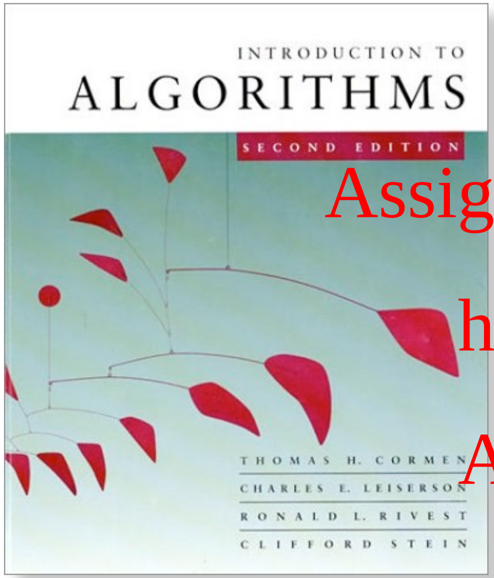
COT5405 Analysis of Algorithms

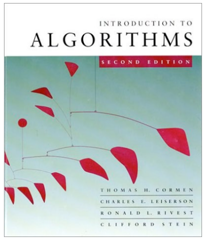


LECTURE 14-15

Dynamic Programming vs Greedy Algorithms

- MatrixChain Multiplication
- Activity Selection Problem
- Optimal substructure
- Greedy Selection
- Knapsack Problem

Prof. Alper Üngör

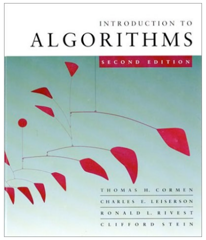


Matrix Chain Multiplication

Given a sequence (chain) of n matrices $A_1, A_2, \ldots,$ A_n, where A_i is a $p_{i-1} \times p_i$ matrix

Compute their product $A_1 \cdot A_2 \cdot \ldots \cdot A_n$ using the minimum number of scalar multiplications

Find a parenthesization that minimizes the number of multiplications



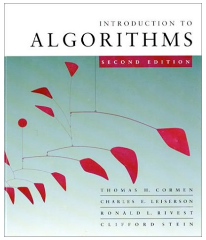
Optimal Substructure

Notation. Let $A_{i,j} = A_i \cdot \ldots \cdot A_j$ for $i \leq j$

- Consider an optimal parenthesization for $A_{i,j}$

Say it splits at k, so $A_{i,j} = (A_i \cdot \ldots \cdot A_k)(A_{k+1} \cdot \ldots \cdot A_j)$

- Then, the parenthesization of the prefix $A_i \cdot \ldots \cdot A_k$ within the optimal parenthesization of $A_{i,j}$ must be an optimal parenthesization of $A_{i,k}$.



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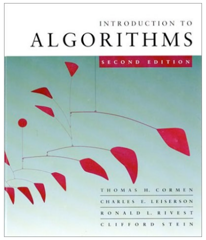
- Then, the parenthesization of the prefix $A_i \cdot \ldots \cdot A_k$ within the optimal parenthesization of $A_{i,j}$ must be an optimal parenthesization of $A_{i,k}$.

(**Proof**. Suppose it is not optimal, then there exists a better parenthesization for $A_{i,k}$. **Copy and paste** this parenthesization into the parenthesization for $A_{i,j}$. This yields a better parenthesization for $A_{i,j}$. Contradiction.)



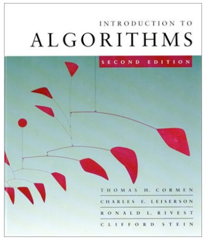
Dynamic programming

$m[i,j]$ = minimum number of scalar multiplications to compute A_{ij}. We want to compute $m[1,n]$

$$A_{i,j} = \underbrace{(A_i \cdots A_k)}_{p_{i-1} \times p_k} \cdot \underbrace{(A_{k+1} \cdots A_j)}_{p_k \times p_j}$$

Recurrence for optimal substructure:

- $m[i,i] = 0$ for $i=1,2,\ldots,n$
- $m[i,j] = \min_{i \le k < j}\{m[i,k]+m[k+1,j] + p_{i-1}\, p_k\, p_j\}$



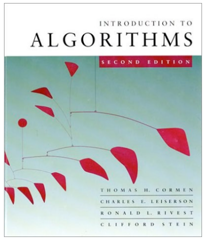
Naive or Recursive Approach

- Enumerate all possible paranthesizations
- Implement the described recursion directly

The runtime of both algorithms is $\Omega(2^n)$

- Overlapping subproblems!

There are only $O(n^2)$ different problems

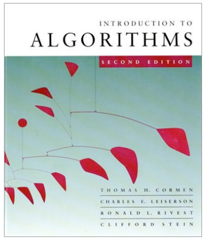


Dynamic Programming

Fill the 2 dimensional $m[i,j]$-table bottom-up

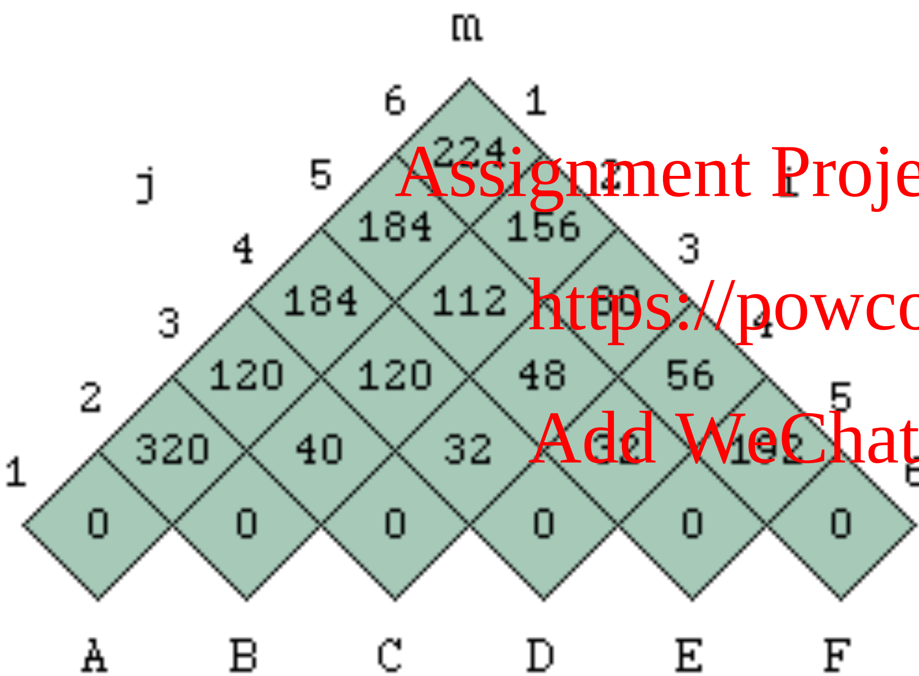
For the construction of the optimal parenthesization, use an additional array $s[i,j]$ that records that value of k for which the minimum is attained and stored in $m[i,j]$

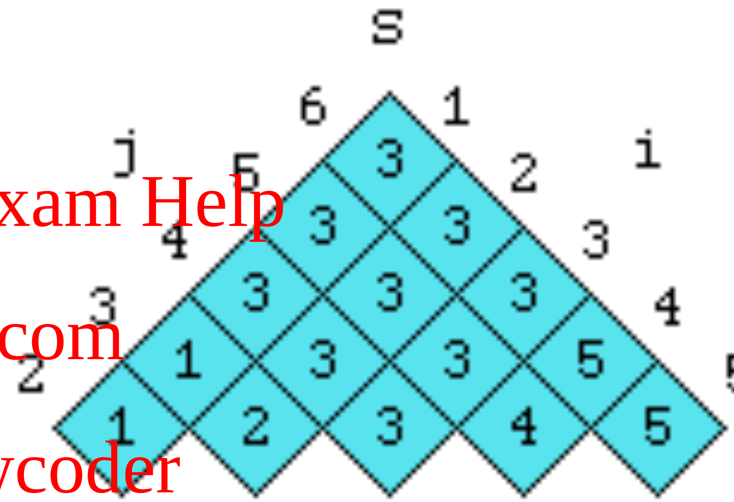
- $m[1,n]$ is the desired value



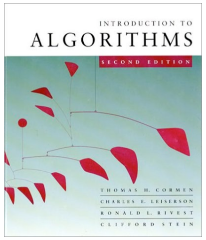
MatrixChain Mult. Example

$A_1, \ldots A_6$ with sizes 8x10, 10x4, 4x1, 1x8, 8x4, 4x6





Nice Visualization/Animaiton of this Algorithm:
http://www.brian-borowski.com/Software/Matrix/
http://www.cs.auckland.ac.nz/software/AlgAnim/mat_chain.html#mat_chain_anim

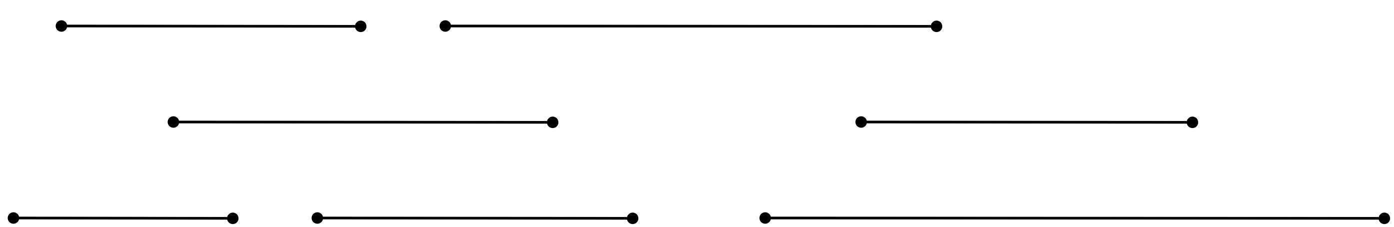


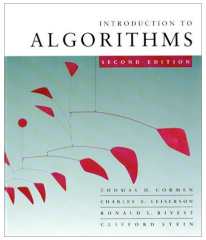
Activity Selection Problem

- Input: Set S of n activities, $a_1, a_2, \ldots, a_n$.
  - » s_i = start time of activity i.
  - » f_i = finish time of activity i.

- Output: Subset A of maximum number of compatible activities.
  - » Two activities are compatible, if their intervals don't overlap.

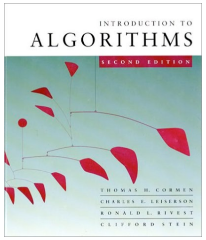
Example:





Optimal Substructure

- Assume activities are sorted by finishing times.
  - » $f_1 \le f_2 \le \ldots \le f_n$.
- Suppose an optimal solution includes activity a_k.
  - » This generates two subproblems.
  - » Selecting from $a_1, \ldots, a_{k-1}$, activities compatible with one another, and that finish before a_k starts (compatible with a_k).
  - » Selecting from $a_{k+1}, \ldots, a_n$, activities compatible with one another, and that start after a_k finishes.
  - » The solutions to the two subproblems must be optimal.
    - Prove using the cut-and-paste approach.



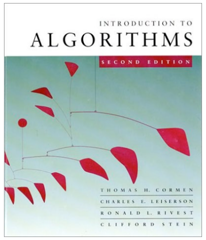
Recursive formulation

- Let S_{ij} = subset of activities in S that start after a_i finishes and finish before a_j starts.

- Subproblems: Selecting maximum number of mutually compatible activities from S_{ij}.

- Let $c[i, j]$ = size of maximum-size subset of mutually compatible activities in S_{ij}.

$$c[i,j] = \begin{cases} 0 & \text{if } S_{ij} = \phi \\ \max\limits_{i<k<j}\{c[i,k] + c[k,j] + 1\} & \text{if } S_{ij} \neq \phi \end{cases}$$

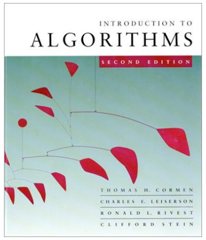


Can we do better?

Theorem. Consider any non-empty subproblem S_{ij} , and a_m be the activity in S_{ij} with earliest finish time. Then,

i) Activity a_m is used in some maximum size subset of mutually compatible activities of S_{ij}

ii) The first subproblem S_{im} is empty.



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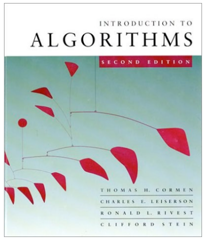
ii) The first subproblem S_{im} is empty.

Proof. (ii) Suppose S_{im} is non-empty. There exists some activity a_k such that $f_i \leq s_k < f_k \leq s_m < f_m$. Then a_k is also in S_{ij} and it has earlier finish time than a_m. Contradiction.



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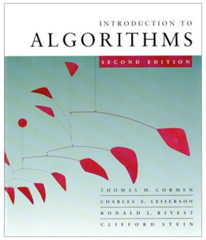
Proof. (i) Let A_{ij} be an opt solution for S_{ij} . let a_k be the activity with earliest finish in A_{ij} . If $a_k = a_m$, we are done.

Otherwise, construct a new solution

$$A'_{ij} = A_{ij} - \{a_k\} + \{a_m\}$$

which is also an optimum feasible solution.



Implication

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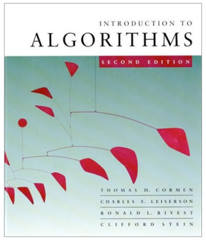
ii) The first subproblem S_{im} is empty.

Implication

ii) solve only one of the two of subproblems.

i) a simple top-down approach. pick the job with the earliest finish time. **Greedy Algorithm**!



Recursive Greedy Algorithm

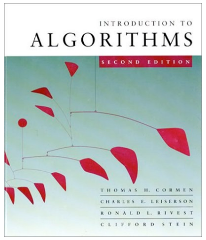
Recursive-Activity-Selector (s, f, i, j)

$m \leftarrow i+1$

while $m < j$ and $s_m < f_i$

  do $m \leftarrow m+1$
if $m < j$

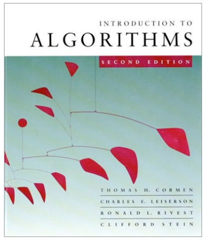
  then return $\{a_m\} \cup$
    Recursive-Activity-Selector(s, f, m, j)
  else return ϕ



Iterative Greedy Algorithm

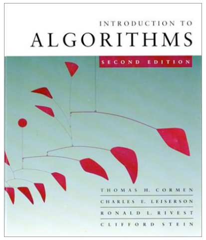
Greedy-Activity-Selector (s, f)

1. $n \leftarrow$ length[s]
2. $A \leftarrow [a_1]$
3. $i \leftarrow 1$
4. **for** $m \leftarrow 2$ **to** n
5. **do if** $s_m < f_i$
6. **then** $A \leftarrow A \cup \{a_m\}$
7. $i \leftarrow m$
8. **return** A



Recap of Greedy Strategy

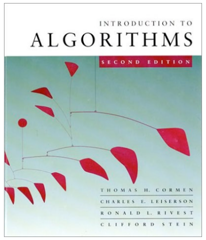
- Cast the optimization problem as one in which we make a choice and are left with one subproblem to solve.
- Prove that there's always an optimal solution that makes the greedy choice, so that the greedy choice is always safe.
- Show that greedy choice and optimal solution to subproblem $\Rightarrow$ optimal solution to the problem.
- Make the greedy choice and **solve top-down**.
- May have to preprocess input to put it into greedy order.
  - » Example: Sorting activities by finish time.



Why not use all the time?

- **Matrix Chain Multiplication Problem.**
  Greedy Strategy: do the leftmost multiplication first;
  do the rightmost multiplication first;
  do the cheapest product first;
  do the product $A_{ik}A_{kj}$ with largest k first;

- **Longest Common Subsequence.**
  Greedy Strategy: ???

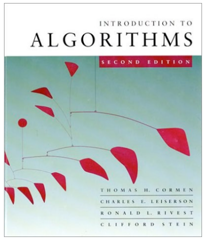


Knapsack Problem

- Given a knapsack with weight $W > 0$.
  A set S of n items with weights $w_i > 0$ and
  benefits $b_i > 0$ for $i = 1, \ldots, n$.

- $S = \{\, (item_1, w_1, b_1), \ (item_2, w_2, b_2),$
  $\ldots, (\, item_n, w_n, b_n) \,\}$

- Find a subset of the items which does not exceed the weight W of the knapsack and maximizes the benefit.



0/1 Knapsack Problem

Determine a subset *A* of { 1, 2, …, *n* } that satisfies the following:

$$\max \sum_{i \in A} b_i \text{ where } \sum_{i \in A} w_i \le W$$

In 0/1 knapsack a specific item is either selected or not

Variations of the Knapsack problem

- Fractions are allowed. This applies to items such as:
  - bread, for which taking half a loaf makes sense
  - gold dust
- No fractions.

  - 0/1 (1 brown pants, 1 green shirt…)

  - Allows putting many items of same type in knapsack

    - 5 pairs of socks
    - 10 gold bricks
  - More than one knapsack, etc.
- First 0/1 *knapsack* problem will be covered then the Fractional *knapsack* problem.

Brute force!

- Generate all 2^n subsets
- Discard all subsets whose sum of the weights exceed *W (not feasible)*

- Select the maximum total benefit of the remaining (feasible) subsets

- What is the run time?
  O(n 2^n), Omega(2^n)
- Lets try the obvious greedy strategy .

Example with “brute force”

S = { ($item_1$, 5, \$70), ($item_2$,10, \$90), ($item_3$, 25, \$140) } , W=25

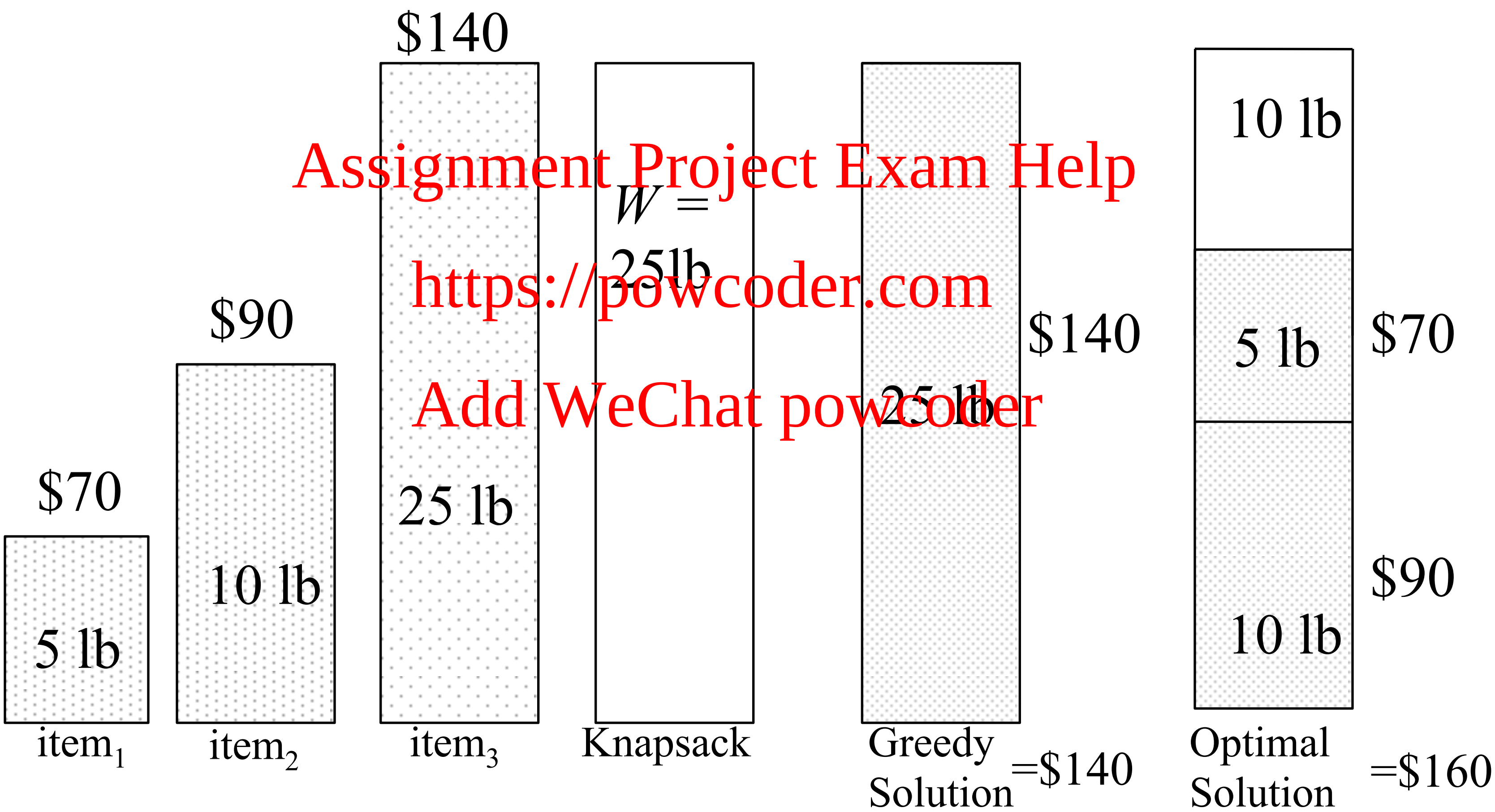
- Subsets:

1. {}
2. { ($item_1$, 5, \$70) } Profit=\$70
3. { ($item_2$,10, \$90) } Profit=\$90
4. { ($item_3$, 25, \$140) } Profit=\$140
5. { ($item_1$, 5, \$70), ($item_2$,10, \$90) } Profit=\$160 ****
6. { ($item_2$,10, \$90), ($item_3$, 25, \$140) } exceeds W
7. { ($item_1$, 5, \$70), ($item_3$, 25, \$140) } exceeds W
8. { ($item_1$, 5, \$70), ($item_2$,10, \$90), ($item_3$, 25, \$140) } exceeds W

Greedy 1: Selection criteria: *Maximum beneficial* item.

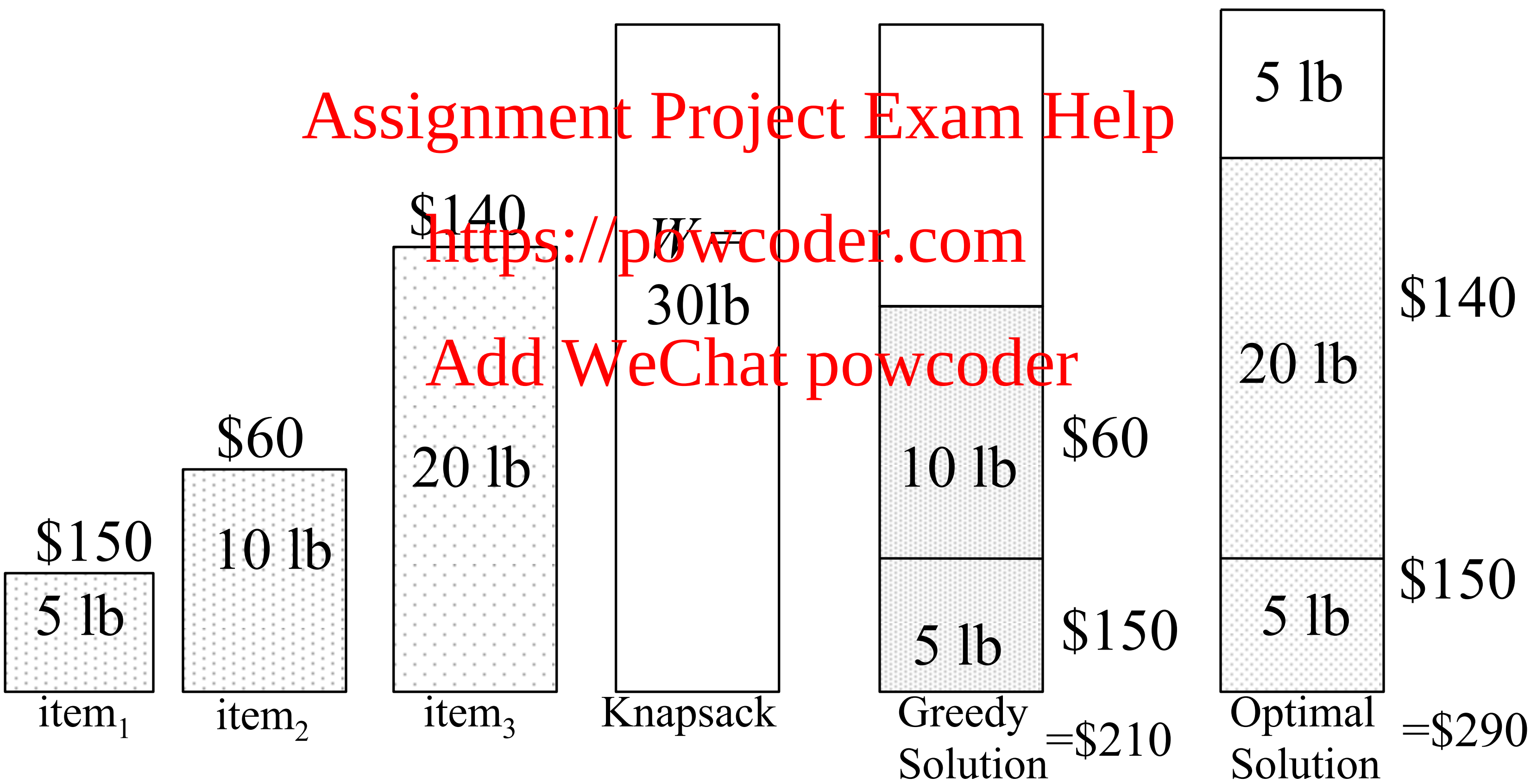
Counter Example:

$S = \{ (item_1 , 5, \$70), (item_2 ,10, \$90), (item_3, 25, \$140) \}$



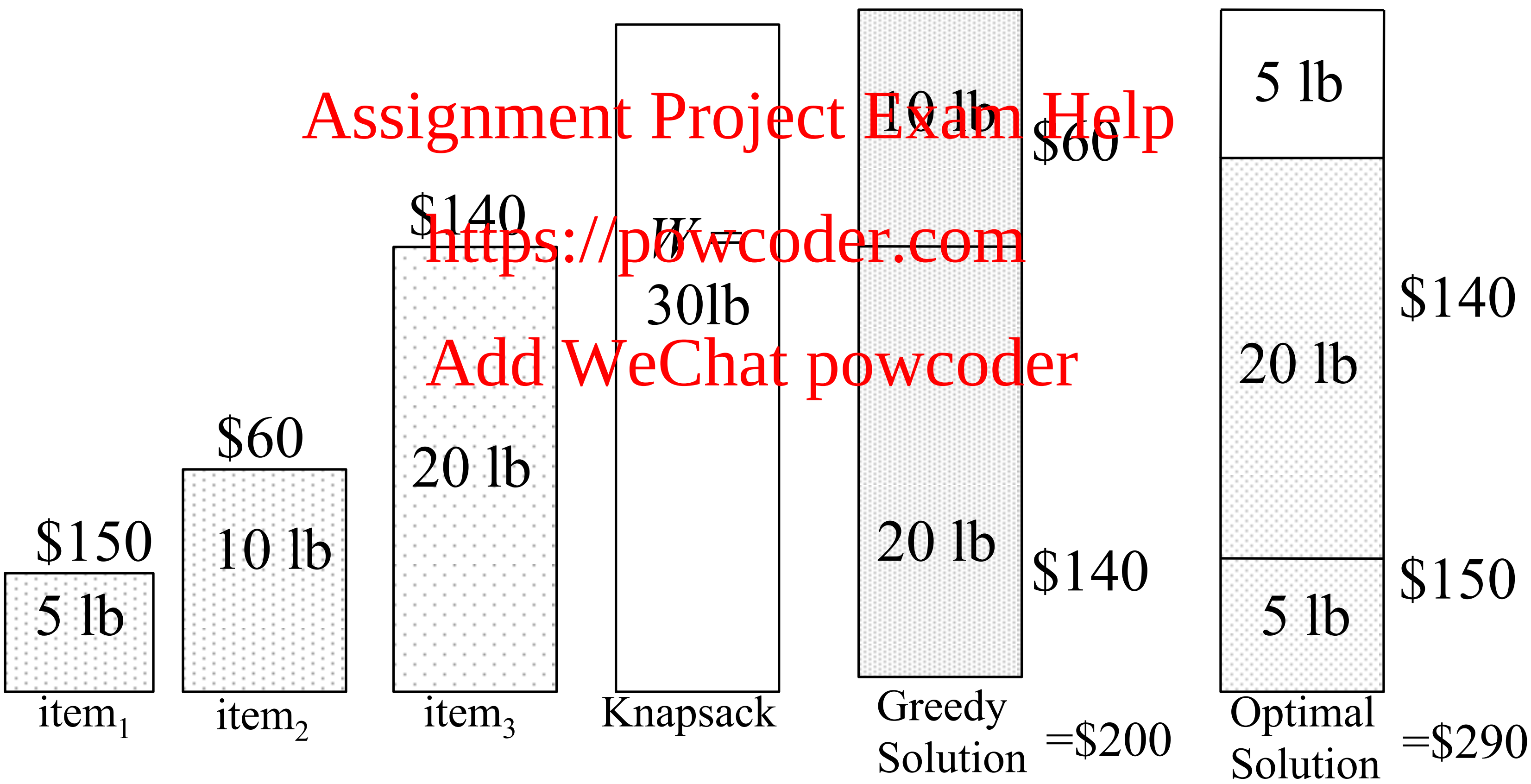
Greedy 2: Selection criteria: *Minimum weight* item
Counter Example:

$S = \{ (item_1 , 5, \$150), (item_2 ,10, \$60), (item_3, 20, \$140) \}$



Greedy 3: Selection criteria: *Maximum weight* item
Counter Example:

$S = \{ (\ item_1\ , 5, \$150\ ), (item_2\ ,10, \$60\ ), (\ item_3, 20, \$140\ ) \}$



Greedy 4: Selection criteria: *Maximum benefit per unit item* Counter Example

$S = \{ (\ item_1\ , 5, \$50\ ), (\ item_2, 20, \$140\ )\ (item_3\ ,10, \$60\ ), \}$