University of California, Los Angeles
Department of Statistics

Statistics 100B **Instructor: Nicolas Christou**

Joint moment generating functions

Let $\mathbf{X} = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix}$, be a random vector and let $\mathbf{t} = \begin{pmatrix} t_1 \\ t_2 \\ \vdots \\ t_n \end{pmatrix}$. The joint moment generating function of $\mathbf{X}$ is defined as $M_{\mathbf{X}}(\mathbf{t}) = Ee^{\mathbf{t}'\mathbf{X}} = Eexp(\sum_{i=1}^n t_i x_i)$.

Theorem
Let $M_i(\mathbf{t}) = \frac{\partial M_{\mathbf{X}}(\mathbf{t})}{\partial t_i}$, $M_{ii}(\mathbf{t}) = \frac{\partial^2 M_{\mathbf{X}}(\mathbf{t})}{\partial {t_i}^2}$, and $M_{ij}(\mathbf{t}) = \frac{\partial^2 M_{\mathbf{X}}(\mathbf{t})}{\partial t_i \partial t_j}$.
Then, $EX_i = M_i(\mathbf{0})$, $EX_i^2 = M_{ii}(\mathbf{0})$, and $EX_iX_j = M_{ij}(\mathbf{0})$.

Corollary
Let $\psi(\mathbf{t}) = logM_{\mathbf{X}}(\mathbf{t})$, $\psi_i(\mathbf{t}) = \frac{\partial \psi_{\mathbf{X}}(\mathbf{t})}{\partial t_i}$, $\psi_{ii}(\mathbf{t}) = \frac{\partial^2 \psi_{\mathbf{X}}(\mathbf{t})}{\partial {t_i}^2}$, and $\psi_{ij}(\mathbf{t}) = \frac{\partial^2 \psi_{\mathbf{X}}(\mathbf{t})}{\partial t_i \partial t_j}$.
Then $EX_i = \psi_i(\mathbf{0})$, $var(X_i) = \psi_{ii}(\mathbf{0})$, and $cov(X_i, X_j) = \psi_{ij}(\mathbf{0})$.

Theorem
Let $\mathbf{X} = \begin{pmatrix} \mathbf{Y} \\ \mathbf{Z} \end{pmatrix}$. The marginal moment generating function of $\mathbf{Y}$ ($\mathbf{Z}$) is the moment generating function of $\mathbf{X}$ ignoring the vector $\mathbf{Z}$ ($\mathbf{Y}$). This is expressed as $M_{\mathbf{Y}}(\mathbf{u}) = \mathbf{M}_{\mathbf{X}}(\mathbf{u}, \mathbf{0})$ and $M_{\mathbf{Z}}(\mathbf{v}) = \mathbf{M}_{\mathbf{X}}(\mathbf{0}, \mathbf{v})$, where $\mathbf{t} = \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix}$.

Proof

Theorem
If $\mathbf{Y}$ and $\mathbf{Z}$ are independent then $M_{\mathbf{X}}(\mathbf{t}) = \mathrm{M}_{\mathbf{Y}}(\mathbf{u})\mathrm{M}_{\mathbf{Z}}(\mathbf{v})$.
Proof

Example 1

$\mathbf{X} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$ have joint moment generating function

$M_{\mathbf{X}}(t_1, t_2, t_3) = (1 - t_1 + 2t_2)^{-4}(1 - t_1 + 3t_3)^{-3}(1 - t_1)^{-2}$.

Use the corollary on page 1 to find:

a. $E(X_1)$, $E(X_2)$, $E(X_3)$.

b. $var(X_1)$, $var(X_2)$, $var(X_3)$.

c. $cov(X_1, X_2)$, $cov(X_1, X_3)$, $cov(X_2, X_3)$.

d. ρ_{X_1, X_3}.

Example 2

Let X and Y be independent normal random variables, each with mean μ and standard deviation σ.

a. Consider the random quantities $X + Y$ and $X - Y$. Find the moment generating function of $X + Y$ and the moment generating function of $X - Y$.

b. Find the joint moment generating function of $(X + Y, X - Y)$.

c. Are $X + Y$ and $X - Y$ independent? Explain your answer using moment generating functions.

Example 3

Let $\mathbf{X} = (X_1, X_2, X_3)$ has joint moment generating function

$$M_{\mathbf{X}}(t_1, t_2, t_3) = (1 - t_1 + 2t_2)^{-4}(1 - t_1 + 3t_3)^{-3}(1 - t_1)^{-2}.$$

Answer the following questions:

a. Find the moment generating function of (X_1, X_3).

b. Find the moment generating function of X_1.

c. Find the moment generating function of X_3.

d. Are X_1, X_3 independent?

e. Find the moment generating function of (X_2, X_3).

f. Are X_2, X_3 independent?