Statistics 100B   **Instructor: Nicolas Christou**

Distributions related to the normal distribution

Three important distributions:

- Chi-square (χ^2) distribution.

- t distribution.

- F distribution.

Before we discuss the χ^2, t, and F distributions here are few important things about the gamma (Γ) distribution. The gamma distribution is useful in modeling skewed distributions for variables that are not negative.

A random variable X is said to have a gamma distribution with parameters α, β if its probability density function is given by

$$f(x) = \frac{x^{\alpha-1}e^{-\frac{x}{\beta}}}{\beta^\alpha \Gamma(\alpha)}, \quad \alpha, \beta > 0, \quad x \geq 0.$$

$E(X) = \alpha\beta$ and $\sigma^2 = \alpha\beta^2$.

A brief note on the *gamma* function:
The quantity $\Gamma(\alpha)$ is known as the gamma function and it is equal to:
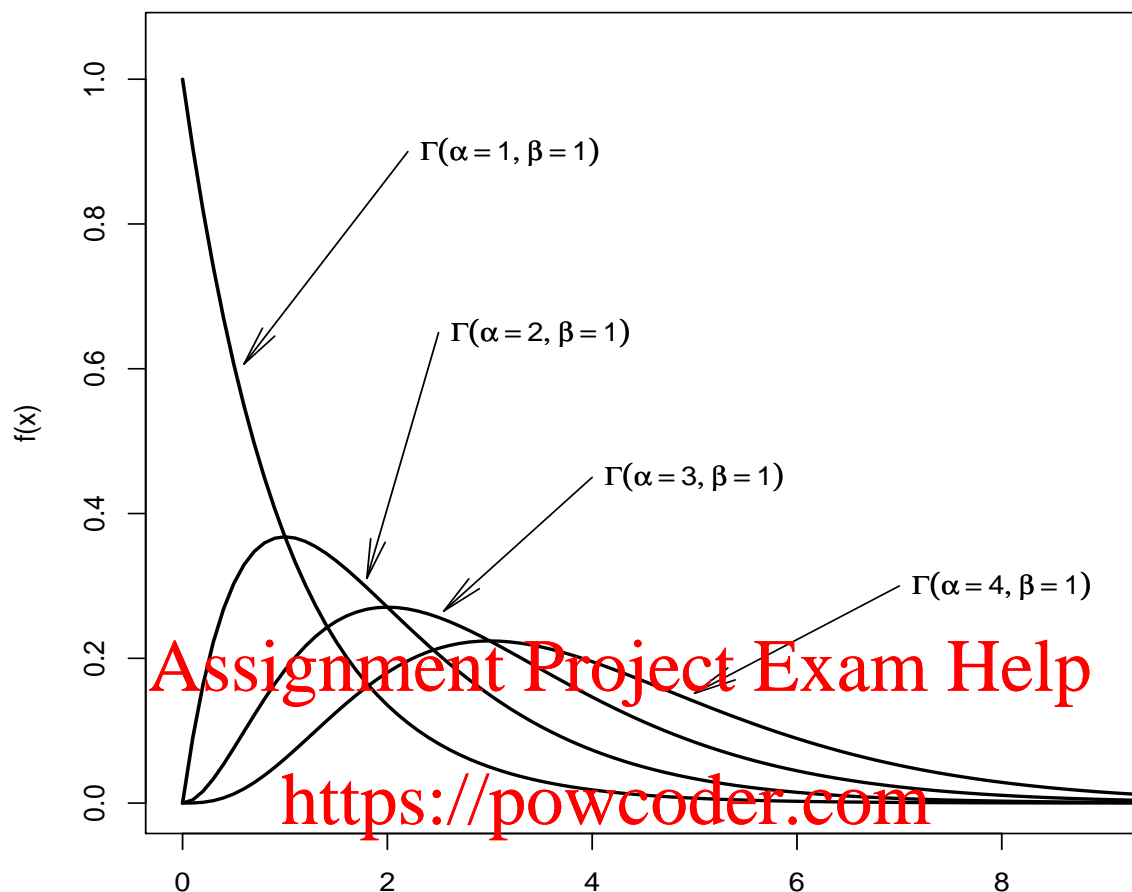
$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-t}dt.$$

Useful result:

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}.$$

If we set $\alpha = 1$ and $\beta = \frac{1}{\lambda}$ we get $f(x) = \lambda e^{-\lambda x}$. We see that the exponential distribution is a special case of the gamma distribution.

The gamma density for $\alpha = 1, 2, 3, 4$ and $\beta = 1$.

Gamma distribution density

Moment generating function of the $X \sim \Gamma(\alpha, \beta)$ random variable:

$$M_X(t) = (1 - \beta t)^{-\alpha}$$

Proof:

$$M_X(t) = Ee^{tX} = \int_0^\infty e^{tx} \frac{x^{\alpha-1} e^{-\frac{x}{\beta}}}{\beta^\alpha \Gamma(\alpha)} dx = \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty x^{\alpha-1} e^{-x(\frac{1-\beta t}{\beta})} dx$$

Let $y = x(\frac{1-\beta t}{\beta}) \Rightarrow x = \frac{\beta}{1-\beta t} y$, and $dx = \frac{\beta}{1-\beta t} dy$. Substitute these in the expression above:

$$M_X(t) = \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty \left(\frac{\beta}{1-\beta t} \right)^{\alpha-1} y^{\alpha-1} e^{-y} \frac{\beta}{1-\beta t} dy$$

$$M_X(t) = \frac{1}{\beta^\alpha \Gamma(\alpha)} \left(\frac{\beta}{1-\beta t} \right)^{\alpha-1} \frac{\beta}{1-\beta t} \int_0^\infty y^{\alpha-1} e^{-y} dy \Rightarrow M_X(t) = (1 - \beta t)^{-\alpha}.$$

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Theorem:
Let $Z \sim N(0,1)$. Then, if $X = Z^2$, we say that X follows the chi-square distribution with 1 degree of freedom. We write, $X \sim \chi_1^2$.

Probability density function of $X \sim \chi_1^2$:
Find the probability density function of $X = Z^2$, where $f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$. Begin with the cdf of X:

$$F_X(x) = P(X \le x) = P(Z^2 \le x) = P(-\sqrt{x} \le Z \le \sqrt{x}) \Rightarrow$$

$$F_X(x) = F_Z(\sqrt{x}) - F_Z(-\sqrt{x}). \quad \text{Therefore:}$$

$$f_X(x) = \frac{1}{2} x^{-\frac{1}{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x} + \frac{1}{2} x^{-\frac{1}{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x} = \frac{1}{2^{\frac{1}{2}}\sqrt{\pi}} x^{-\frac{1}{2}} e^{-\frac{x}{2}}, \quad \text{or}$$

$$f_X(x) = \frac{x^{-\frac{1}{2}} e^{-\frac{x}{2}}}{2^{\frac{1}{2}} \Gamma(\frac{1}{2})}.$$

This is the pdf of $\Gamma(\frac{1}{2}, 2)$, and it is called the chi-square distribution with 1 degree of freedom. We write, $X \sim \chi_1^2$.

The moment generating function of $X \sim \chi_1^2$ is $M_X(t) = (1 - 2t)^{-\frac{1}{2}}$.

Theorem:
Let $Z_1, Z_2, \ldots, Z_n$ be independent random variables with $Z_i \sim N(0,1)$. If $Y = \sum_{i=1}^{n} z_i^2$ then Y follows the chi-square distribution with n degrees of freedom. We write $Y \sim \chi_n^2$.

Proof:
Find the moment generating function of Y. Since $Z_1, Z_2, \ldots, Z_n$ are independent,

$$M_Y(t) = M_{Z_1^2}(t) \times M_{Z_2^2}(t) \times \ldots M_{Z_n^2}(t)$$

Each Z_i^2 follows χ_1^2 and therefore it has mgf equal to $(1 - 2t)^{-\frac{1}{2}}$. Conclusion:

$$M_Y(t) = (1 - 2t)^{-\frac{n}{2}}.$$

This is the mgf of $\Gamma(\frac{n}{2}, 2)$, and it is called the chi-square distribution with n degrees of freedom.

Theorem:
Let $X_1, X_2, \ldots, X_n$ independent random variables with $X_i \sim N(\mu, \sigma)$. It follows directly form the previous theorem that if

$$Y = \sum_{i=1}^{n} \left(\frac{x_i - \mu}{\sigma} \right)^2 \quad \text{then} \quad Y \sim \chi_n^2.$$

We know that the mean of $\Gamma(\alpha, \beta)$ is $E(X) = \alpha\beta$ and its variance $var(X) = \alpha\beta^2$. Therefore, if $X \sim \chi_n^2$ it follows that:

$$E(X) = n, \quad \text{and} \quad var(x) = 2n.$$
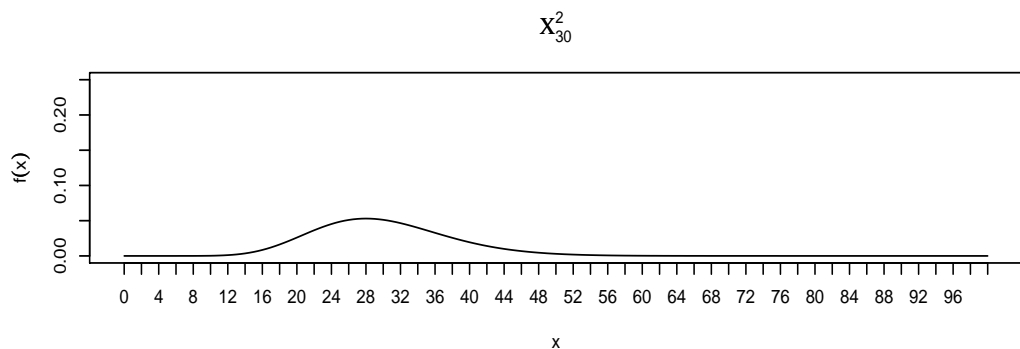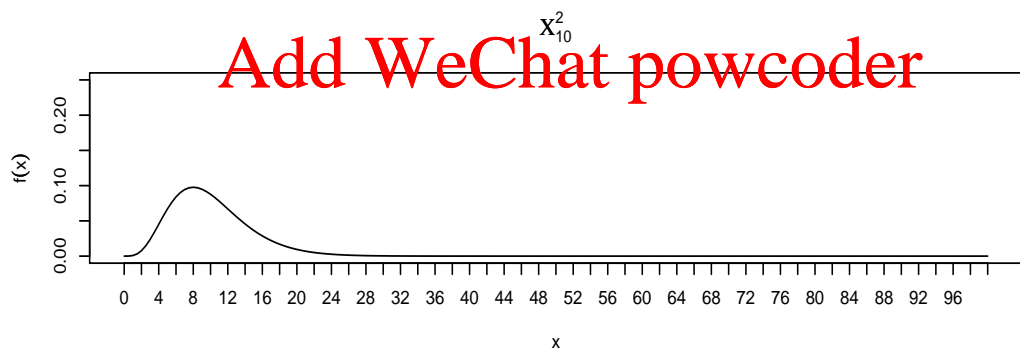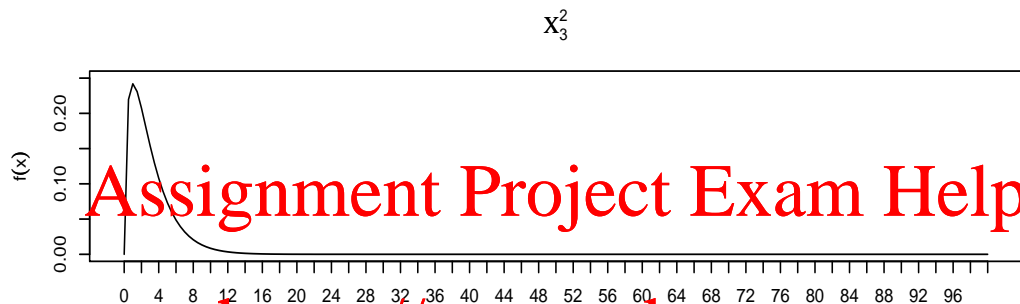
Theorem:
Let $X \sim \chi_n^2$ and $Y \sim \chi_m^2$. If X, Y are independent then

$$X + Y \sim \chi_{n+m}^2.$$

Proof: Use moment generating functions.

Shape of the chi-square distribution:
In general it is skewed to the right but as the degrees of freedom increase it becomes $N(n, \sqrt{2n})$. Here is the graph:

$$\mathbf{X}_3^2$$

$$\mathbf{X}_{10}^2$$

$$\mathbf{X}_{30}^2$$

The χ^2 distribution - examples

Example 1

If $X \sim \chi^2_{16}$, find the following:

    a. $P(X < 28.85)$.

    b. $P(X > 34.27)$.

    c. $P(23.54 < X < 28.85)$.

    d. If $P(X < b) = 0.10$, find b.

    e. If $P(X < c) = 0.950$, find c.

Example 2

If $X \sim \chi^2_{12}$, find constants a and b such that $P(a < X < b) = 0.90$ and $P(X < a) = 0.05$.

Example 3

If $X \sim \chi^2_{30}$, find the following:

    a. $P(13.79 < X < 16.79)$.

    b. Constants a and b such that $P(a < X < b) = 0.95$ and $P(X < a) = 0.025$.

    c. The mean and variance of X.

Example 4

If the moment-generating function of X is $M_X(t) = (1 - 2t)^{-60}$, find:

    a. $E(X)$.

    b. $Var(X)$.

    c. $P(83.85 < X < 163.64)$.

Theorem:

Let $X_1, X_2, \ldots, X_n$ independent random variables with $X_i \sim N(\mu, \sigma)$. Define the sample variance as

$$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2. \quad \text{Then} \quad \frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}.$$

Proof:

Example:

Let $X_1, X_2, \ldots, X_{16}$ i.i.d. random variables from $N(50, 10)$. Find

a. $P \left(796.2 < \sum_{i=1}^{n} (X_i - 50)^2 < 2630 \right).$

b. $P \left(726.1 < \sum_{i=1}^{n} (X_i - \bar{X})^2 < 2500 \right).$

Definition:
Let $Z \sim N(0,1)$ and $U \sim \chi^2_{df}$. If Z, U are independent then the ratio

$$\frac{Z}{\sqrt{\frac{U}{df}}} \quad \text{follows the } t \text{ (or Student's } t) \text{ distribution with degrees of freedom equal to } df.$$

We write $X \sim t_{df}$.

The probability density function of the t distribution with $df = n$ degrees of freedom is

$$f(x) = \frac{\Gamma(\frac{n+1}{2})}{\sqrt{\pi n}\Gamma(\frac{n}{2})} \left(1 + \frac{x^2}{n}\right)^{-\frac{n+1}{2}}, \quad -\infty < x < \infty.$$

Let $X \sim t_n$. Then, $E(X) = 0$ and $var(X) = \frac{n}{n-2}$. The t distribution is similar to the standard normal distribution $N(0,1)$, but it has heavier tails. However as $n \to \infty$ the t distribution converges to $N(0,1)$ (see graph below).
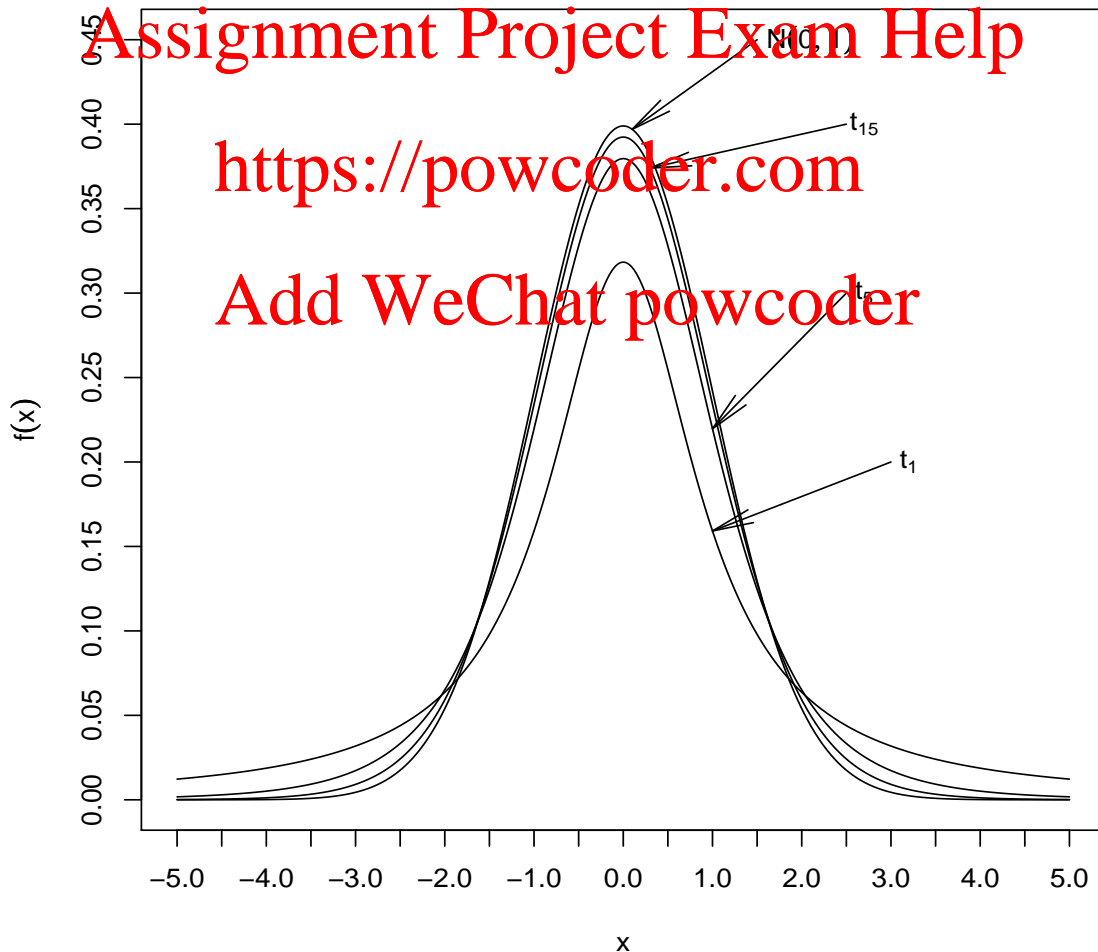
Application:

Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables each one having $N(\mu, \sigma)$. We have seen earlier that $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$. We also know that $\frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1)$. We can apply the definition of the t distribution (see previous page) to get the following:

$$\frac{\frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}}}{\sqrt{\frac{\frac{(n-1)S^2}{\sigma^2}}{n-1}}} = \frac{\bar{X}-\mu}{\frac{s}{\sqrt{n}}}.$$

Therefore $\frac{\bar{X}-\mu}{\frac{s}{\sqrt{n}}} \sim t_{n-1}$.

Compare it with $\frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1)$.

Example:

Let $\bar{X}$ and S^2_X denote the sample mean and sample variance of an independent random sample of size 10 from a normal distribution with mean $\mu = 0$ and variance σ^2. Find c so that

$$P\left(\frac{\bar{X}}{\sqrt{9S^2_X}} < c\right) = 0.95$$

The F distribution

Definition:
Let $U \sim \chi^2_{n_1}$ and $V \sim \chi^2_{n_2}$. If U and V are independent the ratio

$$\frac{\frac{U}{n_1}}{\frac{V}{n_2}}$$ follows the F distribution with numerator d.f. n_1 and denominator d.f. n_2.

We write $X \sim F_{n_1,n_2}$.

The probability density function of $X \sim F_{n_1,n_2}$ is:

$$f(x) = \frac{\Gamma\left(\frac{n_1+n_2}{2}\right)}{\Gamma(\frac{n_1}{2})\Gamma(\frac{n_2}{2})} \left(\frac{n_1}{n_2}\right)^{\frac{n_1}{2}} x^{\frac{n_1}{2}-1} \left(1 + \frac{n_1}{n_2}x\right)^{-\frac{1}{2}(n_1+n_2)}, \quad 0 < x < \infty.$$
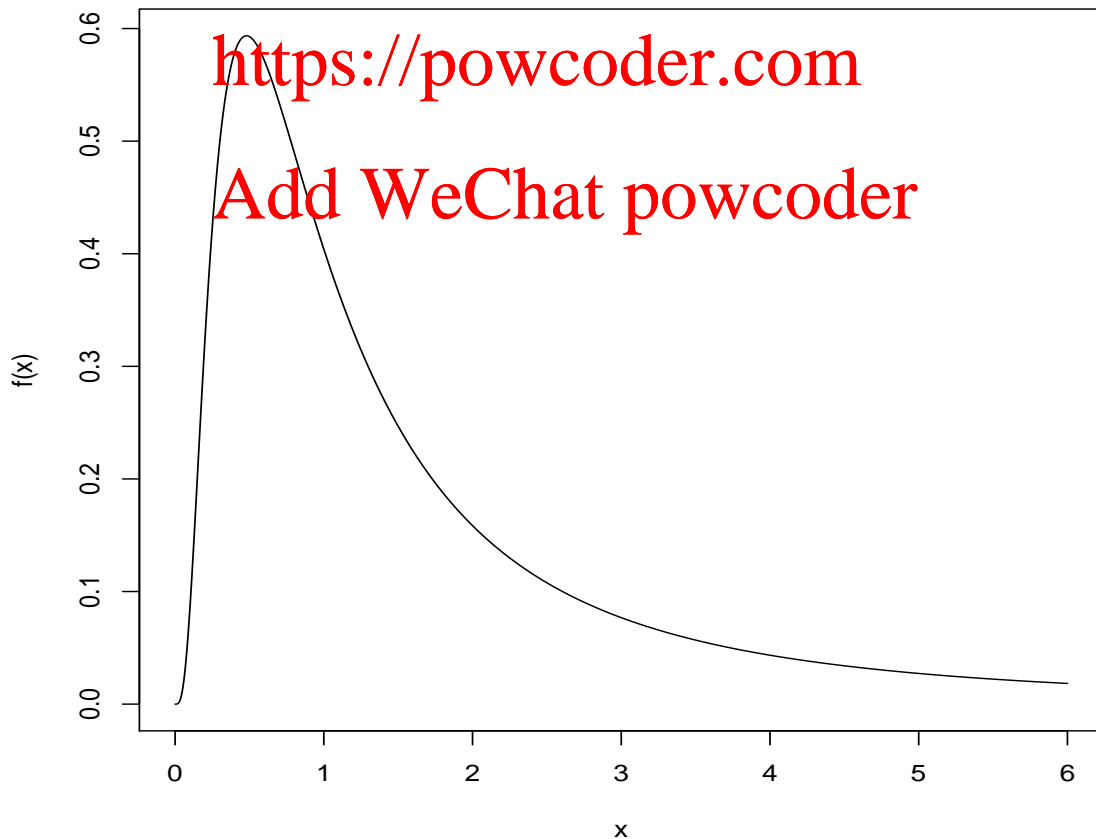
Mean and variance:
Let $X \sim F_{n_1,n_2}$. Then,

$$E(X) = \frac{n_2}{n_2 - 2}, \quad \text{and} \quad var(X) = \frac{2n_2^2(n_1 + n_2 - 2)}{n_1(n_2 - 2)^2(n_2 - 4)}.$$

Shape:
In general the F distribution is skewed to the right. The distribution of $F_{10,3}$ is shown below:

Application:

Let $X_1, X_2, \ldots, X_n$ i.i.d. random variables from $N(\mu_X, \sigma_X)$.

Let $Y_1, Y_2, \ldots, Y_m$ i.i.d. random variables from $N(\mu_Y, \sigma_Y)$.

If X and Y are independent the ratio

$$\frac{\frac{S_X^2}{\sigma_X^2}}{\frac{S_Y^2}{\sigma_Y^2}} \sim F_{n-1,m-1}.$$

Why?

Example:

Two independent samples of size $n_1 = 6, n_2 = 10$ are taken from two normal populations with equal variances. Find b such that $P(\frac{S_1^2}{S_2^2} < b) = 0.95$.

Distribution related to the normal distribution
χ^2, t, F - summary

1. The χ^2 distribution:

   Let $Z \sim N(0,1)$ then $Z^2 \sim \chi_1^2$.

   Let $Z_1, Z_2, \cdots, Z_n$ i.i.d. random variables from $N(0,1)$. Then $\sum_{i=1}^{n} Z_i^2 \sim \chi_n^2$.

   Let $X_1, X_2, \cdots, X_n$ i.i.d. random variables from $N(\mu, \sigma)$. Then $\sum_{i=1}^{n}(\frac{X_i - \mu}{\sigma})^2 \sim \chi_n^2$.

   The distribution of the sample variance:

   $$\frac{(n-1)S^2}{\sigma^2} \sim \chi_{n-1}^2, \quad \text{where} \quad S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2, \quad \bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

   Let $X \sim \chi_n^2$, $Y \sim \chi_m^2$. If X, Y are independent then $X + Y \sim \chi_{n+m}^2$.

2. The t distribution:

   Let $Z \sim N(0,1)$ and $U \sim \chi_n^2$.

   $$\frac{Z}{\sqrt{\frac{U}{n}}} \sim t_n.$$

   Let $X_1, X_2, \cdots, X_n$ i.i.d. random variables from $N(\mu, \sigma)$. Then

   $$\frac{\bar{x} - \mu}{\frac{s}{\sqrt{n}}} \sim t_{n-1}, \quad \text{where} \quad S = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2}, \quad \bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

3. The F distribution:

   Let $U \sim \chi_n^2$ and $V \sim \chi_m^2$. Then

   $$\frac{\frac{U}{n}}{\frac{V}{m}} \sim F_{n,m} \text{ with } n \text{ numerator d.f.}, m \text{ denominator d.f.}$$

   Let $X_1, X_2, \cdots, X_n$ i.i.d. random variables from $N(\mu_X, \sigma_X)$ and
   Let $Y_1, Y_2, \cdots, Y_m$ i.i.d. random variables from $N(\mu_Y, \sigma_Y)$ then:

   $$\frac{\frac{S_X^2}{\sigma_X^2}}{\frac{S_Y^2}{\sigma_Y^2}} \sim F_{n-1,m-1} \quad \text{where}$$

   $$S_X^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2, \quad \bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

   $$S_Y^2 = \frac{1}{m-1}\sum_{i=1}^{m}(y_i - \bar{y})^2, \quad \bar{y} = \frac{1}{n}\sum_{i=1}^{m} y_i$$

   Useful:

   $$t_n^2 = F_{1,n}$$

   and

   $$F_{\alpha;n,m} = \frac{1}{F_{1-\alpha;m,n}}$$

Let $Z_1, Z_2, \cdots, Z_{16}$ be a random sample of size 16 from the standard normal distribution $N(0,1)$.
Let $X_1, X_2, \cdots, X_{64}$ be a random sample of size 64 from the normal distribution $N(\mu, 1)$.
The two samples are independent.

    a. Find $P(Z_1 > 2)$.

    b. Find $P(\sum_{i=1}^{16} Z_i > 2)$.

    c. Find $P(\sum_{i=1}^{16} Z_i^2 > 6.91)$.

    d. Let S^2 be the sample variance of the first sample. Find c such that $P(S^2 > c) = 0.05$.

    e. What is the distribution of Y, where $Y = \sum_{i=1}^{16} Z_i^2 + \sum_{i=1}^{64}(X_i - \mu)^2$?

    f. Find EY.

    g. Find $Var(Y)$.

    h. Approximate $P(Y > 105)$.

    i. Find c such that
$$c\frac{\sum_{i=1}^{16} Z_i^2}{Y} \sim F_{16,80}.$$

    j. Let $Q \sim \chi_{60}^2$. Find c such that
$$P\left(\frac{Z_1}{\sqrt{Q}} < c\right) = 0.95.$$

    k. Use the t table to find the 80_{th} percentile of the $F_{1,30}$ distribution.

    l. Find c such that $P(F_{60,20} > c) = 0.99$.

Central limit theorem, χ^2, t, F distributions - examples

Example 1

Suppose $X_1, \cdots, X_n$ is a random sample from a normal population with mean μ_1 and standard deviation $\sigma = 1$. Another random sample $Y_1, \cdots, Y_m$ is selected from a normal population with mean μ_2 and standard deviation $\sigma = 1$. The two samples are independent.

    a. What is the distribution of W, where W is

$$\sum_{i=1}^{n}(X_i - \bar{X})^2 + \sum_{i=1}^{m}(Y_i - \bar{Y})^2$$

    b. What is the mean of W?

    c. What is the variance of W?

Example 2

Determine which columns in the F tables are squares of which columns in the t table. Clearly explain your answer.

Example 3

The sample $X_1, X_2, \cdots, X_{18}$ comes from a population which is normal $N(\mu_1, \sigma\sqrt{7})$. The sample $Y_1, Y_2, \cdots, Y_{23}$ comes from a population which is also normal $N(\mu_2, \sigma\sqrt{3})$. The two samples are independent. For these samples we compute the sample variances $S_X^2 = \frac{1}{17}\sum_{i=1}^{18}(X_i - \bar{X})^2$ and $S_Y^2 = \frac{1}{22}\sum_{i=1}^{23}(Y_i - \bar{Y})^2$.

For what value of c does the expression $c\frac{S_X^2}{S_Y^2}$ have the F distribution with $(17, 22)$ degrees of freedom?

Example 4

Supply responses *true* or *false* with an explanation to each of the following:

    a. The standard deviation of the sample mean $\bar{X}$ increases as the sample increases.

    b. The Central Limit Theorem allows us to claim, in certain cases, that the distribution of the sample mean $\bar{X}$ is normally distributed.

    c. The standard deviation of the sample mean $\bar{X}$ is usually approximately equal to the unknown population σ.

    d. The standard deviation of the total of a sample of n observations exceeds the standard deviation of the sample mean.

    e. If $X \sim N(8, \sigma)$ then $P(\bar{X} > 4)$ is less than $P(X > 4)$.

Example 5

A selective college would like to have an entering class of 1200 students. Because not all students who are offered admission accept, the college admits 1500 students. Past experience shows that 70% of the students admitted will accept. Assuming that students make their decisions independently, the number who accept X, follows the binomial distribution with $n = 1500$ and $p = 0.70$.

    a. Write an expression for the exact probability that at least 1000 students accept.

    b. Approximate the above probability using the normal distribution.

Example 6

An insurance company wants to audit health insurance claims in its very large database of transactions. In a quick attempt to assess the level of overstatement of this database, the insurance company selects at random 400 items from the database (each item represents a dollar amount). Suppose that the population mean overstatement of the entire database is $8, with population standard deviation $20.

    a. Find the probability that the sample mean of the 400 would be less than $6.50.

    b. The population from where the sample of 400 was selected does not follow the normal distribution. Why?

    c. Why can we use the normal distribution in obtaining an answer to part (a)?

    d. For what value of v can we say that $P(\mu - v < \bar{X} < \mu + v)$ is equal to 80%?

    e. Let T be the total overstatement for the 400 randomly selected items. Find the number b so that $P(T > b) = 0.975$.

Example 7

Next to the cash register of the *Southland market* is a small bowl containing pennies. Customers are invited to take pennies from this bowl to make their purchases easier. For example if a customer has a bill of $2.12 might take two pennies from the bowl. It frequently happens that customers put into the bowl pennies that they receive in change. Thus the number of pennies in the bowl rises and falls. Suppose that the bowl starts with $2.00 in pennies. Assume that the net daily changes is a random variable with mean $-\$0.06$ and standard deviation $0.15. Find the probability that, after 30 days, the value of the pennies in the bowl will be below $1.00.

Example 8

A telephone company has determined that during nonholidays the number of phone calls that pass through the main branch office each hour follows the normal distribution with mean $\mu = 80000$ and standard deviation $\sigma = 35000$. Suppose that a random sample of 60 nonholiday hours is selected and the sample mean $\bar{x}$ of the incoming phone calls is computed.

    a. Describe the distribution of $\bar{X}$.

    b. Find the probability that the sample mean $\bar{X}$ of the incoming phone calls for these 60 hours is larger than 91970.

    c. Is it more likely that the sample average $\bar{X}$ will be greater than 75000 hours, or that one hour's incoming calls will be?

Example 9

Assume that the daily $S\&P$ return follows the normal distribution with mean $\mu = 0.00032$ and standard deviation $\sigma = 0.00859$.

    a. Find the 75_{th} percentile of this distribution.

    b. What is the probability that in 2 of the following 5 days, the daily $S\&P$ return will be larger than 0.01?

    c. Consider the sample average $S\&P$ of a random sample of 20 days.

        i. What is the distribution of the sample mean?

        ii. What is the probability that the sample mean will be larger than 0.005?

        iii. Is it more likely that the sample average $S\&P$ will be greater than 0.007, or that one day's $S\&P$ return will be?

Example 10

Find the mean and variance of

$$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2,$$

where $X_1, X_2, \cdots, X_n$ is a random sample from $N(\mu, \sigma)$.

Example 11

Let X_1, X_2, X_3, X_4, X_5 be a random sample of size $n = 5$ from $N(0, \sigma)$.

    a. Find the constant c so that

$$\frac{c(X_1 - X_2)}{\sqrt{X_3^2 + X_4^2 + X_5^2}}$$

    has a t distribution.

    b. How many degrees of freedom are associated with this t distribution?

Example 12

Let $\bar{X}, \bar{Y}$, and $\bar{W}$ and S_X^2, S_Y^2, and S_W^2 denote the sample means and sample variances of three independent random samples, each of size 10, from a normal distribution with mean μ and variance σ^2. Find c so that

$$P\left(\frac{\bar{X} + \bar{Y} - 2\bar{W}}{\sqrt{9S_X^2 + 9S_Y^2 + 9S_W^2}} < c \right) = 0.95$$

Example 13

If X has an exponential distribution with a mean of λ, show that $Y = \frac{2X}{\lambda}$ has χ^2 distribution with 2 degrees of freedom.

Example 14

Suppose that $X_1, X_2, \cdots, X_{40}$ denotes a random sample of measurements on the proportion of impurities in iron ore samples. Let each X_i have a probability density function given by $f(x) = 3x^2, 0 \le x \le 1$. The ore is to be rejected by the potential buyer if $\bar{X}$ exceeds 0.7. Find $P(\bar{X} > 0.7)$ for the sample of size 40.

Example 15

If Y has a χ^2 distribution with n degrees of freedom, then Y could be represented by

$$Y = \sum_{i=1}^{n} X_i$$

where $X_i's$ are independent, each having a χ^2 distribution with 1 degree of freedom.

a. Show that $Z = \frac{Y-n}{\sqrt{2n}}$ has an asymptotic standard normal distribution.

b. A machine in a heavy-equipment factory produces steel rods of length Y, where Y is a normal random variable with $\mu = 6$ inches and $\sigma^2 = 0.2$. The cost C of repairing a rod that is not exactly 6 inches in length is proportional to the square of the error and is given, in dollars, by $C = 4(Y - \mu)^2$. If 50 rods with independent lengths are produced in a given day, approximate that the total cost for repairs for that day exceeds \$48.

Example 16

Suppose that five random variables $X_1, \cdots, X_5$ are i.i.d., and each has a standard normal distribution. Determine a constant c such that the random variable

$$\frac{c(X_1 + X_2)}{\sqrt{X_3^2 + X_4^2 + X_5^2}}$$

will have a t distribution.

Example 17

Suppose that a random variable X has an F distribution with 3 and 8 degrees of freedom. Determine the value of c such that $P(X < c) = 0.05$.

Example 18

Suppose that a random variable X has an F distribution with 1 and 8 degrees of freedom. Use the table of the t distribution to determine the value of c such that $P(X > c) = 0.2$.

Example 19

Suppose that a point (X, Y, Z) is to be chosen at random in 3-dimensional space, where X, Y, and Z are independent random variables and each has a standard normal distribution. What is the probability that the distance from the origin to the point will be less than 1 unit?

Example 20

Suppose that $X_1, \cdots, X_6$ form a random sample from a standard normal distribution and let

$$Y = (X_1 + X_2 + X_3)^2 + (X_4 + X_5 + X_6)^2.$$

Determine a value of c such that the random variable cY will have a χ^2 distribution.