Algorithms Week 2

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Algorithm design techniques that we will use in this class include:

1. divide-and-conquer
2. backtracking
3. dynamic programming
4. greedy

All of the above techniques rely on reductions.

As your textbook says:

"Reductions are the single most common technique used in designing algorithms.

Reducing one problem X to another problem Y means to write an algorithm for X that uses an algorithm for Y as a black box or subroutine.

Crucially, the correctness of the resulting algorithm for X cannot depend in any way on how the algorithm for Y works. The only thing we can assume is that the black box solves Y correctly. The inner workings of the black box are simply none of our business; they're somebody else's problem. It's often best to literally think of the black box as functioning purely by magic."

Recursion is a special type of reduction that reduces a problem instance to one or more simpler instances of the same problem.

All four algorithm design techniques

1. divide-and-conquer
2. backtracking
3. dynamic programming
4. greedy

that we will use in this class can be described using recursion.

A divide-and-conquer algorithm works as follows:

1. If the problem is small enough, solve it directly (and quickly).
2. Otherwise, divide it into subproblems **(Divide)** that your solve recursively ("as a black box functioning purely by magic"!)
3. Combine the solutions to the subproblems into a solution of the original problem **(Conquer)**.

A fundamental problem is sorting an array of numbers:

Input: An array $A[1..n]$ of n numbers

Output: A re-ordering of the numbers in the array such that $A[1] \leq A[2] \leq \cdots \leq A[n]$

There are many different ways of approaching the problem

InsertionSort works by repeatedly moving elements into their proper relative positions

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MergeSort recursively sorts each half of the array recursively and then merges the two sorted halves

```
MergeSort(A[1..n])
  if n > 1
    m ← ⌊n/2⌋
    MergeSort(A[1..m])
    MergeSort(A[m+1..n])
    Merge(A[1..n], m)
```


We describe `Merge` next

The specification for the Merge procedure is:

Input: An array $A[1..N]$ and index m such that $A[1..m]$ and $A[m+1..n]$ are sorted in non-decreasing order.

Output: A re-ordering of the numbers in the array such that $A[1] \leq A[2] \leq \cdots \leq A[n]$

```
Merge(A[1..n], m)
  i← 1; j← m+1
  for k ← 1 to n
    if j > n
      B[k]← A[i]; i← i+1
    else if i > m
      B[k]← A[j]; j← j+1
    else if A[i] < A[j]
      B[k]← A[i]; i← i+1
    else
      B[k]← A[j]; j← j+1
  for k ← 1 to n
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    else
      B[k]← A[j]; j← j+1
  for k ← 1 to n
    A[k] ← B[k]
```


Correctness: After iteration k, $B[1..k]$ holds the smallest k elements of A in sorted order.

Merge

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  for k ← 1 to n
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  for k ← 1 to n
    A[k] ← B[k]
```


Running time: $O(n)$

Let $T(n)$ be the time needed to sort an array of n elements.

- Lines 1 and 2 take constant time each.
- The first recursive call to MergeSort takes $T(\lfloor \frac{n}{2} \rfloor)$ time.
- The second recursive call to MergeSort takes $T(\lceil \frac{n}{2} \rceil)$ time.
- The call to Merge takes $O(n)$ time.

and we thus we get the recurrence relation:

$$T(n) = T(\lfloor \frac{n}{2} \rfloor) + T(\lceil \frac{n}{2} \rceil) + O(n)$$

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We can safely ignore the floors an ceilings (see textbook).

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and we thus we get the recurrence relation:

$$T(n) = 2T(\frac{n}{2}) + O(n)$$

We can safely ignore the floors and ceilings (see textbook).

A recurrence relation is a recursive way to define a function. It has two parts:

1. The recursive definition, that is, the part that describes $T(n)$ in terms of itself on smaller sized inputs.
   - For example, $T(n) = 2T(n/2) + f(n)$
   - Or, a bit more loosely, $T(n) = 2T(n/2) + O(n)$.

2. The initial conditions, that is, the part that determines when the recurrence stops.
   - For example, $T(1) = 1$
   - Or, $T(1) = O(1)$, which we will in general assume and not write down.

We can solve recurrence relations using recursion trees, which is a rooted tree with one node for each recursive subproblem.

The value of each node is the amount of time spent on the corresponding subproblem excluding recursive calls.

Solving recurrence relations

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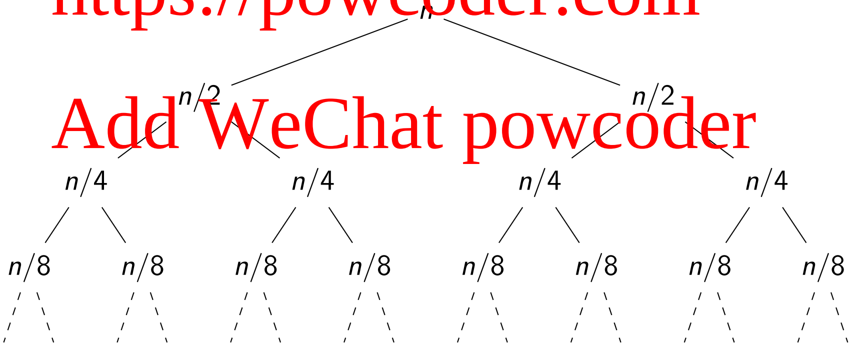
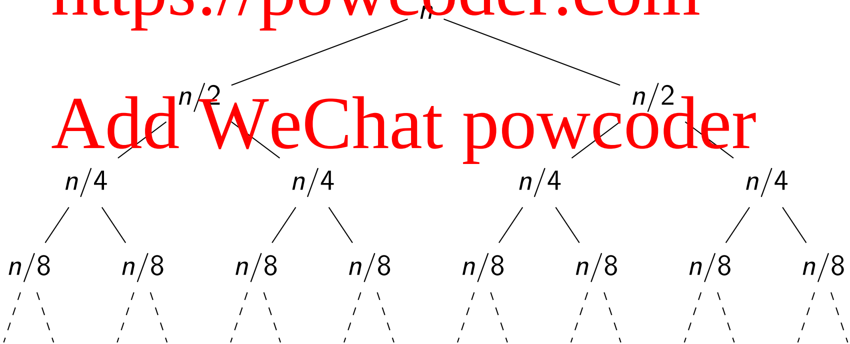
Example: $T(n) = 2T(n/2) + O(n)$



Thus, the overall running time of the algorithm is the sum of the values of all nodes in the tree.

Since each row adds up to n and since there are $\log_2 n$ rows (why?) the total running time $T(n)$ is $O(n \log n)$.

Example: $T(n) = 2T(n/2) + O(n) = O(n \log n)$



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There are many different ways of approaching the problem

QuickSort partitions the array into three parts: an element chosen to be the pivot, a subarray containing elements smaller than the pivot, and a subarray containing elements larger than the pivot. QuickSort then recursively sorts the two subarrays.

```
QuickSort(A[1..n])
  if (n > 1)
    Choose a pivot element A[p]
    r ← Partition(A[1..n], p)
    QuickSort(A[1..r-1])
    QuickSort(A[r+1..n])
```


We describe `Partition` next

The Partition procedure has the following specification:

Input: An array $A[1..N]$ and a pivot element $P = A[p]$ where $1 \leq p \leq N$.

Output: Modified array A with an index r such that entries in $A[1..r-1]$ are less than P, entries in $A[r+1..n]$ are greater than or equal to P, and $A[r] = P$.

```
Partition(A[1..n], p)
  swap A[p] ↔ A[n]
  l ← 0
  for i ← 1 to n - 1
    if A[i] < A[n]
      l ← l+1
      swap A[l] A[i]
  swap A[n] ↔ A[l + 1]
  return l + 1
```

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```


Correctness: At the end of each iteration of the main loop, everything in the subarray $A[1..l]$ is $< A[n]$ and everything in the subarray $A[l+1..i]$ is $\geq A[n]$.

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Running time: $O(n)$.

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- The first call to Quicksort takes $T(r-1)$ time.
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and we thus we get the recurrence relation:

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If we could magically choose the pivot to be the median then

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Running time analysis of QuickSort

If the pivot happens to be the smallest or largest entry then

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If the pivot happens to be the smallest or largest entry then

$T(n) = T(n-1) + O(n) = O(n^2)$

n

$n-1$

$n-2$

$n-3$

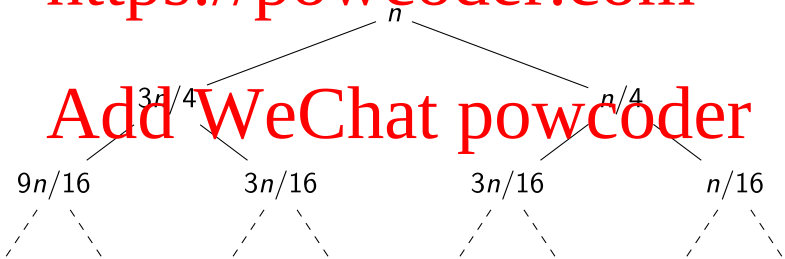
What if we can guarantee that the pivot is in the middle half of the array? Then

$$T(n) \leq T(3n/4) + T(n/4) + O(n)$$

What if we can guarantee that the pivot is in the middle half of the array? Then

$$T(n) = T(3n/4) + T(n/4) + O(n) = O(n \log n)$$

since all rows sum to n and the depth of the tree is $\log_{4/3} n$.



In Section 1.8, your textbook describes an algorithm that can be used to find the pivot in $O(n)$ time.

Therefore the worst-case running time of QuickSort can be made $O(n \log n)$.

Consider the problem of computing a^n:

Input: number a and positive integer n

Output: a^n

Example: given $a = 1.5$ and $n = 2$, $a^n = 1.5^2 = 2.25$

The obvious algorithm


```
SlowPower(a, n)
  res ← a
  for i ← 2 to n
    res ← res * a
  return res
```

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Running time: $T(n) = O(n)$

How could divide-and-conquer possibly help?

Indian scholar Pingala proposed the following recursive formula in 2nd century BCE!

$$a^n = \begin{cases} 1 & \text{if } n = 0 \\ (a^{n/2})^2 & \text{if } n > 0 \text{ and n is even} \\ (a^{\lfloor n/2 \rfloor})^2 a & \text{otherwise} \end{cases}$$

which leads to the following exponentiation algorithm:

```
PingalaPower(a, n)
  if n ← 1
    return a
  tmp ← PingalaPower(a, ⌊n/2⌋)
  if n is even
    return tmp*tmp
  else
    return tmp*tmp*a
```

The running time $T(n)$ for input n satisfies

$$T(n) = T(n/2) + O(1)$$

Running time analysis of PingalaPower

The running time $T(n)$ for input n satisfies

$$T(n) = T(n/2) + O(1) = O(\log n)$$

since the depth of the tree is $\log_2 n$.

The recurrence relation for BinarySearch is

Exercise: What is the running time of BinarySearch?

The recurrence relation for BinarySearch is

$T(n) = T(\frac{n}{2}) + O(1)$

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Multiplying two n-digit numbers x and y using a traditional multiplication algorithm requires $O(n^2)$ single-digit multiplications.

$$\begin{array}{ccccccccc} & x_{n-1} & x_{n-2} & x_{n-3} & \cdots & x_3 & x_2 & x_1 & x_0 \\ \times & y_{n-1} & y_{n-2} & y_{n-3} & \cdots & y_3 & y_2 & y_1 & y_0 \end{array}$$

Multiplying two n-digit numbers x and y using a traditional multiplication algorithm requires $O(n^2)$ time

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Let a, b, c, and d be numbers defined as follows:

$$\begin{array}{cccccccc} a & = & x_{n-1} & x_{n-2} & \cdots & x_{m+1} & x_m \\ b & = & x_{m-1} & x_{m-2} & \cdots & x_1 & x_0 \\ c & = & y_{n-1} & y_{n-2} & \cdots & y_1 & y_0 \\ d & = & y_{m-1} & y_{m-2} & \cdots & y_1 & y_0 \end{array}$$

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Then $xy = (10^m a + b)(10^m c + d) = 10^{2m} ac + 10^m(bc + ad) + bd$.

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Then $xy = (10^m a + b)(10^m c + d) = 10^{2m} ac + 10^m (bc + ad) + bd$.

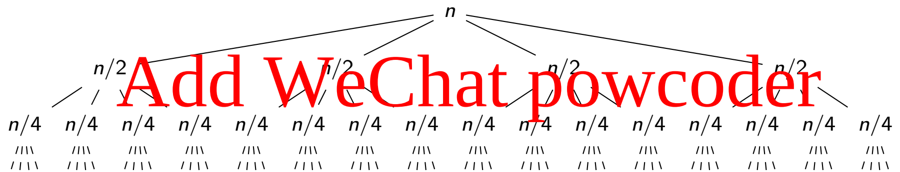
If $m \approx n/2$, multiplying two n-digit numbers is reduced to multiplying 4 $n/2$-digit numbers: ac, bc, ad, and bd.

```
SplitMultiply(x, y, n):
  if n = 1
    return xy
  else
    m ← ⌈n/2⌉
    a ← ⌊x/10^m⌋; b ← x mod 10^m
    c ← ⌊y/10^m⌋; d ← y mod 10^m
    e ← SplitMultiply(a, c, m)
    f ← SplitMultiply(b, d, m)
    g ← SplitMultiply(b, c, m)
    h ← SplitMultiply(a, d, m)
  return 10^(2m) e + 10^m (g + h) + f
```

Running time: $T(n) = 4T(n/2) + O(n)$

The running time of this divide-and-conquer multiplication algorithm is

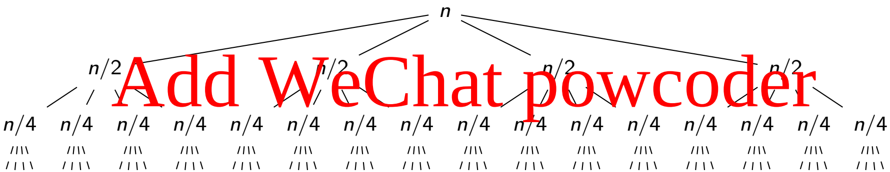
$$T(n) = 4T(n/2) + O(n)$$



The running time of this divide-and-conquer multiplication algorithm is

$$T(n) = 4T(n/2) + O(n) = O(n^2)$$

since the rows add up to the geometric progression $n, 2n, 2^2 n, 2^3 n, \ldots, 2^{\log n} n$ which adds up to $O(n^2)$.



Running time analysis

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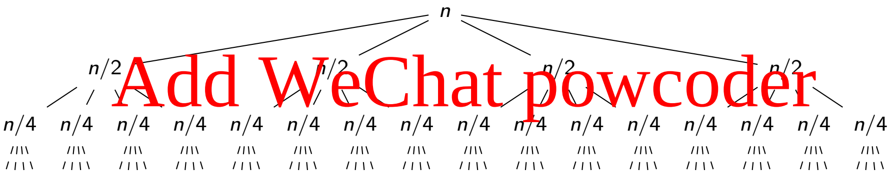

No gain!

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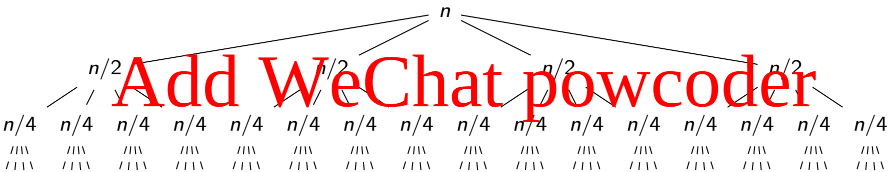

No gain! So why did we bother ..?

$$\begin{array}{ccccccccc} & x_{n-1} & x_{n-2} & x_{n-3} & \cdots & x_3 & x_2 & x_1 & x_0 \\ \times & y_{n-1} & y_{n-2} & y_{n-3} & \cdots & y_3 & y_2 & y_1 & y_0 \end{array}$$

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Then $xy = (10^m a + b)(10^m c + d) = 10^{2m} ac + 10^m (bc + ad) + bd$.

In the mid-1950s, Karatsuba, a 23 year old student, realized that we do not necessary need to compute bc and ad to get $(bc + ad)$. One can compute $(bc + ad)$ as follows:

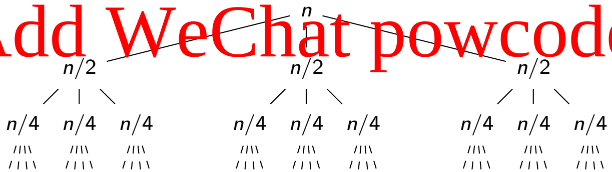
$$ac + bd - (a - b)(c - d) = bc + ad$$

```
FastMultiply(x, y,n)
  if n ← 1
    return xy
  else
    m ← ⌈n/2⌉
    a ← ⌊x/10^m⌋; b ← x mod 10^m
    c ← ⌊y/10^m⌋; d ← y mod 10^m
    e ← FastMultiply(a, c, m)
    f ← FastMultiply(b, d, m)
    g ← FastMultiply(a - b, c - d, m)
    return 10^(2m) e + 10^m (e + f - g) + f
```

Running time: $T(n) = 3T(n/2) + O(n)$.

The running time of the 2nd divide-and-conquer multiplication algorithm is

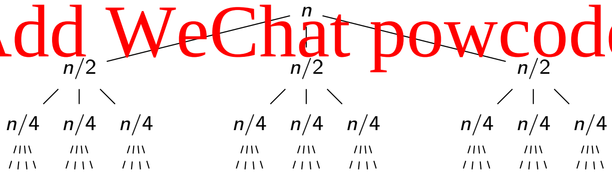
$$T(n) = 3T(n/2) + O(n)$$



The running time of the 2nd divide-and-conquer multiplication algorithm is

$$T(n) = 3T(n/2) + O(n) = O(n^{1.58496})$$

since the rows add up to the geometric progression $n, 3n/2, (3/2)^2 n, (3/2)^3 n, \ldots, (3/2)^{\log_2 n} n$ which adds up to $O(n^{\log_2 3}) = O(n^{1.58496})$.



Given n points on the plane,

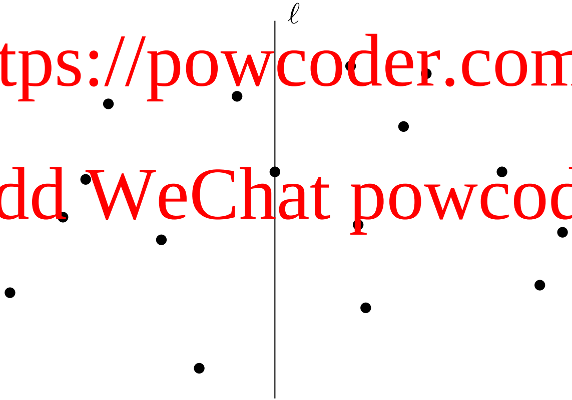
Finding the closest pair of points

Given n points on the plane, find the pair that is closest together.

Input: Points $p_1, \ldots, p_n$ where p_i has coordinates (x_i, y_i).

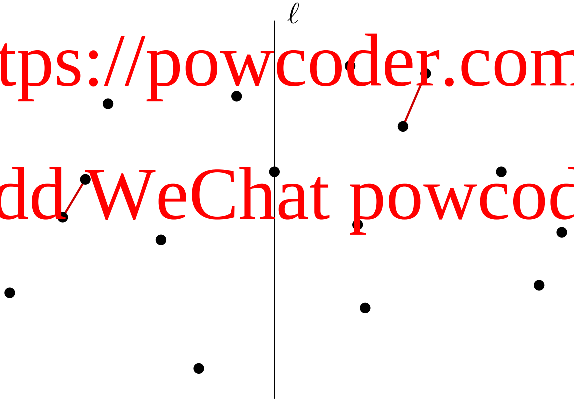
Output: Pair p_i, p_j whose distance $d(p_i, p_j)$ is smallest.

Step 1: Sort the points by x-coordinate



Step 1: Sort the points by x-coordinate

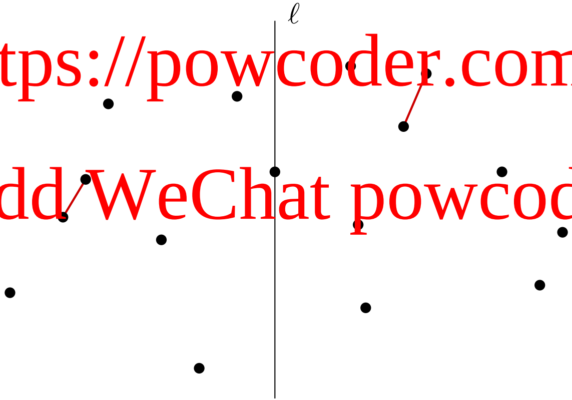
Step 2: Split the plane at the median point and recursively find the closest pair of points in each half



Step 1: Sort the points by x-coordinate

Step 2: Split the plane at the median point and recursively find the closest pair of points in each half

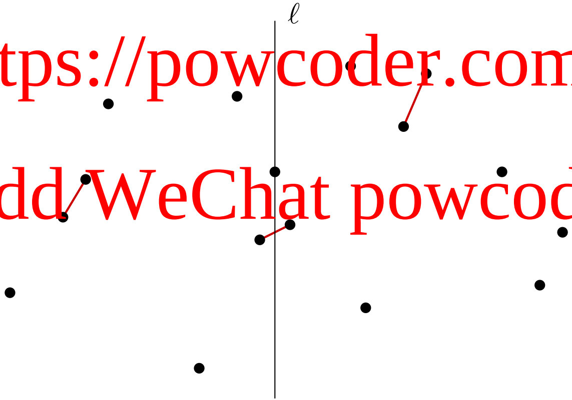
Step 3: Return the closer of the two pairs!



Step 1: Sort the points by x-coordinate

Step 2: Split the plane at the median point and recursively find the closest pair of points in each half

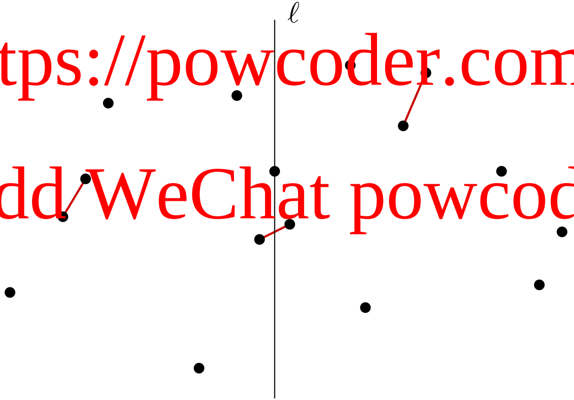
Step 3: ~~Return the closer of the two pairs!~~



Step 1: Sort the points by x-coordinate

Step 2: Split the plane at the median point and recursively find the closest pair of points in each half

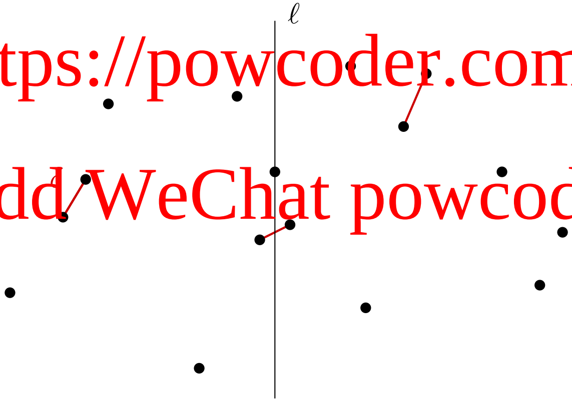
Step 3: Need to also find the closest pair of points that lie on oposite sides of ℓ



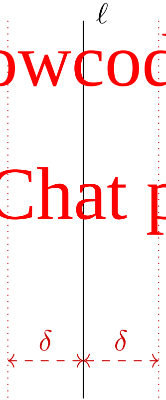
Step 1: Sort the points by x-coordinate

Step 2: Split the plane at the median point and recursively find the closest pair of points in each half

Step 3: Need to also find the closest pair of points that lie on oposite sides of ℓ but only if their distance is less than δ

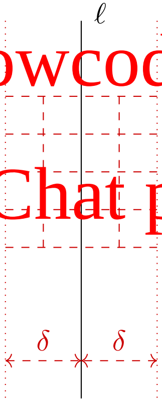


Insight 1: If there is a pair of opposite points whose distance is less than δ their distance from line ℓ must be less than δ

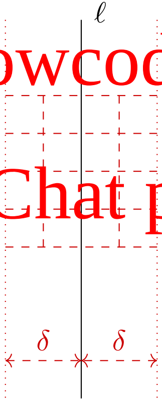


Step 1.1: Sort the points by y-coordinate to get ordered list S_y. Let S'_y be the sublist of S_y consisting of points in the narrow 2δ band.

Insight 2: If two points in S'_y are closer than δ then the points must be within 15 positions of each other in the list.



Step 1.1: Sort the points by y-coordinate to get ordered list S_y.
Step 3: Construct sublist S'_y from S_y and then go through list S'_y and for each point compute the distance to the next 15 points in S'_y to find closest pair of opposite points whose distance is $< \delta$, if any.



- Steps 1 and 1.1 take $O(n\log(n))$ time each and are done only once.
- Step 2 consists of two recursive calls, each taking $T(n/2)$ time.
- Step 3 consists of a sequential search through S_y followed by a sequential search through S'_y which takes a total time of $O(n)$.

If $T(n)$ is the running time of steps 2 and 3 then

$$T(n) = 2T(n/2) + O(n) = O(n\log n).$$

By homework problem 1(c), the running time of the whole algorithm is $O(n\log n)$.