QUIZ 5 SOLUTIONS

EXERCISE 1:

$$f(u,v) = f(u)\cdot f(v) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} \frac{v^{\frac{n}{2}-1} e^{-v/2}}{\Gamma(\frac{n}{2})\, 2^{n/2}}$$

$$J = \begin{vmatrix} \frac{\partial t}{\partial u} & \frac{\partial t}{\partial v} \\ \frac{\partial w}{\partial u} & \frac{\partial w}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{\sqrt{\frac{v}{n}}} & -\frac{1}{2}u\left(\frac{v}{n}\right)^{-3/2} \\ 0 & 1 \end{vmatrix} = \frac{1}{\sqrt{\frac{w}{n}}}$$

JOINT PDF OF t AND w:

$$f(t,w) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2\frac{w}{n}} \frac{w^{\frac{n}{2}-1} e^{-w/2}}{\Gamma(\frac{n}{2})\, 2^{n/2}} \sqrt{\frac{w}{n}}$$

$$f(t,w) = \frac{1}{\sqrt{2\pi n}} \frac{1}{\Gamma(\frac{n}{2})} \frac{1}{2^{n/2}} w^{\frac{n+1}{2}-1} e^{-w\left(\frac{1}{2}\frac{t^2}{n}+\frac{1}{2}\right)}$$

THEREFORE

$$f(t) = \int_0^\infty f(t,w)\,dw = \frac{\Gamma(\frac{n+1}{2})\left(1+\frac{t^2}{n}\right)^{-\frac{n+1}{2}}}{\sqrt{\pi n}\,\Gamma(\frac{n}{2})} \int_0^\infty \frac{w^{\frac{n+1}{2}-1} e^{-w/\frac{2}{(1+\frac{t^2}{n})}}}{\Gamma(\frac{n+1}{2})\left(2/(1+\frac{t^2}{n})\right)^{\frac{n+1}{2}}}\,dw$$

$$= \frac{\Gamma(\frac{n+1}{2})\left(1+\frac{t^2}{n}\right)^{-\frac{n+1}{2}}}{\sqrt{\pi n}\,\Gamma(\frac{n}{2})}.$$

EXERCISE 2

$$M_{Q_i}(t) = (1-2t)^{-\frac{P_i}{2}} e^{\theta_i \frac{t}{1-2t}}$$

MOMENT GENERATING FUNCTION OF NON-CENTRAL χ^2 WITH DEGREES OF FREEDOM P_i AND NON CENTRALITY PARAMETER θ_i.

$$M_Y(t) = M_{Q_1}(t) \dots M_{Q_K}(t)$$

$$= (1-2t)^{-\frac{\Sigma P_i}{2}} e^{\Sigma \theta_i \frac{t}{1-2t}}$$

LET $\psi(t) = \ln M_Y(t) = -\frac{\Sigma P_i}{2} \ln(1-2t) + \frac{t\, \Sigma \theta_i}{1-2t}$

$\psi'(t)\big|_{t=0}$ GIVES THE MEAN $\to \Sigma P_i + \Sigma \theta_i$

$\psi''(t)\big|_{t=0}$ GIVES THE VARIANCE $\to 2(\Sigma P_i + 2\Sigma \theta_i)$

OR $EY = E(Q_1 + \dots + Q_K)$

$= EQ_1 + \dots + EQ_K$

AND $VAR(Y) = VAR(Q_1 + \dots + Q_K)$

$= VAR(Q_1) + \dots + VAR(Q_K)$

BECAUSE $Q_1, \dots, Q_K$ ARE INDEPENDENT.

THEN FIND EQ_i AND $VAR(Q_i)$ USING $M_{Q_i}(t)$.

EXERCISE 3

$X_1, \ldots, X_n \sim \exp(\lambda)$

$\sum X_i \sim \Gamma\left(n, \frac{1}{\lambda}\right)$

$$E \frac{1}{\bar{X}} = E \frac{n}{\sum X_i} = n\, E\left(\sum X_i\right)^{-1}$$

$$= n \frac{\Gamma(n-1)\left(\frac{1}{\lambda}\right)^{-1}}{\Gamma(n)} = \frac{\lambda\, n\, \Gamma(n-1)}{(n-1)\,\Gamma(n-1)}$$

$$= \frac{n}{n-1} \lambda$$

NOT UNBIASED.

$\hat{\lambda}^* = \frac{n-1}{n \bar{X}}$ IS UNBIASED.