QUESTION (9):

$E\bar{X} = \mu = \lambda$ AND $ES^2 = \sigma^2 = \lambda$

(FOR POISSON)

BUT $VAR(\bar{X}) = \frac{\sigma^2}{n} = \frac{\lambda}{n}$

AND THE CRAMÉR-RAO LOWER BOUND IS ALSO $\frac{\lambda}{n}$:

$$p(x) = \frac{\lambda^x e^{-\lambda}}{x!} \qquad \ln p(x) = x\ln\lambda - \lambda - \ln x!$$

$$\frac{\partial \ln p(x)}{\partial \lambda} = \frac{x}{\lambda} - 1$$

$$\frac{\partial^2 \ln p(x)}{\partial \lambda^2} = -\frac{x}{\lambda^2}$$

$$\frac{1}{-nE\left(-\frac{X}{\lambda^2}\right)} = \frac{\lambda}{n}$$

WE CONCLUDE THAT $\bar{X}$ IS AT LEAST AS GOOD AS S^2.

QUESTION (b) :

$$E\left(\alpha\bar{X} + (1-\alpha)cS\right)$$

$$= \alpha E\bar{X} + (1-\alpha) E cS$$

$$= \alpha\theta + (1-\alpha)\theta = \theta.$$ UNBIASED.

$$VAR\left(\alpha\bar{X} + (1-\alpha)cS\right)$$

$$= \alpha^2 VAR(\bar{X}) + (1-\alpha) VAR(cS)$$

$$= \alpha^2 \frac{\theta^2}{n} + (1-\alpha)^2\left[E(cS)^2 - (E cS)^2\right]$$

$$= \alpha^2 \frac{\theta^2}{n} + (1-\alpha)^2\left(c^2\theta^2 - \theta^2\right)$$

MINIMIZE W.R.T. α.

QUESTION (c):

$X_1, \dots, X_n$ iid $\Gamma(\alpha, \beta)$

$$\sum X_i \sim \Gamma(n\alpha, \beta)$$

$$E\left(\frac{1}{\bar{X}}\right) = n\,E\,\frac{1}{\sum X_i} = n\,E\left(\sum X_i\right)^{-1}$$

$$= n\,\Gamma^{-1}(n\alpha, \beta)$$

$$= n\,\frac{\Gamma(n\alpha-1)\,\beta^{-1}}{\Gamma(n\alpha)} = \frac{n\,\Gamma(n\alpha-1)}{(n\alpha-1)\,\Gamma(n\alpha-1)}\,\frac{1}{\beta}$$

$$= \frac{n}{\beta(n\alpha-1)}.$$

WE WANT $E\,c\,\frac{1}{\bar{X}} = \frac{1}{\beta}$

OR $c\,\frac{n}{\beta(n\alpha-1)} = \frac{1}{\beta} \rightarrow c = \frac{n\alpha-1}{n}.$

THEREFORE, UNBIASED ESTIMATOR OF $\frac{1}{\beta}$ IS $\frac{n\alpha-1}{n}\,\frac{1}{\bar{X}}$ OR $\frac{n\alpha-1}{\sum X_i}.$

QUESTION (d):

$X_1, \dots, X_n$ iid BERNOULLI (p) $\quad \boxed{\begin{array}{l} EX_i = p \\ VAR(X_i) = p(1-p) \end{array}}$

LET $\hat\theta = \left(\frac{\sum X_i}{n}\right)^2 = \frac{(\sum X_i)^2}{n^2}$

$$= \frac{1}{n^2}\left(X_1^2 + \dots + X_n^2 + 2X_1X_2 + \dots + 2X_{n-1}X_n\right)$$

$\boxed{EX_1X_2 = p^2}$

$$E\hat\theta = \frac{1}{n^2}\left\{ n(\sigma^2+\mu^2) + 2\binom{n}{2}\mu^2 \right\}$$

$$= \frac{1}{n^2}\left\{ n[p(1-p)+p^2] + n(n-1)p^2 \right\}$$

$$= \frac{1}{n}\left\{ p - p^2 + p^2 + np^2 - p^2 \right\} = \frac{1}{n}\left[np^2 - p^2 + p\right] \neq p^2$$

$$= \frac{p(1-p)}{n} + p^2$$

OR USE ...

$$E\left(\frac{\sum X_i}{n}\right)^2 = VAR\left(\frac{\sum X_i}{n}\right) + \left(E\,\frac{\sum X_i}{n}\right)^2$$

$$= \frac{np(1-p)}{n^2} + \left(\frac{np}{n}\right)^2 = \frac{p(1-p)}{n} + p^2 \neq p^2$$

QUESTION (e):

( ) $\hat{\theta}^{(i)} = \left(\frac{\sum X_r}{n-1}\right)^2$

$$E\,\hat{\theta}^{(i)} = \frac{1}{(n-1)^2}\left\{(n-1)\left[p(1-p)+p^2\right] + (n-1)(n-2)p^2\right\}$$

$$= \frac{1}{n-1}\left\{p(1-p)+p^2+(n-2)p^2\right\}$$

$$\hat{\theta}^* = n\hat{\theta} - \frac{n-1}{n}\sum_{i=1}^{n}\hat{\theta}^{(i)}$$

$$= n\frac{1}{n}\left\{np^2 - p^2 + p\right\} - \frac{n-1}{n}\,n\,\frac{1}{n-1}\left\{p(1-p)+p^2+(n-2)p^2\right\}$$

$$= np^2 - p^2 + p + p^2 - p - p^2 - np^2 + 2p^2 = p^2.$$

THE METHOD REMOVES ALL THE BIAS AND $\hat{\theta}^*$ IS NOW UNBIASED ESTIMATOR OF p^2.

QUESTION (f):

(a). LET $E\hat{\theta} = K(\theta)$

$$\int \cdots \int \hat{\theta} f(x_1) \cdots f(x_n)\, dx_1 \cdots dx_n = K(\theta)$$

TAKE DERIVATIVE WRT θ ON BOTH SIDES:

$$\int \cdots \int \hat{\theta} \left[\sum \frac{1}{f(x_i,\theta)} \frac{\partial f(x_i,\theta)}{\partial \theta} \right] f(x_1) \cdots f(x_n)\, dx_1 \cdots dx_n = K'(\theta)$$

$$\int \cdots \int \hat{\theta} \sum \frac{\partial \ln f(x_i,\theta)}{\partial \theta} f(x_1) \cdots f(x_n)\, dx_1 \cdots dx_n = K'(\theta)$$

LET $Q = \sum \frac{\partial \ln f(x_i,\theta)}{\partial \theta}$ WITH $EQ = 0$, $\mathrm{VAR}(Q) = nI(\theta)$

THEN $E\hat{\theta} Q = K'(\theta)$

$\mathrm{COR}^2(\hat{\theta}, Q) \le 1$

$$\frac{\mathrm{COV}^2(\hat{\theta}, Q)}{\mathrm{VAR}(\hat{\theta})\, \mathrm{VAR}(Q)} \le 1$$

$$\frac{\left(E\hat{\theta} Q - (E\hat{\theta})(EQ)\right)^2}{\mathrm{VAR}(\hat{\theta}) \cdot \mathrm{VAR}(Q)} \le 1 \rightarrow \mathrm{VAR}(\hat{\theta}) \ge \frac{K'(\theta)^2}{nI(\theta)}$$

IF $E\hat{\theta} = \theta$

THEN $\mathrm{VAR}(\hat{\theta}) \ge \frac{1}{nI(\theta)}$

QUESTION (g):

$X_i \sim \Gamma(\alpha, \beta)$

$$\hat{\beta} = \frac{\bar{X}}{\alpha}$$

$$E\hat{\beta} = E\frac{\bar{X}}{\alpha} = \frac{\alpha\beta}{\alpha} = \beta.$$

$$VAR(\hat{\beta}) = VAR\left(\frac{\bar{X}}{\alpha}\right) = \frac{\sigma^2}{n\alpha} = \frac{\alpha\beta^2}{n\alpha^2} = \frac{\beta^2}{n\alpha}.$$

NOW FIND $\frac{1}{nI(\theta)}$ (CRAMER-RAO LOWER BOUND).

$$f(x) = \frac{x^{\alpha-1} e^{-x/\beta}}{\Gamma(\alpha)\beta^{\alpha}} \rightarrow \ln f(x) = (\alpha-1)\ln x - \frac{x}{\beta} - \ln\Gamma(\alpha) - \alpha\ln\beta$$

$$\frac{\partial \ln f(x)}{\partial \beta} = \frac{x}{\beta^2} - \frac{\alpha}{\beta}$$

$$\frac{\partial^2 \ln f(x)}{\partial \beta^2} = -\frac{2x\beta}{\beta^4} + \frac{\alpha}{\beta^2}$$

$$\frac{1}{-nE\left(\frac{-2x\beta}{\beta^4} + \frac{\alpha}{\beta^2}\right)} = \frac{1}{-n\left(-\frac{2\alpha\beta\beta}{\beta^4} + \frac{\alpha}{\beta}\right)} = \frac{1}{-n\left(-\frac{\alpha}{\beta^2}\right)}$$

$$= \frac{\beta^2}{n\alpha}.$$

THEREFORE $\hat{\beta} = \frac{\bar{X}}{\alpha}$ IS EFFICIENT ESTIMATOR OF β.