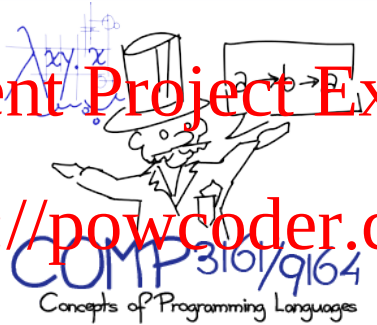

Functional Programming Languages

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University of Edinburgh LFCS
UNSW, Term 3 2020

Functional Programming

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What do they have in common?

Definitions

Unlike imperative languages, functional programming languages are not very crisply defined.

Attempt at a Definition

A *functional programming language* is a programming language derived from or inspired by the λ-calculus, or derived from or inspired by another functional programming language.

The result? If it has λ in it, you can call it functional.

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In this course, we'll consider *purely functional* languages, which have a much better definition.

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Think of a major innovation in the area of programming languages.

Garbage Collection?

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Purely Functional Programming Languages

The term *purely functional* has a very crisp definition.

Definition

A programming language is *purely functional* if β-reduction (or evaluation in general) is actually a confluence.
In other words, functions have to be mathematical functions, and free of *side effects*.

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In other words, functions have to be mathematical functions, and free of *side effects*.

Consider what would happen if we allowed effects in a functional language:

$$count = 0;$$
$$f\ x = \{count := count + x; \textbf{return}\ count\};$$
$$m = (\lambda y.\ y + y)\ (f\ 3)$$

If we evaluate $f\ 3$ first, we will get $m = 6$, but if we β-reduce m first, we will get $m = 9$. $\Rightarrow$ not confluent.

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3. Purely functional (no effects)
4. Call-by-value (strict evaluation)

In your Assignment 1, you will be implementing an evaluator for a slightly less minimal dialect of MinHS.

Syntax

Integers	n	$::=$	$\ldots$
Identifiers	f, x	$::=$	$\ldots$
Literals	b	$::=$	`True` $\mid$ `False`
Types	τ	$::=$	$\mathrm{Bool} \mid \mathrm{Int} \mid \tau_1 \to \tau_2$
Infix Operators	$\circledast$	$::=$	`*` $\mid$ `+` $\mid$ `==` $\mid \ldots$
Expressions	e	$::=$	$x \mid n \mid b \mid (e) \mid e_1 \circledast e_2$
		$\mid$	`if` e_1 `then` e_2 `else` e_3

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		$\mid$	$e_1\ e_2$
		$\mid$	`recfun` $f :: (\tau_1 \to \tau_2)\ x = e$

↑ Like λ, but with recursion.

As usual, this is ambiguous concrete syntax. But all the precedence and associativity rule apply as in Haskell. We assume a suitable parser.

Examples

Example (Stupid division by 5)

$$\begin{array}{l}\textbf{recfun}\ \mathit{divBy5} :: (\mathrm{Int} \to \mathrm{Int})\ x = \\ \quad \textbf{if}\ x < 5 \\ \quad\quad \textbf{then}\ 0 \\ \quad\quad \textbf{else}\ 1 + \mathit{divBy5}\ (x - 5)\end{array}$$

Example (Average Function)

$$\begin{array}{l}\textbf{recfun}\ \mathit{average} :: (\mathrm{Int} \to (\mathrm{Int} \to \mathrm{Int}))\ x = \\ \textbf{recfun}\ \mathit{avX} :: (\mathrm{Int} \to \mathrm{Int})\ y = \\ \quad (x + y)\ /\ 2\end{array}$$

As in Haskell, $(\mathit{average}\ 15\ 5) = ((\mathit{average}\ 15)\ 5)$.

We don't need no let

This language is so minimal, it doesn't even need **let** expressions. How can we do without them?

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$$\textbf{let } x :: \tau_1 = e_1 \textbf{ in } e_2 :: \tau_2 \quad \equiv \quad (\textbf{recfun } f :: (\tau_1 \to \tau_2)\ x = e_2)\ e_1$$

Abstract Syntax

Moving to first order abstract syntax, we get:

1. Things like numbers and boolean literals are wrapped in terms (`Num Int`).
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What changes when we move to higher order abstract syntax?

1. `Var` terms go away – we use the meta-language's variables.
2. (`Recfun` τ_1 τ_2 f x e) now uses meta-language abstraction: (`Recfun` τ_1 τ_2 (f. x. e)).

Working Statically with HOAS

To Code

We're going to write code for an AST and pretty-printer for MinHS with HOAS. Seeing as this requires us to look under abstractions without evaluating the term, we have to extend the AST with special "tag" values.

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To check if a MinHS program is well-formed, we need to check:

1. **Scoping** – all variables used must be well defined
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Our judgement is an extension of the scoping rules to include types:

Under this context of assumptions

$$\Gamma \vdash e : \tau$$

the expression is assigned this type

The context Γ includes typing assumptions for the variables:

$$x : \mathrm{Int}, y : \mathrm{Int} \vdash (\texttt{Plus}\ x\ y) : \mathrm{Int}$$

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$$\frac{}{\Gamma \vdash (\texttt{Num}\ n) : \text{Int}} \qquad \frac{}{\Gamma \vdash (\texttt{Lit}\ b) : \text{Bool}}$$

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Let's implement a *type checker*.

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Structural Operational Semantics (Small-Step)

Initial states:

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Initial states: All well typed expressions.

Final states:

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Initial states: All well typed expressions.

Final states: (`Num` n), (`Lit` b),

Dynamic Semantics

Structural Operational Semantics (Small-Step)

Initial states: All well typed expressions.

Final states: (`Num` n), (`Lit` b), `Recfun too`!

Evaluation of built-in operations

$$\frac{e_1 \mapsto e_1'}{(\texttt{Plus}\ e_1\ e_2) \mapsto (\texttt{Plus}\ e_1'\ e_2)}$$

(and so on as per arithmetic expressions)

Specifying If

$$\frac{e_1 \mapsto e_1'}{(\texttt{If}\ e_1\ e_2\ e_3) \mapsto (\texttt{If}\ e_1'\ e_2\ e_3)}$$

$$\frac{}{(\texttt{If}\ (\texttt{Lit True})\ e_2\ e_3) \mapsto e_2}$$

$$\frac{}{(\texttt{If}\ (\texttt{Lit False})\ e_2\ e_3) \mapsto e_3}$$

How about Functions?

Recall that `Recfun` is a final state – we don't need to evaluate it when it's alone.

Evaluating function application requires us to:

1. Evaluate the left expression to get the function being applied
2. Evaluate the right expression to get the argument value
3. Evaluate the function's body, after supplying substitutions for the abstracted variables.

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$$\frac{v \in F}{(\texttt{Apply}\ (\texttt{Recfun}\ \tau_1\ \tau_2\ (f.x.\ e))\ v) \mapsto e[x := v, f := (\texttt{Recfun}\ \tau_1\ \tau_2\ (f.x.\ e))]}$$