

MATH3075/3975 Financial Derivatives

Tutorial 7: Solutions

Exercise 1 (a) The cumulative distribution function of X reads

$$F_X(x) = \begin{cases} 0, & x < 1, \\ 0.1, & 1 \leq x < 2, \\ 0.2, & 2 \leq x < 3, \\ 0.5, & 3 \leq x < 4, \\ 0.7, & 4 \leq x < 5, \\ 1, & x \geq 5. \end{cases}$$

Equivalently, we may represent the probability distribution of X as follows:

x_j	1	2	3	4	5
p_j	0.1	0.1	0.3	0.2	0.3

(b) We now compute the conditional expectation $\mathbb{E}_{\mathbb{P}}(X | \mathcal{G})$ where the σ -field \mathcal{G} is generated by the partition $\{A_1, A_2, A_3\}$ of Ω . We obtain

$$\mathbb{E}_{\mathbb{P}}(X | \mathcal{G}) = \begin{cases} \frac{1}{0.2} (0.1 \cdot 1 + 0.1 \cdot 2) = 1.5, & \omega \in A_1, \\ \frac{1}{0.3} (0.3 \cdot 3) = 3, & \omega \in A_2, \\ \frac{1}{0.5} (0.2 \cdot 4 + 0.3 \cdot 5) = 4.6, & \omega \in A_3. \end{cases}$$

(c) Let $Y := \mathbb{E}_{\mathbb{P}}(X | \mathcal{G})$. Then the cumulative distribution function of Y satisfies

$$F_Y(y) = \begin{cases} 0, & y < 1.5, \\ 0.2, & 1.5 \leq y < 3, \\ 0.5, & 3 \leq y < 4.6, \\ 1, & y \geq 4.6. \end{cases}$$

This means that the probability distribution of Y equals:

y_l	1.5	3	4.6
\hat{p}_l	0.2	0.3	0.5

(d) We first compute the expectation of X

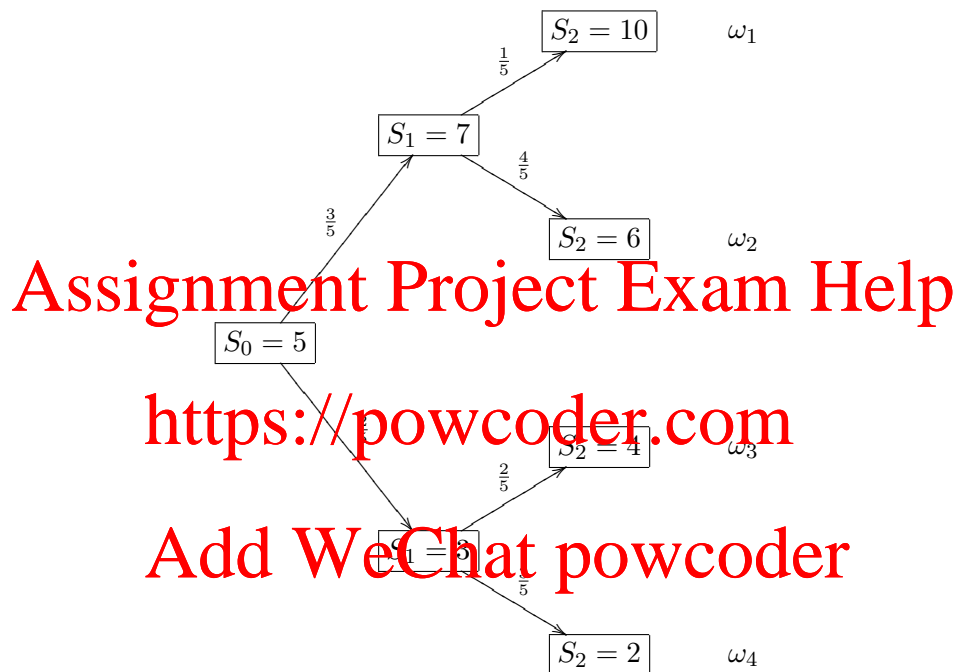
$$\mathbb{E}_{\mathbb{P}}(X) = \int_{-\infty}^{\infty} x dF_X(x) = \sum_{j=1}^5 x_j p_j = 0.1 \cdot 1 + 0.1 \cdot 2 + 0.3 \cdot 3 + 0.2 \cdot 4 + 0.3 \cdot 5 = 3.5.$$

The expected value of Y equals

$$\mathbb{E}_{\mathbb{P}}(Y) = \int_{-\infty}^{\infty} y dF_Y(y) = \sum_{l=1}^3 y_l \hat{p}_l = 0.2 \cdot 1.5 + 0.3 \cdot 3 + 0.5 \cdot 4.6 = 3.5.$$

Hence the equality $\mathbb{E}_{\mathbb{P}}(X) = \mathbb{E}_{\mathbb{P}}(\mathbb{E}_{\mathbb{P}}(X | \mathcal{G}))$ is satisfied.

Exercise 2 We consider the two-period market model $\mathcal{M} = (B, S)$ with the savings account B given by $B_0 = 1$, $B_1 = 1 + r$, $B_2 = (1 + r)^2$ with $r = 0.25$ and the stock price S represented by



(a) The probabilities of the states $\omega_1, \omega_2, \omega_3, \omega_4$ are:

$$\mathbb{P}(\omega_i) = \begin{cases} \frac{3}{5} \cdot \frac{1}{5} = \frac{3}{25}, & i = 1, \\ \frac{3}{5} \cdot \frac{4}{5} = \frac{12}{25}, & i = 2, \\ \frac{2}{5} \cdot \frac{2}{5} = \frac{4}{25}, & i = 3, \\ \frac{2}{5} \cdot \frac{3}{5} = \frac{6}{25}, & i = 4. \end{cases}$$

For instance, $\mathbb{P}(\omega_1)$ is computed as follows

$$\begin{aligned} \mathbb{P}(\omega_1) &= \mathbb{P}(S_0 = 5, S_1 = 7, S_2 = 10) \\ &= \mathbb{P}(S_2 = 10 | S_1 = 7) \mathbb{P}(S_1 = 7 | S_0 = 5) = \frac{3}{5} \cdot \frac{1}{5} = \frac{3}{25} \end{aligned}$$

and $\mathbb{P}(\omega_2)$ satisfies

$$\begin{aligned} \mathbb{P}(\omega_2) &= \mathbb{P}(S_0 = 5, S_1 = 7, S_2 = 6) \\ &= \mathbb{P}(S_2 = 6 | S_1 = 7) \mathbb{P}(S_1 = 7 | S_0 = 5) = \frac{3}{5} \cdot \frac{4}{5} = \frac{12}{25}. \end{aligned}$$

(b1) We observe that $\mathcal{F}_1 = \{\emptyset, A_1, A_2, \Omega\}$ where $A_1 = \{\omega_1, \omega_2\}$ and $A_2 = \{\omega_3, \omega_4\}$. On the event A_1 , we obtain

$$\frac{1}{\mathbb{P}(A_1)} \sum_{\omega \in A_1} S_2(\omega) = \frac{25}{15} \left(\frac{3}{25} \cdot 10 + \frac{12}{25} \cdot 6 \right) = \frac{34}{5}$$

and on A_2 , we get

$$\frac{1}{\mathbb{P}(A_2)} \sum_{\omega \in A_2} S_2(\omega) = \frac{25}{10} \left(\frac{4}{25} \cdot 4 + \frac{6}{25} \cdot 2 \right) = \frac{14}{5}.$$

Hence

$$\mathbb{E}_{\mathbb{P}}(S_2 | \mathcal{F}_1) = \frac{34}{5} \mathbf{1}_{A_1} + \frac{14}{5} \mathbf{1}_{A_2} = \frac{34}{5} \mathbf{1}_{\{S_1=7\}} + \frac{14}{5} \mathbf{1}_{\{S_1=3\}}.$$

(b2) We will now make use of the conditional probabilities $\mathbb{P}(S_2 = s_j | S_1 = 7)$ and $\mathbb{P}(S_2 = s_j | S_1 = 3)$. We obtain

$$\mathbb{E}_{\mathbb{P}}(S_2 | S_1 = 7) = \frac{1}{5} \cdot 10 + \frac{4}{5} \cdot 6 = \frac{34}{5}$$

and

$$\mathbb{E}_{\mathbb{P}}(S_2 | S_1 = 3) = \frac{2}{5} \cdot 4 + \frac{3}{5} \cdot 2 = \frac{14}{5}$$

(c) We first compute $\mathbb{E}_{\mathbb{P}}(S_2)$ directly

$$\mathbb{E}_{\mathbb{P}}(S_2) = \frac{2}{25} \cdot 10 + \frac{12}{25} \cdot 6 + \frac{4}{25} \cdot 4 + \frac{6}{25} \cdot 2 = \frac{130}{25}.$$

Next we compute

$$\mathbb{E}_{\mathbb{P}}(\mathbb{E}_{\mathbb{P}}(S_2 | \mathcal{F}_1)) = \frac{34}{5} \cdot \frac{3}{5} + \frac{14}{5} \cdot \frac{2}{5} = \frac{130}{25}$$

Exercise 3 (MATH3975) Let $\{A_1, A_2, \dots, A_m\}$ be a partition of the space $\Omega = \{\omega_1, \omega_2, \dots, \omega_k\}$. Note that since Ω is finite any σ -field \mathcal{G} is generated by some finite partition of Ω .

(a) Let G be an arbitrary event from the σ -field \mathcal{G} generated by the partition $\{A_1, A_2, \dots, A_m\}$. Then there exists a set $L \subset \{1, 2, \dots, m\}$ such that

$$G = \cup_{l \in L} A_l. \quad (1)$$

Moreover, we know that for every $l \in L$ on the event A_l we have that

$$\mathbb{E}_{\mathbb{P}}(X | \mathcal{G}) = \frac{1}{\mathbb{P}(A_l)} \sum_{\omega \in A_l} X(\omega) \mathbb{P}(\omega). \quad (2)$$

Consequently,

$$\begin{aligned} \sum_{\omega \in G} X(\omega) \mathbb{P}(\omega) &\stackrel{(1)}{=} \sum_{l \in L} \sum_{\omega \in A_l} X(\omega) \mathbb{P}(\omega) \stackrel{(2)}{=} \sum_{l \in L} \mathbb{E}_{\mathbb{P}}(X | \mathcal{G}) \mathbb{P}(A_l) \\ &\stackrel{(3)}{=} \sum_{l \in L} \sum_{\omega \in A_l} \mathbb{E}_{\mathbb{P}}(X | \mathcal{G})(\omega) \mathbb{P}(\omega) \stackrel{(1)}{=} \sum_{\omega \in G} \mathbb{E}_{\mathbb{P}}(X | \mathcal{G})(\omega) \mathbb{P}(\omega) \end{aligned}$$

where equality (3) holds since the conditional expectation $\mathbb{E}_{\mathbb{P}}(X | \mathcal{G})$ is constant on each event A_l , as can be seen from equation (2). If we take $G = \Omega$, then we obtain

$$\mathbb{E}_{\mathbb{P}}(X) = \sum_{\omega \in \Omega} X(\omega) \mathbb{P}(\omega) = \sum_{\omega \in \Omega} \mathbb{E}_{\mathbb{P}}(X | \mathcal{G})(\omega) \mathbb{P}(\omega) = \mathbb{E}_{\mathbb{P}}(\mathbb{E}_{\mathbb{P}}(X | \mathcal{G})).$$

This shows that the equality $\mathbb{E}_{\mathbb{P}}(X) = \mathbb{E}_{\mathbb{P}}(\mathbb{E}_{\mathbb{P}}(X | \mathcal{G}))$ is always valid when Ω is finite and \mathcal{G} is an arbitrary σ -field.

(b) If η is \mathcal{G} -measurable, then it is constant on each event A_l and thus $\eta = \sum_{l=1}^m b_l \mathbb{1}_{A_l}$ for some real numbers b_1, b_2, \dots, b_m . If we take $G = A_l$, then the postulated equality gives

$$\sum_{\omega \in A_l} X(\omega) \mathbb{P}(\omega) = \sum_{\omega \in A_l} \eta(\omega) \mathbb{P}(\omega) = b_l \mathbb{P}(A_l)$$

which implies that

$$\eta = \sum_{l=1}^m \frac{1}{\mathbb{P}(A_l)} \sum_{\omega \in A_l} X(\omega) \mathbb{P}(\omega) \mathbb{1}_{A_l} = \mathbb{E}_{\mathbb{P}}(X | \mathcal{G})$$

where the last equality follows from the definition of the conditional expectation $\mathbb{E}_{\mathbb{P}}(X | \mathcal{G})$.

Exercise 4 (MATH3975) Notice that the \mathcal{F}_t -measurable random variable L_t (respectively, \mathcal{F}_s -measurable random variable L_s) is the Radon-Nikodym density of \mathbb{Q} with respect to \mathbb{P} on the space (Ω, \mathcal{F}_t) (respectively on the space (Ω, \mathcal{F}_s)).

(a) We assume that Ω is finite. The random variable L_s is \mathcal{F}_s -measurable and, by definition, the following equality holds for every B_l from the partition generating \mathcal{F}_s

$$\mathbb{Q}(B_l) = \sum_{\omega \in B_l} L_s(\omega) \mathbb{P}(\omega). \quad (3)$$

Similarly, the random variable L_t is \mathcal{F}_t -measurable and thus constant on every A_j from the partition generating \mathcal{F}_t . Moreover, for every A_j from the partition generating \mathcal{F}_t

$$\mathbb{Q}(A_j) = \sum_{\omega \in A_j} L_t(\omega) \mathbb{P}(\omega). \quad (4)$$

Since $\mathcal{F}_t \subset \mathcal{F}_s$ (recall that $t \leq s$) and the conditional expectation $\mathbb{E}_{\mathbb{P}}(L_s | \mathcal{F}_t)$ is constant on every event A_j and satisfies for every j

$$\begin{aligned} \sum_{\omega \in A_j} \mathbb{E}_{\mathbb{P}}(L_s | \mathcal{F}_t)(\omega) \mathbb{P}(\omega) &= \sum_{\omega \in A_j} L_s(\omega) \mathbb{P}(\omega) = \sum_{\omega \in B_l, B_l \subset A_j} L_s(\omega) \mathbb{P}(\omega) \\ &\stackrel{(3)}{=} \sum_{B_l \subset A_j} \mathbb{Q}(B_l) = \mathbb{Q}(A_j) \stackrel{(4)}{=} \sum_{\omega \in A_j} L_t(\omega) \mathbb{P}(\omega). \end{aligned}$$

Using part (b) in Exercise 3, we conclude that $\mathbb{E}_{\mathbb{P}}(L_s | \mathcal{F}_t) = L_t$.

(b) By applying the abstract Bayes formula to an arbitrary \mathcal{F}_s -measurable random variable Y , we obtain for every $0 \leq t \leq s$

$$\mathbb{E}_{\mathbb{Q}}(Y | \mathcal{F}_t) = \frac{\mathbb{E}_{\mathbb{P}}(Y L_s | \mathcal{F}_t)}{\mathbb{E}_{\mathbb{P}}(L_s | \mathcal{F}_t)} \stackrel{(a)}{\Leftrightarrow} \mathbb{E}_{\mathbb{Q}}(Y | \mathcal{F}_t) = (L_t)^{-1} \mathbb{E}_{\mathbb{P}}(Y L_s | \mathcal{F}_t).$$

(c) It suffices to observe that, for every $0 \leq t \leq s \leq T$

$$\mathbb{E}_{\mathbb{Q}}(M_s | \mathcal{F}_t) = M_t \stackrel{(b)}{\Leftrightarrow} (L_t)^{-1} \mathbb{E}_{\mathbb{P}}(M_s L_s | \mathcal{F}_t) = M_t \Leftrightarrow \mathbb{E}_{\mathbb{P}}(M_s L_s | \mathcal{F}_t) = M_t L_t.$$

Exercise 5 (MATH3975) It is clear that (i) implies (ii). To derive (iii) from (ii) we use the tower property (TP)

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}(M_T | \mathcal{F}_t) &\stackrel{(\text{TP})}{=} \mathbb{E}_{\mathbb{P}}(\mathbb{E}_{\mathbb{P}}(M_T | \mathcal{F}_{T-1}) | \mathcal{F}_t) \stackrel{(\text{ii})}{=} \mathbb{E}_{\mathbb{P}}(M_{T-1} | \mathcal{F}_t) \\ &\stackrel{(\text{TP})}{=} \mathbb{E}_{\mathbb{P}}(\mathbb{E}_{\mathbb{P}}(M_{T-1} | \mathcal{F}_{T-2}) | \mathcal{F}_t) \stackrel{(\text{ii})}{=} \mathbb{E}_{\mathbb{P}}(M_{T-2} | \mathcal{F}_t) = \cdots = M_t.\end{aligned}$$

Finally, we show that (iii) implies (i). We have that, for all $0 \leq t \leq s \leq T$,

$$\mathbb{E}_{\mathbb{P}}(M_T | \mathcal{F}_t) = M_t, \quad \mathbb{E}_{\mathbb{P}}(M_T | \mathcal{F}_s) = M_s.$$

Hence

$$\mathbb{E}_{\mathbb{P}}(M_s | \mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}(\mathbb{E}_{\mathbb{P}}(M_T | \mathcal{F}_s) | \mathcal{F}_t) \stackrel{(\text{TP})}{=} \mathbb{E}_{\mathbb{P}}(M_T | \mathcal{F}_t) = M_t.$$

If X is an \mathcal{F}_T -measurable random variable and $X = M_T$ where M is a martingale, then it is clear that

$$M_t = \mathbb{E}_{\mathbb{P}}(M_T | \mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}(X | \mathcal{F}_t)$$

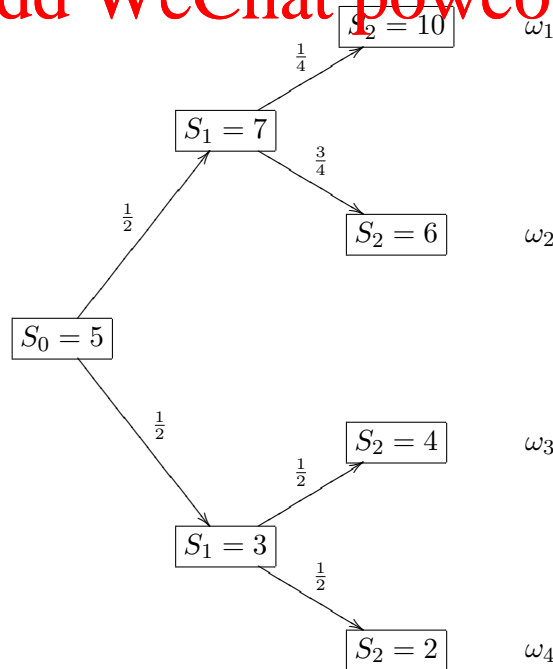
and thus it is unique.

Exercise 6 (MATH3975) (a) It suffices to observe that

$$\mathbb{E}_{\mathbb{P}}(S_1 | \mathcal{F}_0) = \mathbb{E}_{\mathbb{P}}(S_1 | \mathcal{F}_0) = 27/5 \neq 5 = S_0.$$

Similarly, one can check that $\mathbb{E}_{\mathbb{P}}(S_2 | \mathcal{F}_1) \neq S_1$ but, of course, this is not needed to conclude that S is not a martingale under \mathbb{P} .

(b) The unique martingale measure \mathbb{Q} for the process S can be represented as follows



This means that $\mathbb{Q} = (\mathbb{Q}(\omega_1), \mathbb{Q}(\omega_2), \mathbb{Q}(\omega_3), \mathbb{Q}(\omega_4)) = (1/8, 3/8, 1/4, 1/4)$.

(c) This is easy to check using part (b). In particular, on the σ -field \mathcal{F}_2

$$\mathbb{P} = (\mathbb{P}(\omega_1), \mathbb{P}(\omega_2), \mathbb{P}(\omega_3), \mathbb{P}(\omega_4)) = (3/25, 12/25, 4/25, 6/25)$$

and thus on (Ω, \mathcal{F}_2) we obtain

$$L_2 = (L_2(\omega_1), L_2(\omega_2), L_2(\omega_3), L_2(\omega_4)) = 25(1/24, 1/32, 1/16, 1/24).$$

For (Ω, \mathcal{F}_1) , we denote $A_1 = \{\omega_1, \omega_2\}$ and $A_2 = \{\omega_3, \omega_4\}$ and we obtain

$$\mathbb{P}|_{\mathcal{F}_1} = (\mathbb{P}(A_1), \mathbb{P}(A_2)) = (3/5, 2/5)$$

and

$$\mathbb{Q}|_{\mathcal{F}_1} = (\mathbb{Q}(A_1), \mathbb{Q}(A_2)) = (1/2, 1/2).$$

Hence

$$L_1 = (L_1(\omega_1), L_1(\omega_2), L_1(\omega_3), L_1(\omega_4)) = (5/6, 5/6, 5/4, 5/4) = (5/6)\mathbb{1}_{A_1} + (5/4)\mathbb{1}_{A_2}.$$

It is easy to check that

$$\mathbb{E}_{\mathbb{P}}(L_2 | \mathcal{F}_1) = L_1$$

and

$$\mathbb{E}_{\mathbb{P}}(L_1 | \mathcal{F}_0) = \mathbb{E}_{\mathbb{P}}(L_1) = 1 = L_0$$

so that the Radon-Nikodym density process L is a martingale under \mathbb{P} .

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