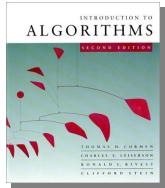


Analysis of Algorithms



LECTURE 25

Network Flows I

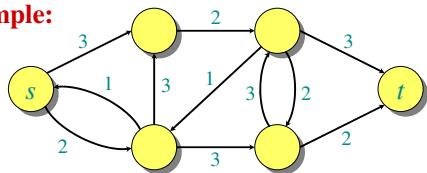
- Flow networks
- Max flow problem
- Residual network
- Augmenting paths
- Max flow-min cut theorem



Flow networks

Definition. A **flow network** is a directed graph $G = (V, E)$ with two distinguished vertices: a **source** s and a **sink** t . Each edge $(u, v) \in E$ has a nonnegative **capacity** $c(u, v)$. If $(u, v) \notin E$, then $c(u, v) = 0$.

Example:



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Flow networks

Definition. A **positive flow** on G is a function $p : V \times V \rightarrow \mathbb{R}$ satisfying the following:

- **Capacity constraint:** For all $u, v \in V$, $0 \leq p(u, v) \leq c(u, v)$.
- **Flow conservation:** For all $u \in V - \{s, t\}$,

$$\sum_{v \in V} p(u, v) - \sum_{v \in V} p(v, u) = 0.$$

The **value** of a flow is the net flow out of the source:

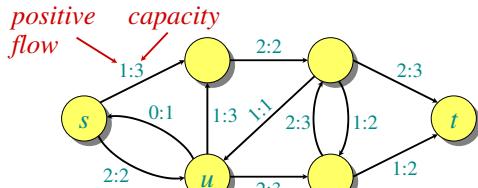
$$\sum_{v \in V} p(s, v) - \sum_{v \in V} p(v, s).$$

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A flow on a network



Flow conservation (like Kirchoff's current law):

- Flow into u is $2 + 1 = 3$.
- Flow out of u is $0 + 1 + 2 = 3$.

The value of this flow is $1 - 0 + 2 = 3$.

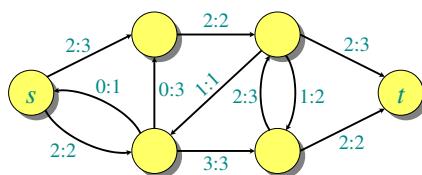
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The maximum-flow problem

Maximum-flow problem: Given a flow network G , find a flow of maximum value on G .



The value of the maximum flow is 4.

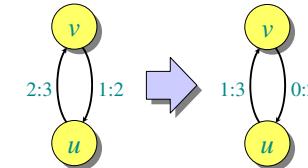
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Flow cancellation

Without loss of generality, positive flow goes either from u to v , or from v to u , but not both.



Net flow from u to v in both cases is 1.

The capacity constraint and flow conservation are preserved by this transformation.

INTUITION: View flow as a **rate**, not a **quantity**.

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Equivalence of definitions

Theorem. The two definitions are equivalent.

Proof. (\Rightarrow) Let $f(u, v) = p(u, v) - p(v, u)$.

- **Capacity constraint:** Since $p(u, v) \leq c(u, v)$ and $p(v, u) \geq 0$, we have $f(u, v) \leq c(u, v)$.

- **Flow conservation:**

$$\begin{aligned} \sum_{v \in V} f(u, v) &= \sum_{v \in V} (p(u, v) - p(v, u)) \\ &= \sum_{v \in V} p(u, v) - \sum_{v \in V} p(v, u) \end{aligned}$$

- **Skew symmetry:**

$$\begin{aligned} f(u, v) &= p(u, v) - p(v, u) \\ &= -(p(v, u) - p(u, v)) \\ &= -f(v, u). \end{aligned}$$

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Proof (continued)

(\Leftarrow) Let

$$p(u, v) = \begin{cases} f(u, v) & \text{if } f(u, v) > 0, \\ 0 & \text{if } f(u, v) \leq 0. \end{cases}$$

- **Capacity constraint:** By definition, $p(u, v) \geq 0$. Since $f(u, v) \leq c(u, v)$, it follows that $p(u, v) \leq c(u, v)$.

- **Flow conservation:** If $f(u, v) > 0$, then $p(u, v) - p(v, u) = f(u, v)$. If $f(u, v) \leq 0$, then $p(u, v) - p(v, u) = -f(v, u) = f(u, v)$ by skew symmetry. Therefore,

$$\sum_{v \in V} p(u, v) - \sum_{v \in V} p(v, u) = \sum_{v \in V} f(u, v). \quad \square$$

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Notation

Definition. The *value* of a flow f , denoted by $|f|$, is given by

$$|f| = \sum_{v \in V} f(s, v) \\ = f(s, V).$$

Implicit summation notation: A set used in an arithmetic formula represents a sum over the elements of the set.

- **Example** — flow conservation:
 $f(u, V) = 0$ for all $u \in V - \{s, t\}$.

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Simple properties of flow

Lemma.

- $f(X, X) = 0$,
- $f(X, Y) = -f(Y, X)$,
- $f(X \cup Y, Z) = f(X, Z) + f(Y, Z)$ if $X \cap Y = \emptyset$. \square

Theorem. $|f| = f(V, t)$.

Proof.

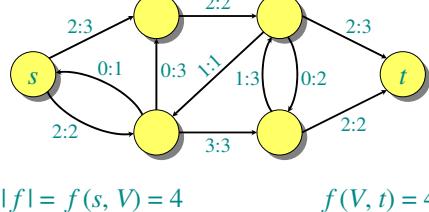
$$|f| = f(s, V) \\ = f(V, V) - f(V-s, V) \quad \text{Omit braces.} \\ = f(V, V-s) \\ = f(V, t) + f(V, V-s-t) \\ = f(V, t). \quad \square$$

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Flow into the sink



$$|f| = f(s, V) = 4$$

$$f(V, t) = 4$$

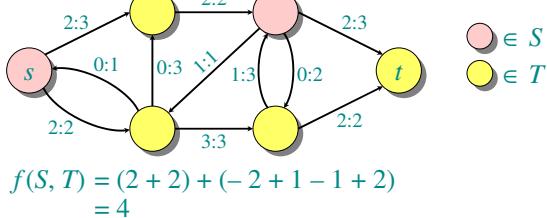
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Cuts

Definition. A *cut* (S, T) of a flow network $G = (V, E)$ is a partition of V such that $s \in S$ and $t \in T$. If f is a flow on G , then the *flow across the cut* is $f(S, T)$.



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Another characterization of flow value

Lemma. For any flow f and any cut (S, T) , we have $|f| = f(S, T)$.

Proof.

$$f(S, T) = f(S, V) - f(S, S) \\ = f(S, V) \\ = f(s, V) + f(S-s, V) \\ = f(s, V) \\ = |f|. \quad \square$$

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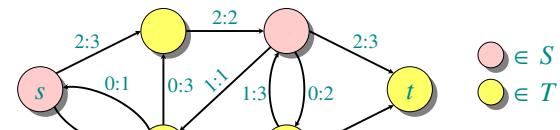
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Capacity of a cut

Definition. The *capacity of a cut* (S, T) is $c(S, T)$.



$$c(S, T) = (3+2) + (1+2+3) \\ = 11$$

Upper bound on the maximum flow value

Theorem. The value of any flow is bounded above by the capacity of any cut.

Proof.

$$|f| = f(S, T) \\ = \sum_{u \in S} \sum_{v \in T} f(u, v) \\ \leq \sum_{u \in S} \sum_{v \in T} c(u, v) \\ = c(S, T). \quad \square$$

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Residual network

Definition. Let f be a flow on $G = (V, E)$. The *residual network* $G_f(V, E_f)$ is the graph with strictly positive *residual capacities*

$$c_f(u, v) = c(u, v) - f(u, v) > 0.$$

Edges in E_f admit more flow.

Example:



Lemma. $|E_f| \leq 2|E|$. \square

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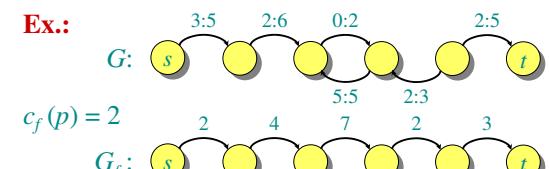
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Augmenting paths

Definition. Any path from s to t in G_f is an *augmenting path* in G with respect to f . The flow value can be increased along an augmenting path p by $c_f(p) = \min_{(u,v) \in p} \{c_f(u, v)\}$.

Ex.:



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Max-flow, min-cut theorem

Theorem. The following are equivalent:

1. f is a maximum flow.
2. f admits no augmenting paths.
3. $|f| = c(S, T)$ for some cut (S, T) .

Proof (and algorithms). Next time.