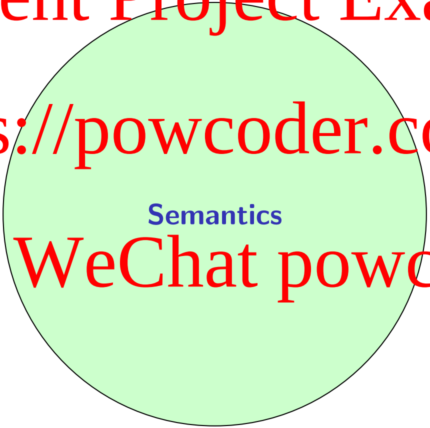
COMP3161/9164
Concepts of Programming Languages

Semantics

Dr. Liam O'Connor
University of Edinburgh LFCS
UNSW, Term 3 2020

Semantics

$\sigma\eta\mu\alpha\nu\tau\iota\chi\omega\varsigma$



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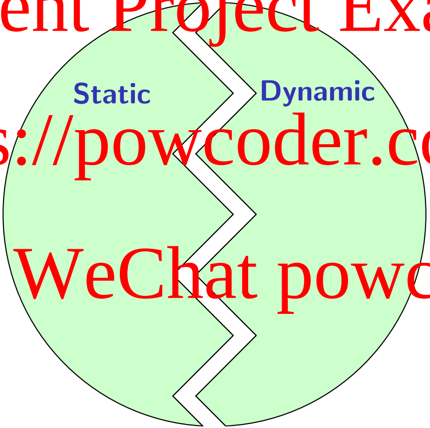

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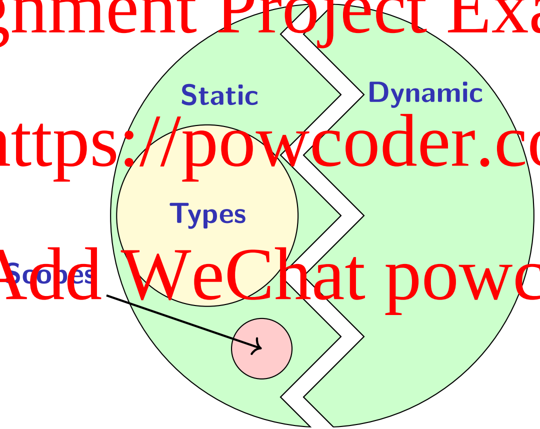

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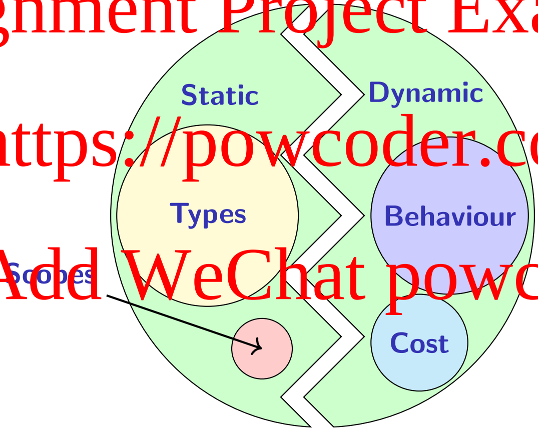

Static Semantics

Definition

The *static semantics* of a program is those significant aspects of the meaning of P that can be determined by the compiler (or an external lint tool) without running the program.

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Recall our arithmetic expression language. What properties might we derive statically about those terms?

The only thing we can check is that the program is well-scoped (assuming FOAS).

Scope-Checking

$$\frac{}{(\texttt{Num}\ n)\ \textbf{ok}} \qquad \frac{e_1\ \textbf{ok} \qquad e_2\ \textbf{ok}}{(\texttt{Times}\ e_1\ e_2)\ \textbf{ok}} \qquad \frac{e_1\ \textbf{ok} \qquad e_2\ \textbf{ok}}{(\texttt{Plus}\ e_1\ e_2)\ \textbf{ok}}$$

$$\frac{}{\Gamma \vdash (\texttt{Var}\ x)\ \textbf{ok}}$$

Key Idea

We keep a *context* Γ, a set of assumptions, on the left hand of our judgement e **ok**, indicating what is required in order for e to be *well-scoped*.
This could be read as hypothetical derivations for the judgement e **ok** or as a binary judgement $\Gamma \vdash e$ **ok**; whichever you prefer.

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Dynamic Semantics

Dynamic Semantics can be specified in many ways:

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In this course

We focus mostly on operational semantics. We will use axiomatic semantics (Hoare Logic) on Thursday in the imperative programming topic. Denotational semantics are mostly an extension topic, except for the very next slide.

Denotational Semantics

$$[\![\cdot]\!] : \textbf{AST} \rightarrow$$

$$
\begin{array}{ll}
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[\![\texttt{Plus}\ e_1\ e_2]\!] & = \\
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[\![\texttt{Let}\ x\ e_1\ e_2]\!] & =
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Denotational Semantics

$$[\![\cdot]\!] : \textbf{AST} \rightarrow \mathbb{Z}$$

$$
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$$[\![\cdot]\!] : \textbf{AST} \rightarrow (\textbf{Var} \nrightarrow \mathbb{Z}) \rightarrow \mathbb{Z}$$

Our denotation for arithmetic expressions is functions from *environments* (mapping from variables to their values) to values.

$$\begin{array}{lcl}
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Where $E[x := n]$ is a new environment just like E except the variable x now maps to n.

Operational Semantics

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Small Step

- Also called *structural operational semantics (SOS)*.
- A judgement that specifies transitions between *states*:

$$e \mapsto e'$$

Big Step

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- One big judgement relating expressions to their values:

$$e \Downarrow v$$

Big-Step Semantics

We need:

- A set of evaluable expressions E
- A set of values V
- A relation $\Downarrow \subseteq E \times V$

Example (Arithmetic Expressions)

E is the set of all closed expressions $\{e \mid e\ \mathbf{ok}\}$. V is the set of integers $\mathbb{Z}$.

$$\frac{}{(\texttt{Num}\ n) \Downarrow n}$$

$$\frac{e_1 \Downarrow v_1 \qquad e_2 \Downarrow v_2}{(\texttt{Plus}\ e_1\ e_2) \Downarrow (v_1 + v_2)} \qquad \frac{e_1 \Downarrow v_1 \qquad e_2 \Downarrow v_2}{(\texttt{Times}\ e_1\ e_2) \Downarrow (v_1 \times v_2)}$$

To Code Let's do it in Haskell!

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Evaluation Strategies

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Any other ways to evaluate Let?

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The above is called *call-by-value* or strict evaluation. Below we have *call-by-name*:

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This can be computationally very expensive, for example:

$$\texttt{let}\ x = \langle \textit{very expensive computation} \rangle\ \texttt{in}\ x + x + x + x$$

In confluent languages like this or λ-calculus, this only matters for performance. In other languages, this is not so. Why?

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Haskell uses *call-by-need* or lazy evaluation, which optimises cases like this.

Small Step Semantics

For small step semantics, we need:

- A set of states Σ
- A set of initial states $I \subseteq \Sigma$
- A set of final states $F \subseteq \Sigma$
- A relation $\mapsto \subseteq \Sigma \times \Sigma$, which specifies only "one step" of the execution.

An *execution* or *trace* $\sigma_1 \mapsto \sigma_2 \mapsto \sigma_3 \mapsto \cdots \mapsto \sigma_n$ is called *maximal* if there exists no σ_{n+1} such that $\sigma_n \mapsto \sigma_{n+1}$; and is called *complete* if it is maximal and $\sigma_n \in F$.

Example

Example (Arithmetic Expressions)

Σ and I are the set of all closed expressions $\{e \mid e\ \textbf{ok}\}$. F is the set of evaluated expressions $\{(\texttt{Num}\ n) \mid n \in \mathbb{Z}\}$.

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To Code Let's do it in Haskell!

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$$\frac{}{(\texttt{Let}\ (\texttt{Num}\ n)\ (x.\ e_2)) \mapsto e_2[x := \texttt{Num}\ n]}$$

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Equivalence

Comparing small step and big step

Small step semantics are lower-level, they clearly specify the order of evaluation. Big step semantics give us a result without telling us explicitly how it was computed.

Having specified the dynamic semantics in these two ways, it becomes desirable to show they are equivalent, that is:

If there exists a trace $e \mapsto \cdots \mapsto (\texttt{Num}\ n)$, *then* $e \Downarrow n$, *and vice versa.*

We will need to define some notation to remove those blasted magic dots.

Notation

Let $\overset{\star}{\mapsto}$ be the *reflexive, transitive closure* of $\mapsto$.

$$\frac{}{e \overset{\star}{\mapsto} e} \qquad \frac{e_1 \mapsto e_2 \qquad e_2 \overset{\star}{\mapsto} e_n}{e_1 \overset{\star}{\mapsto} e_n}$$

We can now state our property formally as:

$$e \overset{\star}{\mapsto} (\texttt{Num}\ n) \iff e \Downarrow n$$

Doing the Proof

The proof will be done on the iPad, with typeset versions being uploaded as usual.
The big-step to small-step direction can be proven by reasonably straightforward rule induction:

$$\frac{e \Downarrow n}{e \mapsto^{\star} n}$$

The other direction requires the lemma:

$$\frac{e \mapsto e' \qquad e' \Downarrow n}{e \Downarrow n}$$

The abridged proof is presented in this lecture, with all cases left for the course website.