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COMP3161/9164
Concepts of Programming Languages

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Abstract Machines

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UNSW, Term 3 2020

Big O

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We all know that MERGESORT has $\mathcal{O}(n \log n)$ time complexity, and that BUBBLESORT has $\mathcal{O}(n^2)$ time complexity, but what does that actually mean?

Big O Notation

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Given functions $f, g : \mathbb{R} \to \mathbb{R}$, $f \in \mathcal{O}(g)$ if and only if there exists a value $x_0 \in \mathbb{R}$ and a coefficient m such that:

$$\forall x > x_0.\ f(x) \leq m \cdot g(x)$$

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What is the codomain of f?

When analysing algorithms, we don't usually time how long they take to run on a real machine.

Cost Models

A *cost model* is a mathematical model that tries to measure of the cost of executing a program.

There exist *denotational* cost models, that assign a cost directly to syntax:

$$[\![\cdot]\!] : \mathrm{Program} \rightarrow \mathrm{Cost}$$

However in this course we will focus on *operational cost models*.

Operational Cost Models

First, we define a program-evaluating *abstract machine*. We can determine the time cost by counting the number of steps taken by the abstract machine.

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Abstract Machines

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Abstract Machines

An *abstract machine* consists of:

1. A set of states Σ,
2. A set of initial states $I \subseteq \Sigma$,
3. A set of final states $F \subseteq \Sigma$, and
4. A transition relation $\mapsto \subseteq \Sigma \times \Sigma$.

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We've seen this before in structural operational (or small-step) semantics.

The M Machine

Is just our usual small-step rules:

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$$\frac{e_1 \mapsto_M e_1'}{(\texttt{Plus}\ e_1\ e_2) \mapsto_M (\texttt{Plus}\ e_1'\ e_2)} \quad \dots$$

$$\frac{e_1 \mapsto_M e_1'}{(\texttt{If}\ e_1\ e_2\ e_3) \mapsto_M (\texttt{If}\ e_1'\ e_2\ e_3)}$$

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$$\frac{}{(\texttt{If}\ (\texttt{Lit}\ \texttt{True})\ e_2\ e_3) \mapsto_M e_2} \quad \frac{}{(\texttt{If}\ (\texttt{Lit}\ \texttt{False})\ e_2\ e_3) \mapsto_M e_3}$$

$$\frac{e_1 \mapsto_M e_1'}{(\texttt{Apply}\ e_1\ e_2) \mapsto_M (\texttt{Apply}\ e_1'\ e_2)}$$

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$$\frac{e_2 \mapsto_M e_2'}{(\texttt{Apply}\ (\texttt{Recfun}\ (f.x.\ e))\ e_2) \mapsto_M (\texttt{Apply}\ (\texttt{Recfun}\ (f.x.\ e))\ e_2')}$$

$$\frac{v \in F}{(\texttt{Apply}\ (\texttt{Recfun}\ (f.x.\ e))\ v) \mapsto_M e[x := v, f := (\texttt{Recfun}\ (f.x.\ e))]}$$

The M Machine is unsuitable as a basis for a cost model. Why?

Performance

One step in our machine should always only be $\mathcal{O}(1)$ in our language implementation. Otherwise, counting steps will not get an accurate description of the time cost.

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This makes for two potential problems:

1. **Substitution** occurs in function application, which is potentially $\mathcal{O}(n)$ time.
2. **Control Flow** is not explicit – which subexpression to reduce is found by recursively descending the abstract syntax tree each time.

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$$
\begin{array}{lcl}
\mathit{eval}\ (\texttt{Num}\ n) & = & n \\
\mathit{eval}\ e & = & \mathit{eval}\ (\mathit{oneStep}\ e)
\end{array}
$$

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$$
\begin{array}{lcl}
\mathit{oneStep}\ (\texttt{Plus}\ (\texttt{Num}\ n)\ (\texttt{Num}\ m)) & = & \texttt{Num}\ (n+m) \\
\mathit{oneStep}\ (\texttt{Plus}\ (\texttt{Num}\ n)\ e_2) & = & \texttt{Plus}\ (\texttt{Num}\ n)\ (\mathit{oneStep}\ e_2) \\
\mathit{oneStep}\ (\texttt{Plus}\ e_1\ e_2) & = & \texttt{Plus}\ (\mathit{oneStep}\ e_1)\ e_2 \\
\ldots & &
\end{array}
$$

The C Machine

We want to define a machine where all the rules are axioms, so there can be no recursive descent into subexpressions. How is recursion typically implemented?

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Stacks!

$$\frac{}{\circ\ \mathsf{Stack}} \qquad \frac{f\ \mathsf{Frame} \quad s\ \mathsf{Stack}}{f \triangleright s\ \mathsf{Stack}}$$

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Key Idea: States will consist of a current expression to evaluate and a stack of computational contexts that situate it in the overall computation. An example stack would be:

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$$(\mathtt{Plus}\ 3\ \square) \triangleright (\mathtt{Times}\ \square\ (\mathtt{Num}\ 2)) \triangleright \circ$$

This represents the computational context:

$$(\mathtt{Times}\ (\mathtt{Plus}\ 3\ \square)\ (\mathtt{Num}\ 2))$$

The C Machine

Our states will consist of two modes:

1. **Evaluate** the current expression within stack s, written $s \succ e$.
2. **Return** a value v (either a function, integer, or boolean) back into the context in s, written $s \prec v$.

Initial states are those that start evaluating an expression from an empty stack, i.e. $\circ \succ e$. Final states are those that return a value to the empty stack, i.e. $\circ \prec v$.

Stack frames are expressions with holes or values in them:

$$\frac{e_2 \ \mathsf{Expr}}{(\mathtt{Plus}\ \Box\ e_2)\ \mathsf{Frame}} \qquad \frac{v_1 \ \mathsf{Value}}{(\mathtt{Plus}\ v_1\ \Box)\ \mathsf{Frame}}$$

$\cdots$

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Evaluating

There are three axioms about `Plus` now:

When evaluating a `Plus` expression, first evaluate the LHS

$$\frac{}{s \succ (\texttt{Plus}\ e_1\ e_2) \quad \mapsto_C \quad (\texttt{Plus}\ \square\ e_2) \triangleright s \succ e_1}$$

Once it is evaluated, switch to the RHS:

$$\frac{}{(\texttt{Plus}\ \square\ e_2) \triangleright s \prec v_1 \quad \mapsto_C \quad (\texttt{Plus}\ v_1\ \square) \triangleright s \succ e_2}$$

Once it is evaluated, return the sum:

$$\frac{}{(\texttt{Plus}\ v_1\ \square) \triangleright s \prec v_2 \quad \mapsto_C \quad s \prec v_1 + v_2}$$

We also have a single rule about `Num` that just returns the value:

$$\frac{}{s \succ (\texttt{Num}\ n) \mapsto_C s \prec n}$$

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Example

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$$
\begin{array}{ll}
 & \circ \succ \texttt{(Plus (Plus (Num 2) (Num 3)) (Num 4))} \\
\mapsto_C & \texttt{(Plus} \ \square \ \texttt{(Num 4))} \triangleright \circ \succ \texttt{(Plus (Num 2) (Num 3))} \\
\mapsto_C & \texttt{(Plus} \ \square \ \texttt{(Num 3))} \triangleright \texttt{(Plus} \ \square \ \texttt{(Num 4))} \triangleright \circ \succ \texttt{(Num 2)} \\
\mapsto_C & \texttt{(Plus} \ \square \ \texttt{(Num 3))} \triangleright \texttt{(Plus} \ \square \ \texttt{(Num 4))} \triangleright \circ \prec 2 \\
\mapsto_C & \texttt{(Plus 2} \ \square\texttt{)} \triangleright \texttt{(Plus} \ \square \ \texttt{(Num 4))} \triangleright \circ \succ \texttt{(Num 3)} \\
\mapsto_C & \texttt{(Plus 2} \ \square\texttt{)} \triangleright \texttt{(Plus} \ \square \ \texttt{(Num 4))} \triangleright \circ \prec 3 \\
\mapsto_C & \texttt{(Plus} \ \square \ \texttt{(Num 4))} \triangleright \circ \prec 5 \\
\mapsto_C & \texttt{(Plus 5} \ \square\texttt{)} \triangleright \circ \succ \texttt{(Num 4)} \\
\mapsto_C & \texttt{(Plus 5} \ \square\texttt{)} \triangleright \circ \prec 4 \\
\mapsto_C & \circ \prec 9
\end{array}
$$

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Other Rules

We have similar rules for the other operators and for booleans. For If:

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$$\frac{}{s \succ (\texttt{If}\ e_1\ e_2\ e_3) \quad \mapsto_C \quad (\texttt{If}\ \square\ e_2\ e_3) \rhd s \succ e_1}$$

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$$\frac{}{(\texttt{If}\ \square\ e_2\ e_3) \rhd s \prec \texttt{True} \quad \mapsto_C \quad s \succ e_2}$$

$$\frac{}{(\texttt{If}\ \square\ e_2\ e_3) \rhd s \prec \texttt{False} \quad \mapsto_C \quad s \succ e_3}$$

Functions

Recfun (here abbreviated to Fun) evaluates to a *function value*:

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$$\frac{}{s \succ (\texttt{Fun}\ (f.x.\ e)) \mapsto_C s \prec \langle\langle f.x.\ e \rangle\rangle}$$

Function application is then handled similarly to Plus.

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$$\frac{}{s \succ (\texttt{Apply}\ e_1\ e_2) \quad \mapsto_C \quad (\texttt{Apply}\ \Box\ e_2) \triangleright s \succ e_1}$$

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$$\frac{}{(\texttt{Apply}\ \Box\ e_2) \triangleright s \prec \langle\langle f.x.\ e \rangle\rangle \quad \mapsto_C \quad (\texttt{Apply}\ \langle\langle f.x.\ e \rangle\rangle\ \Box) \triangleright s \succ e_2}$$

$$\frac{}{(\texttt{Apply}\ \langle\langle f.x.\ e \rangle\rangle\ \Box) \triangleright s \prec v \quad \mapsto_C \quad s \prec e[x := v, f := (\texttt{Fun}\ (f.x.e))]}$$

We are still using substitution for now.

What have we done?

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- All the rules are axioms – we can now implement the evaluator with a simple `while` loop (or a *tail recursive* function).

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- We have a lower-level specification – helps with code generation (e.g. in an assembly language)
- Substitution is still a machine operation – we need to find a way to eliminate that.

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Correctness

While the M-Machine is reasonably straightforward definition of the language's semantics, the C-Machine is much more detailed.

We wish to prove a theorem that tells us that the C-Machine behaves analogously to the M-Machine.

Refinement

A low-level (*concrete*) semantics of a program is a *refinement* of a high-level (*abstract*) semantics if every possible execution in the low-level semantics has a corresponding execution in the high-level semantics. In our case:

$$\forall e, v. \frac{\circ \succ e \overset{\star}{\mapsto}_C \circ \prec v}{e \overset{\star}{\mapsto}_M v}$$

Functional correctness properties are preserved by refinement, but security properties are not (cf. Dining Cryptographers).

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How to Prove Refinement

We can't get away with simply proving that each C machine step has a corresponding step in the M-Machine, because the C-Machine makes multiple steps that are no-ops in the M-Machine:

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$$
\begin{array}{llll}
 & \circ \succ (+\ (+\ (\texttt{N}\ 2)\ (\texttt{N}\ 3))\ (\texttt{N}\ 4)) & & (+\ (+\ (\texttt{N}\ 2)\ (\texttt{N}\ 3))\ (\texttt{N}\ 4)) \\
\mapsto_C & (+\ \square\ (\texttt{N}\ 4)) \triangleright \circ \succ (+\ (\texttt{N}\ 2)\ (\texttt{N}\ 3)) & & \\
\mapsto_C & (+\ \square\ (\texttt{N}\ 3)) \triangleright (+\ \square\ (\texttt{N}\ 4)) \triangleright \circ \succ (\texttt{N}\ 2) & & \\
\mapsto_C & (+\ \square\ (\texttt{N}\ 3)) \triangleright (+\ \square\ (\texttt{N}\ 4)) \triangleright \circ \prec 2 & & \\
\mapsto_C & (+\ 2\ \square) \triangleright (+\ \square\ (\texttt{N}\ 4)) \triangleright \circ \succ (\texttt{N}\ 3) & & \\
\mapsto_C & (+\ 2\ \square) \triangleright (+\ \square\ (\texttt{N}\ 4)) \triangleright \circ \prec 3 & & \\
\mapsto_C & (+\ \square\ (\texttt{N}\ 4)) \triangleright \circ \prec 5 & \mapsto_M & (+\ (\texttt{N}\ 5)\ (\texttt{N}\ 4)) \\
\mapsto_C & (+\ 5\ \square) \triangleright \circ \succ (\texttt{N}\ 4) & & \\
\mapsto_C & (+\ 5\ \square) \triangleright \circ \prec 4 & & \\
\mapsto_C & \circ \prec 9 & \mapsto_M & (\texttt{N}\ 9)
\end{array}
$$

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How to Prove Refinement

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1. Define an *abstraction function* $\mathcal{A} : \Sigma_C \to \Sigma_M$ that relates C-Machine states to M-Machine states, describing how they "correspond".
2. Prove that for all initial states $\sigma \in I_C$, that the corresponding state $\mathcal{A}(\sigma) \in I_M$.
3. Prove for each step in the C-Machine $\sigma_1 \mapsto_C \sigma_2$, either:
   - the step is a no-op in the M-Machine and $\mathcal{A}(\sigma_1) = \mathcal{A}(\sigma_2)$, or
   - the step is replicated by the M-Machine $\mathcal{A}(\sigma_1) \mapsto_M \mathcal{A}(\sigma_2)$.
4. Prove that for all final states $\sigma \in F_C$, that the corresponding state $\mathcal{A}(\sigma) \in F_M$.

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In general this abstraction function is called a *simulation relation* and this type of proof is called a *simulation* proof.

The Abstraction Function

Our abstraction function $\mathcal{A}$ will need to relate states such that each transition that corresponds to a no-op in the M-Machine will move between $\mathcal{A}$-equivalent states:

$$
\begin{array}{lll}
 & \circ \succ (+\ (+\ (\texttt{N}\ 2)\ (\texttt{N}\ 3))\ (\texttt{N}\ 4)) & (+\ (+\ (\texttt{N}\ 2)\ (\texttt{N}\ 3))\ (\texttt{N}\ 4)) \\
\mapsto_C & (+\ \square\ (\texttt{N}\ 4)) \triangleright \circ \succ (+\ (\texttt{N}\ 2)\ (\texttt{N}\ 3)) & \\
\mapsto_C & (+\ \square\ (\texttt{N}\ 3)) \triangleright (+\ \square\ (\texttt{N}\ 4)) \triangleright \circ \succ (\texttt{N}\ 2) & \\
\mapsto_C & (+\ \square\ (\texttt{N}\ 3)) \triangleright (+\ \square\ (\texttt{N}\ 4)) \triangleright \circ \prec 2 & \\
\mapsto_C & (+\ 2\ \square) \triangleright (+\ \square\ (\texttt{N}\ 4)) \triangleright \circ \succ (\texttt{N}\ 3) & \\
\mapsto_C & (+\ 2\ \square) \triangleright (+\ \square\ (\texttt{N}\ 4)) \triangleright \circ \prec 3 & \\
\mapsto_C & (+\ \square\ (\texttt{N}\ 4)) \triangleright \circ \prec 5 & \mapsto_M (+\ (\texttt{N}\ 5)\ (\texttt{N}\ 4)) \\
\mapsto_C & (+\ 5\ \square) \triangleright \circ \succ (\texttt{N}\ 4) & \\
\mapsto_C & (+\ 5\ \square) \triangleright \circ \prec 4 & \\
\mapsto_C & \circ \prec 9 & \mapsto_M (\texttt{N}\ 9)
\end{array}
$$

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Abstraction Function

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Given a C-Machine state with a stack and a current expression (or value), we reconstruct the overall expression to get the corresponding M-Machine state.

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$$
\begin{array}{lcl}
\mathcal{A}(\circ \succ e) & = & e \\
\mathcal{A}(\circ \prec v) & = & (\texttt{Num}\ v) \\
\mathcal{A}((\texttt{Plus}\ \square\ e_2) \triangleright s \succ e_1) & = & \mathcal{A}(s \succ (\texttt{Plus}\ e_1\ e_2)) \\
\textit{etc.} & &
\end{array}
$$

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By definition, all the initial/final states of the C-Machine are mapped to initial/final states of the M-Machine. So all that is left is the requirement for each transition.

Showing Refinement for Plus

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$$\frac{}{s \succ (\texttt{Plus}\ e_1\ e_2) \quad \mapsto_C \quad (\texttt{Plus}\ \Box\ e_2) \triangleright s \succ e_1}$$

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This is a no-op in the M-Machine:

$$\begin{aligned} \mathcal{A}(RHS) &= \mathcal{A}((\texttt{Plus}\ \Box\ e_2) \triangleright s \succ e_1) \\ &= \mathcal{A}(s \succ (\texttt{Plus}\ e_1\ e_2)) \\ &= \mathcal{A}(LHS) \end{aligned}$$

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Showing Refinement for Plus

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$$\frac{}{(\texttt{Plus}\ \square\ e_2) \triangleright s \prec v_1 \quad \mapsto_C \quad (\texttt{Plus}\ v_1\ \square) \triangleright s \succ e_2}$$

Another no-op in the M-Machine:

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$$\begin{array}{rcl}
\mathcal{A}(LHS) & = & \mathcal{A}((\texttt{Plus}\ \square\ e_2) \triangleright s \prec v_1) \\
& = & \mathcal{A}(s \succ (\texttt{Plus}\ (\texttt{Num}\ v_1)\ e_2)) \\
& = & \mathcal{A}((\texttt{Plus}\ v_1\ \square) \triangleright s \succ e_2) \\
& = & \mathcal{A}(RHS)
\end{array}$$

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Showing Refinement for Plus

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$$\overline{(\texttt{Plus}\ v_1\ \square) \triangleright s \prec v_2 \quad \mapsto_C \quad s \prec v_1 + v_2}$$

This corresponds to a M-Machine transition:

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$$\begin{array}{rcll}
\mathcal{A}(LHS) & = & \mathcal{A}((\texttt{Plus}\ v_1\ \square) \triangleright s \prec v_2) & \\
& = & \mathcal{A}(s \succ (\texttt{Plus}\ (\texttt{Num}\ v_1)\ (\texttt{Num}\ v_2))) & \\
& \mapsto_M & \mathcal{A}(s \succ (\texttt{Num}\ v_1 + v_2)) & (*) \\
& = & \mathcal{A}(s \prec v_1 + v_2) & \\
& = & \mathcal{A}(RHS) &
\end{array}$$

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Technically the reduction step $(*)$ requires induction on the stack.