Algorithms Week 10

Ljubomir Perković, DePaul University

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There are two ways to approach the question:

1. Algorithmic (or upper-bound analysis)
2. Computational complexity (or lower-bound analysis)

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This process gives you an upper bound on the running time of the best algorithm solving the problem.

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It gives us a lower bound on the running time of any algorithm solving the problem.

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Running time: $O(\log n)$.

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Proof.

We can do no better than reduce the search space by a half, in the worst case, with each query. Therefore, $\log n$ questions are required to reduce the search space to size 1. □

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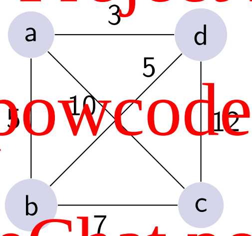
Proof.

There are $n!$ possible orderings of n numbers. Each comparison can only rule out on half of the possibilities, so we need at least $\log(n!) = \Theta(n \log n)$ comparisons. So any algorithm requires $\Omega(n \log n)$ comparisons. $\square$

Example: Travelling Salesman Problem

Given a *complete* weighted graph $G = (V, E)$, find a cycle in G that visits every vertex once such that the sum of the weights of the edges on the cycle is minimized.



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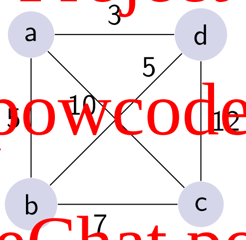

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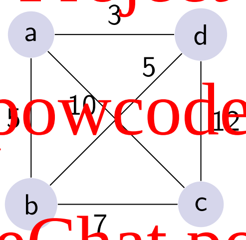

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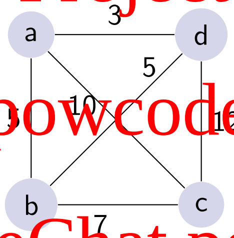

The obvious lower bound for the running time of any algorithm solving the TSP is $\Theta(n)$ (because it takes that much time to list the vertices of the cycle).

Given a graph $G = (V, E)$ and a value k, the problem is to find a valid coloring of the vertices of G using at most k colors.

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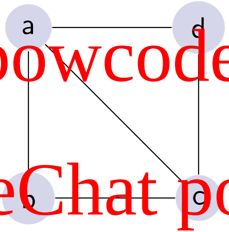

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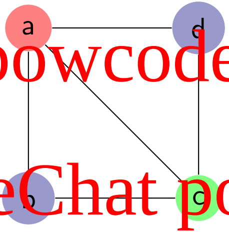

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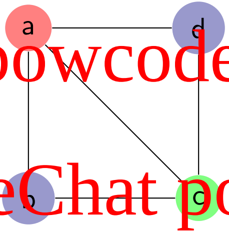

Obvious backtracking algorithm: try every possible k-coloring of the vertices, and check whether any one of is valid.
Running time: $O(k^n n^2)$.

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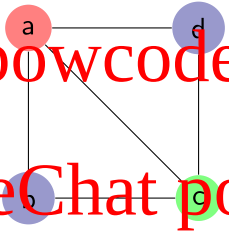

A lower bound for the running time of any algorithm solving the graph coloring problem is $\Theta(n)$ (it takes that much time to assign a color to each vertex).

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Unlike search in a sorted array and sorting, there is a gap between the known lower bound and upper bound on solving these problems.

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So, what we want to do is this: if we are unable to solve a hard problem, we would like to recognize that our problem is known to be equivalent to another problem known to be hard. How can we make this idea more formal?

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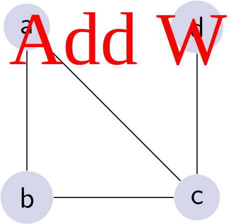

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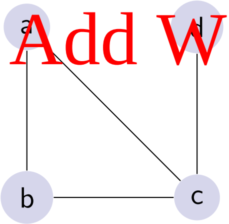

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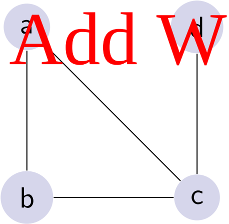

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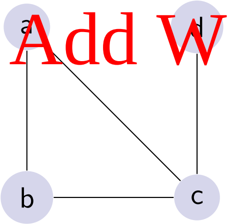

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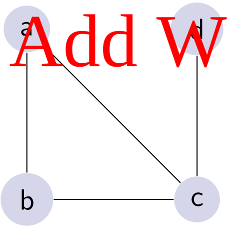

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- Input: A graph $G = (V, E)$; positive weights $w(e)$ for every $e \in E$ and specified nodes $u, v \in V$.

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Shortest path problem as a decision problem

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Input: A weighted graph $G = (V, E)$, positive weights $w(e)$ for every $e \in E$, specified nodes $u, v \in V$ and an integer k.

Output: "YES", if the length of the shortest path from u to v in G is k.

An instance of the decision version of the shortest path problem is $< G, w, u, v, k >$.

Shortest path problem as a decision problem

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For example, suppose that we have an algorithm B that solves the optimization version of the shortest path problem. We want to use B to construct an algorithm A that solves the decision version of the problem. Here is how we do it:


```
Algorithm A(G,w,u,v,k)
  // Using solution to optimization problem to obtain
  // a solution to the decision problem
  if B(G,w,u,v) = k then
    output "YES"
  else
    output "NO"
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Now suppose that we have an algorithm B that solves the decision version of the shortest path problem. We use B to construct an algorithm A for the optimization version of the problem as follows


```
Algorithm B(G,w,u,v)
  // Using solution to decision problem to obtain
  // a solution to the optimization problem
  for k ← 1 to (|V| - 1)M // M ← max edge weight
    if A(G,w,u,v,k) ← "YES" then
      output k
      return
  output "INFINITY"
```

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We introduce our first complexity class:

Definition

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P is the set of all decision problems that can be solved in polynomial time by a deterministic algorithm.

Here polynomial time means time that is a polynomial function of the input, i.e. equal to $\Theta(n^k)$ for some constant $k > 0$, where n is the size of the input.

Among the problems we discussed in class, the following are in P:

1. Polynomial evaluation
2. Searching a sorted array
3. Sorting
4. Maximum sum subvector
5. Longest increasing subsequence
6. Activity selection
7. Huffman code
8. ...

The following problems are not known to be in P (and most experts believe they are not in):

1. Graph coloring
2. TSP
3. Subset-sum

These problems happen to have a property in common:

The following problems are not known to be in P (and most experts believe they are not in):

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These problems happen to have a property in common: a solution for each of them can be verified quickly (in polynomial time).

Consider the graph $G = (V, E)$ where $V = \{A, B, C, D, E, F, G\}$ and $E = \{(A, D), (A, B), (B, D), (B, E), (D, E), (B, G), (E, G), (B, C), (E, F), (C, E)\}$.

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Is G 3-colorable?

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If we have to devise the coloring, this is not easy in general... But if someone claims that it is true, they can quickly convince us by giving a 3-coloring of G:

A	B	C	D	E	F	G
blue	green	blue	red	blue	green	red

Example: verifying a solution in polynomial time

Consider the graph $G = (V, E)$ where $V = \{A, B, C, D, E, F, G\}$ and $E =$
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Verifying that a coloring is valid takes time $\Theta(n^2)$ for a graph with n nodes.

Formalism: verification

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The witness y is a piece of information that "proves" that $x \in T$. We require that $|y|$ be polynomial in $|x|$.

Let us clarify the relationship between solving and verifying:

Solving: Given instance x of problem T of size $|x| = n$ we want decide whether $x \in T$ or $x \notin T$.

Verifying: Given instance x of size $|x| = n$ and a witness y of size $|y| \leq c|x|$ for some constant $c > 0$, we want to verify that the witness y proves that $x \in T$.

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The running time of this verification algorithm is $\Theta(n)$ (n steps to verify that the nodes appear the proper number of times, and n steps to sum the edges in the cycle).

Definition

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The following problems are contained in *NP*:

1. Graph coloring
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Theorem

If a problem can be solved, then it can be verified. So $P \subseteq NP$.

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Proof.

Let $Q \in P$ be a problem. Since $Q \in P$, there must be an algorithm A that solves problem Q in polynomial time. Let x be an instance of Q. Then,

- if $x \in Q$ then $A(x) =$ "*YES*",
- if $x \notin Q$ then $A(x) =$ "*no*".

We need to produce an algorithm B that given x and a witness y verifies that $x \in Q$. Here is is:


```
Algorithm B(x, y)
if A(x) ← "YES" then
  return "YES"
else
  return "NO"
```

The algorithm B simply ignores the witness and solves the problem using the algorithm A. So for any x and for any witness y, $B(x, y)$ returns "YES" if and only if $x \in Q$.

Note that if A runs in polynomial time, then B will run in polynomial time as well. So, $Q \in NP$ and we can conclude that $P \subseteq NP$.

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The question is very much open.

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What we mean by "hardest" is that if we could solve any of the NP-complete problems in polynomial time, then we could solve every problem in NP in polynomial time.

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The *NP*-complete problems are the "hardest" problems in NP.

What we mean by "hardest" is that if we could solve any of the NP-complete problems in polynomial time, then we could solve every problem in *NP* in polynomial time.

In order to develop the theory of *NP*-completeness, we define the notion of reduction.

Consider the problem of scheduling exams:

Input: List of lists $L_1, L_2, \ldots, L_n$ of students in classes $1, 2, \ldots, n$, respectively.

Output: The smallest number of exam periods so that all exams can be scheduled with no conflicts.

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Output: The smallest number of exam periods so that all exams can be scheduled with no conflicts.

Note that this is an optimization problem. Let us rephrase the problem as a decision problem:

Input: A list of students for each class, and an integer k.

Output: Output "YES" if k time periods are sufficient to schedule the exams, and "NO" otherwise.

The exam scheduling problem can be solved by translating it (reducing it) to the graph coloring problem and then solving the graph coloring problem.

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The crucial point is that k exam periods are sufficient if and only if the graph G we constructed is k-colorable.

Example: reduction

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Note that instead of solving the problem T_1 on an instance x, we can, solve the problem T_2 on instance $A(x)$ and return that answer.

Furthermore, if problem T_2 accepts a polynomial time algorithm, i.e. $T_2 \in P$, then $T_1 \in P$ as well.

Supppose that the following is true:

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Then if we have an algorithm B for T, we can solve each one of $T_1, \ldots, T_4$ by translating their instances using the algorithms $A_1, \ldots, A_4$ and returning the answer that B gives on the translated instance.

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Therefore, T is harder than T_1, T_2, T_3, T_4.

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Therefore, T is harder than T_1, T_2, T_3, T_4.

We use this idea above to define NP-completeness.

Definition

A decision problem T is *NP*-complete if:

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A decision problem T is NP-complete if:

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2. For all $T' \in NP$, $T' \leq_P T$.

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In order to do this, we define a literal to be a boolean variable or its negation, and a clause to be a literal or a disjunction ("OR") of literals. For example, $x, y, \neg x$ are literals, $(x \vee y), (\neg x \vee y \vee x)$ are both clauses, but $(x \wedge \neg y \vee z)$ is not.

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A formula in conjunctive normal form (CNF) is the conjunction ("AND") of a set of clauses. For example,

$$(x \vee y) \wedge (\neg x \vee y \vee x) \wedge (\neg y \vee z)$$

is in CNF.

Let *CNF.SAT* be the problem defined by

$\{< f >: f$ is a formula in CNF with some satisfying assignment$\}$

Theorem (Cook, 1971)

CNF.SAT is NP-complete.

The proof is beyond the scope of this course.

Let us now consider the general satisfiability problem defined by:

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Suppose that we want to show that SAT is NP-complete. We need to show:

- $SAT \in NP$.
- For all $A \in NP$, $A \leq_P SAT$.

Proof.

The obvious witness is the a list y of truth values to assign to the variables of f. The verification algotithm A takes f and y as inputs and evaluates f at y. It clearly runs in time that is bounded by a polynomial function of the size of f. $\square$

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An easier way to show that every problem in NP reduces to SAT is to show that some NP-complete problem B reduces to SAT. Then since every A in NP reduces to B, and since the relation $\leq_P$ is transitive, every A also reduces to SAT:

$$A \rightarrow^{T_1} B \rightarrow^{T_2} SAT$$

$$x \in A \iff T_2(T_1(x)) \in SAT$$

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To complete the proof that SAT is NP-complete, we must show that every problem A in NP is reducible to SAT. But there are thousands of known problems in NP, and many more unknown ones! It would be a gigantic task to show that each one reduces to SAT.

An easier way to show that every problem in NP reduces to SAT is to show that some NP-complete problem B reduces to SAT. Then since every A in NP reduces to B, and since the relation $\leq_P$ is transitive, every A also reduces to SAT:

$$A \rightarrow^{T_1} B \rightarrow^{T_2} SAT$$

$$x \in A \iff T_2(T_1(x)) \in SAT$$

We will require that the reductions must be polynomial time. We will show that $CNF.SAT \leq_P SAT$ and then, since every $A \in NP$ has the property that $A \leq_P CNF.SAT$, it must be the case that $A \leq_P SAT$.

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Thus the trivial transformation $T(f) = f$ works! It clearly runs in polynomial time. $\square$

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The End!