Econ/Math C103 (2020) - Problem Set 2

Due: 3:30pm PDT, 9/24/2020

1. Let $X = [0,1] \times [0,1] = \{x = (x_1, x_2) | x_1, x_2 \in [0,1]\}$. Let the preference relation R over X be defined by:

$$(x_1, x_2) R (y_1, y_2) \iff [(x_1 > y_1) \text{ or } (x_1 = y_1 \text{ and } x_2 \geq y_2)],$$

for all $x = (x_1, x_2), y = (y_1, y_2) \in X$. That is, you first compare alternatives $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in terms of their first coordinates, if that does not give a strict ranking, only then you look at the second coordinate. Show that R is rational.

2. Let X^1 be a set of potential chicken dishes and let X^2 be a set of potential vegetarian dishes. Assume that both sets are finite and $X^1 \cap X^2 = \emptyset$. The set of all alternatives are given by $X = X^1 \cup X^2$. For each $i = 1, 2$, the decision maker has a strict preference (linear order) R^i over X^i. Faced with a menu $A \in \mathcal{P}(X)$, she first determines which one of the two categories of meals is more popular: if $|A \cap X^i| > |A \cap X^j|$ then she chooses her favorite dish in $A \cap X^i$ with respect to R^i. In case of a tie, i.e. if there are the same number of dishes available from each category, then she chooses her favorite available chicken dish. Let c denote her choice function. Does there exist a rational preference relation R on X such that $c = c^R$? If your answer is yes show why, otherwise give a counterexample.

3. Consider the choice function c induced by the satisfycing procedure given in text. Does there exist a rational preference R such that $c = c^R$? If your answer is yes show why, otherwise give a counterexample.

4. Let X be a finite set of alternatives. A choice correspondence c satisfies **path-independence** if for any $A, B \in \mathcal{P}(X)$:

$$c(c(A) \cup c(B)) = c(A \cup B).$$

   (a) Briefly explain (in words) how you interpret path independence.

   (b) Show that Sen's α and Sen's β together imply path-independence.

   (c) Show that path-independence implies Sen's α.

   (d) Give a counterexample showing that path-independence does not imply Sen's β.

5. Let X be a finite set of alternatives. Prove that if a choice correspondence c on a finite set of alternatives X satisfies Sen's condition α, then there exists a rational preference relation R such that $c^R \subset c$.

6. Let $X = \mathbb{R}_+$ represent nonnegative monetary prizes and consider an expected utility maximizer with vNM utility function $u(x) = x^2$.

   (a) Would she always make the same choices among lotteries if her vNM utility function were instead:

      i. $v(x) = 5x^2 + 3$?

      ii. $v(x) = (5x + 3)^2$?

      iii. $v(x) = -2x^2 + 3$?

      iv. $v(x) = x^4$?

   (b) Would she make the same choices among degenerate lotteries (i.e. sure prizes) if she were using $v(x) = x^4$? How do you reconcile your answer with your answer to (a.iv)?

   (c) In each of (a.i)-(a.iv), if your answer is negative, find two lotteries p and q such that when the decision-maker is asked to choose from $\{p, q\}$, she strictly prefers p if she has the vNM utility function u and she strictly prefers q if she has the vNM utility function v.

7. Now suppose that X is the set of integers. Would the vNM utility functions $u(x) = 3x + 2\sin(2\pi x)$ and $v(x) = 7x + 5$ yield different choices over lotteries?

8. Let $X = \{x_1, x_2, x_3\}$ be the set of prizes. In each of the following questions, do the following. Check which of the three conditions: rationality, independence, and solvability, the preference R satisfies. Find an expected utility representation if R satisfies all three conditions, otherwise give a counterexample for each condition that it violates.

   (a) Worst-case scenario:

$$pRq \quad \Leftrightarrow \quad \min_{p(x_i)>0} i \geq \min_{q(x_i)>0} i \qquad p, q \in \triangle(X).$$

   That is, our decision-maker compares lotteries by looking at the lowest indexed prize which has positive probability under that lottery.

(b) Let $\alpha_1, \alpha_2, \alpha_3 > 0$ and:

$$pRq \quad \Leftrightarrow \quad \alpha_1^{p(x_1)}\alpha_2^{p(x_2)}\alpha_3^{p(x_3)} \geq \alpha_1^{q(x_1)}\alpha_2^{q(x_2)}\alpha_3^{q(x_3)} \qquad p, q \in \triangle(X).$$

(c) $pRq \quad \Leftrightarrow \quad p(x_1)^2 + p(x_2)^2 + p(x_3)^2 \geq q(x_1)^2 + q(x_2)^2 + q(x_3)^2 \qquad p, q \in \triangle(X).$

(d) Lexicographic:

$$pRq \quad \Leftrightarrow \quad \begin{array}{l} p(x_3) > q(x_3), \text{ or} \\ [p(x_3) = q(x_3) \text{ and } p(x_2) \geq q(x_2)] \end{array} \qquad p, q \in \triangle(X).$$

9. (Relating risk attitudes and the shape of the vNM utility function) Let $X = \mathbb{R}$ represent the set of monetary prizes. A preference R over $\triangle(X)$ is **risk-averse** if

$$\delta_{\alpha x + (1-\alpha)y} \; R \; \alpha\delta_x + (1-\alpha)\delta_y$$

for any $x, y \in \mathbb{R}$ and $\alpha \in [0, 1]$. The interpretation is simple: when the decision-maker is asked to choose between the uncertain gamble that gives x with probability α and y with probability $1-\alpha$ (this is $\alpha\delta_x + (1-\alpha)\delta_y$) and the degenerate lottery that always gives the average value of the other gamble (this is $\delta_{\alpha x+(1-\alpha)y}$), he always weakly prefers the safer option. In the following, suppose that our individual is an expected utility maximizer with the vNM utility function $u\colon \mathbb{R} \to \mathbb{R}$.

(a) Show that our decision-maker is risk-averse if and only if u is concave. (Remember that a function $f : \mathbb{R} \to \mathbb{R}$ is *concave* if $f(\alpha x + (1-\alpha)y) \geq \alpha f(x) + (1-\alpha)f(y)$ for any $x, y \in \mathbb{R}$ and $\alpha \in [0, 1]$.)

(b) Mimicking the above definition of risk-aversion, can you come up with a definition of risk-loving R, and relate the risk-loving behavior to the shape of the vNM utility function? Can you do the same for risk-neutral R?