UCCD1133
Introduction to Computer Organisation and Architecture

Chapter 3
Basic Concept of Logic

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Chapter 3-1

Introduction to concept of logic

Outline

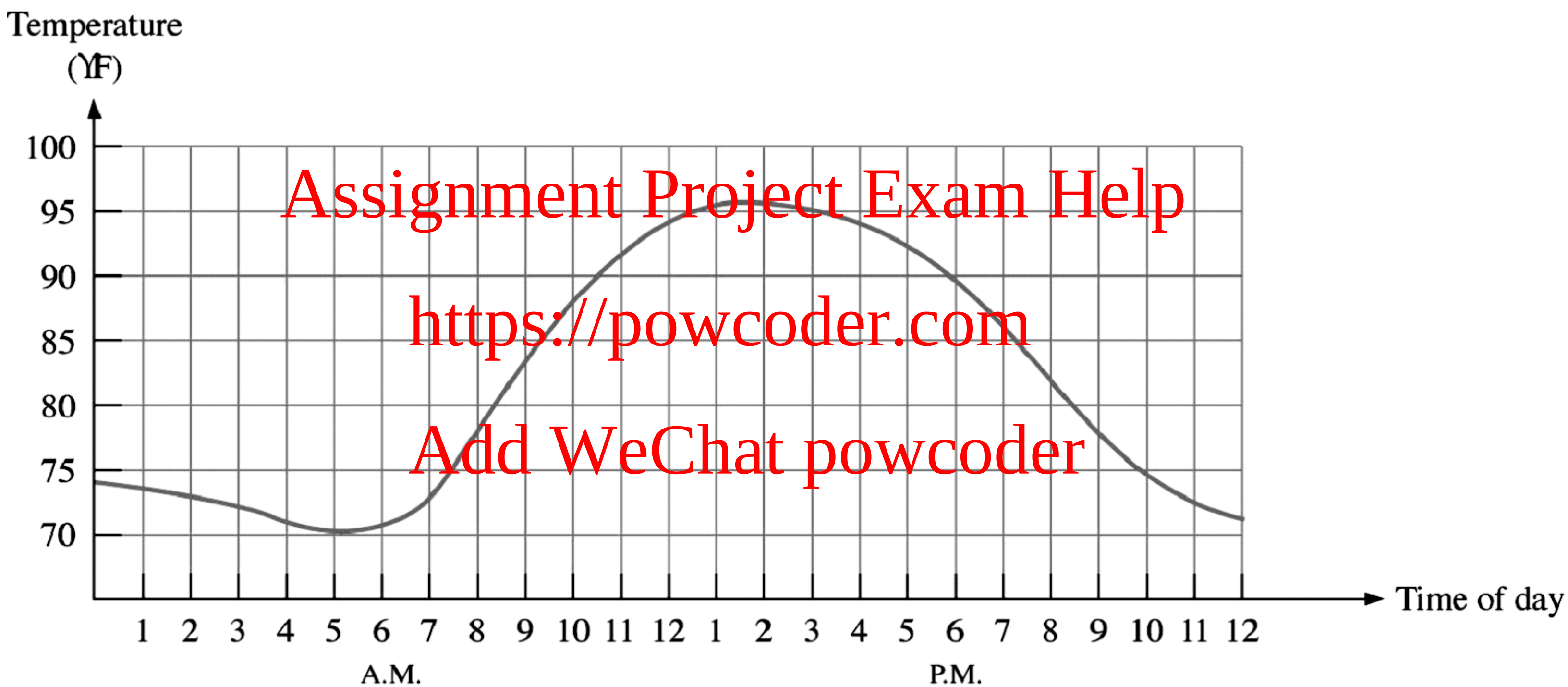
- Analogue and digital quantities
- Binary digits, signal voltage level and logic level
- Data transfer: serial and parallel

Digital versus Analogue

- Electronic circuits can be divided into two broad categories
  - *digital*
  - *Analogue*
- Most physical quantities are analogue in nature.
- However, the electronics inside a computer are *digital*.

Analogue and Digital Quantities

- An analogue quantity - having continuous values over time
- Example: position, velocity, acceleration, force, pressure, temperature and flow rate.

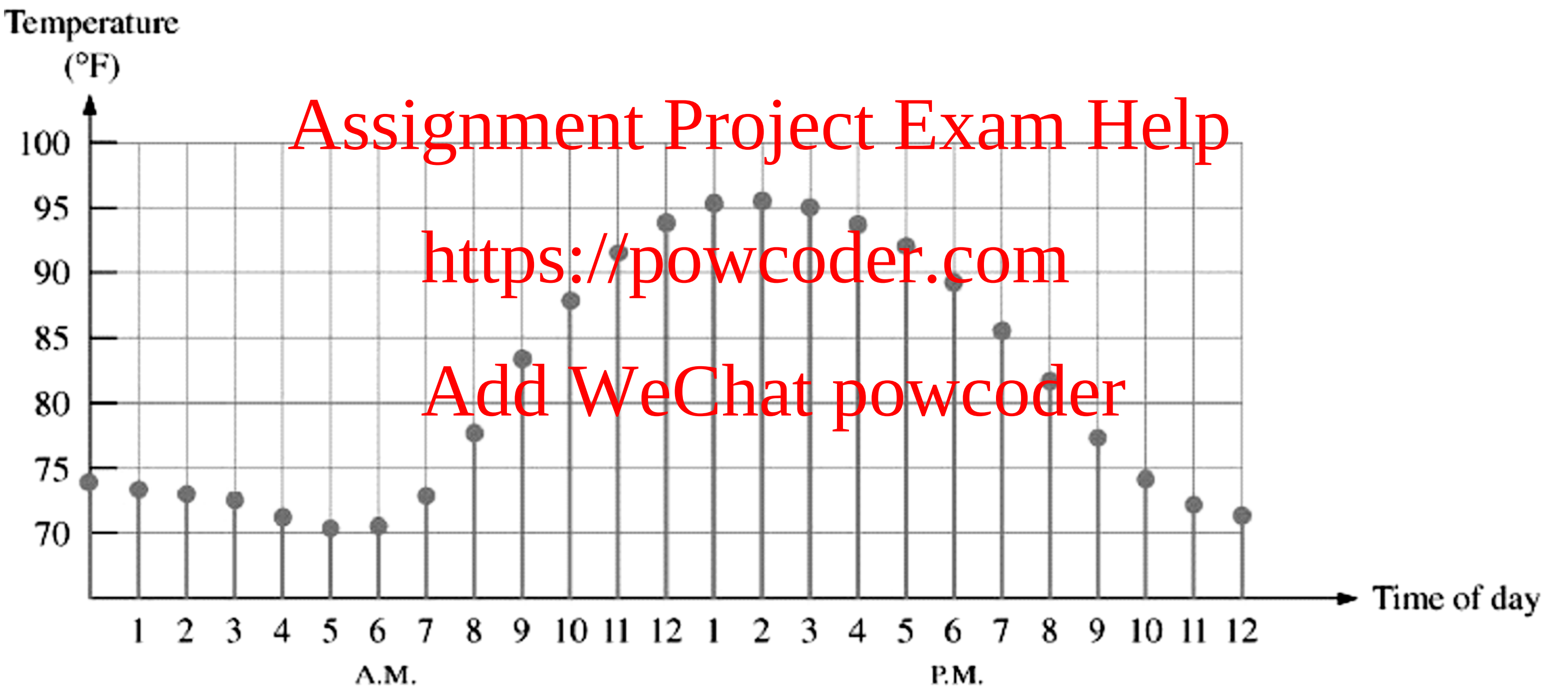


The air temperature values change over a continuous time (analogue quantity)

- No sudden jump in value, e.g. from 92 to 94 degrees.
- In between values are considered as infinite.

Analogue and Digital Quantities

- An digital quantity - having discrete set of values over time
- Example, the air temperature values are read at every hour – at discrete point in time.
  - Can assign a digital value or code to each dot to represent it - digitised.



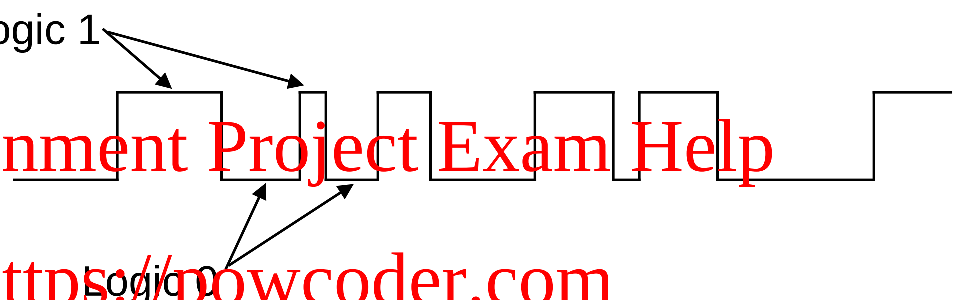
Sampled-value representation of the analogue quantity

- Digital representations of physical quantities are easier to be stored, transferred, and copied.

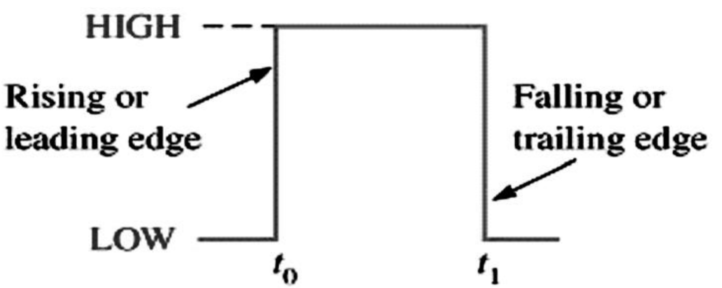
Binary Digits, Signal Voltage Level and Logic Level

- The fact that computers are *digital* is a key reason they use *binary* system.
- Binary system contains only two digits, 0 and 1
  - **Bits** comes from **B**inary dig**its** – 0 and 1.
- When apply in digital electronics, 0 and 1 correspond to logic 0 and logic 1

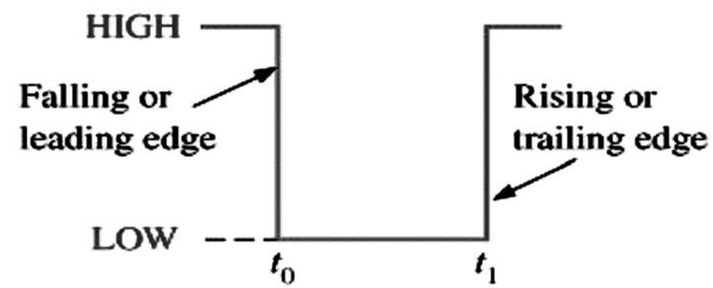
A digital signal/ waveform



- Voltage levels changing back and forth between the HIGH (H) and LOW (L) instantaneously



(a) The digital signal is normally LOW, then a positive-going pulse takes place.

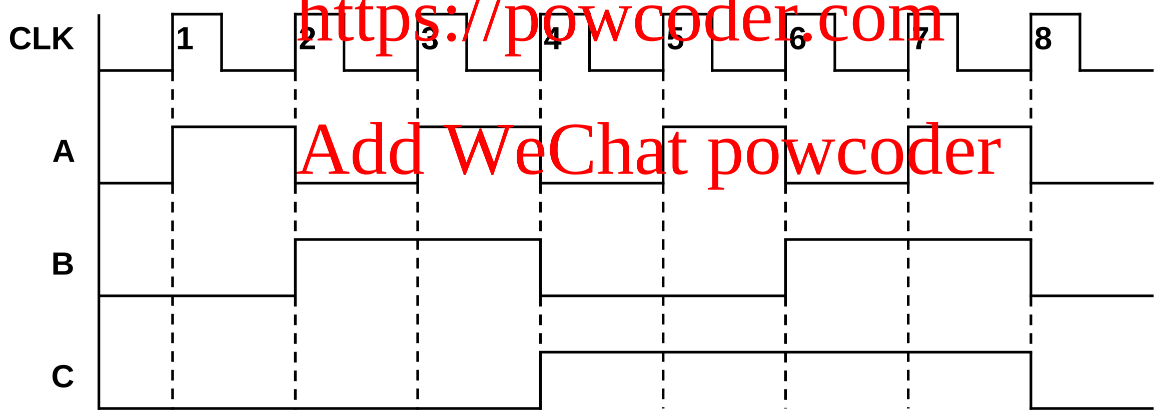


(b) The signal is normally HIGH, then a negative-going pulse takes place.

A Special Digital Signal: The Clock Signal

- Clock circuitry is commonly used in digital systems (analogous to a human heart) to generate periodic clock waveform.
  - To synchronize the generation of other waveforms
  - Thus, synchronizing the transfer activities within the digital system.
    - Creates synchronous digital system

- E.g. waveforms A, B and C are synchronized to the **clock**.
  - The transitions in waveforms A, B and C occur at the rising-edge of the clock



Data Transfer

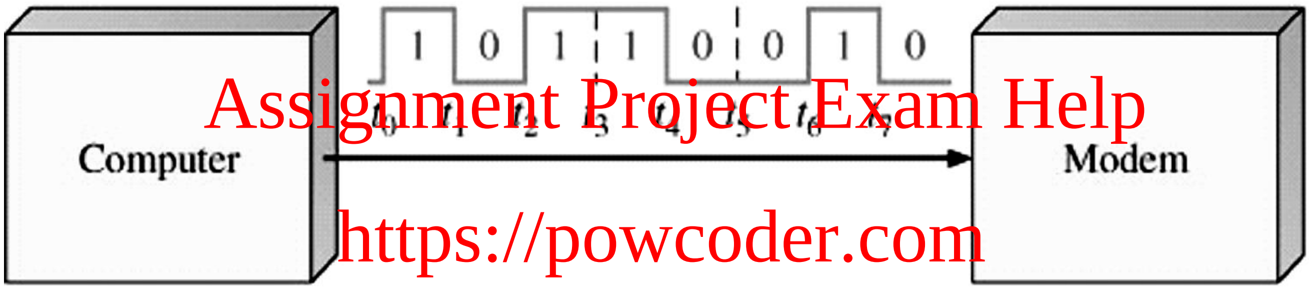
- Grouping of bits is commonly done to represent information in digital system
  - 4 bits => a **nibble**
  - 8 bits => a **byte**
  - 32 bits => a **word**
  - E.g. 2^8 can represent up to 256 information

- Single or group of bits are transferred as binary data from one circuit to another
  - Serial transfer
  - Parallel transfer

- Data transfer involves a transmitting circuit (source) and one or more receiving circuit (destination)

Serial Data Transfer

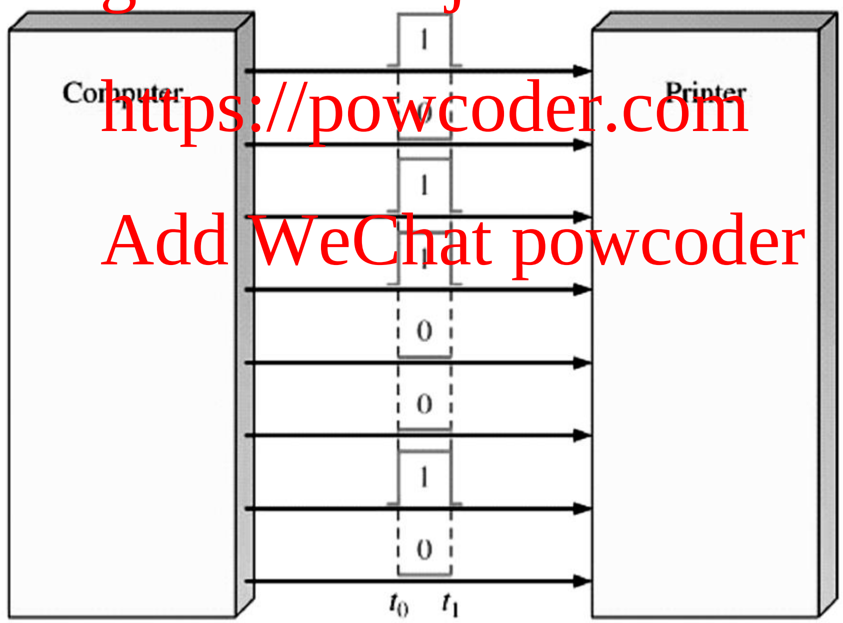
- Data is transferred one bit per clock cycle
  - Require only a single line
- Example, from one digital system (computer) to another (modem)
  - Total clock cycle required to transfer a byte of data: 8 cycles.



Serial transfer of 8 bits of binary data from computer to modem. Interval t_0 to t_1 is first.

Parallel Data Transfer

- Data is transferred as a group (e.g., a byte or word) per clock cycle
  - Require multiple lines
- Example, from the computer to a printer
  - 8 bits are transferred along 8 lines in a clock cycle
  - Total clock cycle required no matter how large the group: 1 cycle.



Chapter 3-2

Boolean algebra & truth table

Outline

- Laws and theorems of Boolean Algebra.
- Apply these laws and theorems to:
  - Simplify expressions
  - Convert any Boolean expression into a sum-of-product (SOP) form
  - Convert non-canonical form to canonical form
- Represent a Boolean expression by truth table.

Fundamentals of Boolean Algebra

- Boolean algebra is mathematics of logic.

- Unlike ordinary algebra; each **variable** represents a logical quantity which can take on only one of two values: 0 and 1.

- The **complement** is the inverse of a variable
  - E.g. the complement of U => U’ or $\bar{U}$

- Boolean algebra logical operators:
  - “ + ” => logical OR operator
  - “ . ” => logical AND operator
  - “ ’ ” or “ ¯ ” => logical NOT operator
  - E.g. on OR, AND and NOT computation

OR function with two variables = x + y	AND function with two variables = x.y	NOT function with a single variable = x'
0 + 0 = 0	0.0 = 0	0’ = 1
0 + 1 = 1	0.1 = 0	1’ = 0
1 + 0 = 1	1.0 = 0	
1 + 1 = 1	1.1 = 1	

- The operations of a Boolean algebra must adhere to certain laws and theorems

Laws of Boolean Algebra

- Law 1: Existence of 1 and 0 element
  - (a) $p + 0 = p$
  - (b) $p \cdot 1 = p$
- Law 2: Existence of complement
  - (a) $p + p' = 1$
  - (b) $p.p' = 0$
- Law 3: Commutativity
  - (a) $p + q = q + p$
  - (b) $p \cdot q = q \cdot p$

- Law 4: Associativity
  - (a) $p + (q + z) = (p + q) + z$
  - (b) $p \cdot (q \cdot z) = (p \cdot q) \cdot z$
- Law 5: Distributivity
  - (a) $p + (q \cdot z) = (p + q) \cdot (p + z)$
  - (b) $p \cdot (q + z) = p \cdot q + p \cdot z$

The laws of Boolean algebra can be used to further develop Theorems

Example

1. $a + b + c + 0 = a + b + c$ L1(a)
2. $a.b.c.1 = a.b.c$ L1(b)
3. $(W' + X' + Y' + Z')(W' + X' + Y' + Z)(W' + X' + Y + Z')(W' + X' + Y + Z)$
   $= ((W' + X' + Y') + Z' Z) ((W' + X' + Y) + Z' Z)$ L5(a)
   $= (W' + X' + Y') (W' + X' + Y)$ L2(b)
   $= (W' + X') + Y' Y$ L5(a)
   $= (W' + X')$ L2(b)

Theorems of Boolean Algebra

- **Theorem 1: Idempotency**
  - (a) x + x = x
  - (b) x.x = x

Example

1. x + x + x + x + x = x   T1(a)
2. x + a + a + x + x + a = x + a   T1(a)
3. x.x.x.x.x.x = x   T1(b)
4. x.a.x.x.a.a.a = x.a   T1(b)

- Let's prove T1(a)

x + x	= (x + x)1	[L1(b)]
	= (x + x)(x + x')	[L2(a)]
	= x + x.x'	[L5(a)]
	= x + 0	[L2(b)]
	= x	[L1(a)]

- **Theorem 2: Null element**
  - (a) x + 1 = 1
  - (b) x.0 = 0

Example

1. a + b + c + d + 1 = 1   T2(a)
2. a.b.c.d.0 = 0   T2(b)

- Let's prove T2(a)

x + 1	= (x + 1)1	[L1(b)]
	= 1(x + 1)	[L3(b)]
	= (x + x')(x + 1)	[L2(a)]
	= x + x' 1	[L5(a)]
	= x + x'	[L1(b)]
	= 1	[L2(a)]

- **Theorem 3: Involution**
  - (x')' = x

Theorems of Boolean Algebra

- **Theorem 4: Absorption**
  - (a) x + xy = x
  - (b) x(x + y) = x

Example

1. (X + Y) + (X + Y)Z = X + Y [T4(a)]
2. AB'(AB' + B'C) = AB' [T4(b)]

- Let's prove T4(a)

x + xy	= x1 + xy	[L1(b)]
	= x(1 + y)	[L5(b)]
	= x(y + 1)	[L3(b)]
	= x1	[T2(a)]
	= x	[L1(b)]

- **Theorem 5**
  - (a) x + x'y = x + y
  - (b) x(x' + y) = xy

Example

1. B + AB'C'D = B + AC'D [T5(a)]
2. (X + Y)((X + Y)' + Z) = (X + Y)Z[T5(b)]
3. wy' + wx'y + wxyz + wxz'
   = w(y' + x'y) + wx(yz + z')
   = w(y' + x') + wx(y + z')
   = wy' + wx' + wxz' + wxy
   ⋮
   = w

- Let's prove T5(a)

x + x'y	= (x + x')(x + y)	[L5(a)]
	= 1(x + y)	[L2(a)]
	= (x + y)1	[L3(b)]
	= (x + y)	[L1(b)]

Theorems of Boolean Algebra

- **Theorem 6: DeMorgan's Theorem**
  - To determine the complement of an expression
    - $(x + y)' = x'y'$
    - $(xy)' = x' + y'$
  - Generalized DeMorgan's Theorem
    - $(x + y + \dots z)' = x'y' \dots z'$
    - $(xy \dots z)' = x' + y' + \dots z'$

Example

$$
\begin{aligned}
(x + yz)' = (x + (yz))' &\neq ((x + y)z)' \\
&= x'(yz)' && [T6(a)] \\
&= x'(y' + z') && [T6(b)] \\
&= x'y' + x'z' && [L5(b)]
\end{aligned}
$$

- Useful in manipulating Boolean expressions into formats suitable for realization with specific types of logic gates
- Shortcut to apply DeMorgan's theorem is to invert the operators
  - E.g. $(x + yz)' = x'(y' + z')$.
  - Note $(x + yz)' \neq x'y' + z'$.

Theorems of Boolean Algebra

- **Theorem 7: Consensus**
  - (a) $xy + x'z + yz = xy + x'z$
  - (b) $(x + y)(x' + z)(y + z) = (x + y)(x' + z)$

- Let’s prove T7(a).

$xy + x'z + yz$	$= xy + x'z + 1yz$	[L1(b)]
	$= xy + x'z + (x + x')yz$	[L2(a)]
	$= xy + x'z + xyz + x'yz$	[L5(b)]
	$= (xy + xyz) + (x'z + x'yz)$	
	$= xy + x'z$	[T4(a)]

Example

1. $AB + A'CD + BCD = AB + A'CD$ [T7(a)]

2.

$ABC + A'D + B'D + CD$	$= ABC + (A' + B')D + CD$	[L5(b)]
	$= ABC + (AB)'D + CD$	[T6(b)]
	$= ABC + (AB)'D$	[T7(a)]
	$= ABC + (A' + B')D$	[T6(b)]
	$= ABC + A'D + B'D$	[L5(b)]

Duality

- The dual of an expression is found by replacing
  - All (+) operators with (.).
  - All (.) operators with (+).
  - All ones with zeros.
  - All zeros with ones.

- Example 1

  p + p' = 1 [L2(a)]

  p.p' = 0 [L2(b)]

  L2(a) is the dual of L2(b).

- Example 2

  Find the dual of the expression x + (yz) = (x + y)(x + z).

  Solution 2

  x + (yz) = (x + y)(x + z)

  x(y + z) = (xy) + (xz)

- Do not alter the location of parenthesis when obtaining a dual.

Algebraic Representation of a Logic Function

variable
compliment

literal (e.g. 6 literals and 3 variables)

$$\overbrace{\underbrace{f(A,B,C)}_{\text{Logic function of 3 variables}} \quad = \quad \underbrace{AB + A'C + AC'}_{\text{Logic expression}}}^{\text{Logic equation}}$$

- Algebra – used as a mathematical representation of a logic function
  - Basically relates the inputs to output
  - How?
    - If any one or more terms AB, A'C or AC' are asserted, the output f will be asserted.
  - Evaluating f:
    - Example: if A = 1, B = 0, C = 0, then

      $f(1,0,0) = 1.0 + 1'.0 + 1.0'$
      $= 0 + 0 + 1$
      $= 1$

- Can also write:

$$Y \quad = \quad AB + A'C + AC'$$

Variables and logic function can be treated as signals

- C' is not a signal name.
  - It is an expression since ' is an operator.

Canonical Forms of Logic Expression

- A logic function can be expressed in a variety of algebraic forms.

  For example,

  $$Y = ab' + ac = a(b' + c) = a(a' + c + b')$$

- In general, a logic expression can be represented in the form of:
  - Sum-of-products (SOP).
  - Product-of-sum (POS).

- To eliminate the possible confusion, logic designers must learn to specify a Boolean function using *canonical* or standardised form

  – everyone will come up with the same expression.

Sum-of-Product (SOP) Forms

- SOP form
  - Example

A special product term called minterm (it has all the variables)

f(A, B, C) = A'BC + AB + C

Product term but not a minterm

- Canonical SOP form
  - f contains minterms only
  - Example

5 minterms

f(A, B, C) = A'B'C + A'BC + AB'C + ABC' + ABC

- A *minterm* is a product involving all of the inputs to the function.

Converting Non-Canonical Form into Canonical Form

- Use of algebra method to convert non-canonical to canonical form

- Example 1

  Expand the following function to canonical SOP in minterm list form:

  $f(A,B,C) = AB + AC' + A'C$

  Solution 1

  $$\begin{aligned} f(A,B,C) &= AB + AC' + A'C \\ &= AB(C + C') + AC'(B + B') + A'C(B + B') \\ &= ABC + ABC' + AB'C' + A'BC + A'B'C \\ &= m7 + m6 + m4 + m3 + m1 \\ &= \sum m(1, 3, 4, 6, 7) \end{aligned}$$

- On the contrary,

  $f = ABC + ABC' + AB'C' + A'BC + A'B'C$

  can be simplified into

  $f = AB + AC' + A'C$

  using the previous theorems and laws.

Sum-of-Product (SOP) Forms

- Canonical form can also be written as *minterm list form*

  Example

  f(A, B, C) = A'B'C + A'BC + AB'C + ABC' + ABC

  Solution

  From the truth table,

  f(A, B, C) = m1 + m3 + m5 + m6 + m7

  = $\sum$ m(1, 3, 5, 6, 7)

Inputs A B C	Minterm	Minterm Full List	
0 0 0	A'B'C'	m0	
0 0 1	A'B'C	m1	✓
0 1 0	A'BC'	m2	
0 1 1	A'BC	m3	✓
1 0 0	AB'C'	m4	
1 0 1	AB'C	m5	✓
1 1 0	ABC'	m6	✓
1 1 1	ABC	m7	✓

- The table shows all possible minterms of f, but only m1, m3, m5, m6 and m7 makes up f.
  - Each minterm is used to **detect** a specific code pattern.
- Careful with the ordering of the variables in the functional notation, f(A, B, C)
  - Example, even with the same minterm list, the following functions are not the same
    - f1(A, B, C) = $\sum$ m(1, 3, 5, 6, 7) *(A is the MSB, C is the LSB)*
    - f2(B, C, A) = $\sum$ m(1, 3, 5, 6, 7) *(B is the MSB, A is the LSB)*

Table Representation of a Logic Function

- A truth table is another type of representation of a logic function
  - It relates the inputs to output
  - How?
    - List the evaluated logic function for all the possible input combinations
      - E.g. the truth table for f(A,B,C) = AB + C

Input	Output
A B C	f(A,B,C)
0 0 0	0
0 0 1	1
0 1 0	0
0 1 1	1
1 0 0	0
1 0 1	1
1 1 0	1
1 1 1	1

- How to get from

  f(A,B,C) = AB + C => truth table?

Solution

Convert non-canonical to canonical form:

f(A,B,C)

= AB + C

= AB(C + C') + (A'B' + A'B + AB' + AB)C

= ABC + ABC' + A'B'C + A'BC + AB'C + ABC

= ABC + ABC' + A'B'C + A'BC + AB'C

- Minterm info can be directly transfer to truth table

- From the truth table
  - f(A, B, C) = $\sum$ m(1, 3, 5, 6, 7)
- Let's list out f' as well:
  - f'(A, B, C) = $\sum$ m(0, 2, 4)

  Each minterm will either appear in f or f'.
  - Meaning, f + f' = 1
    - ORing all minterms will yield a 1.

Optimization of Logic Function using Algebra Method

- Logic optimisation
  - Reduce redundant product or sum terms and literals.

- Let's use Algebra method to reduce a logic expression.

- Example 1

Reduce the canonical SOP of logic function f shown in the table to a simpler version.

Input	Output
A B C	f(A,B,C)
0 0 0	0
0 0 1	1
0 1 0	0
0 1 1	1
1 0 0	0
1 0 1	1
1 1 0	1
1 1 1	1

Solution 1

f = ∑ m(1, 3, 5, 6, 7)

= A'B'C + A'BC + AB'C + ABC' + ABC

= (A'B' + A'B + AB' + AB)C + ABC'

= ((A' + A)(B' + B))C + ABC'

= C + ABC'

= ABC' + C

= AB + C

Optimization of Logic Function using Algebra Method

- To obtain the reduce version of f', can either:
  - Perform the optimization process on f'.
  - Apply DeMorgan's theorem on the reduced f
- The result may or may not be the same.

- Example 2

By optimization process on f,

$f' = \sum m(0, 2, 4)$

$= A'B'C' + A'BC' + AB'C'$

$= (B' + B)\,A'C' + AB'C'$

$= (A' + AB')\,C'$

$= (A' + B')\,C'$

$= A'C' + B'C'$

Inputs A B C	Output f(A, B, C)	Complemented Output f'(A,B,C)
0 0 0	0	1
0 0 1	1	0
0 1 0	0	1
0 1 1	1	0
1 0 0	0	1
1 0 1	1	0
1 1 0	1	0
1 1 1	1	0

By applying DeMorgan's theorem on f,

$f = AB + C$

$f' = (AB + C)'$

$= (A' + B')\,C'$

$= A'C' + B'C'$ (same as above)

Limitations of Algebraic Method for Logic Optimisation

- Limitation of Boolean algebra method for logic optimization:
  - Solution is not guaranteed to be minimum.
    - Non systematic steps to reach a desired minimum solution.
    - Approach is heuristic.
      - Repeated search is based on intuition and experience.
        - Time consuming and error-prone
  - Impractical for large number of variables since relies heavily on the ability of the designer to use theorems and laws.
    - Slow and error-prone
    - Usage is limited to small number of variables.
  - The expression is often made complex due to expansion before it can be simplified.

- Other better methods that can overcome the limitations of Boolean algebra method
  - Karnaugh Map method
  - Quine-McCluskey method