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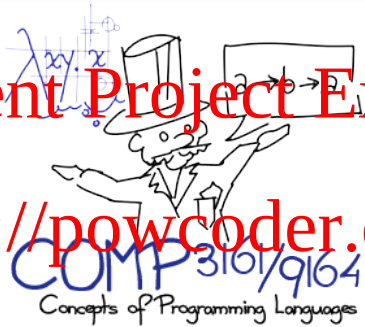

Dr. Liam O'Connor
University of Edinburgh LFCS
UNSW, Term 3 2020

λ-Calculus

The term language we defined for Higher Order Abstract Syntax is almost a full featured programming language.
Just enrich the syntax slightly:

$$
\begin{array}{rcll}
t & ::= & \texttt{Symbol} & \\
 & | & x & \textit{(variables)} \\
 & | & t_1\ t_2 & \textit{(application)} \\
 & | & \lambda x.\ t & \textit{(λ-abstraction)}
\end{array}
$$

There is just one rule to evaluate terms, called β-reduction:

$$(\lambda x.\ t)\ u \quad \mapsto_\beta \quad t[x := u]$$

Just as in Haskell, $(\lambda x.\ t)$ denotes a function that, given an argument for x, will return t.

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Syntax Concerns

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Function application is left-associative:

$$f\ a\ b\ c \quad = \quad ((f\ a)\ b)\ c$$

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λ-abstraction extends as far as possible:

$$\lambda a.\ f\ a\ b \quad = \quad \lambda a.\ (f\ a\ b)$$

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All functions are unary, like Haskell. Multiple argument functions are modelled with nested λ-abstractions:

$$\lambda x.\lambda y.\ x + y$$

β-reduction

β-reduction is a *congruence*:

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$$\frac{}{(\lambda x.\ t)\ u \mapsto_\beta t[x := u]}$$

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$$\frac{t \mapsto_\beta t'}{s\ t \mapsto_\beta s\ t'} \qquad \frac{s \mapsto_\beta s'}{s\ t \mapsto_\beta s'\ t} \qquad \frac{t \mapsto_\beta t'}{\lambda x.\ t \mapsto_\beta \lambda x.\ t'}$$

This means we can pick any reducible subexpression (called a *redex*) and perform β-reduction.

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Example:

$$(\lambda x.\ \lambda y.\ f\ (y\ x))\ 5\ (\lambda x.\ x)$$

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Example:

$$\begin{aligned}(\lambda x.\ \lambda y.\ f\ (y\ x))\ 5\ (\lambda x.\ x) &\mapsto_\beta (\lambda y.\ f\ (y\ 5))\ (\lambda x.\ x)\\ &\mapsto_\beta f\ ((\lambda x.\ x)\ 5)\end{aligned}$$

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$$\begin{aligned}(\lambda x.\ \lambda y.\ f\ (y\ x))\ 5\ (\lambda x.\ x) &\mapsto_\beta (\lambda y.\ f\ (y\ 5))\ (\lambda x.\ x)\\ &\mapsto_\beta f\ ((\lambda x.\ x)\ 5)\\ &\mapsto_\beta f\ 5\end{aligned}$$

Confluence

Supposing we arrive via one reduction path to an expression that cannot be reduced further (called a *normal form*), then any other reduction path will result in the same normal form.

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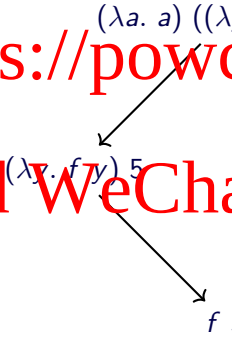

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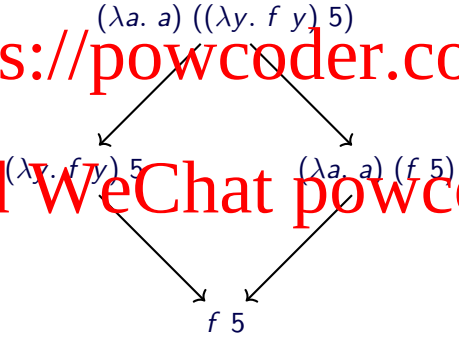

Equivalence

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Confluence means we can define another notion of *equivalence*, which equates more than α-equivalence. Two terms are $\alpha\beta$-*equivalent*, written $s \equiv_{\alpha\beta} t$ if they β-reduce to α-equivalent normal forms.

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Equivalence

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η

There is also another equation that cannot be proven from β-equivalence alone, called η-reduction:

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$$\lambda x.\, f\ x \to_\eta f$$

Adding this reduction to the system preserves confluence and uniqueness of normal forms, so we have a notion of $\alpha\beta\eta$-*equivalence* also.

Normal Forms

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Does every term in λ-calculus have a normal form?

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Normal Forms

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Does every term in λ-calculus have a normal form?

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$(\lambda x.\ x\ x)(\lambda x.\ x\ x)$

Try to β-reduce this! (the answer is that it doesn't have a normal form)

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Why learn this stuff?

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- λ-calculus is a *Turing-complete* programming language.
- λ-calculus is the foundation for every functional programming language and some non-functional ones.
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- λ-calculus is the foundation of *Higher Order Logic* and *Type Theory*, the two main foundations used for mathematics in interactive proof assistants.
- λ-calculus is the smallest example of a usable programming language, so it's good for teaching about programming languages.
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Making λ-Calculus Usable

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In order to demonstrate that λ calculus is actually a usable programming language, we will demonstrate how to encode booleans and natural numbers as λ-terms, along with their operations.

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General Idea

We transform a data type into the type of its *eliminator*. In other words, we make a function that can serve the same purpose as the data type at its use sites.

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Booleans

How do we use booleans?

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Booleans

How do we use booleans? To choose between two results!

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Booleans

How do we use booleans? To choose between two results!

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So, a boolean will be a function that, given two arguments, returns the first one if it is true and the second one if it is false:

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$$\textsc{True} \equiv \lambda a.\ \lambda b.\ a$$
$$\textsc{False} \equiv \lambda a.\ \lambda b.\ b$$

How do we write conjunction? to board

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Example (Test it out!)

Try β-normalising $\textsc{And}\ \textsc{True}\ \textsc{False}$.

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Example (Test it out!)

Try β-normalising $\textsc{And}\ \textsc{True}\ \textsc{False}$.

What about $\textsc{Or}$?

Natural Numbers

How do we use natural numbers?

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Natural Numbers

How do we use natural numbers? To do something n times!

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Natural Numbers

How do we use natural numbers? To do something n times!

So, a natural number will be a function that takes a function f and a value x, and applies the function f to x that number of times:

$$\begin{aligned} \textsc{Zero} &\equiv \lambda f.\ \lambda x.\ x \\ \textsc{One} &\equiv \lambda f.\ \lambda x.\ f\ x \\ \textsc{Two} &\equiv \lambda f.\ \lambda x.\ f\ (f\ x) \\ &\ldots \end{aligned}$$

How do we write $\textsc{Succ}$?

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How do we write $\textsc{Suc}$?

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$$\textsc{Suc} \equiv \lambda n.\ \lambda f.\ \lambda x.\ f\ (n\ f\ x)$$

How do we write $\textsc{Add}$?

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How do we write $\textsc{Suc}$?

$$\textsc{Suc} \equiv \lambda n.\ \lambda f.\ \lambda x.\ f\ (n\ f\ x)$$

How do we write $\textsc{Add}$?

$$\textsc{Add} \equiv \lambda m.\lambda n.\ \lambda f.\ \lambda x.\ m\ f\ (n\ f\ x)$$

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How do we write $\textsc{Suc}$?

$$\textsc{Suc} \equiv \lambda n.\ \lambda f.\ \lambda x.\ f\ (n\ f\ x)$$

How do we write $\textsc{Add}$?

$$\textsc{Add} \equiv \lambda m.\lambda n.\ \lambda f.\ \lambda x.\ m\ f\ (n\ f\ x)$$

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Natural Number Practice

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Example

Try β-normalising $\textsc{Suc}\ \textsc{One}$.

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Example

Try writing a different λ-term for defining $\textsc{Suc}$.

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Example

Try writing a λ-term for defining $\textsc{Multiply}$.