MAST20005/MAST90058: Week 12 Problems

1. Let $X_1, \dots, X_n$ be a random sample from $\mathrm{Bi}(1, p)$.

   (a) Find the Cramér–Rao lower bound for unbiased estimators of p.

   (b) We know that $\bar{X}$ is an unbiased estimator of p.
   Show that $\bar{X}$ attains the Cramér–Rao lower bound.

2. Let $X_1, \dots, X_n$ be a random sample from $\mathrm{N}(\mu, \theta)$ where μ is known.

   (a) Show that the maximum likelihood estimator of θ is,

   $$\hat{\theta} = \frac{1}{n} \sum_{i=1}^{n} (X_i - \mu)^2.$$

   (b) Find the Cramér–Rao lower bound for unbiased estimators of θ.

   (c) What is the approximate distribution of $\hat{\theta}$?

   (d) What is the exact distribution of $n\hat{\theta}/\theta$?

3. Let $X_1, \dots, X_n$ be a random sample from the density:

   $$f(x \mid \theta) = \frac{x}{\theta^2} e^{-x/\theta}, \quad 0 < x < \infty, \quad 0 < \theta < \infty$$

   (a) Find a sufficient statistic for θ.

   (b) Write down the log-likelihood function and the score function.

   (c) Determine the maximum likelihood estimator of θ.

   (d) Find the Cramér–Rao lower bound for unbiased estimators of θ.
   (Hint: some information from previous week's tutorial will help you to find $\mathbb{E}(X)$.)

   (e) A random sample of size $n = 35$ gave $\bar{x} = 10.5$. Determine the maximum likelihood estimate of θ and an approximate 95% confidence interval for θ.

4. Find a sufficient statistic for p when you toss a coin 10 times and p is the probability of a head. Also do this for the case where p is the probability of a head for the first 5 tosses and changes to $(1 - p)$ for the last five tosses.

5. Find sufficient statistics for θ (where $\theta > 0$) when we observe data from:

   (a) $X \sim \mathrm{Unif}(0, \theta)$

   (b) $X \sim \mathrm{Unif}(-\frac{\theta}{2}, \frac{\theta}{2})$

6. Find sufficient statistics for θ (where $\theta > 0$) when we observe X from the following pdfs:

   (a) $f(x \mid \theta) = \frac{1}{\theta} e^{-x/\theta}, \quad 0 < x < \infty$

   (b) $f(x \mid \theta) = e^{-(x-\theta)}, \quad \theta < x < \infty$

   (c) $f(x \mid \theta) = \frac{1}{\theta} e^{-(x-\theta)/\theta}, \quad \theta < x < \infty$

7. Refer back to problem 2 from week 3.

   (a) What is a sufficient statistic for θ?

   (b) What does that suggest about the relative merits of the two estimators we derived?