Algorithms Week 5

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The sequence can be defined recursively:

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The Fibonacci numbers have various interesting properties, including being related to the golden ratio and its conjugate:

$$\phi = \frac{1+\sqrt{5}}{2} \approx 1.61803 \qquad \hat{\phi} = \frac{1-\sqrt{5}}{2} \approx -0.61803$$

You could prove by induction that $F(n) = \frac{\phi^n - \hat{\phi}^n}{\sqrt{5}}$.

Fibonacci number recursive algorithm

The obvious algorithm for computing $F(n)$:


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  if n = 0 or n = 1
    return 1
  else
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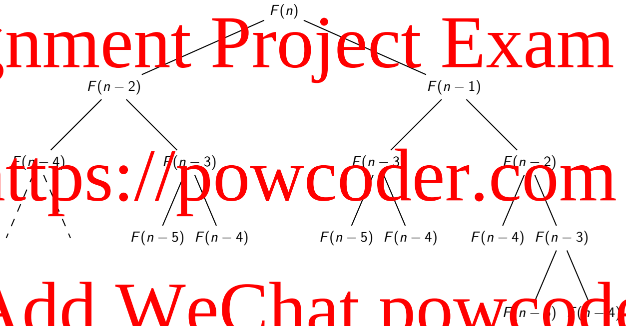
$$T(n) = T(n-1) + T(n-2) + \Theta(1).$$

Consider the recursion tree for $F(n)$:



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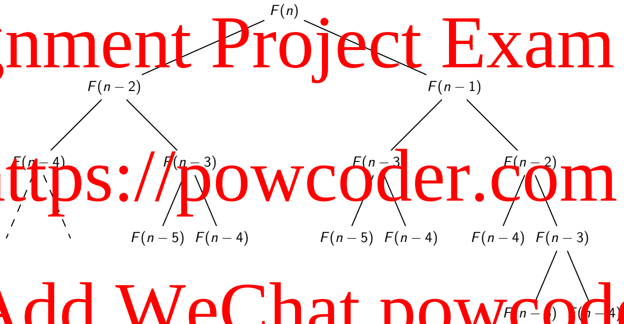

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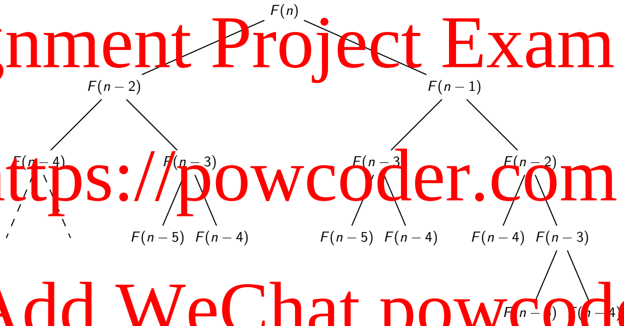

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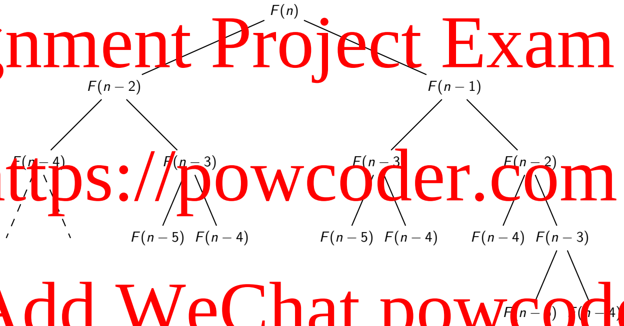

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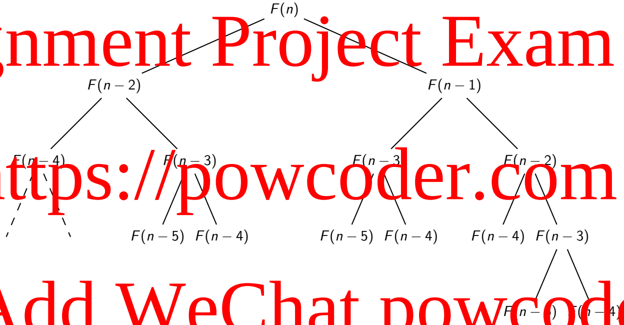

Insights:

- The number of recursive calls at depth i is 2^i
- The shallowest leaf is the leftmost one, with depth $\frac{n}{2}$.

Thus the total number of recursive calls is $\Omega(2^{\frac{n}{2}})$, or exponential.

In fact, the solution to the recursion

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The reason the recursive algorithm is so slow is because the same recursive calls are recomputed over and over.

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This technique is called memoization.

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- Then, at the beginning of each recursive call, check to see if the value we want already exists.
- If it does, then we re-use it. Otherwise we compute it.

This technique is called memoization. It:

- maintains the familiar, recursive, top-down structure of the algorithm
- but without the exponential costs of re-computing all the values.

Memoized algorithm for Fibonacci numbers:


```
// F[0..n] is a global array
MemFib(n)
  if n = 0 or n = 1 then
    F[n] ← 1
  else if F[n] undefined
    F[n] ← MemFib(n-1) + MemFib(n-2)
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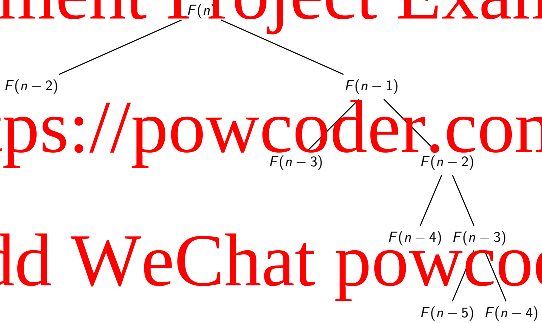
Running Time?

Running time of MemFib



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Running Time? $O(n)$! An exponential speedup!

Rather than using a top-down approach to reach the bottom of the recursion and then compute Fibonacci numbers bottom-up, dynamic programming does the bottom-up approach directly and iteratively.

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To compute $F(n)$:

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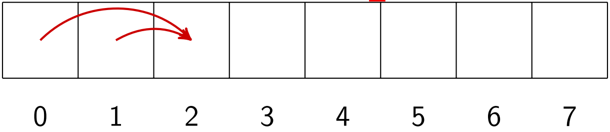
- We start by computing $F(2)$ from $F(0) = 1$ and $F(1) = 1$.



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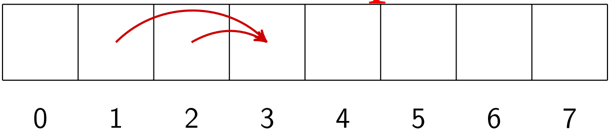

Dynamic Programming

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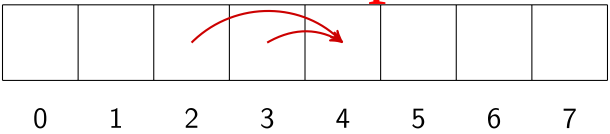
- and then $F(4)$ from $F(3)$ and $F(2)$, and so on . . .



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IterFib(n)
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Note that this algorithm uses $\Theta(n)$ space. Can you modify it so it uses $\Theta(1)$ space?

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   e. Write down the algorithm
   f. Analyze space and running time

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Output: True if A can be segmented into a sequence of words, False otherwise.

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Running time? $O(2^n)$

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For example:

BLUE	**STEM**	**UNIT**	**ROBOT**	HEARTHANDSATURNSPIN

BLUEST	**EMU**	**NITRO**	**BOT**	HEARTHANDSATURNSPIN

Dynamic programming algorithm development step 2

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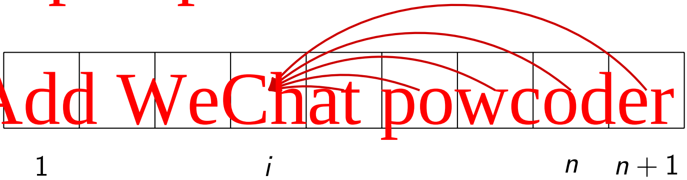
- (b) ... so we can memoize the function `Splittable` into an array `SplitTable[1..n+1]`.
- (c) Each subproblem `Splittable(i)` depends only on results of subproblems `Splittable(j)` where `j > i` ...



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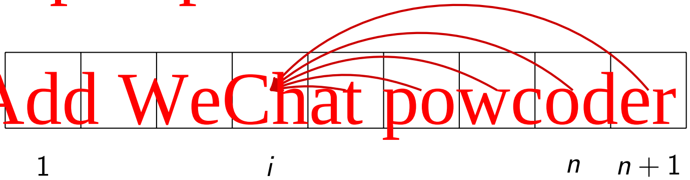

d. ... so we should be filling the array in decreasing index order, starting with `SplitTable[n+1] = True`.

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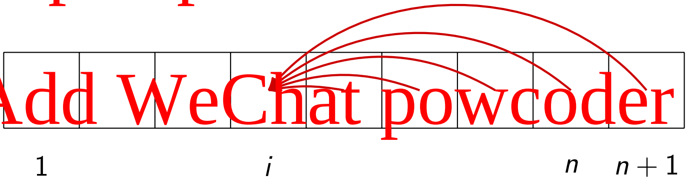

d. ... so we should be filling the array in decreasing index order, starting with `SplitTable[n+1] = True`.
e. The algorithm is on the next slide ...
f. Also on next slide, the running time analysis ...

Text segmentation dynamic programming algorithm

```
// Is A[1..n] splittable?
FastSplittable(A[1..n]):
  SplitTable[n + 1] ← True
  for i ← n down to 1
    SplitTable[i] ← False
    for j ← i to n
      if IsWord(i, j) and SplitTable[j + 1]
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Running time? $O(n^2)$

For any sequence S,

2	6	1	4	2	4	2	9	5	3	5	7	8	3

For any sequence S, a subsequence of S is another sequence obtained from S by deleting zero or more elements, without changing the order of the remaining elements.

2	**6**	**1**	4	2	**4**	2	**9**	5	3	**5**	**7**	8	3

Longest Increasing Subsequence

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Problem: Given a sequence of integers S find the Longest Increasing Subsequence (LIS) of S.

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Problem: Given a sequence of integers S find the Longest Increasing Subsequence (LIS) of S.

Input: Integer array $A[1..n]$.

Output: Longest possible sequence of indices $1 \le i_1 < i_2 < \cdots < i_l \le n$ such that $A[i_1] < A[i_2] < \cdots < A[i_l]$.

Example: See above.

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// every element is larger than A[i]
LISbigger(i, j):
  if j > n
    return 0
  else if A[i] ≥ A[j]
    return LISbigger(i, j + 1)
  else
    skip ← LISbigger(i, j + 1)
    take ← LISbigger(j, j + 1) + 1
    return max{skip, take}
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2. Build solutions to your recurrence from the bottom up
   a. Identify the subproblems
   b. Choose a memoization data structure
   c. Identify dependencies
   d. Find a good evaluation order
   e. Write down the algorithm
   f. Analyze space and running time

Dynamic programming algorithm development step 2

Build solutions to your recurrence from the bottom up

1. The subproblems are `LISbigger(i, j + 1)` and `LISbigger(j, j + 1)` with indices i and j with values between 0 and n

Build solutions to your recurrence from the bottom up

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b. We can memoize the results of these subproblems into a two-dimensional array `LISbigger[0..n, 1..n+1]`.

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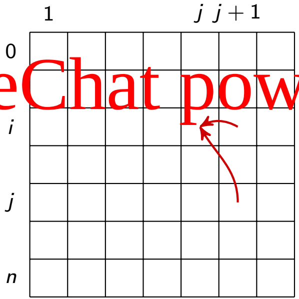

Dynamic programming algorithm development step 2

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d. Fill in the entries of the two-dimensional table column-by-column, right-to-left.

Dynamic programming algorithm development step 2

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e. Iterative algorithm on the slide...
f. Running time analysis on the next slide...

```
// return length of LIS of A[1..n]
FastLIS(A[1 .. n]):
  A[0] ← −∞
  for i ← 0 to n
    LISbigger[i, n + 1] ← 0
  for j ← n down to 1
    for i ← 0 to j − 1
      keep ← 1 + LISbigger[j, j + 1]
      skip ← LISbigger[i, j + 1]
      if A[i] ≥ A[j]
        LISbigger[i, j] ← skip
      else
        LISbigger[i, j] ← max{keep, skip}
  return LISbigger[0, 1]
```

FastLIS

```
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```


Running time?

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```


Running time? $O(n^2)$