

## LECTURE 25

### Network Flows I

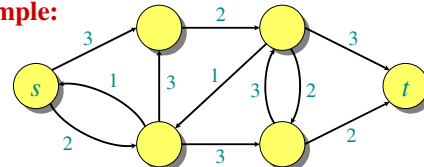
- Flow networks
- Max flow problem
- Residual network
- Augmenting paths
- Max flow-min cut theorem



## Flow networks

**Definition.** A **flow network** is a directed graph  $G = (V, E)$  with two distinguished vertices: a **source**  $s$  and a **sink**  $t$ . Each edge  $(u, v) \in E$  has a nonnegative **capacity**  $c(u, v)$ . If  $(u, v) \notin E$ , then  $c(u, v) = 0$ .

**Example:**



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## Flow networks

**Definition.** A **positive flow** on  $G$  is a function  $p : V \times V \rightarrow \mathbb{R}$  satisfying the following:

- **Capacity constraint:** For all  $u, v \in V$ ,  $0 \leq p(u, v) \leq c(u, v)$ .
- **Flow conservation:** For all  $u \in V - \{s, t\}$ ,  $\sum_{v \in V} p(u, v) - \sum_{v \in V} p(v, u) = 0$ .

The **value** of a flow is the net flow out of the source:

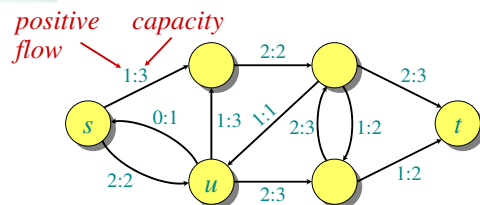
$$\sum_{v \in V} p(s, v) - \sum_{v \in V} p(v, s).$$

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## A flow on a network



**Flow conservation** (like Kirchoff's current law):

- Flow into  $u$  is  $2 + 1 = 3$ .
- Flow out of  $u$  is  $0 + 1 + 2 = 3$ .

The value of this flow is  $1 - 0 + 2 = 3$ .

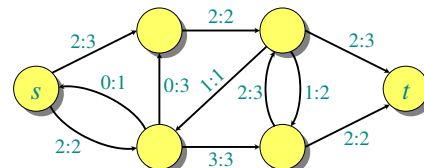
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## The maximum-flow problem

**Maximum-flow problem:** Given a flow network  $G$ , find a flow of maximum value on  $G$ .



The value of the maximum flow is 4.

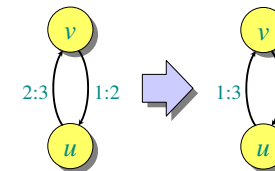
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## Flow cancellation

Without loss of generality, positive flow goes either from  $u$  to  $v$ , or from  $v$  to  $u$ , but not both.



Net flow from  $u$  to  $v$  in both cases is 1.

The capacity constraint and flow conservation are preserved by this transformation.

**INTUITION:** View flow as a **rate**, not a **quantity**.

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## A notational simplification

**IDEA:** Work with the net flow between two vertices, rather than with the positive flow.

**Definition.** A **(net) flow** on  $G$  is a function  $f : V \times V \rightarrow \mathbb{R}$  satisfying the following:

- **Capacity constraint:** For all  $u, v \in V$ ,  $f(u, v) \leq c(u, v)$ .
- **Flow conservation:** For all  $u \in V - \{s, t\}$ ,  $\sum_{v \in V} f(u, v) = 0$ .

← One summation instead of two.

- **Skew symmetry:** For all  $u, v \in V$ ,  $f(u, v) = -f(v, u)$ .

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## Equivalence of definitions

**Theorem.** The two definitions are equivalent.

**Proof.** ( $\Rightarrow$ ) Let  $f(u, v) = p(u, v) - p(v, u)$ .

- **Capacity constraint:** Since  $p(u, v) \leq c(u, v)$  and  $p(v, u) \geq 0$ , we have  $f(u, v) \leq c(u, v)$ .
- **Flow conservation:**

$$\begin{aligned} \sum_{v \in V} f(u, v) &= \sum_{v \in V} (p(u, v) - p(v, u)) \\ &= \sum_{v \in V} p(u, v) - \sum_{v \in V} p(v, u) \end{aligned}$$

- **Skew symmetry:**

$$\begin{aligned} f(u, v) &= p(u, v) - p(v, u) \\ &= -(p(v, u) - p(u, v)) \\ &= -f(v, u). \end{aligned}$$

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## Proof (continued)

( $\Leftarrow$ ) Let

$$p(u, v) = \begin{cases} f(u, v) & \text{if } f(u, v) > 0, \\ 0 & \text{if } f(u, v) \leq 0. \end{cases}$$

- **Capacity constraint:** By definition,  $p(u, v) \geq 0$ . Since  $f(u, v) \leq c(u, v)$ , it follows that  $p(u, v) \leq c(u, v)$ .
- **Flow conservation:** If  $f(u, v) > 0$ , then  $p(u, v) - p(v, u) = f(u, v)$ . If  $f(u, v) \leq 0$ , then  $p(u, v) - p(v, u) = -f(v, u) = f(u, v)$  by skew symmetry. Therefore,

$$\sum_{v \in V} p(u, v) - \sum_{v \in V} p(v, u) = \sum_{v \in V} f(u, v). \quad \square$$

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## Notation

**Definition.** The **value** of a flow  $f$ , denoted by  $|f|$ , is given by

$$|f| = \sum_{v \in V} f(s, v) = f(s, V).$$

**Implicit summation notation:** A set used in an arithmetic formula represents a sum over the elements of the set.

• **Example** — flow conservation:  
 $f(u, V) = 0$  for all  $u \in V - \{s, t\}$ .

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## Simple properties of flow

**Lemma.**

- $f(X, X) = 0$ ,
- $f(X, Y) = -f(Y, X)$ ,
- $f(X \cup Y, Z) = f(X, Z) + f(Y, Z)$  if  $X \cap Y = \emptyset$ .  $\square$

**Theorem.**  $|f| = f(V, t)$ .

*Proof.*

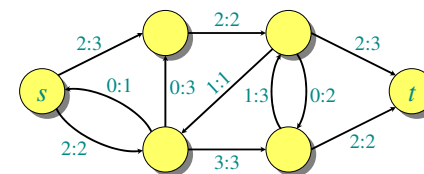
$$\begin{aligned} |f| &= f(s, V) \\ &= f(V, V) - f(V - s, V) \quad \text{Omit braces.} \\ &= f(V, V - s) \\ &= f(V, t) + f(V, V - s - t) \\ &= f(V, t). \quad \square \end{aligned}$$

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## Flow into the sink



$$|f| = f(s, V) = 4$$

$$f(V, t) = 4$$

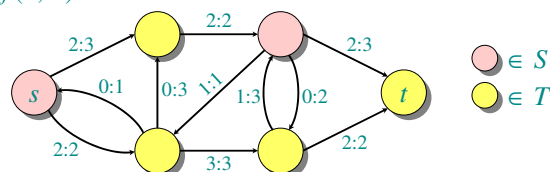
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## Cuts

**Definition.** A **cut**  $(S, T)$  of a flow network  $G = (V, E)$  is a partition of  $V$  such that  $s \in S$  and  $t \in T$ . If  $f$  is a flow on  $G$ , then the **flow across the cut** is  $f(S, T)$ .



$$f(S, T) = (2 + 2) + (-2 + 1 - 1 + 2) = 4$$

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## Another characterization of flow value

**Lemma.** For any flow  $f$  and any cut  $(S, T)$ , we have  $|f| = f(S, T)$ .

*Proof.*

$$\begin{aligned} f(S, T) &= f(S, V) - f(S, S) \\ &= f(S, V) \\ &= f(s, V) + f(S - s, V) \\ &= f(s, V) \\ &= |f|. \quad \square \end{aligned}$$

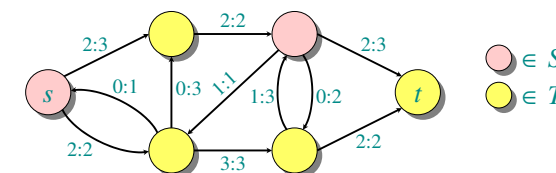
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## Capacity of a cut

**Definition.** The **capacity of a cut**  $(S, T)$  is  $c(S, T)$ .



$$c(S, T) = (3 + 2) + (1 + 2 + 3) = 11$$

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## Upper bound on the maximum flow value

**Theorem.** The value of any flow is bounded above by the capacity of any cut.

*Proof.*

$$\begin{aligned} |f| &= f(S, T) \\ &= \sum_{u \in S} \sum_{v \in T} f(u, v) \\ &\leq \sum_{u \in S} \sum_{v \in T} c(u, v) \\ &= c(S, T). \quad \square \end{aligned}$$

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## Residual network

**Definition.** Let  $f$  be a flow on  $G = (V, E)$ . The **residual network**  $G_f(V, E_f)$  is the graph with strictly positive **residual capacities**

$$c_f(u, v) = c(u, v) - f(u, v) > 0.$$

Edges in  $E_f$  admit more flow.

**Example:**



**Lemma.**  $|E_f| \leq 2|E|$ .  $\square$

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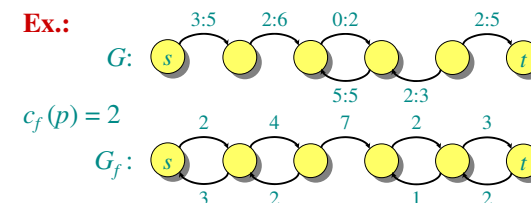
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## Augmenting paths

**Definition.** Any path from  $s$  to  $t$  in  $G_f$  is an **augmenting path** in  $G$  with respect to  $f$ . The flow value can be increased along an augmenting path  $p$  by  $c_f(p) = \min_{(u,v) \in p} \{c_f(u, v)\}$ .

**Ex.:**



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## Max-flow, min-cut theorem

**Theorem.** The following are equivalent:

1.  $f$  is a maximum flow.
2.  $f$  admits no augmenting paths.
3.  $|f| = c(S, T)$  for some cut  $(S, T)$ .

*Proof* (and algorithms). Next time. □