Econ/Math C103 (2020) - Problem Set 3

Due: 3:30pm PDT, 10/15/2020

1. Read the definition of the tournament SWF from the lecture notes. Which of the three properties: Rationality, IIA, and Unanimity, does the tournament SWF satisfy?

2. For the case when $|X| = 2$ and $n \geq 2$, give an example of a non-dictatorial SWF that is complete, unanimous and is different from the majority rule. Briefly interpret the SWF you have constructed.

3. Suppose that the society $N = \{1, \ldots, n\}$ needs to determine a tax rate $t \in [0,1]$. Each individual in the society has an income of \$1. If the tax rate is $t \in [0,1]$, then all the collected tax money $\$tn$ is invested in a public project. If a total of $\$x$ dollars are invested in the public project and if individual $i \in N$ has $\$y_i$ to spend on her private consumption, then her utility is

$$u_i(x, y_i) = x^{\alpha_i} y_i^{1-\alpha_i}$$

for some parameter $\alpha_i \in [0,1]$ that only individual i knows.

   (a) Suppose individuals have no way to save, so each one of them spends all her income remaining from taxes to private consumption. Compute each individual's utility when the tax rate is set to be $t \in [0,1]$.

   (b) Let n be odd. Is there a non-dictatorial and strategy-proof SCF f that sets a tax rate $t = f(\alpha) \in [0,1]$, as a function of the profile of utility parameters $\alpha = (\alpha_1, \ldots, \alpha_n) \in [0,1]^n$ and $\mathrm{Im}(f) = [0,1]$ (i.e. f is onto)?

4. Let n be odd. Show that when $X = [0,1]$ and $R = (R_1, \ldots, R_n)$ is a profile of single-peaked preferences, the pairwise majority outcome $\succsim$ associated with R is also single-peaked.

5. Does there exist a profile of single-peaked preferences $R \in \mathcal{R}^{*5}$ and alternatives $a, b, c, d \in [0,1]$ such that:

R_1	R_2	R_3	R_4	R_5
a	b	c	b	d
b	d	d	a	b
c	a	b	c	a
d	c	a	d	c

6. For any preference profile $R \in \mathcal{R}^n$ and $i \in N$, let $R_{-i} = (R_1, \ldots, R_{i-1}, R_{i+1}, \ldots, R_n)$ denote the preference profile of all agents other than i. For any $i \in N$, $R_i \in \mathcal{R}$, and $x \in X$ let $L(x, R_i) = \{y \in X : xR_iy\}$ denote the set of alternatives that are weakly worse than x with respect to R_i, also known as **the lower contour set of R_i at x**. Given a SCF f, for any $i \in N$ and $R_{-i} \in \mathcal{R}^{n-1}$, define $o_i(R_{-i}) = \{f(R_i, R_{-i}) : R_i \in \mathcal{R}\}$. Note that $o_i(R_{-i})$ is the set of alternatives that are chosen by f when we vary the preferences of individual i and fix the preferences of all others to R_{-i}. Show that a SCF $f : \mathcal{R}^n \to X$ is strategyproof if and only if it satisfies any one of the three properties below:[1]

   - The SCF f is **monotone** is for any $R, R' \in \mathcal{R}^n$ such that $x = f(R)$ and $L(x, R_i) \subset L(x, R'_i)$ for all $i \in N$, we have that $x = f(R')$. *Interpretation:* x is chosen at R, and R' is a new preference profile where x's position is improved for every individual, then monotonicity requires that x also be chosen at R'.

   - The SCF f is **weakly monotone** is for any $R \in \mathcal{R}^n$, $i \in N$, and $R'_i \in \mathcal{R}$, such that $x = f(R)$ and $L(x, R_i) \subset L(x, R'_i)$, we have that $x = f(R'_i, R_{-i})$. The interpretation is similar to that of monotonicity, but here we are only changing one person 's preference: x is chosen at R, and R'_i is a new preference of i where x's position is improved, then weak monotonicity requires that x also be chosen after only individual i's preference is changed to R'_i.

   - The SCF f satisfies $(*)$ if for all $R \in \mathcal{R}^n$ and $i \in N$, and $y \in o_i(R_{-i})$, we have that $f(R)R_iy$. *Interpretation:* For any individual i and preference profile R, $f(R)$ is the best alternative in $o_i(R_{-i})$ with respect to R_i.

7. Consider a marriage market with four men $M = \{m_1, m_2, m_3, m_4\}$ and four women $W = \{w_1, w_2, w_3, w_4\}$ whose preferences are such that:

R_{m_1}	R_{m_2}	R_{m_3}	R_{m_4}	R_{w_1}	R_{w_2}	R_{w_3}	R_{w_4}
w_2	w_3	w_4	w_4	m_1	m_2	m_4	m_2
w_3	w_1	w_2	w_3	m_2	.	m_3	m_1
w_1	w_2	.	w_2	m_3	.	m_1	m_4
.	w_4	.	w_1	m_4	.	.	m_3

[1]Hint: Clearly monotonicity implies weak monotonicity. I would recommend you to first prove that weak monotonicity implies monotonicity to conclude that those two conditions are equivalent, and then show that: $(*) \Rightarrow$ weak monotonicity $\Rightarrow$ strategyproofness $\Rightarrow (*)$.

For each $x \in M \cup W$, I have only listed x's acceptable mates, all unlisted people from the other side of the market are unacceptable to x. Go through the men-proposing deferred acceptance algorithm and find μ_M. Repeat the same exercise for the women-proposing deferred acceptance algorithm and find μ_W. Is there any other stable matchings? Explain your anwser.

8. Consider a model of indivisible objects where each agent can consume exactly one object. There are six agents $\{1, 2, 3, 4, 5, 6\}$ and six objects $\{a, b, c, d, e, f\}$. The initial endowment vector μ_E and the preference profile R are given by:

$$\mu_E = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ a & b & c & d & e & f \end{pmatrix}$$

R_1	R_2	R_3	R_4	R_5	R_6
b	a	b	c	b	a
a	b	f	d	d	f
$\vdots$	$\vdots$	e	$\vdots$	e	$\vdots$
		c			
		$\vdots$			

Find the unique core allocation. Find a price vector that supports it as a Walrasian equilibrium. The unspecified parts of the preference profile do not change the answers.