Econ/Math C103 (2020) - Problem Set 5

Due: 3:30pm PDT, 12/3/2020

1. Consider the first-price auction where the valuations are distributed i.i.d. on $[0,1]$ with uniform density $f(x) = 1$. What is the BNE bid function? What are the expected revenues of the seller and the expected equilibrium payoff of a bidder with valuation $x \in [0,1]$. What happens to the equilibrium bid function, payoffs, and revenues as $n \to \infty$?

2. In the all-pay auction, the highest bidder wins the object, every bidder (even if she does not win the object!) pays her bid. If there are multiple highest bidders, then one of them is chosen at random to win the object. Formally:

$$u_i(b_1, \ldots, b_n; x_i) = \begin{cases} \frac{x_i}{|\{j \in N : b_j = b^1\}|} - b_i & \text{if } b_i = b^1 \\ -b_i & \text{otherwise,} \end{cases}$$

where b^1 denotes the highest bid among $b_1, \ldots, b_n$. Let $[\underline{x}, \bar{x}] = [0,1]$. As usual, the bidders' valuations are i.i.d. with a continuous and strictly positive density f and c.d.f. F. Conjecture that there is a symmetric BNE with a differentiable bid function $b\colon [0,1] \to \mathbb{R}_+$ such that $b'(x) > 0$ for any $x \in [0,1]$.[1]

(a) Write out the expected payoff of bidder i with valuation x when she bids b_i, given that the others bid according to $b(\cdot)$.

(b) Find the necessary first order condition for b_i to maximize i's expected payoff. After you derive the first order condition, simplify it conjecturing that $b_i = b(x)$ is the optimal bid for bidder i with valuation x.

(c) Solve for the bid function $b(\cdot)$. (Explain why the integration constant must be zero.)

(d) Compute the expected payoffs of a bidder with valuation x and the expected revenues of the seller, when bidders use the bid function you solved for in part (c). Show that they are the same as in the first- and second-price auctions.

(e) Show that the bid function $b(\cdot)$ you derived above from the necessary first order conditions, is in fact a BNE.

[1] Note that here we are ruling out negative bids, which could be interpreted as monetary transfers from the seller to the bidder. The reason is that the all-pay auction does not have a BNE when negative bids are allowed for (can you see why?), a concern we did not have for the type of auctions studied in class.

3. Consider the following modification of the second-price action with a reserve price. Suppose that $[\underline{x}, \bar{x}] = [0, 1]$ and fix $r \in [0, 1]$. In the second-price auction with reserve price r, the highest bidder wins the object if his bid exceeds r, and pays the maximum of the second-highest bid and r. No one wins the object if all bids are strictly below the reserve price. If there are multiple highest bids above r, then one of then is chosen at random to win the object. Formally:

$$u_i(b_1, \ldots, b_n; x_i) = \begin{cases} \frac{1}{|\{j \in N : b_j = b^1\}|}(x_i - \max\{b^2, r\}) & \text{if } b_i = b^1 \geq r \\ 0 & \text{otherwise.} \end{cases}$$

When $r = 0$, this reduces to the second-price auction studied in class. Note that even when $r > 0$, it is a weakly dominant strategy for bidders to bid their own valuation.[2] Suppose that the valuations are distributed i.i.d. on $[0, 1]$ with uniform density $f(x) = 1$.

(a) For any reserve price $r \in [0, 1]$, compute the expected payoffs of a bidder with valuation x and the expected revenues of the seller, when bidders bid their valuations. (Hint: When calculating the expected revenues, consider three cases in terms of the realizations of the random variables X^1 and X^2, ignoring ties: (i) $X^1 < r$, (ii) $X^2 < r < X^1$, and (iii) $r < X^2$.)

(b) What is the expected revenue maximizing reserve price for the seller?

4. Consider the allocation of three identical indivisible objects to three agents each of whom can consume multiple units. The types of each agent is given by $\Theta_i = \{\theta_i = (\theta_i^1, \theta_i^2, \theta_i^3) \in \mathbb{R}_+^3 \mid \theta_i^1 \leq \theta_i^2 \leq \theta_i^3\}$ where for each $l \in \{1, 2, 3\}$, θ_i^l denotes agent i's valuation for l units. Consider the type profile:

θ_1	θ_2	θ_3
5	4	7
10	9	10
15	16	12

What is the allocation of the three goods and the transfers according to the pivotal VCG mechanism at the above type profile?

[2] We call this a *weakly* dominant strategy since bidders with valuations $x < r$ are indifferent among all bids strictly less than r which all guarantee a payoff of zero. Therefore, the dominant strategy condition can only be guaranteed with a weak inequality when $r > 0$.

5. Consider a society with n individuals who need to decide on the provision of a public good $k \in K = \mathbb{R}_+$. Each agent i's quasi linear utility is given by $v_i(k, \theta_i) = \theta_i \sqrt{k} - \alpha k$ where her type $\theta_i \in \Theta_i \equiv [0, 1]$ is her private information and $\alpha > 0$ is a fixed common parameter. Derive the pivotal VCG mechanism.

6. There is a perfectly divisible object of unit size to be divided among n individuals in nonnegative amounts. That is, the set K is:

$$K = \left\{ k \in [0, 1]^n : \sum_{i=1}^{n} k_i = 1 \right\}.$$

The value of individual i from consuming $k_i \in [0, 1]$ units of this good is given by $v_i(k_i, \theta_i) = \theta_i k_i^\alpha$ where $\theta_i \in \Theta_i = (0, 1]$ is individual i's private information and $\alpha \in (0, 1)$ is a fixed common parameter. Derive the pivotal VCG mechanism. (Hint: Use the Lagrangian method to solve the constrained optimization problem of finding the ex-post efficient project rule.)