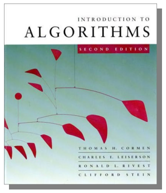
Analysis of Algorithms



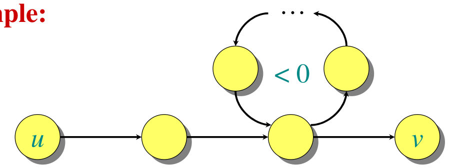
# LECTURE 23
**Shortest Paths II**
- Bellman-Ford algorithm
- Linear programming and difference constraints
- VLSI layout compaction



## Negative-weight cycles

**Recall:** If a graph $G = (V, E)$ contains a negative-weight cycle, then some shortest paths may not exist.

**Example:**

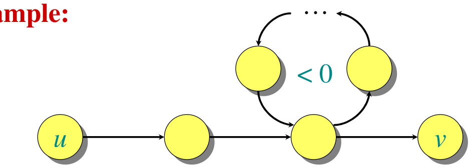





## Negative-weight cycles

**Recall:** If a graph $G = (V, E)$ contains a negative-weight cycle, then some shortest paths may not exist.

**Example:**




***Bellman-Ford algorithm:*** Finds all shortest-path lengths from a ***source*** $s \in V$ to all $v \in V$ or determines that a negative-weight cycle exists.



## Bellman-Ford algorithm

$d[s] \leftarrow 0$
**for** each $v \in V - \{s\}$
  **do** $d[v] \leftarrow \infty$
} initialization

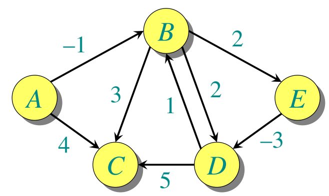
**for** $i \leftarrow 1$ **to** $|V| - 1$
  **do for** each edge $(u, v) \in E$
    **do if** $d[v] > d[u] + w(u, v)$
      **then** $d[v] \leftarrow d[u] + w(u, v)$
} ***relaxation step***

**for** each edge $(u, v) \in E$
  **do if** $d[v] > d[u] + w(u, v)$
    **then** report that a negative-weight cycle exists

At the end, $d[v] = \delta(s, v)$, if no negative-weight cycles. Time $= O(VE)$.

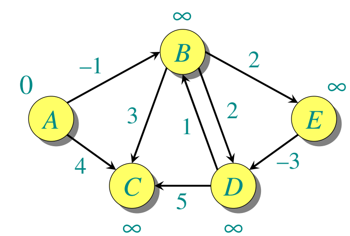


## Example of Bellman-Ford





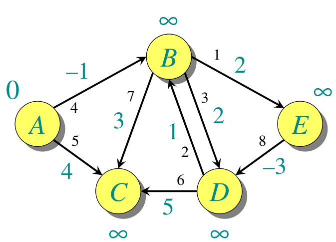

## Example of Bellman-Ford




Initialization.



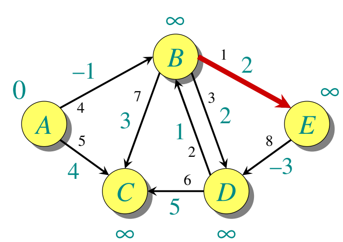
## Example of Bellman-Ford




Order of edge relaxation.

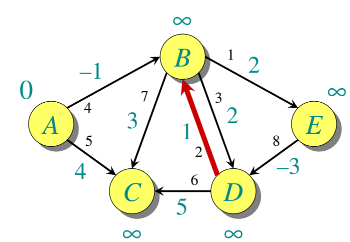


## Example of Bellman-Ford

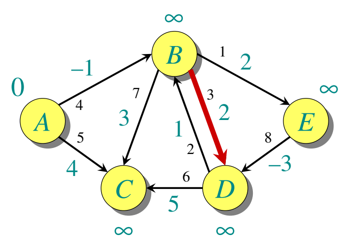





## Example of Bellman-Ford

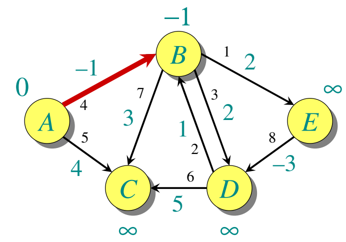


## Example of Bellman-Ford



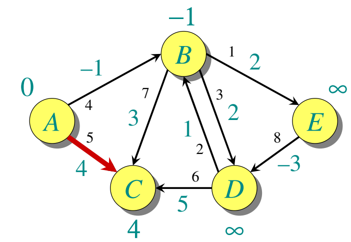



## Example of Bellman-Ford



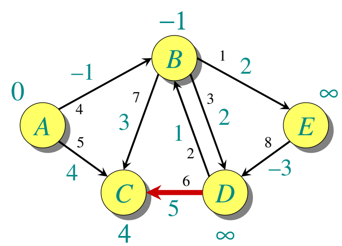



## Example of Bellman-Ford



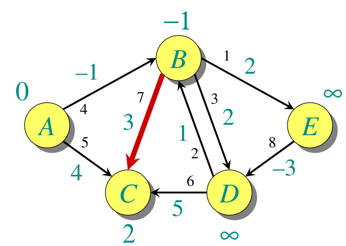



## Example of Bellman-Ford



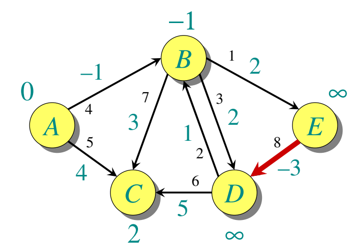



## Example of Bellman-Ford



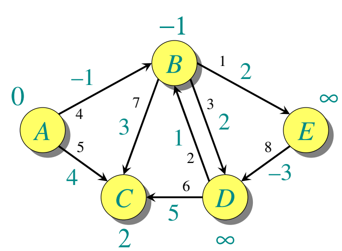



## Example of Bellman-Ford





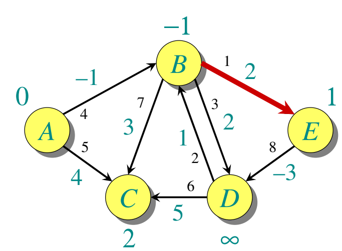

## Example of Bellman-Ford




End of pass 1.

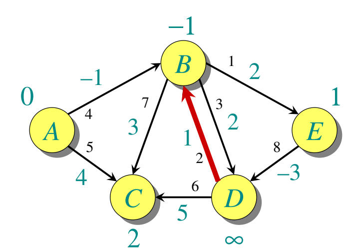


## Example of Bellman-Ford

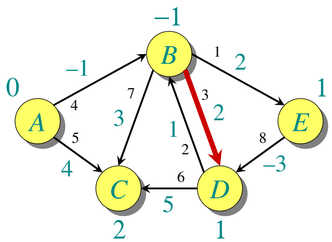





## Example of Bellman-Ford

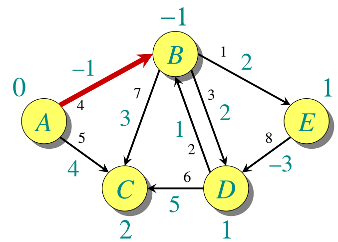


## Example of Bellman-Ford



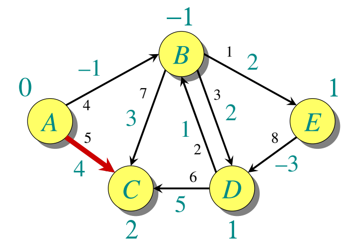



## Example of Bellman-Ford



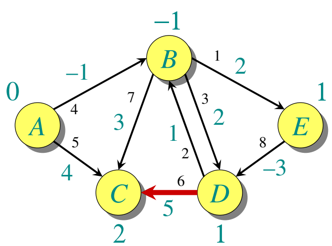



## Example of Bellman-Ford



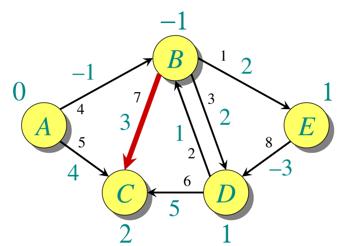



## Example of Bellman-Ford



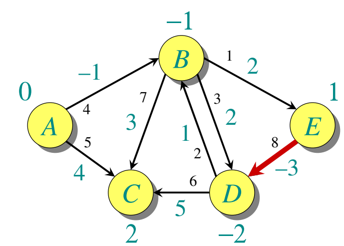



## Example of Bellman-Ford



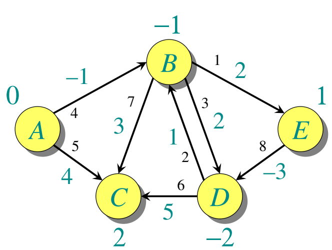



## Example of Bellman-Ford






## Example of Bellman-Ford




End of pass 2 (and 3 and 4).



## Correctness

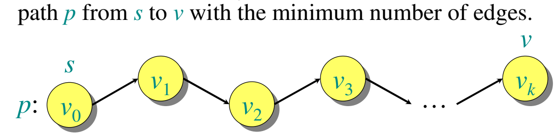
**Theorem.** If $G = (V, E)$ contains no negative-weight cycles, then after the Bellman-Ford algorithm executes, $d[v] = \delta(s, v)$ for all $v \in V$.



## Correctness

**Theorem.** If $G = (V, E)$ contains no negative-weight cycles, then after the Bellman-Ford algorithm executes, $d[v] = \delta(s, v)$ for all $v \in V$.

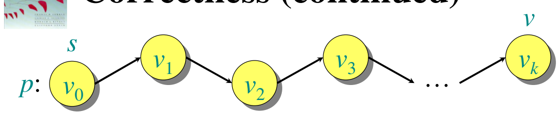
*Proof.* Let $v \in V$ be any vertex, and consider a shortest path $p$ from $s$ to $v$ with the minimum number of edges.




Since $p$ is a shortest path, we have

$$\delta(s, v_i) = \delta(s, v_{i-1}) + w(v_{i-1}, v_i) .$$

## Correctness (continued)




Initially, $d[v_0] = 0 = \delta(s, v_0)$, and $d[v_0]$ is unchanged by subsequent relaxations (because of the lemma from *Shortest Paths I* that $d[v] \geq \delta(s, v)$).

- After 1 pass through $E$, we have $d[v_1] = \delta(s, v_1)$.
- After 2 passes through $E$, we have $d[v_2] = \delta(s, v_2)$.

⋮

- After $k$ passes through $E$, we have $d[v_k] = \delta(s, v_k)$.

Since $G$ contains no negative-weight cycles, $p$ is simple. Longest simple path has $\leq |V| - 1$ edges. □



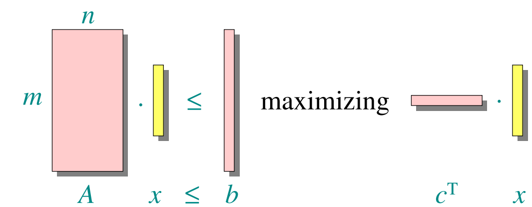
## Detection of negative-weight cycles

**Corollary.** If a value $d[v]$ fails to converge after $|V| - 1$ passes, there exists a negative-weight cycle in $G$ reachable from $s$. □



## Linear programming

Let $A$ be an $m \times n$ matrix, $b$ be an $m$-vector, and $c$ be an $n$-vector. Find an $n$-vector $x$ that maximizes $c^{T}x$ subject to $Ax \leq b$, or determine that no such solution exists.






## Linear-programming algorithms

**Algorithms for the general problem**

- Simplex methods — practical, but worst-case exponential time.
- Interior-point methods — polynomial time and competes with simplex.



## Linear-programming algorithms

**Algorithms for the general problem**

- Simplex methods — practical, but worst-case exponential time.
- Interior-point methods — polynomial time and competes with simplex.

***Feasibility problem:*** No optimization criterion. Just find $x$ such that $Ax \leq b$.

- In general, just as hard as ordinary LP.



## Solving a system of difference constraints

Linear programming where each row of $A$ contains exactly one 1, one –1, and the rest 0's.

**Example:**

$$\left.\begin{array}{l} x_1 - x_2 \leq 3 \\ x_2 - x_3 \leq -2 \\ x_1 - x_3 \leq 2 \end{array}\right\} \quad x_j - x_i \leq w_{ij}$$



## Solving a system of difference constraints

Linear programming where each row of $A$ contains exactly one 1, one –1, and the rest 0's.

**Example:**

$$\left.\begin{array}{l} x_1 - x_2 \leq 3 \\ x_2 - x_3 \leq -2 \\ x_1 - x_3 \leq 2 \end{array}\right\} \quad x_j - x_i \leq w_{ij}$$

**Solution:**

$$x_1 = 3, \quad x_2 = 0, \quad x_3 = 2$$



## Solving a system of difference constraints

Linear programming where each row of $A$ contains exactly one 1, one –1, and the rest 0's.

**Example:**

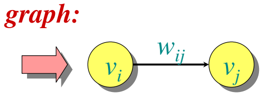
$$\left.\begin{array}{l} x_1 - x_2 \leq 3 \\ x_2 - x_3 \leq -2 \\ x_1 - x_3 \leq 2 \end{array}\right\} \quad x_j - x_i \leq w_{ij}$$

**Solution:**

$$x_1 = 3, \quad x_2 = 0, \quad x_3 = 2$$

***Constraint graph:***

$x_j - x_i \leq w_{ij}$




(The "$A$" matrix has dimensions $|E| \times |V|$.)



## Unsatisfiable constraints

**Theorem.** If the constraint graph contains a negative-weight cycle, then the system of differences is unsatisfiable.

## Unsatisfiable constraints

**Theorem.** If the constraint graph contains a negative-weight cycle, then the system of differences is unsatisfiable.

*Proof.* Suppose that the negative-weight cycle is $v_1 \to v_2 \to \cdots \to v_k \to v_1$. Then, we have

$$\begin{aligned} x_2 - x_1 &\le w_{12} \\ x_3 - x_2 &\le w_{23} \\ &\vdots \\ x_k - x_{k-1} &\le w_{k-1,k} \\ x_1 - x_k &\le w_{k1} \end{aligned}$$



## Unsatisfiable constraints

**Theorem.** If the constraint graph contains a negative-weight cycle, then the system of differences is unsatisfiable.

*Proof.* Suppose that the negative-weight cycle is $v_1 \to v_2 \to \cdots \to v_k \to v_1$. Then, we have

$$\begin{aligned} x_2 - x_1 &\le w_{12} \\ x_3 - x_2 &\le w_{23} \\ &\vdots \\ x_k - x_{k-1} &\le w_{k-1,k} \\ x_1 - x_k &\le w_{k1} \\ \hline 0 &\le \text{weight of cycle} \\ &< 0 \end{aligned}$$

Therefore, no values for the $x_i$ can satisfy the constraints. $\square$



## Satisfying the constraints

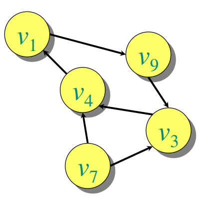
**Theorem.** Suppose no negative-weight cycle exists in the constraint graph. Then, the constraints are satisfiable.



## Satisfying the constraints

**Theorem.** Suppose no negative-weight cycle exists in the constraint graph. Then, the constraints are satisfiable.

*Proof.* Add a new vertex $s$ to $V$ with a $0$-weight edge to each vertex $v_i \in V$.

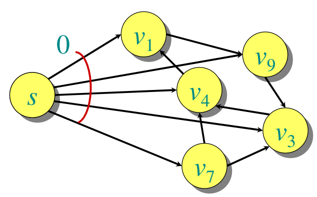





## Satisfying the constraints

**Theorem.** Suppose no negative-weight cycle exists in the constraint graph. Then, the constraints are satisfiable.

*Proof.* Add a new vertex $s$ to $V$ with a $0$-weight edge to each vertex $v_i \in V$.



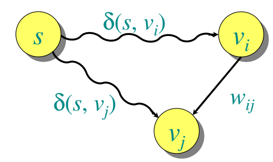

**Note:** No negative-weight cycles introduced $\Rightarrow$ shortest paths exist.



## Proof (continued)

**Claim:** The assignment $x_i = \delta(s, v_i)$ solves the constraints.

Consider any constraint $x_j - x_i \le w_{ij}$, and consider the shortest paths from $s$ to $v_j$ and $v_i$:




The triangle inequality gives us $\delta(s,v_j) \le \delta(s, v_i) + w_{ij}$. Since $x_i = \delta(s, v_i)$ and $x_j = \delta(s, v_j)$, the constraint $x_j - x_i \le w_{ij}$ is satisfied. $\square$



## Bellman-Ford and linear programming

**Corollary.** The Bellman-Ford algorithm can solve a system of $m$ difference constraints on $n$ variables in $O(mn)$ time. $\square$

Single-source shortest paths is a simple LP problem.

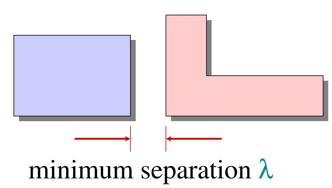
In fact, Bellman-Ford maximizes $x_1 + x_2 + \cdots + x_n$ subject to the constraints $x_j - x_i \le w_{ij}$ and $x_i \le 0$ (exercise).

Bellman-Ford also minimizes $\max_i\{x_i\} - \min_i\{x_i\}$ (exercise).



## Application to VLSI layout compaction

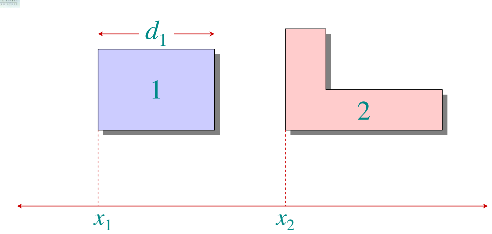
*Integrated-circuit features:*




**Problem:** Compact (in one dimension) the space between the features of a VLSI layout without bringing any features too close together.



## VLSI layout compaction




**Constraint:** $x_2 - x_1 \ge d_1 + \lambda$

Bellman-Ford minimizes $\max_i\{x_i\} - \min_i\{x_i\}$, which compacts the layout in the $x$-dimension.