The Vapnik-Chervonenkis Dimension

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Trace of a Collection of Sets

Definition

Let $\mathcal{C}$ be a collection of sets and let U be a set. The trace of collection $\mathcal{C}$ on the set U is the collection

$$\mathcal{C}_U = \{U \cap C \mid C \in \mathcal{C}\}.$$

If the trace of $\mathcal{C}$ on U, $\mathcal{C}_U$ equals $\mathcal{P}(U)$, then we say that U is shattered by $\mathcal{C}$.

U is shattered by $\mathcal{C}$ if $\mathcal{C}$ can carve *any subset of U* as an intersection with a set in C.

Example

Let $U = \{u_1, u_2\}$ and let $\mathcal{C}$ be the collection of sets

$$\mathcal{C} = \{\{u_3\}, \{u_1, u_3\}, \{u_2, u_3\}, \{u_1, u_2, u_3\}\}.$$

$\mathcal{C}$ shatters U because we can write:

$$\begin{aligned} \emptyset &= U \cap \{u_3\} \\ \{u_1\} &= U \cap \{u_1, u_3\} \\ \{u_2\} &= U \cap \{u_2, u_3\} \\ \{u_1, u_2\} &= U \cap \{u_1, u_2, u_3\} \end{aligned}$$

Definition

The Vapnik-Chervonenkis dimension of the collection $\mathcal{C}$ (called the VC-dimension for brevity) is the largest size of a set K that is shattered by $\mathcal{C}$.

This largest size is denoted by VCD$(\mathcal{C})$.

Example

Note that the previous collection $\mathcal{C}$ cannot shatter the set $U' = \{u_1, u_2, u_3\}$ because this set has 8 subsets and $\mathcal{C}$ has just four sets. Thus, if is impossible to express all subsets of U' as intersections of U' with some set of $\mathcal{C}$.

The VCD dimension of the collection $\mathcal{C}$ is 2.

Note that:

- We have VCD$(\mathcal{C}) = 0$ if and only if $|\mathcal{C}| = 1$.
- If VCD$(\mathcal{C}) = d$, then there exists a set K of size d such that for each subset L of K there exists a set $C \in \mathcal{C}$ such that $L = K \cap C$.
- $\mathcal{C}$ shatters K if and only if the trace of $\mathcal{C}$ on K denoted by $\mathcal{C}_K$ shatters K. This allows us to assume without loss of generality that both the sets of the collection $\mathcal{C}$ and a set K shattered by $\mathcal{C}$ are subsets of a set U.

Collections of Sets as Sets of Hypotheses

Let U be a set, K a subset, and let $\mathcal{C}$ be a collection of sets.
Each $C \in \mathcal{C}$ defines a hypothesis $h_C : U \longrightarrow \{-1, 1\}$ that is a dichotomy, where

$$h_C(u) = \begin{cases} 1 & \text{if } u \in C, \\ -1 & \text{if } u \notin C. \end{cases}$$

K is shattered by $\mathcal{C}$ if and only if for every subset L of K there exists a hypothesis h_C such that the set of positive examples $\{u \in U \mid h_C(u) = 1\}$ equals L.

Finite Collections have Finite VC-Dimension

Let $\mathcal{C}$ be a collection of sets with $\text{VCD}(\mathcal{C}) = d$ and let K be a set shattered by $\mathcal{C}$ with $|K| = d$.
Since there exist 2^d subsets of K, there are at least 2^d subsets of $\mathcal{C}$, so

$$2^d \leqslant |\mathcal{C}|.$$

Consequently, $\text{VCD}(\mathcal{C}) \leqslant \log_2 |\mathcal{C}|$. This shows that if $\mathcal{C}$ is finite, then $\text{VCD}(\mathcal{C})$ is finite.

The converse is false: there exist infinite collections $\mathcal{C}$ that have a finite VC-dimension.

A Tabular Representation of Collections

If $U = \{u_1, \ldots, u_n\}$ is a finite set, then the trace of a collection $\mathcal{C} = \{C_1, \ldots, C_p\}$ of subsets of U on a subset K of U can be presented in an intuitive, tabular form.

Let θ be a table containing the rows $t_1, \ldots, t_p$ and the binary attributes $u_1, \ldots, u_n$.

Each tuple t_k corresponds to a set C_k of $\mathcal{C}$ and is defined by

$$t_k[u_i] = \begin{cases} 1 & \text{if } u_i \in C_k, \\ 0 & \text{otherwise,} \end{cases}$$

for $1 \leqslant i \leqslant n$. Then, $\mathcal{C}$ shatters K if the content of the projection $\mathbf{r}[K]$ consists of $2^{|K|}$ distinct rows.

Example

Let $U = \{u_1, u_2, u_3, u_4\}$ and let
$\mathcal{C} = \{\{u_2, u_3\}, \{u_1, u_3, u_4\}, \{u_2, u_4\}, \{u_1, u_2\}, \{u_2, u_3, u_4\}\}$ represented by:

$T_{\mathcal{C}}$

u_1	u_2	u_3	u_4
0	1	1	0
1	0	1	1
0	1	0	1
1	1	0	0
0	1	1	1

The set $K = \{u_1, u_3\}$ is shattered by the collection $\mathcal{C}$ because the projection on K $((0,1),(1,1),(0,0),(1,0),(0,1))$. contains the all four necessary tuples $(0,1)$, $(1,1)$, $(0,0)$, and $(1,0)$.
No subset K of U that contains at least three elements can be shattered by $\mathcal{C}$ because this would require the projection $\mathbf{r}[K]$ to contain at least eight tuples. Thus, VCD$(\mathcal{C}) = 2$.

Observations:

- Every collection of sets shatters the empty set.
- If $\mathcal{C}$ shatters a set of size n, then it shatters a set of size p, where $p \leqslant n$.

For a collection of sets $\mathcal{C}$ and for $m \in \mathbb{N}$, let

$$\Pi_{\mathcal{C}}[m] = \max\{|\mathcal{C}_K| \mid |K| = m\}$$

be the largest number of distinct subsets of a set having m elements that can be obtained as intersections of the set with members of $\mathcal{C}$.

- We have $\Pi_{\mathcal{C}}[m] \leqslant 2^m$;
- if $\mathcal{C}$ shatters a set of size m, then $\Pi_{\mathcal{C}}[m] = 2^m$.

Definition

A Vapnik-Chervonenkis class (or a VC class) is a collection $\mathcal{C}$ of sets such that $\mathrm{VCD}(\mathcal{C})$ is finite.

Example

Let $\mathbb{R}$ be the set of real numbers and let $\mathcal{I}$ be the collection of sets $\{(-\infty, t) \mid t \in \mathbb{R}\}$.

We claim that any singleton is shattered by $\mathcal{I}$. Indeed, if $S = \{x\}$ is a singleton, then $\mathcal{P}(\{x\}) = \{\emptyset, \{x\}\}$. Thus, if $t \geqslant x$, we have $(-\infty, t) \cap S = \{x\}$; also, if $t < x$, we have $(-\infty, t) \cap S = \emptyset$, so $\mathcal{I}_S = \mathcal{P}(S)$.

There is no set S with $|S| = 2$ that can be shattered by $\mathcal{I}$. Indeed, suppose that $S = \{x, y\}$, where $x < y$. Then, any member of $\mathcal{I}$ that contains y includes the entire set S, so $\mathcal{I}_S = \{\emptyset, \{x\}, \{x, y\}\} \neq \mathcal{P}(S)$.

This shows that $\mathcal{I}$ is a VC class and $\mathrm{VCD}(\mathcal{I}) = 1$.

Example

Consider the collection $\mathcal{I} = \{[a, b] \mid a, b \in \mathbb{R}, a \leqslant b\}$ of closed intervals. We claim that VCD$(\mathcal{I}) = 2$. To justify this claim, we need to show that there exists a set $S = \{x, y\}$ such that $\mathcal{I}_S = \mathcal{P}(S)$ and no three-element set can be shattered by $\mathcal{I}$.

For the first part of the statement, consider the intersections

$$
\begin{aligned}
&[u, v] \cap S = \emptyset, \text{ where } v < x,\\
&[x - \epsilon, \tfrac{x+y}{2}] \cap S = \{x\},\\
&[\tfrac{x+y}{2}, y] \cap S = \{y\},\\
&[x - \epsilon, y + \epsilon] \cap S = \{x, y\},
\end{aligned}
$$

which show that $\mathcal{I}_S = \mathcal{P}(S)$.

For the second part of the statement, let $T = \{x, y, z\}$ be a set that contains three elements. Any interval that contains x and z also contains y, so it is impossible to obtain the set $\{x, z\}$ as an intersection between an interval in $\mathcal{I}$ and the set T.

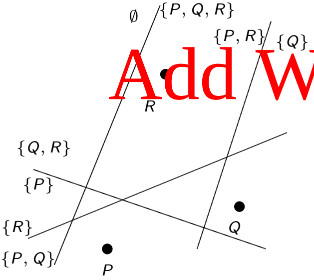
An Example

Let $\mathcal{H}$ be the collection of closed half-planes in $\mathbb{R}^2$ of the form

$$\{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 \mid ax_1 + bx_2 - c \geq 0, a \neq 0 \text{ or } b \neq 0\}.$$

We claim that VCD$(\mathcal{H}) = 3$.

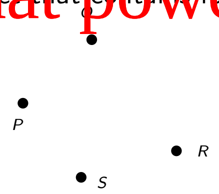
Let P, Q, R be three non-colinear points. Each line is marked with the sets it defines; thus, it is clear that the family of half-planes shatters the set $\{P, Q, R\}$, so VCD$(\mathcal{H})$ is at least 3.



Example (cont'd)

To complete the justification of the claim we need to show that no set that contains at least four points can be shattered by $\mathcal{H}$.

Let $\{P, Q, R, S\}$ be a set that contains four points such that no three points of this set are collinear. If S is located inside the triangle P, Q, R, then every half-plane that contains P, Q, R also contains S, so it is impossible to separate the subset $\{P, Q, R\}$. Thus, we may assume that no point is inside the triangle formed by the remaining three points. Any half-plane that contains two diagonally opposite points, for example, P and R, contains either Q or S, which shows that it is impossible to separate the set $\{P, R\}$. Thus, no set that contains four points may be shattered by $\mathcal{H}$, so $\text{VCD}(\mathcal{H}) = 3$.



CLAIM: the VCD of an arbitrary family of hyperplanes in $\mathbb{R}^d$ is $d+1$.
Consider the set of $d+1$ points $\{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_d\}$ defined as

$$\mathbf{x}_0 = \mathbf{0}_d, \mathbf{x}_i = \mathbf{e}_i \text{ for } 1 \leqslant i \leqslant d.$$

Let $y_0, y_1, \ldots, y_d \in \{-1, 1\}$ and let $\mathbf{w} \in \mathbb{R}^d$ be the vector whose i^{th} coordinate is y_i. We have $\mathbf{w}'\mathbf{x}_i = y_i$ for $1 \leqslant i \leqslant d$. Therefore,

$$\text{sign}\left(\mathbf{w}'\mathbf{x}_i + \frac{y_0}{2}\right) = \text{sign}\left(y_i + \frac{y_0}{2}\right) = y_i.$$

Thus, points $\mathbf{x}_i$ for which $y_i = 1$ are on the positive side of the hyperplane $\mathbf{y}'\mathbf{x} = 0$; the ones for which $y_i = -1$ are on the oposite side, so any family of $d+1$ points in $\mathbb{R}^d$ can be shattered by hyperplanes.

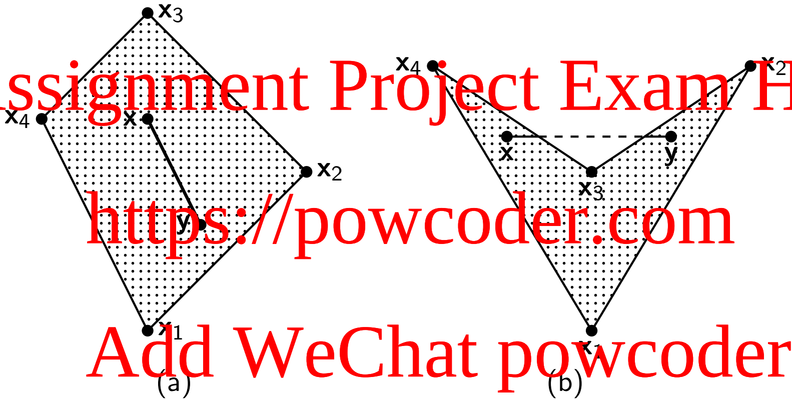
Also we need to show that no set of $d+2$ points can be shattered by hyperplanes. For this we need the notion of convex set and the notion of convex hull.

Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. The *closed segment* determined by $\mathbf{x}$ and $\mathbf{y}$ is the set

$$[\mathbf{x}, \mathbf{y}] = \{(1 - a)\mathbf{x} + a\mathbf{y} \mid 0 \leqslant a \leqslant 1\}.$$

Definition

A subset C of $\mathbb{R}^n$ is *convex* if, for all $\mathbf{x}, \mathbf{y} \in C$ we have $[\mathbf{x}, \mathbf{y}] \subseteq C$.



Convex Set (a) vs. a Non-convex Set (b)

Example

The convex subsets of $\mathbb{R}$ are the intervals of $\mathbb{R}$.
Regular polygons are convex subsets of $\mathbb{R}^2$.
An open sphere $B(\mathbf{x}_0, r)$ or a closed sphere $B[\mathbf{x}_0, r]$ in $\mathbb{R}^n$ is convex.

Definition

Let U be a subset of $\mathbb{R}^n$. A *convex combination* of U is a vector of the form $a_1\mathbf{x}_1 + \cdots + a_k\mathbf{x}_k$, where $\mathbf{x}_1, \ldots, \mathbf{x}_k \in U$, $a_i \geqslant 0$ for $1 \leqslant i \leqslant k$, and $a_1 + \cdots + a_k = 1$.

Theorem

The intersection of any collection of convex sets in $\mathbb{R}^n$ is a convex set.

Proof.

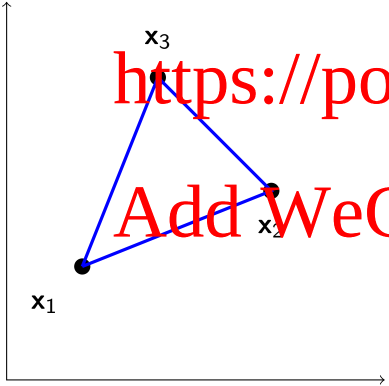
Let $\mathcal{C} = \{C_i \mid i \in I\}$ be a collection of convex sets and let $C = \bigcap \mathcal{C}$. Suppose that $\mathbf{x}_1, \ldots, \mathbf{x}_k \in C$, $a_i \geqslant 0$ for $1 \leqslant i \leqslant k$, and $a_1 + \cdots + a_k = 1$. Since $\mathbf{x}_1, \ldots, \mathbf{x}_k \in C_i$, it follows that $a_1\mathbf{x}_1 + \cdots + a_k\mathbf{x}_k \in C_i$ for every $i \in I$. Thus, $a_1\mathbf{x}_1 + \cdots + a_k\mathbf{x}_k \in C$, which proves the convexity of C. □

Definition

The *convex hull* (or the *convex closure* of a subset U of $\mathbb{R}^n$ is the intersection of all convex sets that contain U.
The convex null of U is denoted by $\mathbf{K}_{\mathrm{conv}}(U)$.

Example

A two-dimensional simplex is defined starting from three points $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ in $\mathbb{R}^2$ such that none of these points is collinear with the others two. Thus, the two-dimensional simplex generated by $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ is the full triangle determined by $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$.



Let S be the n-dimensional simplex generated by the points $\mathbf{x}_1, \ldots, \mathbf{x}_{n+1}$ in $\mathbb{R}^n$ and let $\mathbf{x} \in S$. If $\mathbf{x} \in S$, then $\mathbf{x}$ is a convex combination of $\mathbf{x}_1, \ldots, \mathbf{x}_n, \mathbf{x}_{n+1}$. In other words, there exist $a_1, \ldots, a_n, a_{n+1}$ such that $a_1, \ldots, a_n, a_{n+1} \in (0,1)$, $\sum_{i=1}^{n+1} a_i = 1$, and $\mathbf{x} = a_1\mathbf{x}_1 + \cdots + a_n\mathbf{x}_n + a_{n+1}\mathbf{x}_{n+1}$.

Theorem

(Radon's Theorem) Any set $X = \{x_1, \ldots, x_{d+2}\}$ *of* $d+2$ *points in* $\mathbb{R}^d$ *can be partitioned into two sets* X_1 *and* X_2 *such that the convex hulls of* X_1 *and* X_2 *intersect.*

Proof

Consider the following system with $d+1$ linear equations and $d+2$ variables $\alpha_1, \alpha_2, \ldots, \alpha_{d+2}$:

$$\sum_{i=1}^{d+2} \alpha_i \mathbf{x}_i = \mathbf{0}_d, \sum_{i=1}^{d+2} \alpha_i = 0.$$

Since the number of variables $(d+2)$ is larger than $d+1$, the system has a non-trivial solution $\beta_1, \ldots, \beta_{d+2}$. Since $\sum_{i=1}^{d+2} \beta_i = 0$ both sets

$$I_1 = \{i \mid 1 \leqslant i \leqslant d+2, \beta_i > 0\}, I_2 = \{i \mid 1 \leqslant i \leqslant d+2, \beta_i < 0\}$$

are non-empty sets and

$$X_1 = \{\mathbf{x}_i \mid i \in I_1\}, X_2 = \{\mathbf{x}_i \mid i \in I_2\},$$

form a partition of X.

Proof (cont'd)

Define $\beta = \sum_{i \in I_1} \beta_i$. Since $\sum_{i \in I_1} \beta_i = -\sum_{i \in I_2} \beta_i$, we have

$$\sum_{i \in I_1} \frac{\beta_i}{\beta} \mathbf{x}_i = \sum_{i \in I_2} \frac{-\beta_i}{\beta} \mathbf{x}_i .$$

Also,

$$\sum_{i \in I_1} \frac{\beta_i}{\beta} = \sum_{i \in I_2} \frac{-\beta_i}{\beta} = 1,$$

$\frac{\beta_i}{\beta} \geqslant 0$ for $i \in I_1$ and $\frac{-\beta_i}{\beta} \geqslant 0$ for $i \in I_2$. This implies that

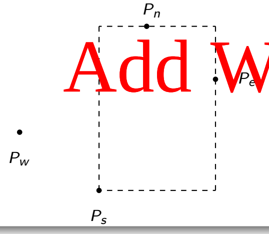
$$\sum_{i \in I_1} \frac{\beta_i}{\beta} \mathbf{x}_i$$

belongs both to the convex hulls of X_1 and X_2.

Let X be a set of $d+2$ points in $\mathbb{R}^d$. By Radon's Theorem it can be partitioned into X_1 and X_2 such that the two convex hulls intersect. When two sets are separated by a hyperplane, their convex hulls are also separated by the hyperplane. Thus, X_1 and X_2 cannot be separated by a hyperplane and X is not shattered.

Example

Let $\mathcal{R}$ be the set of rectangles whose sides are parallel with the axes x and y. There is a set S with $|S| = 4$ that is shattered by $\mathcal{R}$. Let S be a set of four points in $\mathbb{R}^2$ that contains a unique "northernmost point" P_n, a unique "southernmost point" P_s, a unique "easternmost point" P_e, and a unique "westernmost point" P_w. If $L \subseteq S$ and $L \neq \emptyset$, let R_L be the smallest rectangle that contains L. For example, we show the rectangle R_L for the set $\{P_n, P_s, P_e\}$.



Example (cont'd)

This collection cannot shatter a set of points that contains at least five points. Indeed, let S be such that $|S| \geqslant 5$. If the set contains more than one "northernmost" point, then we select exactly one to be P_n. Then, the rectangle that contains the set $K = \{P_n, P_e, P_s, P_w\}$ contains the entire set S, which shows the impossibility of separating S.

The Class of All Convex Polygons

Example

Consider the system of all convex polygons in the plane.
For any positive integer m, place m points on the unit circle. Any subset of the points are the vertices of a convex polygon. Clearly that polygon will not contain any of the points not in the subset. This shows that we can shatter arbitrarily large sets, so the VC-dimension of the class of all convex polygons is infinite.

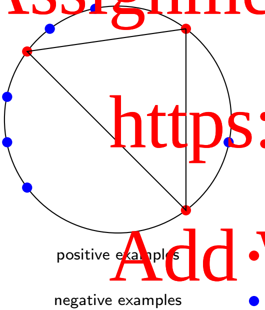
The Case of Convex Polygons with d Vertices

Example

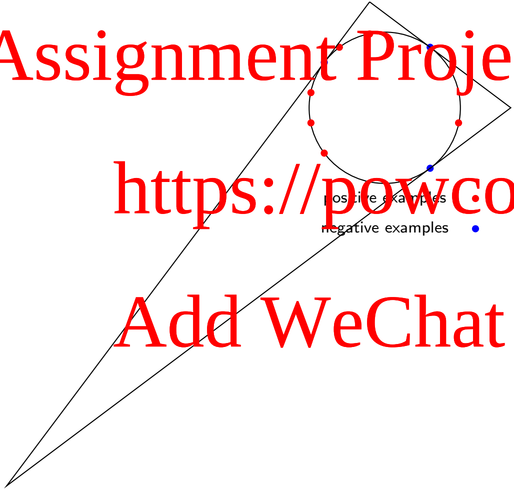
Consider the class of convex polygons that have no more than d vertices in $\mathbb{R}^2$ and place $2d + 1$ points on a circle.

- Label a subset of these points as positive, and the remaining points as negative. Since we have an odd number of points there exists a majority in one of the classes (positive or negative).
- If the negative point are in majority, there are at most d positive points; these are contained by the convex polygon formed by joining the positive points.
- If the positive are in majority, consider the polygon formed by the tangents of the negative points.

Negative Points in the Majority



Positive Points in the Majority



Example cont'd

- Since a set with $2d+1$ points can be shattered, the *VC* dimension of the set of convex polygons with at most d vertices is at least $2d+1$.

- If all labeled points are located on a circle then it is impossible for a point to be in the convex closure of a subset of the remaining points. Thus, placing the points on a circle maximizes the number of sets required to shatter the set, so the VC-dimension is indeed $2d+1$.

Definition

Let H be a set of hypotheses and let $(x_1, \ldots, x_m)$ be a sequence of examples of length m. A hypothesis $h \in H$ induces a classification

$$(h(x_1), \ldots, h(x_m))$$

of the components of this sequence. The growth function of H is the function $\Pi_H : \mathbb{N} \longrightarrow \mathbb{N}$ gives the number of ways a sequence of examples of length m can be classified by a hypothesis in H:

$$\Pi_H(m) = \max_{(x_1,\ldots,x_m)\in\mathcal{X}^m} |\{(h(x_1), \ldots, h(x_m)) \mid h \in H\}|$$

Dichotomies

Definition

A dichotomy is a hypothesis $h : \mathcal{X} \longrightarrow \{-1, 1\}$.

If H consists of dichotomies, then $(x_1, \ldots, x_m)$ can be classified in at most 2^m ways.

Theorem

Let $S = \{s_1, \ldots, s_n\}$ be a set and let $\mathcal{C}$ be a collection of subsets of S, $\mathcal{C} \subseteq \mathcal{P}(S)$. If $SH(\mathcal{C})$ is the family of subsets of S that are shattered by $\mathcal{C}$ then $|SH(\mathcal{C})| \geqslant |\mathcal{C}|$.

Proof

The argument is by induction on $|\mathcal{C}|$.
Consider the subfamilies

$$\begin{aligned}
\mathcal{C}_0 &= \{U \in \mathcal{C} \mid s_1 \notin U\} \\
\mathcal{C}_1 &= \{U \in \mathcal{C} \mid s_1 \in U\}
\end{aligned}$$

By the inductive hypothesis, $|\mathsf{SH}(\mathcal{C}_0)| \geqslant |\mathcal{C}_0|$ that is, $\mathcal{C}_0$ shatters at least as many subsets of $S' = \{s_2, s_3, \ldots, s_n\}$ as $|\mathcal{C}_0|$.
Next, consider the family

$$\mathcal{C}_1' = \{U - \{s_1\} \mid U \in \mathcal{C}_1\}.$$

Note that:

- The families $\mathcal{C}_0$ and $\mathcal{C}_1$ of subsets of S are disjoint and $|\mathcal{C}| = |\mathcal{C}_0| + |\mathcal{C}_1|$.
- $\mathcal{C}_0$ and $\mathcal{C}_1'$ are families of subsets of S' and $|\mathcal{C}_1'| = |\mathcal{C}_1|$.

Proof (cont'd)

By induction, $\mathcal{C}_1'$ shatters at least as many subsets of $S' = \{s_2, s_3, \ldots, s_n\}$ as its cardinality, that is, $|\mathsf{SH}(\mathcal{C}_1')| \geqslant |\mathcal{C}_1'|$.

The number of subsets of S' shattered by $\mathcal{C}_0$ and $\mathcal{C}_1'$ sum up to at least $|\mathcal{C}_0| + |\mathcal{C}_1'| = |\mathcal{C}_0| + |\mathcal{C}_1| = |\mathcal{C}|$, and every subset of S' shattered by $\mathcal{C}_1'$ is shattered by $\mathcal{C}_1 \subseteq \mathcal{C}$. Note that there may be subsets V of S' shattered by both $\mathcal{C}_0$ and $\mathcal{C}_1'$. In this case both V and $V \cup \{s_1\}$ are shattered by $\mathcal{C}$.

For $n, k \in \mathbb{N}$ and $0 \leqslant k \leqslant n$ define the number $\binom{n}{\leqslant k}$ as

$$\binom{n}{\leqslant k} = \sum_{i=0}^{k} \binom{n}{i}.$$

Clearly, $\binom{n}{\leqslant 0} = 1$ and $\binom{n}{\leqslant n} = 2^n$.

Theorem

(Sauer-Shelah Theorem) *Let S be a set with $|S| = n$ and let $\mathcal{C}$ be a collection of subsets of S such that*

$$|\mathcal{C}| > \binom{n}{\leqslant k}$$

Then, there exists a subset T of S having at least $k+1$ elements such that $\mathcal{C}$ shatters T.

Proof

Let $|\mathsf{SH}(\mathcal{C})|$ be the number of sets shattered by $\mathcal{C}$. We have $|\mathsf{SH}(\mathcal{C})| \geqslant |\mathcal{C}|$ by the previous theorem.

Let $\mathcal{P}_k(S)$ be the collection of subsets of S that contain k or fewer elements.

The inequality of the theorem means that $|\mathcal{C}| > |\mathcal{P}_k(S)|$, hence $|\mathsf{SH}(\mathcal{C})| > |\mathcal{P}_k(S)|$. Therefore, there exists a subset T of S with at least $k+1$ elements that is shattered by $\mathcal{C}$.

Theorem

Let $\phi : \mathbb{N}^2 \longrightarrow \mathbb{N}$ *be the function defined by*

$$\phi(d, m) = \begin{cases} 1 & \text{if } m = 0 \text{ or } d = 0 \\ \phi(d, m-1) + \phi(d-1, m-1), & \text{otherwise} \end{cases}$$

We have

$$\phi(d, m) = \binom{m}{\leqslant d}$$

for $d, m \in \mathbb{N}$.

Proof

The argument is by strong induction on $s = d + m$.
The base case, $s = 0$, implies $m = 0$ and $d = 0$, and the equality is immediate.

Suppose that the equality holds for $\phi(d', m')$, where $d' + m' < d + m$. We have:

$$
\begin{aligned}
\phi(d, m) &= \phi(d, m-1) + \phi(d-1, m-1) \\
&\quad \text{(by definition)} \\
&= \sum_{i=0}^{d} \binom{m-1}{i} + \sum_{i=0}^{d-1} \binom{m-1}{i} \\
&\quad \text{(by inductive hypothesis)} \\
&= \sum_{i=0}^{d} \binom{m-1}{i} + \sum_{i=1}^{d} \binom{m-1}{i-1} \\
&\quad \text{(by changing the summation index in the second sum)} \\
&= \sum_{i=0}^{d} \binom{m-1}{i} + \sum_{i=0}^{d} \binom{m-1}{i-1} \\
&\quad \text{(because } \binom{m-1}{-1} = 0\text{)} \\
&= \sum_{i=0}^{d} \left(\binom{m-1}{i} + \binom{m-1}{i-1} \right) \\
&= \sum_{i=0}^{d} \binom{m}{i} = \binom{m}{\leqslant d},
\end{aligned}
$$

which gives the desired conclusion.

Another Inequality

Suppose that VCD$(\mathcal{C}) = d$ and $|S| = n$. Then SH$(\mathcal{C}) \subseteq \mathcal{P}_d(S)$, hence

$$|\mathcal{C}| \leqslant |\mathrm{SH}(\mathcal{C})| \leqslant \sum_{i=1}^{d} \binom{n}{i} = \binom{n}{\leqslant d}.$$

Together with the previous inequality we obtain:

$$2^d \leqslant |\mathcal{C}| \leqslant \binom{n}{\leqslant d} = \phi(n, d).$$

Lemma

For $d \in \mathbb{N}$ and $d \geqslant 2$ we have

$$2^{d-1} \leqslant \frac{d^d}{d!}.$$

Proof: The argument is by induction on d. In the basis step, $d = 2$ both members are equal to 2.

Suppose the inequality holds for d. We have

$$\begin{aligned}\frac{(d+1)^{d+1}}{(d+1)!} &= \frac{(d+1)^d}{d!} = \frac{d^d}{d!} \cdot \frac{(d+1)^d}{d^d} \\ &= \underbrace{\frac{d^d}{d!}}_{\text{(by inductive hypothesis)}} \cdot \left(1+\frac{1}{d}\right)^d \geqslant 2^d \cdot \left(1+\frac{1}{d}\right)^d \geqslant 2^d\end{aligned}$$

because

$$\left(1+\frac{1}{d}\right)^d \geqslant 1 + d\frac{1}{d} = 2.$$

This concludes the proof of the inequality.

Lemma

We have $\phi(d, m) \leqslant 2\frac{m^d}{d!}$ for every $m \geqslant d$ and $d \geqslant 1$.

Proof: The argument is by induction on d and m. If $d = 1$, then $\phi(1, m) = m + 1 \leqslant 2m$ for $m \geqslant 1$, so the inequality holds for every $m \geqslant 1$, when $d = 1$.

Proof (cont'd)

If $m = d \geqslant 2$, then $\phi(d, m) = \phi(d, d) = 2^d$ and the desired inequality follows immediately from a previous Lemma.
Suppose that the inequality holds for $m > d \geqslant 1$. We have

$$\begin{aligned}
\phi(d, m+1) &= \phi(d, m) + \phi(d-1, m) \\
&\quad \text{(by the definition of } \phi\text{)} \\
&\leqslant 2\frac{m^d}{d!} + 2\frac{m^{d-1}}{(d-1)!} \\
&\quad \text{(by inductive hypothesis)} \\
&= 2\frac{m^{d-1}}{(d-1)!}\left(1 + \frac{m}{d}\right).
\end{aligned}$$

Proof (cont'd)

It is easy to see that the inequality

$$2\frac{m^{d-1}}{(d-1)!}\left(1+\frac{m}{d}\right) \leqslant 2\frac{(m+1)^d}{d!}$$

is equivalent to

$$\frac{d}{m}+1 \leqslant \left(1+\frac{1}{m}\right)^d$$

and, therefore, is valid. This yields immediately the inequality of the lemma.

The Asymptotic Behavior of the Function ϕ

Theorem

The function ϕ satisfies the inequality:

$$\phi(d, m) \leqslant \left(\frac{em}{d}\right)^d$$

for every $m \geqslant d$ and $d \geqslant 1$.

Proof: By a previous Lemma, $\phi(d, m) \leqslant 2\frac{m^d}{d!}$. Therefore, we need to show only that

$$2\left(\frac{d}{e}\right)^d < d!.$$

Proof (cont'd)

The argument is by induction on $d \geqslant 1$. The basis case, $d = 1$ is immediate. Suppose that $2\left(\frac{d}{e}\right)^d < d!$. We have

$$\begin{aligned} 2\left(\frac{d+1}{e}\right)^{d+1} &= 2\left(\frac{d}{e}\right)^d \left(\frac{d+1}{d}\right)^d \frac{d+1}{e} \\ &= \left(1+\frac{1}{d}\right)^d \frac{1}{e} \cdot 2\left(\frac{d}{e}\right)^d (d+1) < 2\left(\frac{d}{e}\right)^d (d+1), \end{aligned}$$

because

$$\left(1+\frac{1}{d}\right)^d < e.$$

The last inequality holds because the sequence $\left(\left(1+\frac{1}{d}\right)^d\right)_{d\in\mathbb{N}}$ is an increasing sequence whose limit is e. Since $2\left(\frac{d+1}{e}\right)^{d+1} < 2\left(\frac{d}{e}\right)^d (d+1)$, by inductive hypothesis we obtain:

$$2\left(\frac{d+1}{e}\right)^{d+1} < (d+1)!.$$

Corollary

If m is sufficiently large we have $\phi(d, m) = O(m^d)$.

The statement is a direct consequence of the previous theorem.

Denote by $\oplus$ the symmetric difference of two sets.

Theorem

Let $\mathcal{C}$ a family of sets and $C_0 \in \mathcal{C}$. Define the family Δ_{C_0} as

$$\Delta_{C_0}(\mathcal{C}) = \{T \mid T = C_0 \oplus C \text{ where } C \in \mathcal{C}\}.$$

We have $VCD(\mathcal{C}) = VCD(\Delta_{C_0}(\mathcal{C}))$.

Proof

Let S be a set, $\mathcal{S} = \mathcal{C}_S$ and $\mathcal{S}_0 = (\Delta_{C_0}(\mathcal{C}))_S$.
Define $\psi : \mathcal{S} \longrightarrow \mathcal{S}_0$ as $\psi(S \cap C) = S \cap (C_0 \oplus C)$. We claim that ψ is a bijection.
If $\psi(S \cap C) = \psi(S \cap C')$ for $C, C' \in \mathcal{C}$, then $S \cap (C_0 \oplus C) = S \cap (C_0 \oplus C')$. Therefore,

$$(S \cap C_0) \oplus (S \cap C) = (S \cap C_0) \oplus (S \cap C'),$$

which implies $S \cap C = S \cap C'$, so ψ is injective.
On other hand, if $U \in \mathcal{S}_0$ we have $U = S \cap (C_0 \oplus C)$, so $U = \psi(S \cap C)$, hence ψ is a surjection. Thus, $\mathcal{S}$ and $\mathcal{S}_0$ have the same number of sets, which implies that a set S is shattered by $\mathcal{C}$ if and only if it is shattered by $\Delta_{C_0}(\mathcal{C})$.

Classes with Infinite VCDs are not PAC-learnable

Theorem

A class $\mathcal{H}$ with VCD$(\mathcal{H}) = \infty$ is not PAC-learnable.

Proof: Assume that $\mathcal{H}$ is PAC-learnable. Let $\mathcal{A}$ be a training algorithm and let m be the sample size needed to learn $\mathcal{H}$ with accuracy ϵ and certainty $1-\delta$. In other words, after seeing m examples, $\mathcal{A}$ produces a hypothesis $h \in \mathcal{H}$ with $P(L_{\mathcal{D}}(h) \leqslant \epsilon) \geqslant 1-\delta$.

Since VCD$(\mathcal{H}) = \infty$, for every $m \in \mathbb{N}$ there exists a sample S of length $2m$ that is shattered by $\mathcal{H}$. Let $\mathcal{D}$ be such that the probability of each example x_i of S is $\frac{1}{2m}$ and the probability of other examples is 0.

Since S is shattered, we can choose a target hypothesis $h_t \in \mathcal{H}$ such that

$$P(h_t(x_i) = 0) = P(h_t(x_i) = 1) = \frac{1}{2}$$

for every x_i in S (as if the labels $h_t(x_i)$ are determined by a coin flip).

Proof (cont'd)

$\mathcal{A}$ selects an iid sample of m instances S' such that $S' \subseteq S$ and outputs a consistent hypothesis h. The probability of error for each $x_i \notin S'$ is:

$$P(h_t(x_i) \neq h(x_i)) = \frac{1}{2}$$

because we could select the labels of the points not seen by $\mathcal{A}$ (which produces h) arbitrarily.
Regardless of h we have:

$$E(L_{\mathcal{D}}(h)) = m \cdot 0 \cdot \frac{1}{2m} + m \cdot \frac{1}{2} \cdot \frac{1}{2m} = \frac{1}{4}.$$

(We have $2m$ points to sample such that the error of half of them is 0 as h is consistent on S').

Proof (cont'd)

Thus, for any sample size m, if $\mathcal{A}$ produces a consistent hypothesis, then the expectation of the error will be $\frac{1}{4}$.

However, since with probability at least $1-\delta$ we have that $L_{\mathcal{D}}(h) \leqslant \epsilon$, it follows that

$$E(L_{\mathcal{D}}(h)) \leqslant (1-\delta)\epsilon + \delta \cdot \beta,$$

where β is such that $\epsilon < \beta \leqslant 1$. Note that

$$(1-\delta)\epsilon + \delta \cdot \beta \leqslant (1-\delta)\epsilon + \delta = \epsilon + \delta - \epsilon\delta < \epsilon + \delta.$$

It suffices to take

$$\epsilon + \delta < \frac{1}{4}$$

to obtain a contradition!

Hypothesis Consistency in Set-Theoretical Terms

Let C be a concept over the set of examples $\mathcal{X}$ and let S be a sample drawn from $\mathcal{X}$ according to a probability distribution $\mathcal{D}$.

- A hypothesis C_0 regarded here as a set, is consistent with S if $C_0 \cap S = C \cap S$. Equivalently, $S \cap (C_0 \oplus C) = \emptyset$.
- C_0 is inconsistent with S if $S \cap (C_0 \oplus C) \neq \emptyset$.

On slide 58 we established that $\text{VCD}(\mathcal{C}) = \text{VCD}(\Delta_{C_0}(\mathcal{C}))$, where

$$\Delta_{C_0}(\mathcal{C}) = \{T \mid T = C_0 \oplus C \mid C \in \mathcal{C}\}.$$

Define now

$$\begin{aligned}\Delta_{C_0,\epsilon}(\mathcal{C}) &= \{T \mid T \in \Delta_{C_0}(\mathcal{C}) \mid P(T) \geqslant \epsilon\} \\ &= \{T \mid T = C_0 \oplus C, C \in \mathcal{C} \text{ and } P(T) \geqslant \epsilon\}.\end{aligned}$$

- $\Delta_{C_0}(\mathcal{C})$ is the set of error regions relative to the hypothesis C_0.
- $\Delta_{C_0,\epsilon}(\mathcal{C})$ is the set of error regions relative to the hypothesis C_0 having the probability not smaller than ϵ.

Definition

A set S is an ϵ-net for $\Delta_{C_0}(\mathcal{C})$ if every set T in $\Delta_{C_0,\epsilon}(\mathcal{C})$ is hit by a point in S, that is, for every error region $T \in \Delta_{C_0,\epsilon}(\mathcal{C})$ we have $T \cap S \neq \emptyset$.

Claim:

If the sample S forms an ϵ-net for $\Delta_{C_0}(\mathcal{C})$ and the learning algorithm outputs a hypothesis (represented here by a set $C_0 \in \mathcal{C}$) that is consistent with S, then this hypothesis must have error less than ϵ.

Indeed, since

- $C_0 \oplus C \in \Delta_{C_0}(\mathcal{C})$ was not hit by S (otherwise, C_0 would not be consistent with S), and
- S is an ϵ-net for $\Delta_{C_0}(\mathcal{C})$,

we must have $C_0 \oplus C \notin \Delta_{C_0,\epsilon}(\mathcal{C})$ and therefore $L_{\mathcal{D}}(C_0) \leqslant \epsilon$.

Thus, if we can bound the probability that a random sample S does not form an ϵ-net for $\Delta_{C_0,\epsilon}(\mathcal{C})$, then we have bounded the probability that for a hypothesis C_0 consistent with S we have $L_{\mathcal{D}}(C_0) > \epsilon$.

Example

Suppose that $\mathcal{C}$ is finite. For any fixed set $C_0 \oplus C \in \Delta_{C_0,\epsilon}(\mathcal{C})$, the probability that we fail to hit $C_0 \oplus C$ in m random examples is at most $(1-\epsilon)^m$. Thus, the probability that we fail to hit some $C_0 \oplus C \in \Delta_{C_0,\epsilon}(\mathcal{C})$ is bounded above by $|\mathcal{C}|(1-\epsilon)^m$.

The Double Sample Theorem

Theorem

Let $\mathcal{C}$ be a concept class with $VCD(\mathcal{C}) = d$.
Let $\mathcal{A}$ be any algorithm that given a set S of m labeled examples $\{(x_i, c(x_i)) \mid 1 \leqslant i \leqslant m\}$ sampled iid according to some fixed but unknown distribution $\mathcal{D}$ over the instance space $\mathcal{X}$ produces as output a hypothesis h that is consistent with c. Then, $\mathcal{A}$ is a PAC algorithm and

$$m \geqslant k_0 \left(\frac{1}{\epsilon} \log \frac{1}{\delta} + \frac{d}{\epsilon} \log \frac{1}{\epsilon} \right)$$

for some positive constant k_0.

Proof

- Draw a sample S_1 of size m from $\mathcal{D}$ and let A be the event that the elements of S_1 fail to form an ϵ-net for $\Delta_{C_0,\epsilon}(\mathcal{C})$.
- If A occurs, then S_1 misses some region T, where

$$T \in \Delta_{C_0,\epsilon}(\mathcal{C}).$$

Fix this region T and draw an additional sample S_2 of size m from $\mathcal{D}$.

Let V be a binomial random variable that gives the number of hits of T by the sample S_2. We have $E(V) = m\epsilon$ and $\mathrm{var}(V) = m\epsilon(1-\epsilon)$ because the probability of an element of S_2 hitting T is ϵ.

By Chebyshev's Inequality applied to V we have

$$P(|V - m\epsilon| \geqslant a) \leqslant \frac{m\epsilon(1-\epsilon)}{a^2}.$$

Taking $a = \frac{\epsilon m}{2}$ it follows that

$$\begin{aligned} P\left(|V - m\epsilon| \geqslant \frac{\epsilon m}{2}\right) &\leqslant \frac{4(1-\epsilon)}{\epsilon m} \\ &\leqslant \frac{4}{\epsilon m} \leqslant \frac{1}{2}, \end{aligned}$$

provided that $m \geqslant \frac{8}{\epsilon}$.

Thus, if $m \geqslant \frac{8}{\epsilon}$

$$P(|V - \epsilon m| \leqslant \frac{\epsilon m}{2}) \geqslant \frac{1}{2}.$$

The inequality

$$|V - \epsilon m| \leqslant \frac{\epsilon m}{2}$$

is equivalent to $\frac{\epsilon m}{2} \leqslant V \leqslant \frac{3\epsilon m}{2}$, which implies $P(V \geqslant \frac{\epsilon m}{2}) \geqslant \frac{1}{2}$.

Proof (cont'd)

To summarize: we have calculated the probability that S_2 will hit T many times given that T was fixed using the previous sampling, that is, given that S_1 does not form an ϵ-net.

Let B be the event that S_1 does not form an ϵ-net and that S_2 hits T at least $\frac{\epsilon m}{2}$ times. Then, we have shown that for $m = O(1/\epsilon)$ we have $P(B|A) \geqslant \frac{1}{2}$.

Proof (cont'd)

Since $P(B|A) \geqslant \frac{1}{2}$ we have

$$P(B) = P(B|A)P(A) \geqslant \frac{1}{2}P(A).$$

Our goal of bounding $P(A)$ is equivalent to finding δ such that $P(B) \leqslant \frac{\delta}{2}$ because this would imply $P(A) \leqslant \delta$.

Proof (cont'd)

Let $S = S_1 \cup S_2$ be a random sample of $2m$. Note that since the samples are iid obtaining S is equivalent of sampling S_1 and S_2 separately and let T be a fixed set such that $|T| \geqslant \frac{\epsilon m}{2}$.

Consider a random partition of S into S_1 and S_2 and consider the probability that $S_1 \cap T = \emptyset$.

An Equivalent Problem: we have 2m balls each colored red or blue with exactly ℓ red balls, where $\ell \geqslant \frac{\epsilon m}{2}$. Divide the 2m balls into groups of equal size S_1 and S_2. Find an upper bound on the probability that all ℓ balls fall in S_2 (that is, the probability that $S_1 \cap R = \emptyset$).

Proof (cont'd)

Yet Another Equivalent Problem: Divide $2m$ non-colored balls into S_1 and S_2, choose ℓ to be colored red, and compute the probability that all red balls fall in S_2. The probability of this taking place is:

$$\frac{\binom{m}{l}}{\binom{2m}{\ell}}$$

Note that

$$\frac{\binom{m}{l}}{\binom{2m}{\ell}} = \prod_{i=0}^{\ell-1} \frac{m-i}{2m-i} \leqslant \prod_{i=0}^{\ell-1} \frac{1}{2} = \frac{1}{2^{\ell}} = 2^{-\frac{\epsilon m}{2}}.$$

This is the probability for a fixed S and T. The probability that this occurs for some $T \in \Delta_{C_0,\epsilon}(S)$ such that $|T| \geqslant \frac{\epsilon m}{2}$ can be computed by summing over all T and applying the union bound:

$$\begin{aligned} P(B) &\leqslant |\Pi_{\Delta_{C_0,\epsilon}(S)}(\frac{\epsilon m}{2})| 2^{-\frac{\epsilon m}{2}} \leqslant |\Pi_{\Delta_{C_0}(S)}(\frac{\epsilon m}{2})| 2^{-\frac{\epsilon m}{2}} \\ &\leqslant \left(\frac{2\epsilon m}{d}\right)^{d} 2^{-\frac{\epsilon m}{2}} \leqslant \frac{\delta}{2}. \end{aligned}$$

Proof (cont'd)

The last inequality implies

$$m \geqslant k_0 \left(\frac{1}{\epsilon} \log \frac{1}{\delta} + \frac{d}{\epsilon} \log \frac{1}{\epsilon} \right).$$

for some positive constant k_0.

Optional Material

Let $u : B_2^k \longrightarrow B_2$ be a Boolean function of k arguments and let $C_1, \ldots, C_k$ be k subsets of a set U. Define the set $u(C_1, \ldots, C_k)$ as the subset C of U whose indicator function is $I_C = u(I_{C_1}, \ldots, I_{C_k})$.

Example

If $u : B_2^2 \longrightarrow B_2$ is the Boolean function $u(a_1, a_2) = a_1 \vee a_2$, then $u(C_1, C_2)$ is $C_1 \cup C_2$; similarly, if $u(x_1, x_2) = x_1 \oplus x_2$, then $u(C_1, C_2)$ is the symmetric difference $C_1 \oplus C_2$ for every $C_1, C_2 \in \mathcal{P}(U)$.

Let $u : B_2^k \longrightarrow B_2$ and $\mathcal{C}_1, \ldots, \mathcal{C}_k$ are k family of subsets of U, the family of sets $u(\mathcal{C}_1, \ldots, \mathcal{C}_k)$ is

$$u(\mathcal{C}_1, \ldots, \mathcal{C}_k) = \{u(C_1, \ldots, C_k) \mid C_1 \in \mathcal{C}_1, \ldots, C_k \in \mathcal{C}_k\}.$$

Theorem

*Let $\alpha(k)$ be the least integer a such that $\frac{a}{\log(ea)} > k$.
If $\mathcal{C}_1, \ldots, \mathcal{C}_k$ are k collections of subsets of the set U such that $d = \max\{VCD(\mathcal{C}_i) \mid 1 \leqslant i \leqslant k\}$ and $u : B_2^2 \longrightarrow B_2$ is a Boolean function, then*

$$VCD(u(\mathcal{C}_1, \ldots, \mathcal{C}_k)) \leqslant \alpha(k) \cdot d.$$

Proof

Let S be a subset of U that consists of m elements. The collection $(\mathcal{C}_i)_S$ is not larger than $\phi(d, m)$. For a set in the collection $W \in u(\mathcal{C}_1, \ldots, \mathcal{C}_k)_S$ we can write $W = S \cap u(C_1, \ldots, C_k)$, or, equivalently,
$1_W = 1_S \cdot u(1_{C_1}, \ldots, 1_{C_k})$.

There exists a Boolean function g_S such that

$$1_S \cdot u(1_{C_1}, \ldots, 1_{C_k}) = g_S(1_S \cdot 1_{C_1}, \ldots, 1_S \cdot 1_{C_k}) = g_S(1_{S \cap C_1}, \ldots, 1_{S \cap C_k}).$$

Since there are at most $\phi(d, m)$ distinct sets of the form $S \cap C_i$ for every i, $1 \leqslant i \leqslant k$, it follows that there are at most $(\phi(d, m))^k$ distinct sets W, hence $u(\mathcal{C}_1, \ldots, \mathcal{C}_k)[m] \leqslant (\phi(d, m))^k$.

Proof (cont'd)

By a previous theorem,

$$u(\mathcal{C}_1, \ldots, \mathcal{C}_k)[m] \leqslant \left(\frac{em}{d}\right)^{kd}.$$

We observed that if $\Pi_{\mathcal{C}}[m] < 2^m$, then VCD$(\mathcal{C}) < m$. Therefore, to limit the Vapnik-Chervonenkis dimension of the collection $u(\mathcal{C}_1, \ldots, \mathcal{C}_k)$ it suffices to require that $\left(\frac{em}{d}\right)^{kd} < 2^m$.

Let $a = \frac{m}{d}$. The last inequality can be written as $(ea)^{kd} < 2^{ad}$; equivalently, we have $(ea)^k < 2^a$, which yields $k < \frac{a}{\log(ea)}$. If $\alpha(k)$ is the least integer a such that $k < \frac{a}{\log(ea)}$, then $m \leqslant \alpha(k)d$, which gives our conclusion.

Example

If $k = 2$, the least integer a such that $\frac{a}{\log(ea)} > 2$ is $k = 10$, as it can be seen by graphing this function; thus, if $\mathcal{C}_1, \mathcal{C}_2$ are two collection of concepts with $\mathsf{VCD}(\mathcal{C}_1) = \mathsf{VCD}(\mathcal{C}_2) = d$, the Vapnik-Chervonenkis dimension of the collections $\mathcal{C}_1 \vee \mathcal{C}_2$ or $\mathcal{C}_1 \wedge \mathcal{C}_2$ is not larger than $10d$.

Lemma

Let S, T be two sets and let $f : S \longrightarrow T$ be a function. If $\mathcal{D}$ is a collection of subsets of T, U is a finite subset of S and $\mathcal{C} = f^{-1}(\mathcal{D})$ is the collection $\{f^{-1}(D) \mid D \in \mathcal{D}\}$, then $|\mathcal{C}_U| \leqslant |\mathcal{D}_{f(U)}|$.

Proof: Let $V = f(U)$ and denote $f\restriction U$ by g. For $D, D' \in \mathcal{D}$ we have

$$
\begin{aligned}
&(U \cap f^{-1}(D)) \oplus (U \cap f^{-1}(D')) \\
&= U \cap (f^{-1}(D) \oplus f^{-1}(D')) = U \cap (f^{-1}(D \oplus D')) \\
&= g^{-1}(V \cap (D \oplus D')) = g^{-1}(V \cap D) \oplus g^{-1}(V \cap D')
\end{aligned}
$$

Thus, $C = U \cap f^{-1}(D)$ and $C' = U \cap f^{-1}(D')$ are two distinct members of $\mathcal{C}_U$, then $V \cap D$ and $V \cap D'$ are two distinct members of $\mathcal{D}_{f(U)}$. This implies $|\mathcal{C}_U| \leqslant |\mathcal{D}_{f(U)}|$.

Theorem

Let S, T be two sets and let $f : S \longrightarrow T$ be a function. If $\mathcal{D}$ is a collection of subsets of T and $\mathcal{C} = f^{-1}(\mathcal{D})$ is the collection $\{f^{-1}(D) \mid D \in \mathcal{D}\}$, then $VCD(\mathcal{C}) \leqslant VCD(\mathcal{D})$. Moreover, if f is a surjection, then $VCD(\mathcal{C}) = VCD(\mathcal{D})$.

Proof

Suppose that $\mathcal{C}$ shatters an n-element subset $K = \{x_1, \ldots, x_n\}$ of S, so $|\mathcal{C}_K| = 2^n$ By a previous Lemma we have $|\mathcal{C}_K| \leqslant |\mathcal{D}_{f(U)}|$, so $|\mathcal{D}_{f(U)}| \geqslant 2^n$, which implies $|f(U)| = n$ and $|\mathcal{D}_{f(U)}| = 2^n$, because $f(U)$ cannot have more than n elements. Thus, $\mathcal{D}$ shatters $f(U)$, so VCD$(\mathcal{C}) \leqslant$ VCD$(\mathcal{C})$. Suppose now that f is surjective and $H = \{t_1, \ldots, t_m\}$ is an m element set that is shattered by $\mathcal{D}$. Consider the set $L = \{u_1, \ldots, u_m\}$ such that $u_i \in f^{-1}(t_i)$ for $1 \leqslant i \leqslant m$. Let U be a subset of L. Since H is shattered by $\mathcal{D}$, there is a set $D \in \mathcal{D}$ such that $f(U) = H \cap D$, which implies $U = L \cap f^{-1}(D)$. Thus, L is shattered by $\mathcal{C}$ and this means that VCD$(\mathcal{C}) =$ VCD$(\mathcal{D})$.

Definition

The *density* of $\mathcal{C}$ is the number

$$\mathrm{dens}(\mathcal{C}) = \inf\{s \in \mathbb{R}_{\geqslant 0} \mid \Pi_{\mathcal{C}}[m] \leqslant c \cdot m^s \text{ for every } m \in \mathbb{N}\},$$

for some positive constant c.

Theorem

Let S, T be two sets and let $f : S \longrightarrow T$ be a function. If $\mathcal{D}$ is a collection of subsets of T and $\mathcal{C} = f^{-1}(\mathcal{D})$ is the collection $\{f^{-1}(D) \mid D \in \mathcal{D}\}$, then $\text{denss}(\mathcal{C}) \leqslant \text{denss}(\mathcal{D})$. Moreover, if f is a surjection, then $\text{denss}(\mathcal{C}) = \text{denss}(\mathcal{D})$.

Proof: Let L be a subset of S such that $|L| = m$. Then, $|\mathcal{C}_L| \leqslant |\mathcal{D}_{f(L)}|$. In general, we have $|f(L)| \leqslant m$, so $|\mathcal{D}_{f(L)}| \leqslant \mathcal{D}[m] \leqslant cm^s$. Therefore, we have $|\mathcal{C}_L| \leqslant |\mathcal{D}_{f(L)}| \leqslant \mathcal{D}[m] \leqslant cm^s$, which implies $\text{denss}(\mathcal{C}) \leqslant \text{denss}(\mathcal{D})$. If f is a surjection, then, for every finite subset M of T such that $|M| = m$ there is a subset L of S such that $|L| = |M|$ and $f(L) = M$. Therefore, $\mathcal{D}[m] \leqslant \Pi_{\mathcal{C}}[m]$ and this implies $\text{denss}(\mathcal{C}) = \text{denss}(\mathcal{D})$.

If $\mathcal{C}, \mathcal{D}$ are two collections of sets such that $\mathcal{C} \subseteq \mathcal{D}$, then $\mathsf{VCD}(\mathcal{C}) \leqslant \mathsf{VCD}(\mathcal{D})$ and $\mathsf{denss}(\mathcal{C}) \leqslant \mathsf{denss}(\mathcal{D})$.

Theorem

Let $\mathcal{C}$ be a collection of subsets of a set S and let $\mathcal{C}' = \{S - C \mid C \in \mathcal{C}\}$. Then, for every $K \in \mathcal{P}(S)$ we have $|\mathcal{C}_K| = |\mathcal{C}'_K|$.

Proof

We prove the statement by showing the existence of a bijection $f : \mathcal{C}_K \longrightarrow \mathcal{C}'_K$. If $U \in \mathcal{C}_K$, then $U = K \cap C$, where $C \in \mathcal{C}$. Then $S - C \in \mathcal{C}'$ and we define $f(U) = K \cap (S - C) = K - C \in \mathcal{C}'_K$. The function f is well-defined because if $K \cap C_1 = K \cap C_2$, then $K - C_1 = K - (K \cap C_1) = K - (K \cap C_2) = K - C_2$.

It is clear that if $f(U) = f(V)$ for $U, V \in \mathcal{C}_K$, $U = K \cap C_1$, and $V = K \cap C_2$, then $K - C_1 = K - C_2$, so $K \cap C_1 = K \cap C_2$ and this means that $U = V$. Thus, f is injective. If $W \in \mathcal{C}'_K$, then $W = K \cap C'$ for some $C' \in \mathcal{C}$. Since $C' = S - C$ for some $C \in \mathcal{C}$, it follows that $W = K - C$, so $W = f(U)$, where $U = K \cap C$.

Corollary

Let $\mathcal{C}$ be a collection of subsets of a set S and let $\mathcal{C}' = \{S - C \mid C \in \mathcal{C}\}$. We have $dens(\mathcal{C}) = dens(\mathcal{C}')$ and $\mathit{VCD}(\mathcal{C}) = \mathit{VCD}(\mathcal{C}')$.

Theorem

For every collection of sets we have $\text{denss}(\mathcal{C}) \leqslant \text{VCD}(\mathcal{C})$. *Furthermore, if* $\text{denss}(\mathcal{C})$ *is finite, then* $\mathcal{C}$ *is a VC-class.*

Proof: If $\mathcal{C}$ is not a VC-class the inequality $\text{denss}(\mathcal{C}) \leqslant \text{VCD}(\mathcal{C})$ is clearly satisfied. Suppose now that $\mathcal{C}$ is a VC-class and $\text{VCD}(\mathcal{C}) = d$. By Sauer-Shelah Theorem we have $\Pi_{\mathcal{C}}[m] \leqslant \phi(d, m)$; then, we obtain $\Pi_{\mathcal{C}}[m] \leqslant \left(\frac{em}{d}\right)^d$, so $\text{denss}(\mathcal{C}) \leqslant d$.

Suppose now that $\text{denss}(\mathcal{C})$ is finite. Since $\Pi_{\mathcal{C}}[m] \leqslant cm^s \leqslant 2^m$ for m sufficiently large, it follows that $\text{VCD}(\mathcal{C})$ is finite, so $\mathcal{C}$ is a VC-class.

Let $\mathcal{D}$ be a finite collection of subsets of a set S. The partition $\pi_{\mathcal{D}}$ was defined as consisting of the nonempty sets of the form $\{D_1^{a_1} \cap D_2^{a_2} \cap \cdots \cap D_r^{a_r}$, where $(a_1, a_2, \ldots, a_r) \in \{0,1\}^r$.

Definition

A collection $\mathcal{D} = \{D_1, \ldots, D_r\}$ of subsets of a set S is *independent* if the partition $\pi_{\mathcal{D}}$ has the maximum numbers of blocks, that is, it consists of 2^r blocks.

If $\mathcal{D}$ is independent, then the Boolean subalgebra generated by $\mathcal{D}$ in the Boolean algebra $(\mathcal{P}(S), \{\cap, \cup, \bar{\ }, \emptyset, S\})$ contains 2^{2^r} sets, because this subalgebra has 2^r atoms. Thus, if $\mathcal{D}$ shatters a subset T with $|T| = p$, then the collection $\mathcal{D}_T$ contains 2^p sets, which implies $2^p \leqslant 2^{2^r}$, or $p \leqslant 2^r$.

Definition

Let $\mathcal{C}$ be a collection of subsets of a set S. The independence number of $\mathcal{C}$ $I(\mathcal{C})$ is:

$$\begin{aligned} I(\mathcal{C}) &= \sup\{r \mid \{C_1, \ldots, C_r\} \\ &\quad \text{is independent for some finite } \{C_1, \ldots, C_r\} \subseteq \mathcal{C}\}. \end{aligned}$$

Theorem

Let S, T be two sets and let $f : S \longrightarrow T$ be a function. If $\mathcal{D}$ is a collection of subsets of T and $\mathcal{C} = f^{-1}(\mathcal{D})$ is the collection $\{f^{-1}(D) \mid D \in \mathcal{D}\}$, then $I(\mathcal{C}) \leqslant I(\mathcal{D})$. Moreover, if f is a surjection, then $I(\mathcal{C}) = I(\mathcal{D})$.

Proof: Let $\mathcal{E} = \{D_1, \ldots, D_p\}$ be an independent finite subcollection of $\mathcal{D}$. The partition $\pi_{\mathcal{E}}$ contains 2^r blocks. The number of atoms of the subalgebra generated by $\{f^{-1}(D_1), \ldots, f^{-1}(D_p)\}$ is not greater than 2^r. Therefore $I(\mathcal{C}) \leqslant I(\mathcal{D})$; from the same supplement it follows that if f is surjective, then $I(\mathcal{C}) = I(\mathcal{D})$.

Theorem

If $\mathcal{C}$ is a collection of subsets of a set S such that $\mathrm{VCD}(\mathcal{C}) \geqslant 2^n$, then $I(\mathcal{C}) \geqslant n$.

Proof: Suppose that $\mathrm{VCD}(\mathcal{C}) \geqslant 2^n$, that is, there exists a subset T of S that is shattered by $\mathcal{C}$ and has at least 2^n elements. Then, the collection $\mathcal{H}_t$ contains at least 2^{2^n} sets, which means that the Boolean subalgebra of $\mathcal{P}(T)$ generated by $\mathcal{T}_C$ contains at least 2^n atoms. This implies that the subalgebra of $\mathcal{P}(S)$ generated by $\mathcal{C}$ contains at least this number of atoms, so $I(\mathcal{C}) \geqslant n$.