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COMP3161/9164
Concepts of Programming Languages

# Natural Deduction and Rule Induction

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UNSW, Term 3 2020

## Formalisation

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To talk about languages in a mathematical way, we need to formalise them.

**Formalisation**

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*Formalisation* is the process of giving a language a formal, mathematical description.

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# Formalisation

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To talk about languages in a mathematical way, we need to formalise them.

**Formalisation**

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*Formalisation* is the process of giving a language a formal, mathematical description.

Typically, we describe the language in another language, called the *meta-language*. For implementations, it may be a programming language such as Haskell, but for formalisations it is usually a minimal logic called a *meta-logic*.

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## Learning from History

What sort of meta logic should we use? There are a number of things to formalise:

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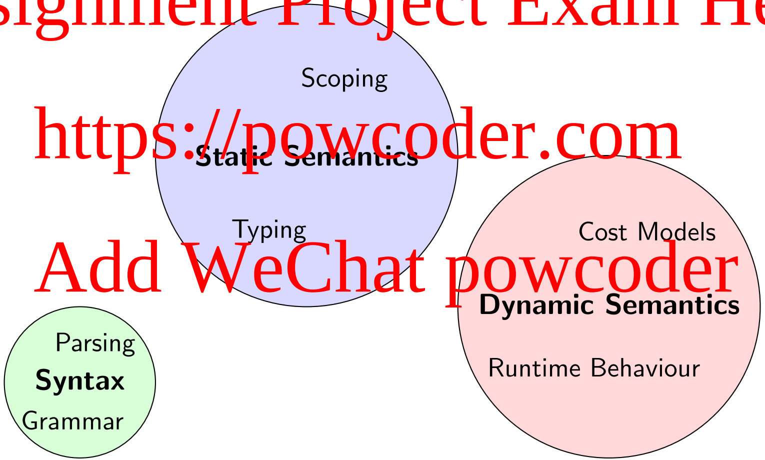
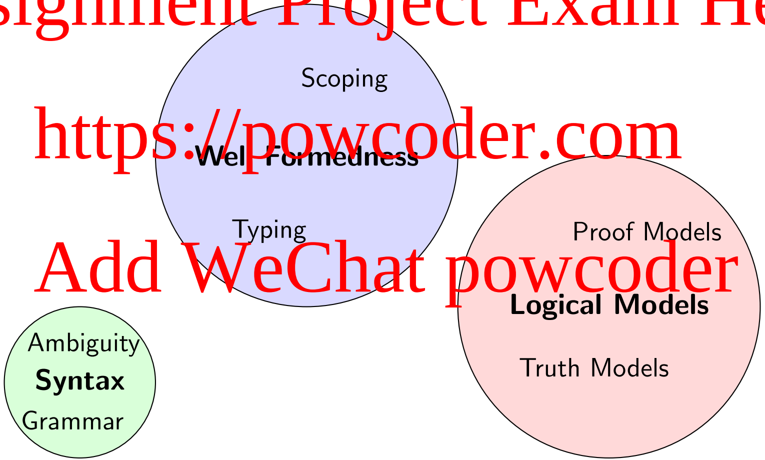


# Learning from History

Logicians in the early 20th century had much the same desire to formalise *logics*.

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# Learning from History

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In this course, we will use a meta-logic based on *Natural Deduction* and inductive inference rules, originally invented for formalising logics by Gerhard Gentzen in the mid 1930s.

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**Der Kalkül des natürlichen Schließens.**

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$$\frac{\mathfrak{A} \quad \mathfrak{B}}{\mathfrak{A} \,\&\, \mathfrak{B}} \qquad \frac{\mathfrak{A} \,\&\, \mathfrak{B}}{\mathfrak{A}} \qquad \frac{\mathfrak{A} \,\&\, \mathfrak{B}}{\mathfrak{B}}$$

# Judgements

A *judgement* is a statement asserting a certain property for an object.

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**Example (Informal Judgements)**

- $3 + 4 \times 5$ is a valid arithmetic expression.
- The string *madam* is a palindrome.
- The string *snooze* is a palindrome

$\Longrightarrow$ Judgements do not have to hold.

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# Judgements

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**Unary Judgements**

Formally, we denote the judgement that a property A holds for an object $s$ by writing $s$ A.

Typically, $s$ is a string when describing syntax, and $s$ is a term when describing semantics.

# Proving Judgements

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We define how a judgement may be *proven* by providing a set of *inference rules*.

## Inference Rules

An inference rule is written as:

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$$\frac{J_1 \qquad J_2 \qquad \ldots \qquad J_n}{J}$$

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This states that in order to prove judgement $J$ (the *conclusion*), it suffices to prove all judgements $J_1$ through to $J_n$ (the *premises*).

Rules with no premises are called *axioms*. Their conclusions always hold.

# Examples

**Example (Natural Numbers)**

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$\boxed{n \text{ Nat}}$

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$$\frac{}{0 \text{ Nat}}N_1 \qquad \frac{n \text{ Nat}}{(\mathsf{S}\ n) \text{ Nat}}N_2$$

0 is a natural number

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if $n$ is a natural number, then the successor of $n$ is a natural number.

What terms are in the set $\{n \mid n \text{ Nat}\}$?

## Examples

**Example (Natural Numbers)**

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$$\frac{}{0 \text{ Nat}}N_1 \qquad \frac{n \text{ Nat}}{(\mathsf{S}\ n) \text{ Nat}}N_2$$

0 is a natural number

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if $n$ is a natural number, then the successor of $n$ is a natural number.

What terms are in the set $\{n \mid n \text{ Nat}\}$?

$$\{0, (\mathsf{S}\ 0), (\mathsf{S}\ (\mathsf{S}\ 0)), (\mathsf{S}\ (\mathsf{S}\ (\mathsf{S}\ 0))), \dots\}$$

# Examples

## Example (Even and Odd Numbers)

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$\boxed{n \text{ Even}}$ $\boxed{n \text{ Odd}}$

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$$\frac{}{0\ \text{Even}}\,E_1 \qquad \frac{n\ \text{Even}}{(\text{S}\ (\text{S}\ n))\ \text{Even}}\,E_2 \qquad \frac{n\ \text{Even}}{(\text{S}\ n)\ \text{Odd}}\,O_1$$

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## The Proof Video Game

To show that a judgement $s$ A holds:

1. Find a rule whose conclusion matches $s$ A.
2. The preconditions of the applied rules become new proof obligations.
3. Rince and repeat until all obligations are proven up to axioms.

## Examples

**Example (Even and Odd Numbers)**

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$\boxed{n \text{ Even}}$ $\boxed{n \text{ Odd}}$

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$$\frac{}{0 \text{ Even}}E_1 \qquad \frac{n \text{ Even}}{(\mathsf{S}\ (\mathsf{S}\ n)) \text{ Even}}E_2 \qquad \frac{n \text{ Even}}{(\mathsf{S}\ n) \text{ Odd}}O_1$$

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$$\frac{}{(\mathsf{S}\ (\mathsf{S}\ (\mathsf{S}\ (\mathsf{S}\ (\mathsf{S}\ 0))))) \text{ Odd}}$$

# Examples

**Example (Even and Odd Numbers)**

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$\boxed{n \text{ Even}}$ $\boxed{n \text{ Odd}}$

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$$\frac{}{0 \text{ Even}} E_1 \qquad \frac{n \text{ Even}}{(\mathsf{S}\ (\mathsf{S}\ n)) \text{ Even}} E_2 \qquad \frac{n \text{ Even}}{(\mathsf{S}\ n) \text{ Odd}} O_1$$

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$$\frac{\overline{(\mathsf{S}\ (\mathsf{S}\ (\mathsf{S}\ (\mathsf{S}\ 0)))) \text{ Even}}}{(\mathsf{S}\ (\mathsf{S}\ (\mathsf{S}\ (\mathsf{S}\ (\mathsf{S}\ 0))))) \text{ Odd}} O_1$$

# Examples

**Example (Even and Odd Numbers)**

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$\boxed{n \text{ Even}}$ $\boxed{n \text{ Odd}}$

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$$\frac{}{0 \text{ Even}}\,E_1 \qquad \frac{n \text{ Even}}{(\mathsf{S}\ (\mathsf{S}\ n)) \text{ Even}}\,E_2 \qquad \frac{n \text{ Even}}{(\mathsf{S}\ n) \text{ Odd}}\,O_1$$

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$$\cfrac{\cfrac{\overline{(\mathsf{S}\ (\mathsf{S}\ 0)) \text{ Even}}}{(\mathsf{S}\ (\mathsf{S}\ (\mathsf{S}\ (\mathsf{S}\ 0)))) \text{ Even}}\,E_2}{(\mathsf{S}\ (\mathsf{S}\ (\mathsf{S}\ (\mathsf{S}\ (\mathsf{S}\ 0))))) \text{ Odd}}\,O_1$$

# Examples

**Example (Even and Odd Numbers)**

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$$\boxed{n\ \mathsf{Even}} \qquad \boxed{n\ \mathsf{Odd}}$$

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$$\frac{}{0\ \mathsf{Even}}\,E_1 \qquad \frac{n\ \mathsf{Even}}{(\mathsf{S}\ (\mathsf{S}\ n))\ \mathsf{Even}}\,E_2 \qquad \frac{n\ \mathsf{Even}}{(\mathsf{S}\ n)\ \mathsf{Odd}}\,O_1$$

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$$\cfrac{\cfrac{\cfrac{\overline{0\ \mathsf{Even}}}{(\mathsf{S}\ (\mathsf{S}\ 0))\ \mathsf{Even}}\,E_2}{(\mathsf{S}\ (\mathsf{S}\ (\mathsf{S}\ (\mathsf{S}\ 0))))\ \mathsf{Even}}\,E_2}{(\mathsf{S}\ (\mathsf{S}\ (\mathsf{S}\ (\mathsf{S}\ (\mathsf{S}\ 0)))))\ \mathsf{Odd}}\,O_1$$

## Examples

### Example (Even and Odd Numbers)

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$\boxed{n \text{ Even}}$ $\boxed{n \text{ Odd}}$

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$$\frac{}{0 \text{ Even}}E_1 \qquad \frac{n \text{ Even}}{(\mathsf{S}\ (\mathsf{S}\ n))\text{ Even}}E_2 \qquad \frac{n \text{ Even}}{(\mathsf{S}\ n)\text{ Odd}}O_1$$

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$$\cfrac{\cfrac{\cfrac{\cfrac{}{0 \text{ Even}}E_1}{(\mathsf{S}\ (\mathsf{S}\ 0))\text{ Even}}E_2}{(\mathsf{S}\ (\mathsf{S}\ (\mathsf{S}\ (\mathsf{S}\ 0))))\text{ Even}}E_2}{(\mathsf{S}\ (\mathsf{S}\ (\mathsf{S}\ (\mathsf{S}\ (\mathsf{S}\ 0)))))\text{ Odd}}O_1$$

# Defining Languages

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**Example (Bracket Matching Language)**

$$\mathsf{M} ::= \varepsilon \mid \mathsf{MM} \mid (\ \mathsf{M}\ )$$

Examples of strings: $\varepsilon$, (), (()), ()(), (()()), ...

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Three rules:

**Axiom** The empty string is in M

**Nesting** Any string in M can be surrounded by parentheses, giving a new string in M

**Juxtaposition** Any two strings in M can be concatenated to give a new string in M

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# With Rules

**The Language M**

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$$\boxed{s\ \mathsf{M}}$$

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$$\frac{}{\varepsilon\ \mathsf{M}}\,M_E \qquad \frac{s\ \mathsf{M}}{(\,s\,)\ \mathsf{M}}\,M_N \qquad \frac{s_1\ \mathsf{M} \quad s_2\ \mathsf{M}}{s_1 s_2\ \mathsf{M}}\,M_J$$

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$$\frac{}{(\,)\,(\,(\,)\,)\ \mathsf{M}}$$

# With Rules

## The Language M

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$\boxed{s\ \mathsf{M}}$

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$$\frac{}{\varepsilon\ \mathsf{M}}M_E \qquad \frac{s\ \mathsf{M}}{(\,s\,)\ \mathsf{M}}M_N \qquad \frac{s_1\ \mathsf{M} \quad s_2\ \mathsf{M}}{s_1 s_2\ \mathsf{M}}M_J$$

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$$\frac{\dfrac{}{(\,)\ \mathsf{M}} \qquad \dfrac{}{(\,(\,)\,)\ \mathsf{M}}}{(\,)\,(\,(\,)\,)\ \mathsf{M}}M_J$$

# With Rules

## The Language M

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$$\boxed{s \ \mathsf{M}}$$

$$\frac{}{\varepsilon \ \mathsf{M}}\,M_E \qquad \frac{s \ \mathsf{M}}{(\, s \,) \ \mathsf{M}}\,M_N \qquad \frac{s_1 \ \mathsf{M} \quad s_2 \ \mathsf{M}}{s_1 s_2 \ \mathsf{M}}\,M_J$$

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$$\dfrac{\dfrac{\dfrac{}{\varepsilon \ \mathsf{M}}\,M_E}{(\,) \ \mathsf{M}}\,M_N \qquad \dfrac{}{(\,(\,)\,) \ \mathsf{M}}}{(\,)\,(\,(\,)\,) \ \mathsf{M}}\,M_J$$

# With Rules

**The Language M**

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$$\boxed{s \ \mathsf{M}}$$

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$$\dfrac{}{\varepsilon \ \mathsf{M}}\, M_E \qquad \dfrac{s \ \mathsf{M}}{(\, s\,) \ \mathsf{M}}\, M_N \qquad \dfrac{s_1 \ \mathsf{M} \quad s_2 \ \mathsf{M}}{s_1 s_2 \ \mathsf{M}}\, M_J$$

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$$\dfrac{\dfrac{\dfrac{}{\varepsilon \ \mathsf{M}}\, M_E}{(\,) \ \mathsf{M}}\, M_N \qquad \dfrac{\dfrac{\dfrac{}{\varepsilon \ \mathsf{M}}\, M_E}{(\,) \ \mathsf{M}}\, M_N}{(\,(\,)\,) \ \mathsf{M}}\, M_N}{(\,)\,(\,(\,)\,) \ \mathsf{M}}\, M_J$$

# Getting Stuck

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If we had started with rule $M_N$ instead, we would have gotten stuck:

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$$\dfrac{\dfrac{???}{)\,(()\ \ \mathsf{M}}}{()\,(())\ \ \mathsf{M}}\,M_N$$

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**Takeaway**

Getting stuck does not mean what you're trying to prove is false!

## Derivability

Consider the following rule:

$$\frac{s\ \mathsf{M}}{(\,(s)\,)\ \mathsf{M}}$$

Does adding this rule change M? (i.e. is it not *admissible* to M)?

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## Derivability

Consider the following rule:

$$\frac{s \ \mathsf{M}}{(\,(s)\,) \ \mathsf{M}}$$

Does adding this rule change M? (i.e. is it not *admissible* to M)?

No, because we could always use rule $M_N$ twice instead. Rules that are compositions of existing rules are called *derivable*:

$$\dfrac{\dfrac{s \ \mathsf{M}}{(s) \ \mathsf{M}}\,M_N}{(\,(s)\,) \ \mathsf{M}}\,M_N$$

We can prove rules as well as judgements, by deriving the conclusion of the rule while taking the premises as local axioms.

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# Derivability

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Is this rule derivable?

$$\frac{s\ \mathsf{M}}{(s)\ s\ \mathsf{M}}$$

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## Derivability

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Is this rule derivable?

$$\frac{s\ \mathrm{M}}{(s)\,s\ \mathrm{M}}$$

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We can derive it like so:

$$\dfrac{\dfrac{\dfrac{}{s\ \mathrm{M}}}{(s)\ \mathrm{M}}\,M_N \qquad \dfrac{}{s\ \mathrm{M}}}{(s)\,s\ \mathrm{M}}\,M_J$$

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## Derivability

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Is this rule derivable?

$$\frac{(s)\ \mathsf{M}}{s\ \mathsf{M}}\,Q$$

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## Derivability

Assignment Project Exam Help

Is this rule derivable?

$$\frac{(s)\ \mathsf{M}}{s\ \mathsf{M}}\,Q$$

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It is not admissible, let alone derivable, as it adds strings to M

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\varepsilon\ \mathsf{M}}\,M_E}{()\ \mathsf{M}}\,M_N \qquad \dfrac{\dfrac{}{\varepsilon\ \mathsf{M}}\,M_E}{()\ \mathsf{M}}\,M_N}{()\,()\ \mathsf{M}}\,M_J}{)\,(\ \mathsf{M}}\,Q$$

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# Derivability

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Is this rule admissible? If so, is it derivable?

$$\frac{(\,)\ s \quad M}{s \quad M}$$

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## Derivability

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Is this rule admissible? If so, is it derivable?

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$$\frac{()\, s\ \mathsf{M}}{s\ \mathsf{M}}$$

- It is admissible, as it doesn't let us prove any new judgements about $\mathsf{M}$.
- It is not derivable, as it is not made up of the composition of existing rules.
- We will see how to prove these sorts of rules are admissible later on.

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## Hypothetical Derivations

We can write a rule in a horizontal format as well:

$$\frac{A}{B} \text{ is the same as } A \vdash B$$

This allows us to neatly make *rules* premises of other rules, called *hypothetical derivations*:

**Example**

$$\frac{A \vdash B}{C}$$

Read as: *If assuming $A$ we can derive $B$, then we can derive $C$.*

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## Specifying Logic

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With hypotheticals we can specify logic, which was the original purpose of natural deduction. Let $A$ True be the judgement that the proposition $A$ is true.

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**Example (And and Implies)**

$$\frac{A \text{ True} \qquad B \text{ True}}{A \wedge B \text{ True}}\wedge_I \qquad \frac{A \wedge B \text{ True}}{A \text{ True}}\wedge_{E1} \qquad \frac{A \wedge B \text{ True}}{B \text{ True}}\wedge_{E2}$$

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$$\frac{A \text{ True} \vdash B \text{ True}}{A \Rightarrow B \text{ True}}\Rightarrow_I \qquad \frac{A \Rightarrow B \text{ True} \qquad A \text{ True}}{B \text{ True}}\Rightarrow_E$$

# Specifying Logic

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With hypotheticals we can specify logic, which was the original purpose of natural deduction. Let $A$ True be the judgement that the proposition $A$ is true.

**Example (And and Implies)**

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$$\frac{A\ \text{True} \qquad B\ \text{True}}{A \wedge B\ \text{True}}\wedge_I \qquad \frac{A \wedge B\ \text{True}}{A\ \text{True}}\wedge_{E1} \qquad \frac{A \wedge B\ \text{True}}{B\ \text{True}}\wedge_{E2}$$

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$$\frac{A\ \text{True} \vdash B\ \text{True}}{A \Rightarrow B\ \text{True}}\Rightarrow_I \qquad \frac{A \Rightarrow B\ \text{True} \qquad A\ \text{True}}{B\ \text{True}}\Rightarrow_E$$

# Specifying Logic, Continued

**Example (Or, True, False and Not)**

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$$\frac{A \text{ True}}{A \lor B \text{ True}}\lor_{I1} \qquad \frac{B \text{ True}}{A \lor B \text{ True}}\lor_{I2}$$

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$$\frac{A \text{ True} \vdash C \text{ True} \qquad B \text{ True} \vdash C \text{ True} \qquad A \lor B \text{ True}}{C \text{ True}}\lor_{E}$$

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$$\frac{}{\top \text{ True}}\top_{I} \qquad \frac{\bot \text{ True}}{A \text{ True}}\bot_{E}$$

$$\frac{A \text{ True} \vdash \bot \text{ True}}{\neg A \text{ True}}\neg_{I} \qquad \frac{\neg A \text{ True} \qquad A \text{ True}}{B \text{ True}}\neg_{E}$$

# Specifying Logic, Continued

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**Example (Or, True, False and Not)**

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$$\frac{A \text{ True}}{A \vee B \text{ True}}\vee_{I1} \qquad \frac{B \text{ True}}{A \vee B \text{ True}}\vee_{I2}$$

$$\frac{A \text{ True} \vdash C \text{ True} \quad B \text{ True} \vdash C \text{ True} \quad A \vee B \text{ True}}{C \text{ True}}\vee_E$$

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$$\frac{}{\top \text{ True}}\top_I \qquad \frac{\bot \text{ True}}{A \text{ True}}\bot_E$$

$$\frac{A \text{ True} \vdash \bot \text{ True}}{\neg A \text{ True}}\neg_I \qquad \frac{\neg A \text{ True} \quad A \text{ True}}{B \text{ True}}\neg_E$$

# Minimal Definitions

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$$\boxed{s \ \mathrm{M}}$$

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$$\frac{}{\varepsilon \ \mathrm{M}}\, M_E \qquad \frac{s \ \mathrm{M}}{(s) \ \mathrm{M}}\, M_N \qquad \frac{s_1 \ \mathrm{M} \quad s_2 \ \mathrm{M}}{s_1 s_2 \ \mathrm{M}}\, M_J$$

The above rules are the smallest set of rules to define every string in M.

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**Therefore**

If we know that a string $s$ M, it must have been through one of these rules.

This is called an *inductive definition* of M.

# Rule Induction

Suppose we want to show that a property $P(s)$ of strings $s$ holds for any string $s$ M.
We will use *rule induction*.

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If we show that

$$\frac{}{\varepsilon\ \mathrm{M}}\,M_E \qquad P(\varepsilon) \text{ holds, and}$$

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$$\frac{s\ \mathrm{M}}{(s)\ \mathrm{M}}\,M_N \qquad P(s) \text{ implies } P((s))\text{, and}$$

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$$\frac{s_1\ \mathrm{M} \quad s_2\ \mathrm{M}}{s_1 s_2\ \mathrm{M}}\,M_J \qquad P(s_1) \text{ and } P(s_2) \text{ implies } P(s_1 s_2)$$

Then we have shown $P(s)$ for all $s$ M.

These assumptions are called *inductive hypotheses*.

# Rule Induction

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### Example (Counting Parens)

Let $op(s)$ denote the number of opening parentheses in $s$, and $cl(s)$ denote the number of closing parentheses. We shall prove that

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$$s \text{ M} \implies op(s) = cl(s)$$

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by doing rule induction on $s$ M.

# Rule Induction

**Example (Counting Parens)**

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$$\frac{}{\varepsilon \ \mathsf{M}} M_E$$

**Base Case:** $op(\varepsilon) = 0 = cl(\varepsilon)$

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# Rule Induction

**Example (Counting Parens)**

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$$\frac{}{\varepsilon \ \mathrm{M}} M_E \qquad \frac{s \ \mathrm{M}}{(s) \ \mathrm{M}} M_N$$

**Base Case:** $op(\varepsilon) = 0 = cl(\varepsilon)$

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**Inductive Case:** Assuming I.H.:

$$op(s) = cl(s)$$

$$op((s)) = op(s) + 1 = cl(s) + 1 = cl((s))$$

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# Rule Induction

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**Example (Counting Parens)**

$$\frac{}{\varepsilon \ \mathsf{M}}\, M_E$$

**Base Case:** $op(\varepsilon) = 0 = cl(\varepsilon)$

$$\frac{s \ \mathsf{M}}{(s) \ \mathsf{M}}\, M_N$$

**Inductive Case:** Assuming I.H.:

$$op(s) = cl(s)$$

$$op((s)) = op(s) + 1 = cl(s) + 1 = cl((s))$$

$$\frac{s_1 \ \mathsf{M} \quad s_2 \ \mathsf{M}}{s_1 s_2 \ \mathsf{M}}\, M_J$$

**Inductive Case:** Assuming I.Hs:

$op(s_1) = cl(s_1)$ and $op(s_2) = cl(s_2)$

$$op(s_1 s_2) = op(s_1) + op(s_2) = cl(s_1 s_2)$$

# Rule Induction in General

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## Rule Induction Method

Given a set of rules $R$, we may prove a property $P$ inductively for all judgements that can be inferred with $R$ by showing, for each rule of the form

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$$\frac{J_1 \quad J_2 \quad \dots \quad J_n}{J}$$

that if $P$ holds for each of $J_1 \dots J_n$, then $P$ holds for $J$.

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Therefore, axioms are the base cases of the induction, all other rules form inductive cases, and the premises of each rule give rise to inductive hypotheses.

# Structural Induction

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Conventional *structural induction* such as that on natural numbers, which we have encountered before, is a special case of rule induction.

## Natural Number Induction

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To show a property $P(n)$ for all $n \in \mathbb{N}$, it suffices to:

$$\frac{}{0 \ \mathsf{Nat}}$$

Show that $P(0)$ holds, and

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$$\frac{n \ \mathsf{Nat}}{(\mathsf{S}\ n)\ \mathsf{Nat}}$$

Assuming $P(n)$, show $P(n+1)$.

# Another Example

Recall our definition of even numbers:

$$\boxed{n \text{ Even}}$$

$$\frac{}{0 \text{ Even}}\,E_1 \qquad \frac{n \text{ Even}}{(\mathsf{S}\ (\mathsf{S}\ n))\ \text{Even}}\,E_2$$

We could define odd numbers differently:

$$\boxed{n \text{ Odd}'}$$

$$\frac{}{(\mathsf{S}\ 0)\ \text{Odd}'}\,O'_1 \qquad \frac{n \text{ Odd}'}{(\mathsf{S}\ (\mathsf{S}\ n))\ \text{Odd}'}\,O'_2$$

Let's prove the original Odd rule, but for Odd′ (to whiteboard):

$$\frac{n \text{ Even}}{(\mathsf{S}\ n)\ \text{Odd}'}$$

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# Arithmetic

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**Example (Arithmetic Expression)**

$$\text{Arith} ::= i \mid \text{Arith} \times \text{Arith} \mid \text{Arith} + \text{Arith} \mid (\text{Arith}) \quad (i \in \mathbb{Z})$$

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## Arithmetic

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**Example (Arithmetic Expression)**

$$\mathsf{Arith} ::= i \mid \mathsf{Arith} \times \mathsf{Arith} \mid \mathsf{Arith} + \mathsf{Arith} \mid (\mathsf{Arith}) \quad (i \in \mathbb{Z})$$

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$$\frac{i \in \mathbb{Z}}{i \ \mathsf{Arith}}L \qquad \frac{a \ \mathsf{Arith} \qquad b \ \mathsf{Arith}}{a \times b \ \mathsf{Arith}}P \qquad \frac{a \ \mathsf{Arith} \qquad b \ \mathsf{Arith}}{a + b \ \mathsf{Arith}}S \qquad \frac{a \ \mathsf{Arith}}{(a) \ \mathsf{Arith}}$$

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Infer $1 + 2 \times 3 \ \mathsf{Arith}$ (both ways) to whiteboard

## Ambiguity

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Arith is *ambiguous*, which means that there are multiple ways to derive the same judgement.

For syntax, this is a big problem, as different interpretations of syntax can lead to semantic inconsistency:

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$$\dfrac{\dfrac{1 \in \mathbb{Z}}{1\ \mathsf{Arith}} \qquad \dfrac{\dfrac{2 \in \mathbb{Z}}{2\ \mathsf{Arith}} \quad \dfrac{3 \in \mathbb{Z}}{3\ \mathsf{Arith}}}{2 \times 3\ \mathsf{Arith}}}{1 + 2 \times 3\ \mathsf{Arith}} \qquad \dfrac{\dfrac{\dfrac{1 \in \mathbb{Z}}{1\ \mathsf{Arith}} \quad \dfrac{2 \in \mathbb{Z}}{2\ \mathsf{Arith}}}{1 + 2\ \mathsf{Arith}} \qquad \dfrac{3 \in \mathbb{Z}}{3\ \mathsf{Arith}}}{1 + 2 \times 3\ \mathsf{Arith}}$$

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# Second Attempt

We want to specify Arith in such a way that enforces order of operations.
Here we will use multiple judgements:

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**Example (Arithmetic Expression)**

$$\begin{array}{lcl} \text{Atom} & ::= & i \mid (\text{SExp}) \quad (i \in \mathbb{Z}) \\ \text{PExp} & ::= & \text{Atom} \mid \text{PExp} \times \text{PExp} \\ \text{SExp} & ::= & \text{PExp} \mid \text{SExp} + \text{SExp} \end{array}$$

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## Second Attempt

We want to specify Arith in such a way that enforces order of operations.
Here we will use multiple judgements:

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### Example (Arithmetic Expression)

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$$
\begin{array}{lcl}
\mathsf{Atom} & ::= & i \mid (\mathsf{SExp}) \quad (i \in \mathbb{Z}) \\
\mathsf{PExp} & ::= & \mathsf{Atom} \mid \mathsf{PExp} \times \mathsf{PExp} \\
\mathsf{SExp} & ::= & \mathsf{PExp} \mid \mathsf{SExp} + \mathsf{SExp}
\end{array}
$$

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$$
\frac{i \in \mathbb{Z}}{i \ \mathsf{Atom}} \qquad \frac{a \ \mathsf{SExp}}{(a) \ \mathsf{Atom}} \qquad \frac{e \ \mathsf{Atom}}{e \ \mathsf{PExp}} \qquad \frac{e \ \mathsf{PExp}}{e \ \mathsf{SExp}}
$$

$$
\frac{a \ \mathsf{PExp} \qquad b \ \mathsf{PExp}}{a \times b \ \mathsf{PExp}} \qquad \frac{a \ \mathsf{SExp} \qquad b \ \mathsf{SExp}}{a + b \ \mathsf{SExp}}
$$

Consider: Is there still any ambiguity here?

## More ambiguity

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$$
\dfrac{\dfrac{\dfrac{1 \in \mathbb{Z}}{1\ \mathsf{Atom}}}{1\ \mathsf{PExp}} \qquad \dfrac{\dfrac{\dfrac{2 \in \mathbb{Z}}{2\ \mathsf{Atom}}}{2\ \mathsf{PExp}} \quad \dfrac{\dfrac{3 \in \mathbb{Z}}{3\ \mathsf{Atom}}}{3\ \mathsf{PExp}}}{2 \times 3\ \mathsf{PExp}}}{1 \times 2 \times 3\ \mathsf{PExp}}
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{1 \in \mathbb{Z}}{1\ \mathsf{Atom}}}{1\ \mathsf{PExp}} \quad \dfrac{\dfrac{2 \in \mathbb{Z}}{2\ \mathsf{Atom}}}{2\ \mathsf{PExp}}}{1 \times 2\ \mathsf{PExp}} \qquad \dfrac{\dfrac{3 \in \mathbb{Z}}{3\ \mathsf{Atom}}}{3\ \mathsf{PExp}}}{1 \times 2 \times 3\ \mathsf{PExp}}
$$

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This ambiguity seems harmless, but it would not be harmless for some other operations. Which ones?

## More ambiguity

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$$\dfrac{\dfrac{\dfrac{1 \in \mathbb{Z}}{1\ \mathsf{Atom}}}{1\ \mathsf{PExp}} \qquad \dfrac{\dfrac{\dfrac{2 \in \mathbb{Z}}{2\ \mathsf{Atom}}}{2\ \mathsf{PExp}} \quad \dfrac{\dfrac{3 \in \mathbb{Z}}{3\ \mathsf{Atom}}}{3\ \mathsf{PExp}}}{2 \times 3\ \mathsf{PExp}}}{1 \times 2 \times 3\ \mathsf{PExp}} \qquad \dfrac{\dfrac{\dfrac{\dfrac{1 \in \mathbb{Z}}{1\ \mathsf{Atom}}}{1\ \mathsf{PExp}} \quad \dfrac{\dfrac{2 \in \mathbb{Z}}{2\ \mathsf{Atom}}}{2\ \mathsf{PExp}}}{1 \times 2\ \mathsf{PExp}} \qquad \dfrac{\dfrac{3 \in \mathbb{Z}}{3\ \mathsf{Atom}}}{3\ \mathsf{PExp}}}{1 \times 2 \times 3\ \mathsf{PExp}}$$

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This ambiguity seems harmless, but it would not be harmless for some other operations. Which ones? Operators that are not *associative*.

We have to specify the *associativity* of operators. How?

# Associativities

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Operators have various *associativity* constraints:

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**Associative** All associativities are equal.

**Left-Associative** $A \odot B \odot C = (A \odot B) \odot C$

**Right-Associative** $A \odot B \odot C = A \odot (B \odot C)$

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Try to think of some examples!

# Enforcing associativity

We force the grammar to accept a smaller set of expressions on one side of the operator only. Show how this works on the whiteboard.

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**Example (Arithmetic Expression)**

$$
\begin{array}{lcl}
\text{Atom} & ::= & i \mid (\text{SExp}) \quad (i \in \mathbb{Z}) \\
\text{PExp} & ::= & \text{Atom} \mid \text{Atom} \times \text{PExp} \\
\text{SExp} & ::= & \text{PExp} \mid \text{PExp} + \text{SExp}
\end{array}
$$

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# Enforcing associativity

We force the grammar to accept a smaller set of expressions on one side of the operator only. Show how this works on the whiteboard.

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### Example (Arithmetic Expression)

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$$\begin{array}{rcl} \text{Atom} & ::= & i \mid (\text{SExp}) \quad (i \in \mathbb{Z}) \\ \text{PExp} & ::= & \text{Atom} \mid \text{Atom} \times \text{PExp} \\ \text{SExp} & ::= & \text{PExp} \mid \text{PExp} + \text{SExp} \end{array}$$

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$$\frac{i \in \mathbb{Z}}{i \ \text{Atom}} \qquad \frac{a \ \text{SExp}}{(a) \ \text{Atom}} \qquad \frac{e \ \text{Atom}}{e \ \text{PExp}} \qquad \frac{e \ \text{PExp}}{e \ \text{SExp}}$$

$$\frac{a \ \text{Atom} \qquad b \ \text{PExp}}{a \times b \ \text{PExp}} \qquad \frac{a \ \text{PExp} \qquad b \ \text{SExp}}{a + b \ \text{SExp}}$$

Here we made multiplication and addition right associative. How would we do left?

## Bring Back Parentheses

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### The Parenthetical Language

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$$\boxed{s\ \mathsf{M}}$$

$$\frac{}{\varepsilon\ \mathsf{M}}\,M_E \qquad \frac{s\ \mathsf{M}}{(s)\ \mathsf{M}}\,M_N \qquad \frac{s_1\ \mathsf{M} \qquad s_2\ \mathsf{M}}{s_1 s_2\ \mathsf{M}}\,M_J$$

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Is this language ambiguous? to whiteboard

## Ambiguity in Parentheses

Not only is it ambiguous, it is infinitely so. Strings like ()()() could be split at two different locations by rule $M_J$, but if we use $\varepsilon$, then even the string () is ambiguous:

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$$\dfrac{\dfrac{}{\varepsilon\ \mathrm{M}}M_E}{(\,)\ \mathrm{M}}M_N \qquad \dfrac{\dfrac{}{\varepsilon\ \mathrm{M}}M_E \quad \dfrac{\dfrac{}{\varepsilon\ \mathrm{M}}M_E}{(\,)\ \mathrm{M}}M_N}{(\,)\ \mathrm{M}}M_J$$

$$\dfrac{\dfrac{}{\varepsilon\ \mathrm{M}}M_E \quad \dfrac{\dfrac{}{\varepsilon\ \mathrm{M}}M_E \quad \dfrac{\dfrac{}{\varepsilon\ \mathrm{M}}M_E}{(\,)\ \mathrm{M}}M_N}{(\,)\ \mathrm{M}}M_J}{(\,)\ \mathrm{M}}M_J$$

We will eliminate the ambiguity by once again splitting M into two judgements, N and L.

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The crucial observation is that terms in M are a list (L) of terms nested within parentheses (N).

**Example (Unambiguous Parentheses)**

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$$\boxed{s\ \mathsf{L}} \qquad \boxed{s\ \mathsf{N}}$$

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$$\frac{}{\varepsilon\ \mathsf{L}}\,L_E \qquad \frac{s\ \mathsf{L}}{(s)\ \mathsf{N}}\,N_N \qquad \frac{s_1\ \mathsf{N} \qquad s_2\ \mathsf{L}}{s_1 s_2\ \mathsf{L}}\,L_J$$

# Proving Equivalence

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Now we shall prove $\mathsf{M} = \mathsf{L}$. There are two cases, each dispatched with rule induction:

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$$\frac{s\ \mathsf{M}}{s\ \mathsf{L}} \qquad \frac{s\ \mathsf{L}}{s\ \mathsf{M}}$$

The first case requires proving a *lemma*. The second requires *simultaneous induction*. These proofs will be carried out on the "board" (iPad). A properly typeset PDF of the proof will also be uploaded.

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