HOMEWORK 1 SOLUTIONS

EXERCISE 1:

$$X \sim \Gamma(\alpha, \beta) \rightarrow f(x) = \frac{x^{\alpha-1} e^{-x/\beta}}{\Gamma(\alpha)\beta^{\alpha}} \qquad x > 0,\ \alpha > 0,\ \beta > 0$$

$$f(x) = \frac{1}{x} \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha} e^{-x/\beta}$$

$$f(x) = \frac{1}{x} \frac{1}{\Gamma(\alpha)\beta^{\alpha}} e^{\ln x^{\alpha}} e^{-x/\beta}$$

$$f(x) = \frac{1}{x} \frac{1}{\Gamma(\alpha)\beta^{\alpha}} e^{\alpha \ln x - \frac{1}{\beta} x}$$

$$f(x) = \frac{1}{x} \frac{1}{\Gamma(\alpha)\beta^{\alpha}} e^{\dots}$$

$$h(x) = \frac{1}{x}, \quad c(\theta) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}}$$

$$w_1(\theta) = \alpha, \quad t_1(x) = \ln x$$

$$w_2(\theta) = \frac{1}{\beta}, \quad t_2(x) = -x$$

EXERCISE 2:

$$F_Y(y) = P(Y \le y) = P(CX \le y) = P\left(X \le \frac{y}{c}\right)$$

$$F_Y(y) = F_X\left(\frac{y}{c}\right)$$

THEN $f_Y(y) = \frac{1}{c} f_X\left(\frac{y}{c}\right)$

$$f_Y(y) = \frac{1}{c} \frac{\left(\frac{y}{c}\right)^{\alpha-1} e^{-y/c\beta}}{\Gamma(\alpha)\beta^{\alpha}}$$

$$f_Y(y) = \frac{y^{\alpha-1} e^{-y/c\beta}}{\Gamma(\alpha)(c\beta)^{\alpha}} \quad \therefore \quad Y \sim \Gamma(\alpha, c\beta)$$

EXERCISE 3:

$$EX^k = \int_0^\infty \frac{x^{\alpha+k-1} e^{-x/\beta}}{\Gamma(\alpha)\beta^\alpha}\, dx = \frac{1}{\Gamma(\alpha)\beta^\alpha}\int_0^\infty x^{\alpha+k-1} e^{-x/\beta}\, dx$$

$$= \frac{\Gamma(\alpha+k)\,\beta^{\alpha+k}}{\Gamma(\alpha)\,\beta^\alpha}\int_0^\infty \frac{x^{\alpha+k-1} e^{-x/\beta}}{\Gamma(\alpha+k)\,\beta^{\alpha+k}}\, dx = \frac{\Gamma(\alpha+k)\,\beta^k}{\Gamma(\alpha)}$$

$$k=1 \rightarrow EX = \frac{\Gamma(\alpha+1)\,\beta}{\Gamma(\alpha)} = \frac{\alpha\,\Gamma(\alpha)\,\beta}{\Gamma(\alpha)} = \alpha\beta$$

$$k=2 \rightarrow EX^2 = \frac{\Gamma(\alpha+2)\,\beta^2}{\Gamma(\alpha)} = \frac{\alpha(\alpha+1)\,\Gamma(\alpha)\,\beta^2}{\Gamma(\alpha)}$$

$$EX^2 = \alpha(\alpha+1)\beta^2$$

$$\mathrm{var}(X) = EX^2 - (EX)^2 = \alpha(\alpha+1)\beta^2 - \alpha^2\beta^2$$

$$\mathrm{var}(X) = \alpha\beta^2.$$

EXERCISE 4:

$X \sim b(n,p) \qquad p(x) = \binom{n}{x} p^x (1-p)^{n-x}$

IN EXPONENTIAL FAMILY FORM:

$$p(x) = \binom{n}{x} (1-p)^n e^{x \ln \frac{p}{1-p}}$$

$h(x) = \binom{n}{x}, \quad c(\theta) = (1-p)^n, \quad w_1(\theta) = \ln \frac{p}{1-p}$

$t_1(x) = x$

FROM STATEMENT (a) OF THE THEOREM:

$$E\left[\frac{1}{p(1-p)} X\right] = -\left(-\frac{n}{1-p}\right) \rightarrow EX = np$$

FROM STATEMENT (b) OF THE THEOREM:

$$\frac{d^2 \ln (1-p)^n}{d p^2} = -\frac{n}{(1-p)^2}$$

$$\frac{d^2 \ln \frac{p}{1-p}}{d p^2} = \frac{-1+2p}{p^2(1-p)^2}$$

$$\mathrm{VAR}\left(\frac{1}{p(1-p)} X\right) = \frac{n}{(1-p)^2} + \frac{(1-2p)\, np}{p^2(1-p)^2}$$

$$\mathrm{VAR}(X) = np(1-p)$$