Algorithms Week 9

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A directed graph is strongly connected if and only if every pair of vertices is strongly connected.

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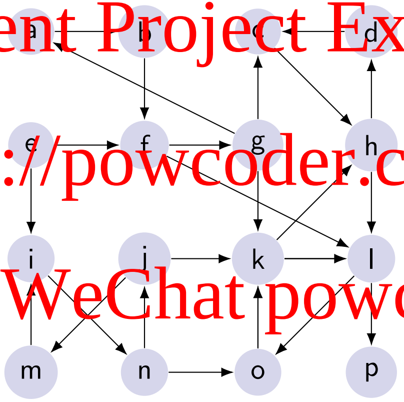
$scc(G)$ is always a dag.

Strongly connected components



Finding all strongly connected components of a directed graph

To find the strongly connected component that a vertex v is part of:

- First we compute $reach(v)$ via whatever-first search
- Then compute $reach^{-1}(v) = \{u | v \in reach(u)\}$ by searching, from v, the directed graph obtained by reversing the direction of edges of G.

The strongly connected component of v is $reach(v) \cap reach^{-1}(v)$.

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To find all strongly connected components in a directed graph we can repeat the above using the standard DFS wrapper function. However, the resulting algorithm runs in $O(VE)$ time: there are at most V strong components, and each requires $O(E)$ time to discover.

Algorithms to compute all strongly connected components in $O(V + E)$ time rely on the following observation:

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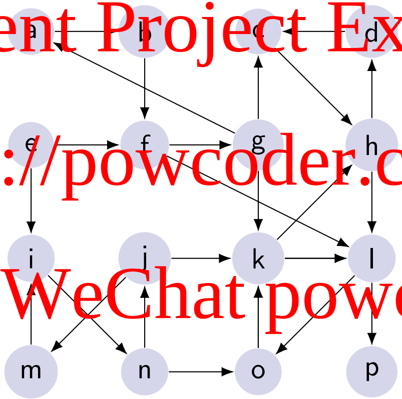
The lemma implies that each strongly connected component of a directed graph G defines a connected subtree of any depth-first forest of G. In particular, for any strongly connected component C, the vertex in C with the earliest preorder time is the lowest common ancestor of all vertices in C; we call this vertex the root of C.

Strongly connected components in linear time



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until no vertices remain.

Strongly connected components in linear time


```
StrongComponents(G):
  count ← 0
  while G is non-empty
    C ← ∅
    count ← count + 1
    v ← any vertex in a sink component of G
    for all vertices w in reach(v):
      w.label ← count
      add w to C
    remove C and its incoming edges from G
```

How to find a vertex in a sink component?

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The last vertex in any postordering of the graph obtained by reversing the edges of G lies in a sink component of G.

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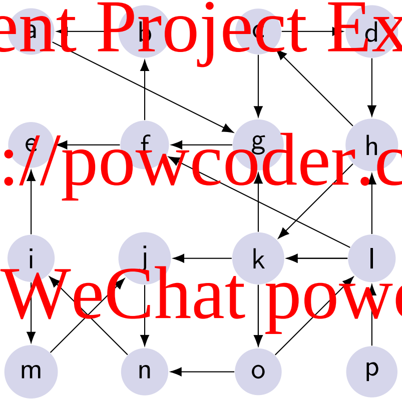
Then, if we traverse the graph a second time, where the wrapper function follows a reverse postordering of rev(G), each call to DFS in the wrapper function visits exactly one strongly connected component.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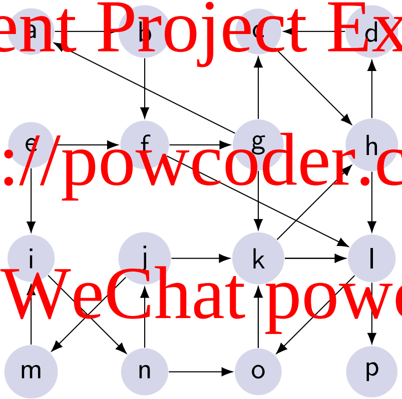
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... we are interested in finding the minimum spanning tree of G, that is, the spanning tree T that minimizes the function

$$w(T) = \sum_{e \in T} w(e).$$

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Some edges of $G \setminus F$ are neither safe nor useless; we call these edges undecided.

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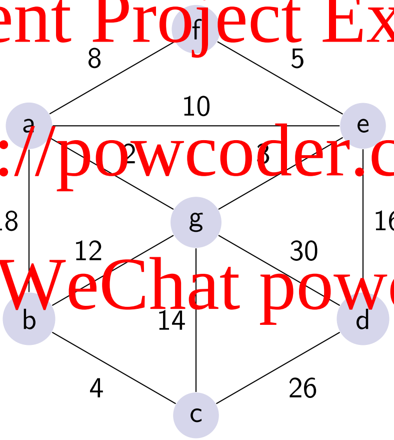
To fully specify a particular algorithm, we must decide which safe edge(s) to add in each iteration and how to find them.

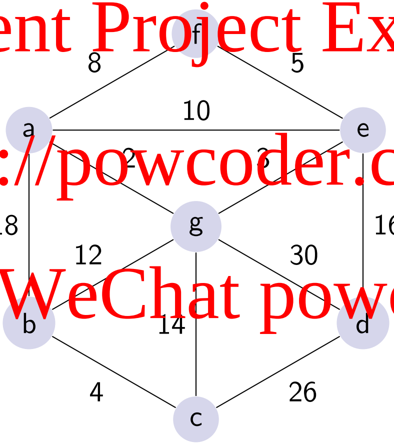
Boruvka's algorithm

Add ALL the safe edges and recurse.



Jarnik's ("Prim's") Algorithm

Start with T being an arbitrary vertex, then repeatedly add T's safe edge to T.



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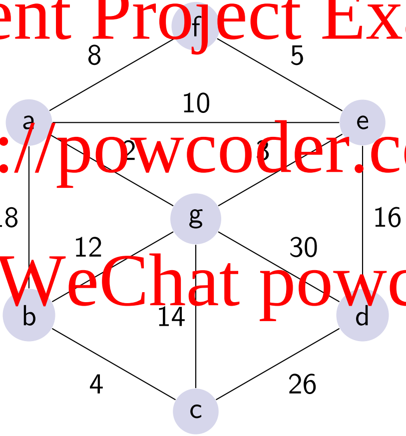
Jarnik's algorithm is a variant of "best-first search" which runs in $O(E \log E) = O(E \log V)$ time if a binary heap is used to implemenent the priority queue.

Scan all edges by increasing weight; if an edge is safe, add it to F.



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> *SSSP: Find the shortest paths from the source vertex s to every other vertex in the graph.*

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At the beginning of the algorithm, we initialize the distances and predecessors as follows:


```
InitSSSP(s):
  dist(s) ← 0
  pred(s) ← Null
  for all vertices v ≠ s
    dist(v) ← ∞
    pred(v) ← Null
```

During the execution of the algorithm, an edge $u \to v$ is said to be **tense** if $dist(u) + w(u \to v) < dist(v)$.

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We improve this overestimate by relaxing the edge as follows:

```
Relax(u→v):
  dist(v) ← dist(u) + w(u→v)
  pred(v) ← u
```

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There are several ways to do that, depending on the structure of the input graph, which leads to different algorithms.

If all edges have weight 1 and the length of a path is just the number of edges then we can just use BFS:

Unweighted Graphs: Breadth-First Search

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```
BFS(s):
  InitSSSP(s)
  Push(s)
  while the queue is not empty
    u ← Pull()
    for all edges u→v
      if dist(v) > dist(u) + 1
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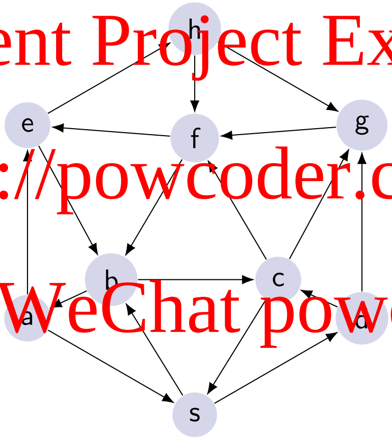
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No negative edges: Dijkstra's algorithm

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Works well for arbitrary weighted graphs as longs as the edge weights are not negative.

```
NonnegativeDijkstra(s):
  InitSSSP(s)
  for all vertices v
    Insert(v,dist(v))
  while the priority queue is not empty
    u ← ExtractMin( )
    for all edges u→v
      if u→v is tense
        Relax(u→v)
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Running time: $O(E \log V)$

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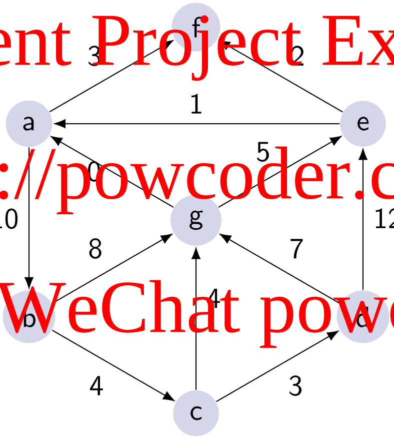

Directed Acyclic Graphs: DFS + dynamic programming

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... but it is only a recurrence for directed acyclic graphs.

Why? If the input graph G contained a cycle, a recursive evaluation of this function would fall into an infinite loop; if G is a dag, each recursive call visits an earlier vertex in topological order.

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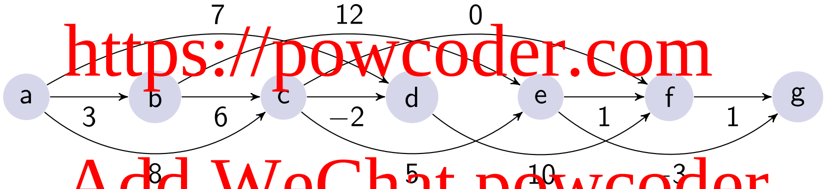
Thus, we compute the distance from s of every vertex v in G by first computing (using DFS) a topological ordering of the vertices in G and then applying dynamic programming to compute $dist(v)$ in topological order.

```
DagSSSP(s):
  for all vertices v in topological order
    if v = s
      dist(v) ← 0
    else
      dist(v) ← ∞
      for all edges u→v
        if dist(v) > dist(u) + w(u→v)
          dist(v) ← dist(u) + w(u→v)
```

Directed Acyclic Graphs: DFS + dynamic programming



A refactoring of the algorithm to fit the SSSP algorithm format:


```
PushDagSSSP(s):
  InitSSSP(s)
  for all vertices u in topological order
    for all outgoing edges u→v
      if u→v is tense
        Relax(u→v)
```