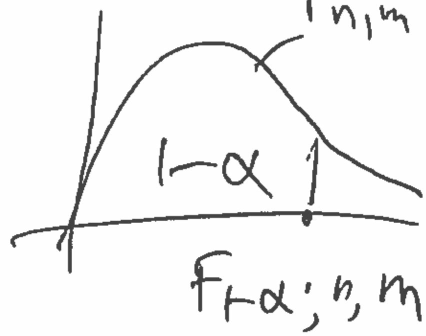
Homework 5 solutions

(a). $X \sim F_{m,n} \rightarrow \frac{1}{X} \sim F_{n,m}$

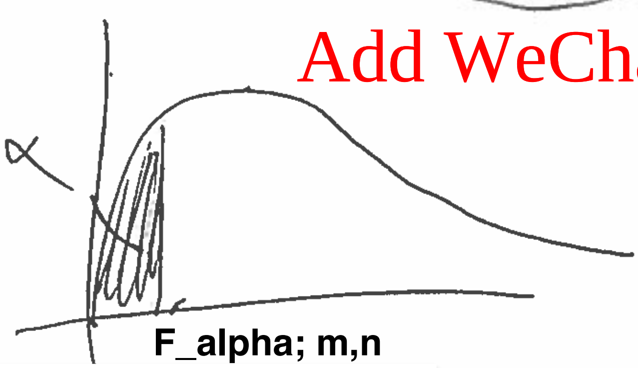
$$P\left(F_{n,m} < F_{1-\alpha;n,m}\right) = 1-\alpha$$



$$P\left(F_{m,n} > \frac{1}{F_{1-\alpha;n,m}}\right) = 1-\alpha$$

$$P\left[F_{m,n} < \frac{1}{F_{1-\alpha;n,m}}\right] = \alpha$$

THIS IS THE PERCENTILE of $F_{m,n}$.



F_alpha; m,n

Let X ~ F_1,n, then P(X<F_1-alpha;1,n)=1-a

But X=t_n^2, so P(t_n^2 < F_1-alpha;1,n)=1-alpha

Or P[-sqrt(F_1-alpha;1,n) < t_n < +sqrt(F_1-alpha;1,n)]=1-alpha

Therefore, sqrt(F_1-alpha;1,n)=t_1-alpha/2;n

Finally, t^2_1-alpha/2;n = F_1-alpha;1,n (b). $\bar{X}-\bar{Y} \sim N\left(\mu_1-\mu_2, \sqrt{\frac{\sigma_1^2}{13}+\frac{\sigma_2^2}{16}}\right)$

OR $\bar{X}-\bar{Y} \sim N\left(\mu_1-\mu_2, \sigma_1\sqrt{\frac{1}{13}+\frac{5}{16}}\right)$

AND $\frac{(13-1)S_x^2}{\sigma_1^2} + \frac{(16-1)S_y^2}{\sigma_2^2} \sim \chi^2_{27}$

OR $\frac{12 S_x^2}{\sigma_1^2} + \frac{15 S_y^2}{5\sigma_1^2} \sim \chi^2_{27}$

OR $\frac{12 S_x^2}{\sigma_1^2} + \frac{3 S_y^2}{\sigma_1^2} \sim \chi^2_{27}$

THEN

$$\frac{\dfrac{\bar{X}-\bar{Y}-(\mu_1-\mu_2)}{\sigma_1\sqrt{\frac{1}{13}+\frac{5}{16}}}}{\sqrt{\dfrac{\frac{12S_x^2+3S_y^2}{\sigma_1^2}}{27}}} \sim t_{27}$$

1(c). $\sum X_i \sim \Gamma(n\alpha, \beta) \rightarrow M_{\sum X_i}(t) = (1-\beta t)^{-n\alpha}$

$$M_{\bar{X}}(t) = M_{\frac{\sum X_i}{n}}(t) = M_{\sum X_i}\left(\frac{t}{n}\right)$$

$$= \left(1 - \frac{\beta t}{n}\right)^{-n\alpha}$$

LET $Q = \frac{2n}{\beta}\bar{X}$

THEN

$$M_Q(t) = (1-2t)^{-n\alpha}$$

$$\text{OR } M_Q(t) = (1-2t)^{-\frac{2n\alpha}{2}}$$

$$\therefore Q = \frac{2n\bar{X}}{\beta} \sim \chi^2_{2n\alpha}$$

(ASSUME α IS INTEGER).

(f). $X \sim F_{n,m}$

OR $\dfrac{\chi^2_n/n}{\chi^2_m/m} =$

$$EX = \left(E\,\frac{\chi^2_n}{n}\right) E\left(\frac{\chi^2_m}{m}\right)^{-1}$$

BECAUSE χ^2_n, χ^2_m ARE IND.

BUT $\chi^2_m \sim \Gamma\left(\frac{m}{2}, 2\right)$

AND $E\,\frac{\chi^2_n}{n} = 1$

SO, $EX = E\left(\Gamma\left(\frac{m}{2}, 2\right)\right)^{-1}$

USE PROPERTY OF $\Gamma(\alpha, \beta)$:

$$E X^k = \frac{\Gamma(\alpha + k)\, \beta^k}{\Gamma(\alpha)}$$

HERE $k = -1$.

(e). $X \sim t_n$

$$X = \frac{Z}{\sqrt{U/n}} \qquad Z \sim N(0,1) \qquad U \sim \chi^2_n$$

Z, U ARE INDEPENDENT

$$EX = \sqrt{n}\, E Z\, E U^{-1/2} = 0$$ BECAUSE $EZ = 0$

$$VAR(X) = n\left[E Z^2\right]\left(E U^{-1}\right) = \cancel{X}$$

BUT $U \sim \chi^2_n$ OR $U \sim \Gamma\left(\frac{n}{2}, 2\right)$

$$= n\left(\sigma^2 + \mu^2\right) \frac{\Gamma\left(\frac{n}{2} - 1\right) 2^{-1}}{\Gamma\left(\frac{n}{2}\right)}$$

$$= n(1+0) \frac{\Gamma\left(\frac{n}{2} - 1\right)}{\left(\frac{n}{2} - 1\right)\Gamma\left(\frac{n}{2} - 1\right)} \cdot 2^{-1}$$

$$= \frac{n}{n-2}.$$

(d)

$$y'\Sigma^{-1}y = (y_1 \quad y_2)\begin{pmatrix}\sigma_1^2 & \sigma_{12}\\ \sigma_{21} & \sigma_2^2\end{pmatrix}^{-1}\begin{pmatrix}y_1\\ y_2\end{pmatrix}$$

$$= (y_1 \quad y_2)\begin{pmatrix}\frac{\sigma_2^2}{\sigma_1^2\sigma_2^2-\sigma_{12}^2} & -\frac{\sigma_{12}}{\sigma_1^2\sigma_2^2-\sigma_{12}^2}\\ \frac{-\sigma_{12}}{\sigma_1^2\sigma_2^2-\sigma_{12}^2} & \frac{\sigma_1^2}{\sigma_1^2\sigma_2^2-\sigma_{12}^2}\end{pmatrix}\begin{pmatrix}y_1\\ y_2\end{pmatrix}$$

$$= \left(\frac{y_1\sigma_2^2 - y_2\sigma_{12}}{\sigma_1^2\sigma_2^2-\sigma_{12}^2} \quad \frac{-y_1\sigma_{12} + y_2\sigma_1^2}{\sigma_1^2\sigma_2^2-\sigma_{12}^2}\right)\begin{pmatrix}y_1\\ y_2\end{pmatrix}$$

$$= \frac{y_1^2\sigma_2^2 - y_1y_2\sigma_{12} - y_1y_2\sigma_{12} + y_2^2\sigma_1^2}{\sigma_1^2\sigma_2^2-\sigma_{12}^2}$$

$$= \frac{y_1^2\sigma_2^2 - 2y_1y_2\rho\sigma_1\sigma_2 + y_2^2\sigma_1^2}{\sigma_1^2\sigma_2^2 - \rho^2\sigma_1^2\sigma_2^2} = \frac{y_1^2\sigma_2^2 - 2y_1y_2\rho\sigma_1\sigma_2 + y_2^2\sigma_1^2}{\sigma_1^2\sigma_2^2(1-\rho^2)}$$

Now: $y'\Sigma^{-1}y - \frac{y_1^2}{\sigma_1^2} = \frac{y_1^2\sigma_2^2 - 2y_1y_2\rho\sigma_1\sigma_2 + y_2^2\sigma_1^2 - \sigma_2^2(1-\rho^2)y_1^2}{\sigma_1^2\sigma_2^2(1-\rho^2)}$

$$= \frac{\cancel{y_1^2\sigma_2^2} - 2y_1y_2\rho\sigma_1\sigma_2 + y_2^2\sigma_1^2 - \cancel{y_1^2\sigma_2^2} + \rho^2y_1^2\sigma_2^2}{\sigma_1^2\sigma_2^2(1-\rho^2)}$$

$$= \frac{1}{1-\rho^2}\left[\frac{\rho^2y_1^2}{\sigma_1^2} - \frac{2y_1y_2\rho}{\sigma_1\sigma_2} + \frac{y_2^2}{\sigma_2^2}\right] = \frac{1}{1-\rho^2}\left(\frac{\rho y_1}{\sigma_1} - \frac{y_2}{\sigma_2}\right)^2$$

$$= \frac{Q^2}{1-\rho^2}, \text{ where } Q \sim N(0, \sqrt{1-\rho^2})$$

$$\therefore \sim \chi^2_1$$

(g). FROM CLASS NOTES

$$(\underset{\sim}{Y}-\underset{\sim}{\mu})' \underset{\sim}{\Sigma}^{-1} (\underset{\sim}{Y}-\underset{\sim}{\mu}) \sim \chi^2_n \quad (\mathrm{I})$$

$(x_1, y_1), \ldots, (x_n, y_n)$ IS A RANDOM SAMPLE FROM $N_n(\underset{\sim}{\mu}, \underset{\sim}{\Sigma})$

THEREFORE

$$M_{\bar{X}, \bar{Y}}(t_1, t_2) = E\, e^{t_1 \bar{X} + t_2 \bar{Y}}$$

$$= E\, e^{t_1\left(\frac{x_1 + \cdots + x_n}{n}\right) + t_2\left(\frac{y_1 + \cdots + y_n}{n}\right)}$$

$$= \left\{E\, e^{\frac{t_1}{n} x_1 + \frac{t_2}{n} y_1}\right\} \times \cdots \times \left\{E\, e^{\frac{t_1}{n} x_n + \frac{t_2}{n} y_n}\right\}$$

$$= \left\{M_{x_i, y_i}\left(\frac{t_1}{n}, \frac{t_2}{n}\right)\right\}^n$$

$$= \left\{e^{\frac{\underset{\sim}{t}'}{n} \underset{\sim}{\mu} + \frac{1}{2} \frac{\underset{\sim}{t}'}{n} \underset{\sim}{\Sigma} \frac{\underset{\sim}{t}}{n}}\right\}^n = e^{\underset{\sim}{t}' \underset{\sim}{\mu} + \frac{1}{2} \underset{\sim}{t}' \frac{\underset{\sim}{\Sigma}}{n} \underset{\sim}{t}}$$

$$\therefore \begin{pmatrix} \bar{X} \\ \bar{Y} \end{pmatrix} \sim N_2\left(\underset{\sim}{\mu}, \frac{\underset{\sim}{\Sigma}}{n}\right) \qquad \text{USING } (\mathrm{I})$$

WE GET $(\bar{X}-\mu_1, \bar{Y}-\mu_2) \left(\frac{\underset{\sim}{\Sigma}}{n}\right)^{-1} \begin{bmatrix} \bar{X}-\mu_1 \\ \bar{Y}-\mu_2 \end{bmatrix} \sim \chi^2_2.$

(h) $Z \sim N(0,1)$ $\quad U \sim \chi^2_n$

Z AND U ARE INDEPENDENT.

$$f(z,u) = f(z)\cdot f(u) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \frac{u^{\frac{n}{2}-1} e^{-u/2}}{\Gamma(\frac{n}{2})\, 2^{n/2}}$$

$$\left.\begin{aligned} x &= \frac{z}{\sqrt{u/n}} \\ w &= u \end{aligned}\right\} \begin{aligned} &\rightarrow z = x\sqrt{\frac{w}{n}} \\ &\rightarrow u = w \end{aligned}$$

$$J = \begin{vmatrix} \frac{dx}{dz} & \frac{dx}{dw} \\ \frac{dw}{dz} & \frac{dw}{du} \end{vmatrix} = \begin{vmatrix} \frac{1}{\sqrt{u/n}} & -\frac{1}{2}\sqrt{n}\, z\, u^{-3/2} \\ 0 & 1 \end{vmatrix} = \frac{1}{\sqrt{w/n}}$$

THEREFORE,

$$f(x,w) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2 \frac{w}{n}} \frac{w^{\frac{n}{2}-1} e^{-w/2}}{\Gamma(\frac{n}{2})\, 2^{n/2}} \left(\frac{w}{n}\right)^{1/2}$$

NOW INTEGRATE THE JOINT PDF OF X, W W.R.T. W TO FIND $f(x)$:

$$f(x) = \int_0^\infty f(x, w)\, dw$$

$$= \frac{1}{\sqrt{2\pi}\, \Gamma\left(\frac{n}{2}\right) 2^{n/2} \sqrt{n}} \int_0^\infty \underbrace{w^{\frac{n+1}{2}-1} e^{-w\left(\frac{1}{2}\frac{x^2}{n}+\frac{1}{2}\right)}}_{\text{KERNEL FUNCTION}} dw$$

OF $\Gamma\left(\frac{n+1}{2}, \frac{2}{\frac{x^2}{n}+1}\right)$

$$= \frac{\left(\frac{2}{\frac{x^2}{n}+1}\right)^{\frac{n+1}{2}} \Gamma\left(\frac{n+1}{2}\right)}{\sqrt{2\pi}\, \Gamma\left(\frac{n}{2}\right) 2^{n/2} \sqrt{n}} = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi n}\, \Gamma\left(\frac{n}{2}\right)} \left(1 + \frac{x^2}{n}\right)^{-\frac{n+1}{2}}$$

(i). $Z \sim N(\delta, 1)$

$U \sim \chi^2_n$

THEN $t = \dfrac{Z}{\sqrt{\frac{U}{n}}} \sim t_n$ $(NCP = \delta)$

$$E t = \sqrt{n}\,(E Z)\left(E\, U^{-1/2}\right)$$

$$= \sqrt{n} \cdot \delta \cdot \frac{\Gamma\left(\frac{n}{2} - \frac{1}{2}\right) 2^{-1/2}}{\Gamma\left(\frac{n}{2}\right)}$$

$$\mathrm{VAR}(t) = E t^2 - (E t)^2$$

(j). $U \sim \chi^2_{n_1}$ (ncp $= \theta$)

$V \sim \chi^2_{n_2}$

$$\frac{U/n_1}{V/n_2} \sim F_{n_1, n_2} \text{ (ncp} = \theta)$$

$$E \, \frac{U/n_1}{V/n_2} = \frac{n_2}{n_1} (E\,U)(E\,V^{-1})$$

NOTE: $M_U(t) = (1-2t)^{-\frac{n_1}{2}} e^{\theta \frac{t}{1-2t}}$