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Statistics 100B **Instructor: Nicolas Christou**

Functions of random variables

Functions of one random variable

a. Method of cdf:
Let $X \sim \Gamma(\alpha, \beta)$. Find the distribution of $Y = cX$, $c > 0$. With the method of cdf we begin with the cdf of Y as follows.

$$
\begin{aligned}
F_Y(y) &= P(Y \leq y) \\
F_Y(y) &= P(cX \leq y) \\
F_Y(y) &= P(X \leq \frac{y}{c}) \\
F_Y(y) &= F_X(\frac{y}{c}) \text{ Now differentiate on both sides w.r.t. } y \\
f_Y(y) &= \frac{1}{c} f_X(\frac{y}{c}) \\
f_Y(y) &= \frac{1}{c} \frac{(\frac{y}{c})^{\alpha-1} e^{-\frac{y}{\beta c}}}{\Gamma(\alpha)\beta^\alpha} \\
f_Y(y) &= \frac{y^{\alpha-1} e^{-\frac{y}{\beta c}}}{\Gamma(\alpha)(c\beta)^\alpha}
\end{aligned}
$$

Therefore, $Y \sim \Gamma(\alpha, c\beta)$.

b. **Method of transformations**
It is originated from the method of cdf. In general, to find the pdf of a function of a random variable we use the following theorem.

Let X be a continuous random variable with pdf $f(x)$. Let $Y = g(X)$, either increasing or decreasing. Then the pdf of Y is given by

$$f_Y(y) = f_X[w(y)] \left| \frac{d}{dy} w(y) \right|,$$

where $w(y)$ is the inverse function of g (the value of x such that $g(x) = y$). We can also use the following notation, by defining $g^{-1}(y)$ as the value of x such that $g(x) = y$.

$$f_Y(y) = f_X[g^{-1}(y)] \left| \frac{d}{dy} g^{-1}(y) \right|,$$

Apply the theorem to the example above:

$$\begin{aligned} Y &= cX, \text{ here } g(X) = cX, \text{ and therefore } w(y) = g^{-1}(y) = \frac{y}{c}. \\ f_Y(y) &= f_X[w(y)] \left| \frac{d}{dy} w(y) \right| \\ f_Y(y) &= f_X\left(\frac{y}{c}\right) \frac{1}{c} \\ f_Y(y) &= \frac{y^{\alpha-1} e^{-\frac{y}{\beta c}}}{\Gamma(\alpha)(c\beta)^{\alpha}} \end{aligned}$$

c. **Method of MGF**
Using the uniqueness theorem. Let $X \sim \Gamma(\alpha, \beta)$. Find the distribution of $Y = cX$, $c > 0$. Then

$$M_Y(t) = M_X(ct) = (1 - \beta t)^{-\alpha}.$$

Therefore, $Y \sim \Gamma(\alpha, c\beta)$.

Joint probability distribution of functions of random variables

We can extend the idea of the distribution of a function of a random variable to bivariate and multivariate random vectors as follows.

Let X_1, X_2 be jointly continuous random variables with pdf $f_{X1X2}(x_1, x_2)$. Suppose $Y_1 = g_1(X_1, X_2)$ and $Y_2 = g_2(X_1, X_2)$. We want to find the joint pdf of Y_1, Y_2. We follow this procedure:

1. Solve the equations $y_1 = g_1(x_1, x_2)$ and $y_2 = g_2(x_1, x_2)$ for x_1 and x_2 in terms of y_1 and y_2 to get $x_1 = h_1(y_1, y_2)$ and $x_2 = h_2(y_1, y_2)$.
2. Compute the Jacobian: $\mathbf{J} = \begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{vmatrix}$. ($\mathbf{J}$ is the determinant of the matrix of partial derivatives.)

To find the joint pdf of Y_1, Y_2 use the following result: $f_{Y_1,Y_2}(y_1, y_2) = f_{X_1,X_2}(x_1, x_2)|\mathbf{J}|^{-1}$, where $|\mathbf{J}|$ is the absolute value of the Jacobian. Here, x_1, x_2 are the expressions obtained from step (1) above, $x_1 = h_1(y_1, y_2)$ and $x_2 = h_2(y_1, y_2)$.

Example 1

Let X_1 and X_2 be independent exponential random variables with parameters λ_1 and λ_2 respectively. Find the joint probability density function of $X_1 + X_2$ and $X_1 - X_2$.

Solution:
Since X_1 and X_2 are independent the joint pdf of X_1 and X_2 is

$$f_{X_1,X_2}(x_1, x_2) = f_{X_1}(x_1) f_{X_2}(x_2) = \lambda_1 e^{-\lambda_1 x_1} \lambda_2 e^{-\lambda_2 x_2}$$

Let $U = X_1 + X_2$ and $V = X_1 - X_2$. We solve for x_1 and x_2 to get $x_1 = \frac{u+v}{2}$ and $x_2 = \frac{u-v}{2}$.

We compute now the Jacobian: $\mathbf{J} = \begin{vmatrix} \frac{\partial u}{\partial x_1} & \frac{\partial u}{\partial x_2} \\ \frac{\partial v}{\partial x_1} & \frac{\partial v}{\partial x_2} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2.$

Finally, we find the joint pdf of U and V:

$$f_{U,V}(u, v) = \lambda_1 e^{-\lambda_1 \frac{u+v}{2}} \lambda_2 e^{-\lambda_2 \frac{u-v}{2}} \times \frac{1}{2} = \frac{\lambda_1 \lambda_2}{2} e^{-\lambda_1 \frac{u+v}{2} - \lambda_2 \frac{u-v}{2}}$$

Example 2

Suppose X and Y are independent random variables with $X \sim \Gamma(\alpha_1, \beta)$ and $Y \sim \Gamma(\alpha_2, \beta)$. Compute the joint pdf of $U = X + Y$ and $V = \frac{X}{X+Y}$ and find the distribution of U and the distribution of V. Also show that U, V are independent.

Solution:

A random variable X is said to have a gamma distribution with parameters α, β if its probability density function is given by

$$f(x) = \frac{x^{\alpha-1} e^{-\frac{x}{\beta}}}{\Gamma(\alpha)\beta^{\alpha}}, \quad \alpha, \beta > 0, x \geq 0.$$

Here $X \sim \Gamma(\alpha_1, \beta)$ and $Y \sim \Gamma(\alpha_2, \beta)$, therefore,

$$f_X(x) = \frac{x^{\alpha_1-1} e^{-\frac{x}{\beta}}}{\Gamma(\alpha_1)\beta^{\alpha_1}}, \text{ and } f_Y(y) = \frac{y^{\alpha_2-1} e^{-\frac{y}{\beta}}}{\Gamma(\alpha_2)\beta^{\alpha_2}}$$

Because X, Y are independent, the joint pdf of X and Y is the product of the two marginal pdfs:

$$f_{XY}(x, y) = f_X(x) f_Y(y) = \frac{x^{\alpha_1-1} e^{-\frac{x}{\beta}}}{\Gamma(\alpha_1)\beta^{\alpha_1}} \frac{y^{\alpha_2-1} e^{-\frac{y}{\beta}}}{\Gamma(\alpha_2)\beta^{\alpha_2}} = \frac{x^{\alpha_1-1} y^{\alpha_2-1} e^{-\frac{x+y}{\beta}}}{\Gamma(\alpha_1)\Gamma(\alpha_2)\beta^{\alpha_1+\alpha_2}}.$$

Now follow the two steps above:

1. Solve the equations $u = x + y$ and $v = \frac{x}{x+y}$ in terms of x and y. We get: $x = uv$ and $y = u(1 - v)$.

2. Compute the Jacobian: $\mathbf{J} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ \frac{y}{(x+y)^2} & -\frac{x}{(x+y)^2} \end{vmatrix} = -\frac{1}{x+y} = -\frac{1}{u}$.

Finally to find the joint pdf of U, V use $x = uv$ and $y = u(1 - v)$ in the joint pdf of X, Y: $f_{UV}(u, v) = \frac{(uv)^{\alpha_1-1}[u(1-v)]^{\alpha_2-1} e^{-\frac{u}{\beta}} u}{\Gamma(\alpha_1)\Gamma(\alpha_2)\beta^{\alpha_1+\alpha_2}}$, multiply by $\frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1+\alpha_2)}$ and rearrange to get :

$$f_{UV}(u, v) = \frac{u^{\alpha_1+\alpha_2-1} e^{-\frac{u}{\beta}}}{\Gamma(\alpha_1+\alpha_2)\beta^{\alpha_1+\alpha_2}} \times \frac{v^{\alpha_1-1}(1-v)^{\alpha_2-1}\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)}.$$

Therefore,

$$f_{UV}(u, v) = \frac{u^{\alpha_1+\alpha_2-1} e^{-\frac{u}{\beta}}}{\Gamma(\alpha_1+\alpha_2)\beta^{\alpha_1+\alpha_2}} \times \frac{v^{\alpha_1-1}(1-v)^{\alpha_2-1}}{B(\alpha_1, \alpha_2)},$$

where, $B(\alpha_1, \alpha_2) = \int_0^1 v^{\alpha_1-1}(1-v)^{\alpha_2-1} dv = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1+\alpha_2)}$ is the Beta function.

We observe that

a. U, V are independent.

b. $U \sim \Gamma(\alpha_1 + \alpha_2, \beta)$.

c. $V \sim \text{Beta}(\alpha_1, \alpha_2)$.

Example 3

Suppose X_1, X_2, X_3 be independent random variables that follow $\Gamma(\alpha_i, 1), i = 1, 2, 3$ distribution. Let

$$\begin{aligned} Y_1 &= \frac{X_1}{X_1 + X_2 + X_3} \\ Y_2 &= \frac{X_2}{X_1 + X_2 + X_3} \\ Y_3 &= X_1 + X_2 + X_3 \end{aligned}$$

denote 3 new random variables. Show that the joint pdf of Y_1, Y_2, Y_3 is given by

$$f(y_1, y_2, y_3) = \frac{\Gamma(\alpha_1 + \alpha_2 + \alpha_3)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)} y_1^{\alpha_1 - 1} y_2^{\alpha_2 - 1} (1 - y_1 - y_2)^{\alpha_3 - 1}.$$

(Random variables that have a joint pdf of this form follow the Dirichlet distribution.)