MATH3075/3975 Financial Mathematics

Tutorial 11: Solutions

Exercise 1 We consider the Black-Scholes model $\mathcal{M} = (B, S)$ with the initial stock price $S_0 = 9$, the continuously compounded interest rate $r = 0.01$ per annum and the stock price volatility $\sigma = 0.1$ per annum. Recall that $dB_t = rB_t\,dt$ with $B_0 = 1$ (equivalently, $B(t,T) = e^{-r(T-t)}$ and

$$dS_t = S_t\big(r\,dt + \sigma\,dW_t\big), \quad S_0 > 0,$$

where W is a standard Brownian motion under the martingale measure $\widetilde{\mathbb{P}}$.

(a) Using the Black-Scholes call option pricing formula

$$C_0 = S_0 N\big(d_+(S_0,T)\big) - Ke^{-rT} N\big(d_-(S_0,T)\big)$$

we compute the price C_0 of the European call option with strike price $K = 10$ and maturity $T = 5$ years. We find that

$$d_+(S_0,T) = -0.13578, \quad d_-(S_0,T) = -0.35938$$

and thus $C_0 = 0.59285$.

(b) Using the Black-Scholes put option pricing formula

$$P_0 = Ke^{-rT} N\big(-d_-(S_0,T)\big) - S_0 N\big(-d_+(S_0,T)\big)$$

we find that the price $P_0 = 1.10514$.

(c) The put-call parity relationship holds since

$$C_0 - P_0 = 0.59285 - 1.10514 = -0.51229 = 9 - 10e^{-0.05} = S_0 - Ke^{-rT}.$$

(d) We now recompute the prices of call and put options for modified maturities $T = 5$ months and $T = 5$ days.

- We note that 5 months is equivalent to $T = 0.416667$ and thus

$$d_+(S_0,T) = -1.53541, \quad d_-(S_0,T) = -1.59996.$$

  Hence $C_0 = 0.015315$ and $P_0 = 0.973735$.

- We note that 5 days is equivalent to $T = 0.013699$ and thus

$$d_+(S_0,T) = -8.98455, \quad d_-(S_0,T) = -8.99615.$$

  Hence $C_0 = 1.49E - 21$ and $P_0 = 0.99863$.

(e) The call option (respectively, put option) price decreases to zero (respectively, increases to $K - S_0 = 1$) when the time to maturity tends to zero. This is related to the fact that $S_0 < K$ and thus for short maturities it is unlikely (respectively, very likely) that the call option (respectively, put option) will be exercised at expiration.

Exercise 2 Assume that the stock price S is governed under the martingale measure $\widetilde{\mathbb{P}}$ by the Black-Scholes stochastic differential equation

$$dS_t = S_t\big(r\,dt + \sigma\,dW_t\big)$$

where $\sigma > 0$ is a constant volatility and r is a constant short-term interest rate. Let $0 < L < K$ be real numbers. We consider a path-independent contingent claim with the payoff X at maturity date $T > 0$ given as

$$X = \min\big(|S_T - K|, L\big).$$

(a) It is easy to sketch the profile of the payoff X as the function of the stock price S_T. The decomposition of X in terms of the payoffs of standard call and put options reads

$$X = L - C_T(K-L) + 2C_T(K) - C_T(K+L).$$

Note that other decompositions are possible.

(b) The arbitrage price $\pi_t(X)$ satisfies, for every $t \in [0,T]$,

$$\pi_t(X) = Le^{-r(T-t)} - C_t(K-L) + 2C_t(K) - C_t(K+L).$$

(c) We will now find the limits of the arbitrage price $\lim_{L\to 0}\pi_0(X)$ and $\lim_{L\to\infty}\pi_0(X)$. We observe the payoff X increases when L increases. Hence the price $\pi_0(X)$ is also an increasing function of L. Moreover,

$$\lim_{L\to 0}\pi_0(X) = -C_0(K) + 2C_0(K) - C_0(K) = 0.$$

By analysing the payoff X when L tends to infinity (obviously, we no longer assume here that the inequality $L < K$ holds since K is fixed and L tends to infinity), we obtain

$$\lim_{L\to\infty}\min\big(|S_T - K|, L\big) = |S_T - K| = (K - S_T)^+ + (S_T - K)^+ = P_T(K) + C_T(K)$$

and thus

$$\lim_{L\to\infty}\pi_0(X) = P_0(K) + C_0(K).$$

(d) To find the limit $\lim_{\sigma\to\infty}\pi_0(X)$, we observe that

$$\lim_{\sigma\to\infty} d_+(S_0,T) = \infty, \qquad \lim_{\sigma\to\infty} d_-(S_0,T) = -\infty,$$

so that

$$\lim_{\sigma\to\infty} N\big(d_+(S_0,T)\big) = 1, \qquad \lim_{\sigma\to\infty} N\big(d_-(S_0,T)\big) = 0.$$

Hence the price of the call option satisfies, for all strikes $K \in \mathbb{R}_+$,

$$\lim_{\sigma\to\infty} C_0(K) = S_0.$$

This in turn implies that $\lim_{\sigma\to\infty}\pi_0(X) = Le^{-rT} = \pi_0(L)$.

Exercise 3 We denote by v the Black-Scholes call option pricing, that is, the function $v : \mathbb{R}_+ \times [0,T] \to \mathbb{R}$ such that $C_t = v(S_t, t)$ for all $t \in [0,T]$.

(a) We need to show that, for every $s \in \mathbb{R}_+$,

$$\lim_{t \to T} v(s,t) = (s-K)^+$$

For this purpose, we observe that $d_+(s,K)$ and $d_-(s,K)$ tend to ∞ (respectively, $-\infty$) when $t \to T$ and $s > K$ (respectively, $s < K$). Consequently, $N(d_+(s,K))$ and $N(d_-(s,K))$ tend to 1 (respectively, 0) when $t \to T$ and $s > K$ (respectively, $s < K$). This in turn implies that $v(s,T)$ tends to either $s-K$ or 0 depending on whether $s > K$ or $s < K$. The case when $s = K$ is also easy to analyse and to check that $\lim_{t \to T} v(s,t) = 0$ when $s = K$.

(b) **(MATH3975)** Observe that $v(s,t) = c(s, T-t)$ where the function c is such that $C_t = c(S_t, T-t)$. Our goal is to check that the pricing function of the European call option satisfies the Black-Scholes partial differential equation (PDE)

$$\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 v}{\partial s^2} + rs\frac{\partial v}{\partial s} - rv = 0, \quad \forall\, (s,t) \in (0,\infty) \times (0,T), \tag{1}$$

with the terminal condition $v(s,T) = (s-K)^+$. Equivalently, the function c satisfies

$$-\frac{\partial c}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 c}{\partial s^2} + rs\frac{\partial c}{\partial s} - rc = 0, \quad \forall\, (s,t) \in (0,\infty) \times (0,T),$$

with the initial condition $c(s,0) = (s-K)^+$. From the Black-Scholes theorem, we know that v is given by the following expression

$$v(s,t) = sN(d_+(s,T-t)) - Ke^{-r(T-t)}N(d_-(s,T-t)). \tag{2}$$

Straightforward computations show that the partial derivatives are:

$$\begin{aligned}
v_s(s,t) &= N(d_+(s,T-t)),\\
v_{ss}(s,t) &= \frac{n(d_+(s,T-t))}{\sigma s\sqrt{T-t}},\\
v_t(s,t) &= -\frac{\sigma s}{2\sqrt{T-t}}\, n(d_+(s,T-t)) - Kre^{-r(T-t)}N(d_-(s,T-t))
\end{aligned}$$

where $n(x)$ is the density function of the standard normal distribution. Hence

$$\begin{aligned}
&-\frac{s\sigma}{2\sqrt{T-t}}\, n(d_+(s,T-t)) - Kre^{-r(T-t)}N(d_-(s,T-t))\\
&+ \frac{1}{2}\sigma^2 s^2 \frac{n(d_+(s,T-t))}{s\sigma\sqrt{T-t}} + rsN(d_+(s,T-t)) - rv(s,t) = 0
\end{aligned}$$

where we have also used the equality (2).

It is worth noting that the pricing function $w(s,t) = p(s,T-t)$ for the put option also satisfies the Black-Scholes PDE but with the terminal condition $w(s,T) = (K-s)^+$. This can be checked either by computing directly the partial derivatives or by combining already established PDE (1) with the put-call parity relationship, which reads

$$v(s,t) - w(s,t) = s - Ke^{-r(T-t)}.$$

Exercise 4 (MATH3975) We consider the stock price process S given by the Black and Scholes model.

(a) We will first show that $\widehat{S}_t = e^{-rt} S_t$ is a martingale with respect to the filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ generated by the stock price process S. We observe that this filtration is also generated by W. Using the properties of the conditional expectation, we obtain, for all $s \leq t$,

$$
\begin{aligned}
\mathbb{E}_{\widetilde{\mathbb{P}}}\big(\widehat{S}_t \,\big|\, \mathcal{F}_s\big) &= \mathbb{E}_{\widetilde{\mathbb{P}}}\left(\widehat{S}_s\, e^{\sigma\left(W_t - W_s - \frac{1}{2}\sigma^2(t-s)\right)} \,\big|\, \mathcal{F}_s\right) \\
&= \widehat{S}_s\, e^{-\frac{1}{2}\sigma^2(t-s)}\, \mathbb{E}_{\widetilde{\mathbb{P}}}\left(e^{\sigma(W_t - W_s)} \,|\, \mathcal{F}_s\right) \\
&= \widehat{S}_s\, e^{-\frac{1}{2}\sigma^2(t-s)}\, \mathbb{E}_{\widetilde{\mathbb{P}}}\left(e^{\sigma(W_t - W_s)} \,|\, \mathcal{F}_s\right) \\
&= \widehat{S}_s\, e^{-\frac{1}{2}\sigma^2(t-s)}\, \mathbb{E}_{\widetilde{\mathbb{P}}}\left(e^{\sigma(W_t - W_s)}\right)
\end{aligned}
$$

where in the last equality we used the independence of increments of the Wiener process. Recall also that $W_t - W_s = \sqrt{t-s}\, Z$ where $Z \sim N(0,1)$, and thus

$$
\mathbb{E}_{\widetilde{\mathbb{P}}}\big(\widehat{S}_t \,|\, \mathcal{F}_s\big) = \widehat{S}_s\, e^{-\frac{1}{2}\sigma^2(t-s)}\, \mathbb{E}_{\widetilde{\mathbb{P}}}\big(e^{\sigma\sqrt{t-s}Z}\big).
$$

It is known (and easy to check by integration) that if $Z \sim N(0,1)$ then for any real number a

$$
\mathbb{E}_{\widetilde{\mathbb{P}}}\big(e^{aZ}\big) = e^{\frac{1}{2}a^2}. \tag{3}
$$

By setting $a = \sigma\sqrt{t-s}$, we obtain

$$
\mathbb{E}_{\widetilde{\mathbb{P}}}\big(\widehat{S}_t \,|\, \widehat{S}_u, u \leq s\big) = \widehat{S}_s\, e^{-\frac{1}{2}\sigma^2(t-s)}\, e^{\frac{1}{2}\sigma^2(t-s)} = \widehat{S}_s,
$$

which shows that $\widehat{S}$ is a martingale under $\widetilde{\mathbb{P}}$.

(b) To compute the expectation $\mathbb{E}_{\widetilde{\mathbb{P}}}(S_t)$, we observe that

$$
\mathbb{E}_{\widetilde{\mathbb{P}}}(S_t) = e^{rt}\, \mathbb{E}_{\widetilde{\mathbb{P}}}(\widehat{S}_t) = e^{rt}\, \mathbb{E}_{\widetilde{\mathbb{P}}}(\widehat{S}_0) = e^{rt}\widehat{S}_0 = e^{rt} S_0.
$$

To compute the variance $\operatorname{Var}_{\widetilde{\mathbb{P}}}(S_t)$, we recall that

$$
\operatorname{Var}_{\widetilde{\mathbb{P}}}(S_t) = \mathbb{E}_{\widetilde{\mathbb{P}}}(S_t^2) - \big[\mathbb{E}_{\widetilde{\mathbb{P}}}(S_t)\big]^2
$$

where in turn

$$
\begin{aligned}
\mathbb{E}_{\widetilde{\mathbb{P}}}(S_t^2) &= S_0^2 e^{2rt}\, \mathbb{E}_{\widetilde{\mathbb{P}}}\left[e^{2\sigma W_t - \sigma^2 t}\right] \\
&= S_0^2 e^{2rt} e^{\sigma^2 t}\ \mathbb{E}_{\widetilde{\mathbb{P}}}\left[e^{2\sigma W_t - \frac{1}{2}(2\sigma\sqrt{t})^2}\right] \\
&= S_0^2 e^{2rt} e^{\sigma^2 t}\ \mathbb{E}_{\widetilde{\mathbb{P}}}\left[e^{aZ - \frac{1}{2}a^2}\right]
\end{aligned}
$$

where we denote $a = 2\sigma\sqrt{t}$ and $Z \sim N(0,1)$. Since (see (3))

$$
\mathbb{E}_{\widetilde{\mathbb{P}}}\left[e^{aZ - \frac{1}{2}a^2}\right] = 1
$$

we conclude that

$$
\mathbb{E}_{\widetilde{\mathbb{P}}}(S_t^2) = S_0^2 e^{2rt} e^{\sigma^2 t}
$$

and thus

$$
\operatorname{Var}_{\widetilde{\mathbb{P}}}(S_t) = S_0^2 e^{2rt}\big(e^{\sigma^2 t} - 1\big).
$$