

University of California, Los Angeles  
Department of Statistics

Statistics 100B

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Covariance and correlation

Let random variables  $X, Y$  with means  $\mu_X, \mu_Y$  respectively. The covariance, denoted with  $cov(X, Y)$ , is a measure of the association between  $X$  and  $Y$ .

Definition:

$$\sigma_{XY} = cov(X, Y) = E(X - \mu_X)(Y - \mu_Y)$$

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Note: If  $X, Y$  are independent then  $E(XY) = (EX)E(Y)$  Therefore  $cov(X, Y) = 0$ .

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Let  $W, X, Y, Z$  be random variables, and  $a, b, c, d$  be constants,

- Find  $cov(a + X, Y)$

- Find  $cov(aX, bY)$

- Find  $cov(X, Y + Z)$

- Find  $cov(aW + bX, cY + dZ)$

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- Important:

$$var(X + Y) = var(X) + var(Y) + 2cov(X, Y)$$

Proof:

- Find  $\text{var}(aX + bY)$

- In general: Let  $X_1, X_2, \dots, X_n$  be random variables, and  $a_1, a_2, \dots, a_n$  be constants. Find the variance of the linear combination  $Y = a_1X_1 + a_2X_2 + \dots + a_nX_n$ .

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- Example: Let  $X_1, X_2, X_3$  be random variables with  $EX_1 = 1, EX_2 = 2, EX_3 = -1, \text{var}(X_1) = 1, \text{var}(X_2) = 3, \text{var}(X_3) = 5, \text{cov}(X_1, X_2) = -0.4, \text{cov}(X_1, X_3) = 0.5, \text{cov}(X_2, X_3) = 2$ . Let  $U = X_1 - 2X_2 + X_3$ . Find (a)  $E(U)$ , and (b)  $\text{var}(U)$ .

However, the covariance depends on the scale of measurement and so it is not easy to say whether a particular covariance is small or large. The problem is solved by standardize the value of covariance (divide it by  $\sigma_X\sigma_Y$ ), to get the so called coefficient of correlation  $\rho_{XY}$ .

$$\rho = \frac{\text{cov}(X, Y)}{\sigma_X\sigma_Y}, \quad \text{Always, } -1 \leq \rho \leq 1, \text{ (see proof below).}$$

$$\text{cov}(X, Y) = \rho\sigma_X\sigma_Y$$

If  $X, Y$  are independent then  $\dots$

Show that  $-1 \leq \rho \leq 1$ :

Let  $X, Y$  be random variables with variances  $\sigma_X^2, \sigma_Y^2$  respectively. Examine the following random expressions:

$$\frac{X}{\sigma_X} + \frac{Y}{\sigma_Y}$$

$$\frac{X}{\sigma_X} - \frac{Y}{\sigma_Y}$$

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Example:

$X$  and  $Y$  are random variables with joint probability density function

$f_{XY}(x, y) = x + y, 0 \leq x \leq 1, 0 \leq y \leq 1$ . Find  $\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2, cov(X, Y), \rho_{XY}$ .

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Example:

Let  $X_1, X_2, \dots, X_n$  be independent and identically distributed random variables having variance  $\sigma^2$ . Show that  $\text{cov}(X_i - \bar{X}, \bar{X}) = 0$ .

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## Portfolio risk and return

An investor has a certain amount of dollars to invest into two stocks (*IBM* and *TEXACO*). A portion of the available funds will be invested into IBM (denote this portion of the funds with  $a$ ) and the remaining funds into *TEXACO* (denote it with  $b$ ) - so  $a + b = 1$ . The resulting portfolio will be  $aX + bY$  where  $X$  is the monthly return of *IBM* and  $Y$  is the monthly return of *TEXACO*. The goal here is to find the most efficient portfolios given a certain amount of risk. Using market data from January 1980 to February 2001 we compute that  $E(X) = 0.010$ ,  $E(Y) = 0.013$ ,  $Var(X) = 0.0061$ ,  $Var(Y) = 0.0046$ , and  $Cov(X, Y) = 0.00062$ . We first want to minimize the variance of the portfolio. This will be:

$$\begin{aligned} &\text{Minimize } Var(aX + bY) \\ &\text{subject to } a + b = 1 \end{aligned}$$

Or

$$\begin{aligned} &\text{Minimize } a^2 Var(X) + b^2 Var(Y) + 2abCov(X, Y) \\ &\text{subject to } a + b = 1 \end{aligned}$$

Therefore our goal is to find  $a$  and  $b$ , the percentage of the available funds that will be invested in each stock. Substituting  $b = 1 - a$  into the equation of the variance we get

$$a^2 Var(X) + (1 - a)^2 Var(Y) + 2a(1 - a)Cov(X, Y)$$

To minimize the above expression we take the derivative with respect to  $a$ , set it equal to zero and solve for  $a$ . The result is:

$$a = \frac{Var(Y) - Cov(X, Y)}{Var(X) + Var(Y) - 2Cov(X, Y)}$$

and therefore

$$b = \frac{Var(X) - Cov(X, Y)}{Var(X) + Var(Y) - 2Cov(X, Y)}$$

The values of  $a$  and  $b$  are:

$$a = \frac{0.0046 - 0.00062}{0.0061 + 0.0046 - 2(0.00062)} = 0.42$$

and  $b = 1 - a = 1 - 0.42 \Rightarrow b = 0.58$ . Therefore if the investor invests 42% of the available funds into *IBM* and the remaining 58% into *TEXACO* the variance of the portfolio will be minimum and equal to:

$$Var(0.42X + 0.58Y) = 0.42^2(0.0061) + 0.58^2(0.0046) + 2(0.42)(0.58)(0.00062) = 0.003926$$

The corresponding expected return of this portfolio will be

$$E(0.42X + 0.58Y) = 0.42(0.010) + 0.58(0.013) = 0.01174$$

We can try many other combinations of  $a$  and  $b$  (but always  $a + b = 1$ ) and compute the risk and return for each resulting portfolio. This is shown in the table below and the graph of return against risk on the next page.

$a$	$b$	Risk	Return
1.00	0.00	0.006100	0.01000
0.95	0.05	0.005576	0.01015
0.90	0.10	0.005099	0.01030
0.85	0.15	0.004669	0.01045
0.80	0.20	0.004286	0.01060
0.75	0.25	0.003951	0.01075
0.70	0.30	0.003663	0.01090
0.65	0.35	0.003423	0.01105
0.60	0.40	0.003230	0.01120
0.55	0.45	0.003084	0.01135
0.50	0.50	0.002985	0.01150
0.42	0.58	0.002926	0.01174
0.40	0.60	0.002930	0.01180
0.35	0.65	0.002973	0.01195
0.30	0.70	0.003063	0.01210
0.25	0.75	0.003201	0.01225
0.20	0.80	0.003386	0.01240
0.15	0.85	0.003619	0.01255
0.10	0.90	0.003899	0.01270
0.05	0.95	0.004226	0.01285
0.00	1.00	0.004600	0.01300

The scatter plot illustrates the relationship between Expected return and Risk (portfolio standard deviation). The y-axis, labeled 'Expected return', ranges from 0.0100 to 0.0130. The x-axis, labeled 'Risk (portfolio standard deviation)', ranges from 0.055 to 0.075. The data points show a positive correlation, with a red watermark 'Assignment Project Exam Help' and 'https://powcoder.com' overlaid on the plot.

Risk (portfolio standard deviation)	Expected return
0.054	0.0117
0.054	0.0118
0.055	0.0110
0.055	0.0119
0.056	0.0113
0.056	0.0120
0.057	0.0111
0.057	0.0123
0.058	0.0109
0.058	0.0124
0.059	0.0107
0.059	0.0126
0.060	0.0109
0.060	0.0127
0.061	0.0107
0.061	0.0128
0.062	0.0105
0.062	0.0129
0.063	0.0105
0.063	0.0130
0.064	0.0103
0.064	0.0131
0.065	0.0102
0.065	0.0132
0.066	0.0101
0.066	0.0133
0.067	0.0100
0.067	0.0134
0.068	0.0099
0.068	0.0135
0.069	0.0098
0.069	0.0136
0.070	0.0097
0.070	0.0137
0.071	0.0096
0.071	0.0138
0.072	0.0095
0.072	0.0139
0.073	0.0094
0.073	0.0140
0.074	0.0093
0.074	0.0141
0.075	0.0092
0.075	0.0142
0.076	0.0091
0.076	0.0143
0.077	0.0090
0.077	0.0144
0.078	0.0089
0.078	0.0145
0.079	0.0088
0.079	0.0146
0.080	0.0087
0.080	0.0147
0.081	0.0086
0.081	0.0148
0.082	0.0085
0.082	0.0149
0.083	0.0084
0.083	0.0150
0.084	0.0083
0.084	0.0151
0.085	0.0082
0.085	0.0152
0.086	0.0081
0.086	0.0153
0.087	0.0080
0.087	0.0154
0.088	0.0079
0.088	0.0155
0.089	0.0078
0.089	0.0156
0.090	0.0077
0.090	0.0157
0.091	0.0076
0.091	0.0158
0.092	0.0075
0.092	0.0159
0.093	0.0074
0.093	0.0160
0.094	0.0073
0.094	0.0161
0.095	0.0072
0.095	0.0162
0.096	0.0071
0.096	0.0163
0.097	0.0070
0.097	0.0164
0.098	0.0069
0.098	0.0165
0.099	0.0068
0.099	0.0166
0.100	0.0067
0.100	0.0167
0.101	0.0066
0.101	0.0168
0.102	0.0065
0.102	0.0169
0.103	0.0064
0.103	0.0170
0.104	0.0063
0.104	0.0171
0.105	0.0062
0.105	0.0172
0.106	0.0061
0.106	0.0173
0.107	0.0060
0.107	0.0174
0.108	0.0059
0.108	0.0175
0.109	0.0058
0.109	0.0176
0.110	0.0057
0.110	0.0177
0.111	0.0056
0.111	0.0178
0.112	0.0055
0.112	0.0179
0.113	0.0054
0.113	0.0180
0.114	0.0053
0.114	0.0181
0.115	0.0052
0.115	0.0182
0.116	0.0051
0.116	0.0183
0.117	0.0050
0.117	0.0184
0.118	0.0049
0.118	0.0185
0.119	0.0048
0.119	0.0186
0.120	0.0047
0.120	0.0187
0.121	0.0046
0.121	0.0188
0.122	0.0045
0.122	0.0189
0.123	0.0044
0.123	0.0190
0.124	0.0043
0.124	0.0191
0.125	0.0042
0.125	0.0192
0.126	0.0041
0.126	0.0193
0.127	0.0040
0.127	0.0194

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### Efficient frontier with three stocks

```
> summary(returns)
      ribm      rxom      rboeing
Min.   :-0.2264526 Min.   :-0.5219233 Min.   :-0.34570
1st Qu.: -0.0515524 1st Qu.: -0.0172273 1st Qu.: -0.04308
Median : -0.0089916 Median :  0.0007013 Median :  0.01843
Mean   :  0.0003073 Mean    :-0.0011666 Mean    :  0.01079
3rd Qu.:  0.0462550 3rd Qu.:  0.0337488 3rd Qu.:  0.07357
Max.   :  0.3537987 Max.   :  0.2269380 Max.   :  0.17483

> cov(returns)
      ribm      rxom      rboeing
ribm   9.930174e-03 0.001798962 3.020685e-05
rxom   1.798962e-03 0.006743820 1.781462e-03
rboeing 3.020685e-05 0.001781462 8.282167e-03
```

Portfolio possibilities curve with 3 stocks

