Algorithms Week 6

Ljubomir Perković, DePaul University

Dynamic Programming

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   2. Give a clear recursive formula or algorithm

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   - d. Find a good evaluation order
   - e. Write down the algorithm
   - f. Analyze space and running time

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Output: A subset of X that sums to T.

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- $6 + 3 + 1 = 10$, so $\{6, 3, 1\}$ is a valid solution.

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- $4 + 6 = 10$, so $\{4, 6\}$ is a valid solution.
- $6 + 3 + 1 = 10$, so $\{6, 3, 1\}$ is a valid solution.
- $4 + 3 + 1 = 8$, so $\{4, 3, 1\}$ is **not** a valid solution.

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Define $SS(i, t)$ to be *True* if some subset of $X[i..n]$ sums to t.

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Define $SS(i, t)$ to be *True* if some subset of $X[i..n]$ sums to t. Then

$$SS(i,t) = \begin{cases} \mathit{True}, & \text{if } t = 0 \\ \mathit{False}, & \text{if } i > n \\ SS(i+1,t), & \text{if } t < X[i] \\ SS(i+1,t) \vee SS(i+1,t-X[i]), & \text{otherwise} \end{cases}$$

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We want to compute $SS(1, T)$.

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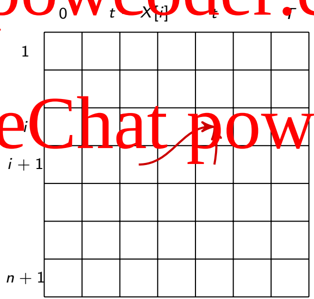

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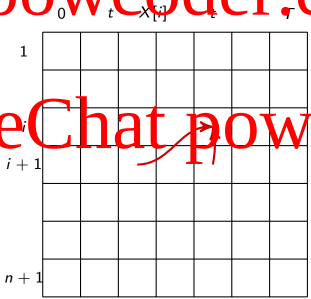

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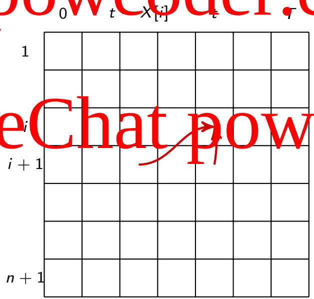

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FastSubsetSum (via dynamic programming)

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  S[n+1, 0] ← True
  for t ← 1 to T
    S[n+1, t] ← False
  for i ← n downto 1
    S[i, 0] ← True
    for t ← 1 to X[i] - 1
      S[i, t] ← S[i+1, t]
    for t ← X[i] to T
      S[i, t] ← S[i+1, t] ∨ S[i+1, t-X[i]]

return S[1, T]
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Running time? $O(nT)$

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Edit distance

Formal problem specification:

Input: Two strings $A[1..m]$ and $B[1..n]$

Output: The edit distance between A and B

The edit distance between **ALGORITHM** and **ALTRUISTIC**, using another kind of visualization:

Edit distance

The edit distance between **ALGORITHM** and **ALTRUISTIC**, using another kind of visualization:

A	L	G	O	R		I		T	H	M
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- a gap in the top word represent the insertion of a letter

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- a gap in the bottom word represents the deletion of a letter
- with two different characters correspond to substitutions

Recursive structure

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Let $Edit(i, j)$ be the edit distance between $A[1..i]$ and $B[1..j]$.

So what can happen in the last column in general?

Insertion

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ALGOR

ALTRU

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An insertion, i.e. the last entry in top row is empty.

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An insertion, i.e. the last entry in top row is empty. Then

$$Edit(i,j) = 1 + Edit(i, j-1)$$

So what can happen in the last column in general?

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Deletion

So what can happen in the last column in general?

ALGO	R
ALTRU	

A deletion, i.e. the last entry in bottom row is empty.

Deletion

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A deletion, i.e. the last entry in bottom row is empty. Then

$$Edit(i,j) = 1 + Edit(i-1,j)$$

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A substitution, i.e. the last entry in both rows is non-empty.

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A substitution, i.e. the last entry in both rows is non-empty. Then

$$Edit(i, j) = 1 + Edit(i - 1, j - 1)$$

if the characters are different.

So what can happen in the last column in general?

Substitution

So what can happen in the last column in general?

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ALTR

So what can happen in the last column in general?

ALGO	R
ALT	R

A substitution, i.e. the last entry in both rows is non-empty.

Substitution

So what can happen in the last column in general?

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ALT	R

A substitution, i.e. the last entry in both rows is non-empty. Then

$$Edit(i, j) = Edit(i - 1, j - 1)$$

if the characters are the same. The substitution is free.

Edit function recurrence

To develop a dynamic algorithm:

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Dynamic programming algorithm development step 2

Build solutions to your recurrence from the bottom up

- (a) Recursive subproblems are `Edit(i,j)` with $0 \leq i \leq m$ and $0 \leq j \leq n$ ...

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a. Recursive subproblems are `Edit(i,j)` with $0 \leq i \leq m$ and $0 \leq j \leq n$ ...
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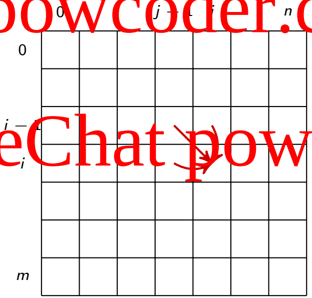

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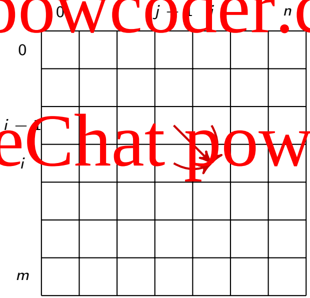

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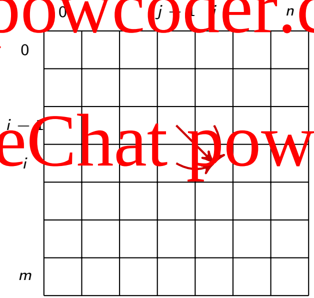

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Edit distance dynamic programming algorithm

```
EditDistance(A[1 .. m], B[1 .. n]):
  for j ← 0 to n
    Edit[0, j] ← j
  for i ← 1 to m
    Edit[i, 0] ← i
    for j ← 1 to n
      ins ← Edit[i, j-1] + 1
      del ← Edit[i-1, j] + 1
      if A[i] = B[j]
        rep ← Edit[i-1, j-1]
      else
        rep ← Edit[i-1, j-1] + 1
      Edit[i, j] ← min{ins, del, rep}
  return Edit[m, n]
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Running time?

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Running time? $O(mn)$

Activity Selection Problem

Suppose activities $1, 2, \ldots, n$ are scheduled at different times of a single day

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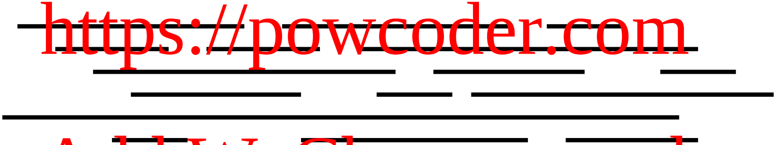
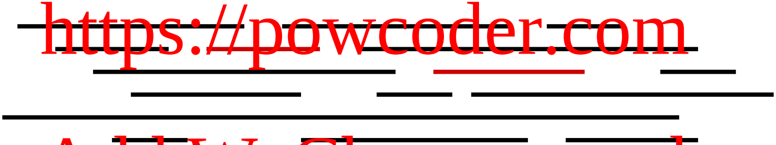
Each activity i is has start time s_i and finish time f_i such that $s_i < f_i$; activity i can be represented as (s_i, f_i).



Activity Selection Problem

Suppose activities $1, 2, \ldots, n$ are scheduled at different times of a single day

Each activity i is has start time s_i and finish time f_i such that $s_i < f_i$; activity i can be represented as (s_i, f_i).



Two activities (s_i, f_i) and (s_j, f_j) are compatible if they do not overlap, i.e. if either $f_j < s_i$ or $f_i < s_j$.

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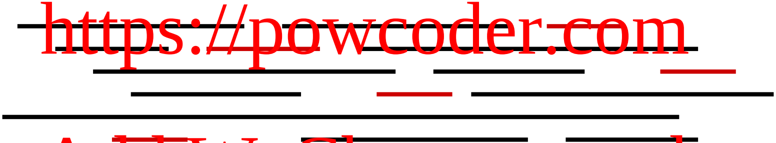

Two activities (s_i, f_i) and (s_j, f_j) are compatible if they do not overlap, i.e. if either $f_j < s_i$ or $f_i < s_j$.

Goal: choose the largest possible set of compatible activities.

Activity Selection Problem

Input: List of activities $1, 2, 3, ..., n$ scheduled at time intervals $(s_1, f_1), (s_2, f_2), \ldots, (s_n, f_n)$.

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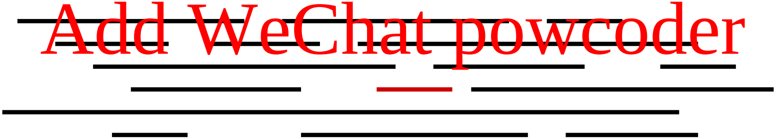
Solution via recursive backtracking:

If activity i is in an optimal solution



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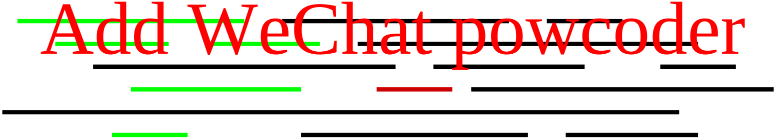
If activity i is in an optimal solution then so is the optimal solution for the set of activities that end before s_i



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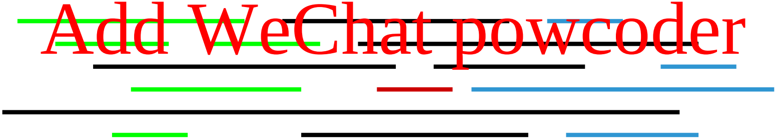
If activity i is in an optimal solution then so is the optimal solution for the set of activities that end before s_i and the optimal solution the set of activities that start after f_i.



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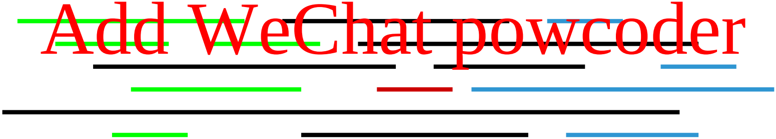

Running time: $O(2^n)$

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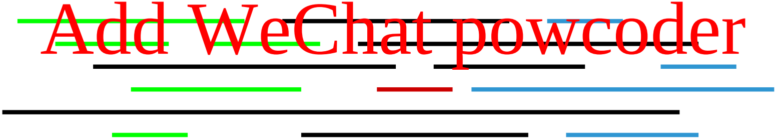
Solution via dynamic programming:

If activity i is in an optimal solution then so is the optimal solution for the set of activities that end before s_i and the optimal solution the set of activities that start after f_i.



Running time: $O(n^3)$

Activity Selection Problem

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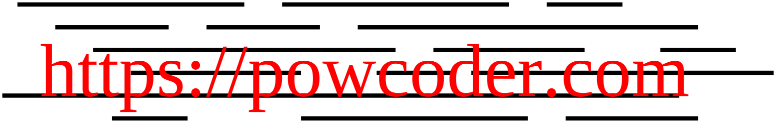
Output: The largest possible set of compatible activities.

Solution via the greedy method:

Running time: $O(n \log n)$

The greedy method

We try “greedy” approaches:



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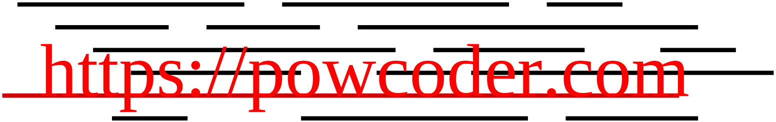
We try “greedy” approaches:

- Choose activity that starts earliest…



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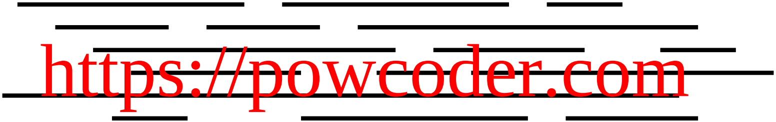

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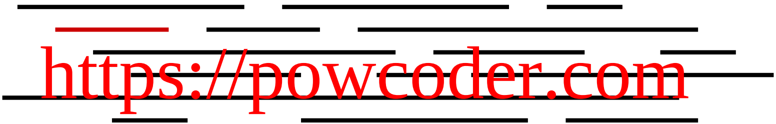

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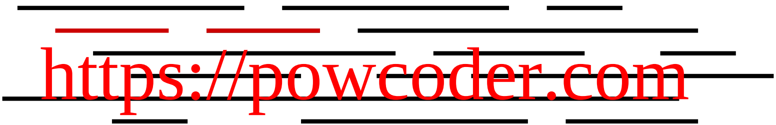

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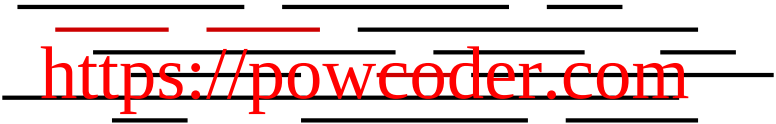

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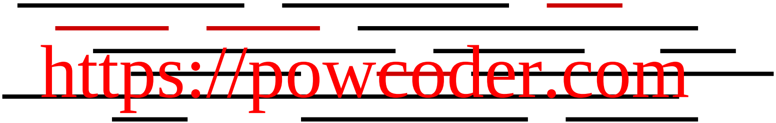

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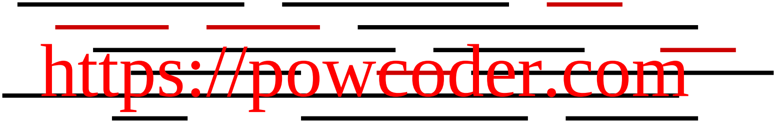

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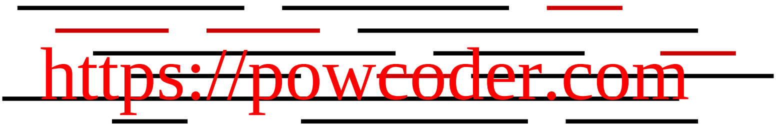

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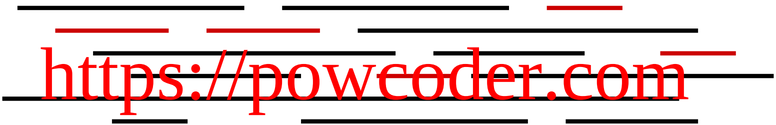

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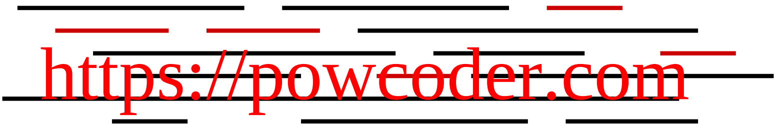

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1. There exists a largest set A of compatible activities containing activity 1.

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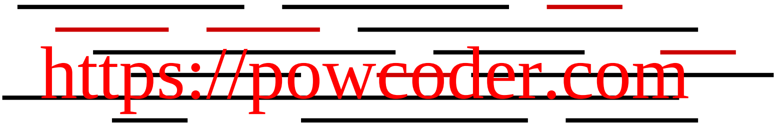

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1. There exists a largest set A of compatible activities containing activity 1.
2. If A is a largest set of activities containing 1 then $A - \{1\}$ is a largest set of compatible activities for $\{(s_i, f_i) : s_i \geq f_1\}$.

Condition 1 proof

1. There exists a largest set A of compatible activities containing activity 1.

Condition 2 proof

1. There exists a largest set A of compatible activities containing activity 1.

2. If A is a largest set of activities containing 1 then $A - \{1\}$ is a largest set of compatible activities for $\{(s_i, f_i) : s_i \geq f_1\}$.

GreedyASP

Let s be an array of activity starting times and let f be an array of activity finishing times, both sorted by finishing time.

In other words, $f[1] \leq f[2] \leq \cdots \leq f[n]$.

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GreedyASP(s,f,n)
  A ← {1}
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  for i ← 2 to n
    if s[i] ≥ f[j] then
      A ← A ∪ {i}
      j ← i
  return A
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Running time?

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Running time? $\Theta(n)$, but only if not including the $\Theta(n \log n)$ time required to sort s and f.

Greedy algorithm

A problem can be solved using a greedy algorithm if it satisfies these conditions:

Greedy choice property: the optimal solution can be constructed from locally optimal choices. This must always be proved before you construct a greedy algorithm for the problem.

Optimal substructure: The optimal solutions contain the optimal solutions to its subproblems.