \sim \chi^2_{2m}$$

d. Let $s = \lambda\pi\sum_{i=1}^{m} X_i^2$ and $t = \lambda\pi\sum_{i=1}^{m}(Y_i^2 - X_i^2)$. If s and t are independent show that $\frac{\sum_{i=1}^{m} X_i^2}{\sum_{i=1}^{m} Y_i^2} \sim \text{beta}(m, m)$.

$s \sim \Gamma(m,1)$, $t \sim \Gamma(m,1)$

LET $u = \frac{s}{s+t}$ AND $v = s+t$ NOTE: $\frac{s}{s+t} = \frac{\sum X_i^2}{\sum Y_i^2}$.

s,t ARE IND. $\rightarrow$ $f_{ST}(s,t) = \frac{s^{m-1}e^{-s}}{\Gamma(m)} \cdot \frac{t^{m-1}e^{-t}}{\Gamma(m)}$

$$\boxed{\begin{aligned} s &= uv \\ t &= v(1-u) \end{aligned}}$$

$$f_{UV}(u,v) = f_{ST}(s = g_1(u,v), t = g_2(u,v))\,|J|^{-1}$$

$$J = \begin{vmatrix} \frac{du}{ds} & \frac{du}{dt} \\ \frac{dv}{ds} & \frac{dv}{dt} \end{vmatrix} = \frac{1}{s+t}$$

$$= \frac{(uv)^{m-1}e^{-uv}}{\Gamma(m)} \cdot \frac{(v(1-u))^{m-1}e^{-v(1-u)}}{\Gamma(m)} \cdot v \quad \frac{\Gamma(2m)}{\Gamma(2m)}$$

$$= \frac{v^{2m-1}e^{-v}}{\Gamma(2m)} \cdot \frac{u^{m-1}(1-v)^{m-1}\,\Gamma(2m)}{\Gamma(m)\Gamma(m)}$$

$\therefore$ $v \sim \Gamma(2m,1)$ AND

$U \sim beta(m,m)$