HOMEWORK 2 SOLUTIONS

Exercise 1

$$\begin{pmatrix} X_1 - \bar{X} \\ X_2 - \bar{X} \\ \vdots \\ X_n - \bar{X} \end{pmatrix} = \underset{\sim}{X} - \underset{\sim}{1}\bar{X}$$

WHERE

$$\underset{\sim}{1} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

WRITE $\bar{X} = \frac{1}{n}\underset{\sim}{1}'\underset{\sim}{X}$

THEN $\underset{\sim}{X} - \underset{\sim}{1}\bar{X} = \underset{\sim}{X} - \frac{1}{n}\underset{\sim}{1}\underset{\sim}{1}'\underset{\sim}{X}$

$$EAX = AEX = \mu\left(I - \frac{1}{n}\underset{\sim}{1}\underset{\sim}{1}'\right)$$

$$\text{VAR}(AX) = \sigma^2 AA' = \sigma^2\left(I - \frac{1}{n}\right.$$

$$= \sigma^2\left(I - \frac{1}{n}\underset{\sim}{1}\underset{\sim}{1}\right)$$

TY
VA
COV

Exercise 2

(a) $X \sim U(0,1)$ $\quad EX = \frac{1}{2}$, $\quad \mathrm{var}(X) = \frac{1}{12}$.

$$EX^2 = \sigma^2 + \mu^2 = \frac{1}{12} + \frac{1}{4} = \frac{1}{3}.$$

$$\therefore E\begin{pmatrix} X \\ X^2 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/3 \end{pmatrix}.$$

$$\mathrm{var}(X) = \frac{1}{12}$$

$$\mathrm{var}(X^2) = EX^4 - (EX^2)^2$$

$$= \int_0^1 x^4 dx - \left(\frac{1}{3}\right)^2 = \frac{1}{5} - \frac{1}{9} = \frac{4}{45}.$$

$$\mathrm{Cov}(X, X^2) = EX^3 - (EX)(EX^2) = \int_0^1 x^3 dx - \frac{1}{2}\cdot\frac{1}{3}$$

$$= \frac{1}{4} - \frac{1}{6} = \frac{1}{12}.$$

$$\therefore \mathrm{var}\begin{pmatrix} X \\ X^2 \end{pmatrix} = \begin{pmatrix} \frac{1}{12} & \frac{1}{12} \\ \frac{1}{12} & \frac{4}{45} \end{pmatrix}$$

PART (b): $X_1 \sim \Gamma(\alpha, 1)$, $X_2 \sim \Gamma(\alpha+\frac{1}{2}, 1)$

$Y = 2\sqrt{X_1 X_2}$

IF $X \sim \Gamma(\alpha, \beta)$

THEN $EX^K = \boxed{\frac{\Gamma(\alpha+K)\beta^K}{\Gamma(\alpha)}}$

$EY = 2E(\sqrt{X_1 X_2})$

$= 2 E X_1^{1/2} E X_2^{1/2}$

$= 2 \frac{\Gamma(\alpha+\frac{1}{2})}{\Gamma(\alpha)} 1^{1/2} \cdot \frac{\Gamma(\alpha+\frac{1}{2}+\frac{1}{2})}{\Gamma(\alpha+\frac{1}{2})} 1^{1/2} = \frac{2\Gamma(\alpha+1)}{\Gamma(\alpha)} = 2\alpha$

$VAR(Y) = VAR(2\sqrt{X_1 X_2})$

$= 4\, VAR\sqrt{X_1 X_2}$

$= 4\left(E(X_1 X_2) - \left(E\sqrt{X_1 X_2}\right)^2\right)$

$= 4(EX_1 EX_2 - - - \cdot$

Exercise 3 (a)

a. Let $\mathbf{X} = (X_1, \ldots, X_n)'$ be a random vector with joint moment generating function $M_{\mathbf{X}}(\mathbf{t})$. In class we discuss this theorem: Let $M_i(\mathbf{t}) = \frac{\partial M_{\mathbf{X}}(\mathbf{t})}{\partial t_i}$, $M_{ii}(\mathbf{t}) = \frac{\partial^2 M_{\mathbf{X}}(\mathbf{t})}{\partial t_i^2}$, and $M_{ij}(\mathbf{t}) = \frac{\partial^2 M_{\mathbf{X}}(\mathbf{t})}{\partial t_i \partial t_j}$. Then, $EX_i = M_i(\mathbf{0})$, $EX_i^2 = M_{ii}(\mathbf{0})$, and $EX_iX_j = M_{ij}(\mathbf{0})$. Prove this theorem when $n = 2$.

$$M_{\underline{X}}(\underline{t}) = E\, e^{t_1X_1+t_2X_2} = \iint e^{t_1x_1+t_2x_2} f(x_1,x_2)\,dx_1\,dx_2$$

$$M_1(\underline{t}) = \frac{\partial M_X(\underline{t})}{\partial t_1}$$

$$= E\, X_1 \left(e^{t_1X_1+t_2X_2}\right) = EX_1 \quad (\text{when } t_1=0,\ t_2=0)$$

$$M_{11}(\underline{t}) = \frac{\partial^2 M_X(\underline{t})}{\partial t_1^2} = E\, X_1^2 \left(e^{t_1X_1+t_2X_2}\right) = X_1^2 \quad (\text{when } t_1=0,\ t_2=0)$$

$$M_{12}(\underline{t}) = \frac{\partial^2 M_X(\underline{t})}{\partial t_1 \partial t_2} = E\, X_1X_2 \left(e^{t_1X_1+t_2X_2}\right) = E\, X_1X_2$$

$(\text{when } t_1=0,\ t_2=0)$

Exercise 3 (b)

$$f(u) = \frac{u^{\alpha-1} e^{-u/\beta}}{\Gamma(\alpha)\,\beta^{\alpha}}$$

a. Suppose $U \sim \Gamma(\alpha, \beta)$, with $\alpha > 0, \beta > 0$ and let $Y = e^U$. Find the probability density function of Y.

$$F_Y(y) = P(Y \le y) = P(e^U \le y) = P(U \le \ln y) = F_U(\ln y)$$

$$f(y) = \frac{1}{y} f_U(\ln y) = \frac{1}{y} \frac{(\ln y)^{\alpha-1} e^{-\ln y/\beta}}{\Gamma(\alpha)\,\beta^{\alpha}}$$

$$= \frac{(\ln y)^{\alpha-1}\, y^{1/\beta - 1}}{\Gamma(\alpha)\,\beta^{\alpha}}$$

b. Refer to part (a). Find EY and $var(Y)$.

$$M_U(t) = E e^{tU} = (1-\beta t)^{-\alpha}$$

$$t=1 \rightarrow E e^{U} = EY = (1-\beta)^{-\alpha}$$

$$t=2 \rightarrow E e^{2U} = EY^2 = (1-2\beta)^{-\alpha}$$

$$var(Y) = (1-2\beta)^{-\alpha} - (1-\beta)^{-2\alpha}$$

EXERCISE 4 :

$$M_{X,Y}(t_1, t_2) = e^{8t_1 + 3t_2 + \frac{1}{2}\left(5t_1^2 + 2t_1t_2 + 2t_2^2\right)}$$

(a). FROM CLASS NOTES USING THE COROLLARY:

$$\psi(\underline{t}) = \ln M_{X,Y}(t_1, t_2)$$

$$= 8t_1 + 3t_2 + \frac{1}{2}\left(5t_1^2 + 2t_1t_2 + 2t_2^2\right)$$

TAKE THE DERIVATIVE OF $\psi(\underline{t})$ W.R.T. t_1

TO GET $\psi_1(\underline{t}) = 8 + 5t_1 + t_2 \Big|_{\substack{t_1=0 \\ t_2=0}}$

$\hookrightarrow EX = 8$

SIMILARLY $\psi_2(\underline{t}) = 3 + t_1 + 2t_2 \Big|_{\substack{t_1=0 \\ t_2=0}}$

$\hookrightarrow EY = 3$

$$\therefore \quad E\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{pmatrix} 8 \\ 3 \end{pmatrix}$$

(b). USE THE COROLLARY AGAIN.

$$\psi_{11}(\underset{\sim}{t}) = 5 = \sigma_x^2$$

$$\psi_{22}(\underset{\sim}{t}) = 4 = \sigma_y^2$$

$$\left.\begin{aligned}\psi_{12}(\underset{\sim}{t}) &= 2\\ \psi_{21}(\underset{\sim}{t}) &= 2\end{aligned}\right\} = \sigma_{xy}$$

$$\therefore \ \mathrm{var}\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \sigma_x^2 & \sigma_{xy} \\ \sigma_{yx} & \sigma_y^2 \end{pmatrix} = \begin{pmatrix} 5 & 2 \\ 2 & 4 \end{pmatrix}.$$

(c). $$\rho = \frac{\sigma_{xy}}{\sigma_x \sigma_y} = \frac{2}{\sqrt{5}\sqrt{4}} = \frac{1}{\sqrt{5}}.$$

(d) $$M_X(t_1) = e^{8t_1 + \frac{1}{2}5t_1^2}$$

$$M_Y(t) = e^{3t_2 + 2t_2^2}$$

$$M_{X,Y}(t_1, t_2) \neq M_X(t_1) \cdot M_Y(t_2)$$

NOT INDEPENDENT.

EXERCISE 5

(a). $X \sim \Gamma\left(\frac{1}{2}, 2\right) \rightarrow f(x) = \dfrac{x^{-\frac{1}{2}} e^{-x/2}}{\Gamma\left(\frac{1}{2}\right) 2^{1/2}}$

$Y = X^{1/2}$ USE THE METHOD OF CDF:

$F_Y(y) = P(Y \le y) = P(X^{1/4} \le y) = P(X \le y^4)$

SO FAR $F_Y(y) = F_X(y^4)$

$\therefore f_Y(y) = 4y^3 f_X(y^4)$

OR $f_Y(y) = \dfrac{4y^3 (y^4)^{-1/2} e^{-y^4/2}}{\Gamma\left(\frac{1}{2}\right) 2^{1/2}} = \dfrac{4y e^{-y^4/2}}{\sqrt{2\pi}}$

(b). $X \sim \exp(\lambda) \rightarrow M_X(t) = \left(1 - \frac{t}{\lambda}\right)^{-1}$

$M_{\Sigma X_i}(t) = M_{X_1}(t) \cdot M_{X_2}(t) \cdot \ldots \cdot M_{X_n}(t)$

$= \left(M_{X_i}(t)\right)^n = \left(1 - \frac{t}{\lambda}\right)^{-n}$

$\therefore \Sigma X_i \sim \Gamma\left(n, \frac{1}{\lambda}\right)$ UNIQUENESS THEOREM.

(c)

$$f(y) = \theta y^{\theta-1} \quad 0<y<1, \quad \theta>0$$

$$W = -\ln(Y) \qquad F_W(w) = P(W \le w)$$

$$= P(-\ln Y \le w) = P(\ln Y \ge -w)$$

$$= P(Y \ge e^{-w}) = 1 - P(Y \le e^{-w}) = 1 - F_Y(e^{-w})$$

$$\therefore f(w) = e^{-w}\,\theta\, e^{-w(\theta-1)} = \theta e^{-\theta w}$$

$$\therefore W \sim \mathrm{Exp}(\theta)$$

(d)

$$M_{2\theta\sum W_i}(t) = M_{2\theta W_1}(t) \cdots M_{2\theta W_n}(t)$$

$$= \left(M_{W_i}(2\theta t)\right)^n = \left(1 - \frac{2\theta t}{\theta}\right)^{-n}$$

$$= (1 - 2t)^{-n}$$

NOTE: FROM (d) $M_{W_i}(t) = \left(1 - \frac{t}{\theta}\right)^{-1}$

$$\therefore 2\theta \sum W_i \sim \Gamma(n, 2).$$

(e)

$$E\left[\frac{n-1}{\sum W_i}\right] = E\left[\frac{2(n-1)\theta}{2\theta\sum W_i}\right]$$

$$= 2(n-1)\theta \, E\left(2\theta \sum w_i\right)^{-1}$$

WITH $2\theta \sum w_i \sim \Gamma(n, 2)$

$$= 2(n-1)\theta \, \frac{\Gamma(n-1)\, 2^{-1}}{\Gamma(n)}$$

$$= 2(n-1)\theta \, \frac{\Gamma(n-1)}{(n-1)\Gamma(n-1)} \, 2^{-1} = \theta .$$

NOTE: WE USED HERE THE PROPERTY OF $\Gamma(\alpha, \beta)$: $EX^k = \frac{\Gamma(\alpha+k)\beta^k}{\Gamma(\alpha)}$.