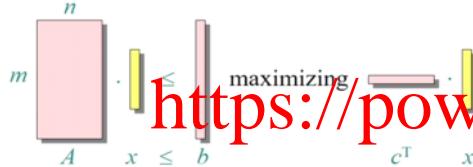


## Lecture19\_ShortestPaths3

Tuesday, October 20, 2020 9:20 PM

### Linear Programming

Let  $A$  be an  $m \times n$  matrix,  $b$  be an  $m$ -vector, and  $c$  be an  $n$ -vector. Find an  $n$ -vector  $x$  that maximizes  $c^T x$  subject to  $Ax \leq b$ , or determine that no such solution exists.



subject to  $Ax \leq b$

$\max c^T x$

$x_2$

$Ax \leq b$

$A_2 x_2 \leq b_2$

$A_3 x_3 \leq b_3$

feasible region (bounded region)

not necessary to be a bounded

unbounded

empty

corner will be one of the corner of the

convex region

### Algorithms for the general problem

- Simplex method: practical but worst-case exponential time
- Interior-point methods: polynomial time and competes with simplex.

**Feasibility problem:** No optimization criterion.

Just find  $x$  such that  $Ax \leq b$ .

- In general, just as hard as ordinary LP.

### Solving a system of difference constraints

Linear programming where each row of  $A$  contains exactly one 1, one  $-1$ , and the rest 0's.

**Example:**

$$\begin{cases} x_1 - x_2 \leq 3 \\ x_2 - x_3 \leq 1 \\ x_1 - x_3 \leq 2 \end{cases}$$

**Solution:**

$$\begin{cases} x_1 = 3 \\ x_2 = 0 \\ x_3 = 2 \end{cases}$$

**Constraint graph:**



(The "A" matrix has dimensions  $|E| \times |V|$ )

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- For each unknown create a vertex.
- For each inequality / constrain put an edge from the one with negative sign to the one with positive sign. The weight of edge will be the right hand side of the inequality.

### Unsatisfiable constraints

**Theorem.** If the constraint graph contains a negative-weight cycle, then the system of differences is unsatisfiable.

**Proof.** Suppose that the negative-weight cycle is  $v_1 \rightarrow v_2 \rightarrow \dots \rightarrow v_k \rightarrow v_1$ . Then, we have

$$\begin{aligned} x_2 - x_1 &\leq w_{12} \\ x_3 - x_2 &\leq w_{23} \\ &\vdots \\ x_k - x_{k-1} &\leq w_{k-1,k} \\ x_1 - x_k &\leq w_{k1} \\ \hline 0 &\leq \text{weight of cycle} \\ &< 0 \end{aligned}$$

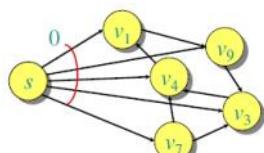
Therefore, no values for the  $x_i$  can satisfy the constraints.

- Right side: negative number  $\rightarrow$  contradiction

### Satisfying the constraints

**Theorem.** Suppose no negative-weight cycle exists in the constraint graph. Then, the constraints are satisfiable.

**Proof.** Add a new vertex  $s$  to  $V$  with a 0-weight edge to each vertex  $v_i \in V$ .

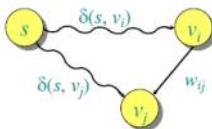


**Note:**

No negative-weight cycles introduced  $\Rightarrow$  shortest paths exist.

- $s$ : dummy variable, a vertex that connect to all vertices.

**Claim:** The assignment  $x_i = \delta(s, v_i)$  solves the constraints.  
 Consider any constraint  $x_j - x_i \leq w_{ij}$ , and consider the shortest paths from  $s$  to  $v_j$  and  $v_i$ :



The triangle inequality gives us  $\delta(s, v_j) \leq \delta(s, v_i) + w_{ij}$ .  
 Since  $x_i = \delta(s, v_i)$  and  $x_j = \delta(s, v_j)$ , the constraint  $x_j - x_i \leq w_{ij}$  is satisfied.  $\square$

### Bellman-Ford and linear programming

Corollary. The Bellman-Ford algorithm can solve a system of  $m$  difference constraints on  $n$  variables in  $O(mn)$  time.

Single-source shortest paths is simple LP problem.

In fact, Bellman-Ford maximizes  $x_1 + x_2 + \dots + x_n$   
 subject to the constraints  $x_j - x_i \leq w_{ij}$  and  $x_i \leq 0$   
 (exercise).

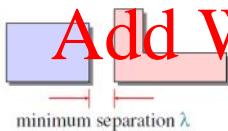
Bellman-Ford also minimizes  $\max_i \{x_i\} - \min_i \{x_i\}$   
 (exercise).

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### Application to VLSI layout compaction

Integrated circuit features:

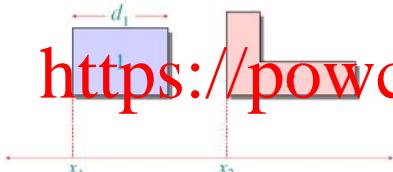


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minimum separation  $\lambda$

**Problem:** Compact (in one dimension) the space between the two IC features without moving any feature to close together.

- Subject to the second optimization function above



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**Constraint:**  $x_1 \geq d_1$

Bellman-Ford minimizes  $\max_i \{x_i\} - \min_i \{x_i\}$ , which compacts the layout in the  $x$ -dimension.

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