

# MATH3075/3975 Financial Derivatives

## Tutorial 4: Solutions

**Exercise 1** We consider the elementary market model  $\mathcal{M} = (B, S)$  with  $S_0 > 0$  and  $0 < d < 1 + r < u$ .

(a) Our aim is to find the probability measure  $\hat{\mathbb{P}}$  such that  $\mathbb{E}_{\hat{\mathbb{P}}}(\hat{B}_T) = \hat{B}_0$  where  $\hat{B}_t = B_t/S_t$  for  $t = 0, 1$ . We will also compute the Radon-Nikodym density  $L$  of  $\hat{\mathbb{P}}$  with respect to  $\mathbb{P}$  and we will show that  $\mathbb{E}_{\hat{\mathbb{P}}}(L) = 1$ .

• We denote  $\hat{\mathbb{P}}(\omega_1) = \hat{p}$  and  $\hat{\mathbb{P}}(\omega_2) = \hat{q} = 1 - \hat{p}$ . The postulated equality  $\mathbb{E}_{\hat{\mathbb{P}}}(\hat{B}_T) = \hat{B}_0$  means that

$$\mathbb{E}_{\hat{\mathbb{P}}}\left(\frac{B_T}{S_T}\right) = \frac{B_0}{S_0},$$

which can be expanded to the following equation for  $\hat{p}$

$$\hat{p} \frac{1+r}{S^u} + \hat{q} \frac{1+r}{S^d} = \frac{1}{S_0}$$

We obtain

$$\hat{p} = \left( \frac{1}{S^d} - \frac{1}{(1+r)S_0} \right) \frac{S^u S^d}{S^u - S^d} = \frac{S^u - (1+r)S_0}{(1+r)S_0 S^d} \frac{S^u S^d}{S^u - S^d} \\ = \frac{1+r-d}{u-d} \frac{u}{1+r} = \tilde{p} \frac{u}{1+r}$$

and

$$\hat{q} = \left( \frac{1}{(1+r)S_0} - \frac{1}{S^u} \right) \frac{S^u S^d}{S^u - S^d} = \frac{S^u - (1+r)S_0}{(1+r)S_0 S^u} \frac{S^u S^d}{S^u - S^d} \\ = \frac{u - (1+r)}{u - d} \frac{d}{1+r} = \tilde{q} \frac{d}{1+r}$$

where we denote  $\tilde{q} = 1 - \tilde{p}$ . It is easy to see that  $\hat{p} > 0, \hat{q} > 0$  and  $\hat{p} + \hat{q} = 1$ . Hence  $\hat{\mathbb{P}} = (\hat{\mathbb{P}}(\omega_1), \hat{\mathbb{P}}(\omega_2)) = (\hat{p}, \hat{q})$  is a probability on  $\Omega = (\omega_1, \omega_2)$  and it is equivalent to the risk-neutral probability measure  $\mathbb{P}$  (hence also equivalent to the probability measure  $\mathbb{P}$ ).

The Radon-Nikodym density  $L$  of  $\hat{\mathbb{P}}$  with respect to  $\mathbb{P}$  equals

$$L(\omega_1) = \frac{d\hat{\mathbb{P}}}{d\mathbb{P}}(\omega_1) = \frac{\hat{\mathbb{P}}(\omega_1)}{\mathbb{P}(\omega_1)} = \frac{\hat{p}}{\tilde{p}} = \frac{u}{1+r}, \quad L(\omega_2) = \frac{d\hat{\mathbb{P}}}{d\mathbb{P}}(\omega_2) = \frac{\hat{\mathbb{P}}(\omega_2)}{\mathbb{P}(\omega_2)} = \frac{\hat{q}}{\tilde{q}} = \frac{d}{1+r}.$$

It is important to notice that

$$L = \frac{d\hat{\mathbb{P}}}{d\mathbb{P}} = \frac{S_T B_0}{S_0 B_T} = \frac{\hat{S}_T}{\hat{S}_0}.$$

Furthermore,

$$\mathbb{E}_{\hat{\mathbb{P}}}(L) = \tilde{p} \frac{u}{1+r} + (1 - \tilde{p}) \frac{d}{1+r} = \hat{p} + \hat{q} = 1.$$

(b) We wish to show that the price  $\pi_0(X)$  of any contingent claim  $X = g(S_T)$  satisfies

$$\pi_0(X) = S_0 \mathbb{E}_{\hat{\mathbb{P}}} \left( \frac{X}{S_T} \right) = S_0 \mathbb{E}_{\hat{\mathbb{P}}} \left( \frac{g(S_T)}{S_T} \right).$$

• We already know from lectures that any contingent claim  $X$  can be replicated in the elementary market model  $\mathcal{M} = (B, S)$  and its arbitrage price at time 0 can be computed using the risk-neutral valuation formula

$$\pi_0(X) = B_0 \mathbb{E}_{\hat{\mathbb{P}}}((1+r)^{-1}X). \quad (1)$$

*First method.* To show that  $\pi_0(X)$  satisfies also the equality

$$\pi_0(X) = S_0 \mathbb{E}_{\hat{\mathbb{P}}}(S_T^{-1}X), \quad (2)$$

we may use Radon-Nikodym density  $L$  of  $\hat{\mathbb{P}}$  with respect to  $\tilde{\mathbb{P}}$ . It suffices to observe that

$$S_0 \mathbb{E}_{\hat{\mathbb{P}}}(S_T^{-1}X) = S_0 \mathbb{E}_{\tilde{\mathbb{P}}}(LS_T^{-1}X) = S_0 \mathbb{E}_{\tilde{\mathbb{P}}}(\hat{S}_T \hat{S}_0^{-1} S_T^{-1}X) = B_0 \mathbb{E}_{\tilde{\mathbb{P}}}(B_T^{-1}X) = \mathbb{E}_{\tilde{\mathbb{P}}}((1+r)^{-1}X).$$

*Second method.* Alternatively, we consider any portfolio  $\varphi = (\varphi^0, \varphi^1)$  where  $\varphi^0 = x - \varphi^1 S_0 = V_0(\varphi) - \varphi^1 S_0$  and

$$V_T(\varphi) = \varphi^0(1+r) + \varphi^1 S_T = \varphi^0 B_T + \frac{1+r}{S_0} V_0(\varphi) + \varphi^1 S_T$$

Then (recall that  $B_0 = 1$ )

$$\frac{V_T(\varphi)}{S_T} = \frac{V_0(\varphi)}{S_0} + \varphi^0 \left( \frac{B_T}{S_T} - \frac{B_0}{S_0} \right),$$

so that, using the definition of the probability  $\hat{\mathbb{P}}$ , we obtain

$$\mathbb{E}_{\hat{\mathbb{P}}}(S_T^{-1}V_T(\varphi)) = \frac{V_0(\varphi)}{S_0} + \mathbb{E}_{\hat{\mathbb{P}}} \left[ \varphi^0 \left( \frac{B_T}{S_T} - \frac{B_0}{S_0} \right) \right] = \frac{V_0(\varphi)}{S_0} + \varphi^0 \mathbb{E}_{\hat{\mathbb{P}}}(\hat{B}_T - \hat{B}_0) = \frac{V_0(\varphi)}{S_0}$$

since  $\mathbb{E}_{\hat{\mathbb{P}}}(\hat{B}_T - \hat{B}_0) = 0$ . Hence if a portfolio  $\varphi$  replicates  $X$  so that  $V_T(\varphi) = X$ , then

$$\mathbb{E}_{\hat{\mathbb{P}}}(S_T^{-1}X) = \frac{V_0(\varphi)}{S_0} = \frac{\pi_0(X)}{S_0}.$$

(c) We consider the put option with the payoff  $P_T(K) = (K - S_T)^+$  for some  $K > 0$ . We will show that the arbitrage price  $P_0(K)$  admits the following representation

$$P_0(K) = K(1+r)^{-1} \tilde{\mathbb{P}}(S_T < K) - S_0 \hat{\mathbb{P}}(S_T < K).$$

• We denote  $A = \{S_T < K\}$  so that

$$P_T(K) = (K - S_T)^+ = (K - S_T)\mathbb{1}_A = \mathbb{1}_A K - \mathbb{1}_A S_T = X_1 - X_2.$$

where, by definition,  $\mathbb{1}_A = 1$  on the event  $A$  and it equals 0 on the complement of  $A$ . Therefore, by applying (1) to  $X_1$  and (2) to  $X_2$ , we obtain

$$\begin{aligned} P_0(K) &= \pi_0(X_1) - \pi_0(X_2) = \pi_0(\mathbb{1}_A K) - \pi_0(\mathbb{1}_A S_T) \\ &= B_0 \mathbb{E}_{\tilde{\mathbb{P}}}((1+r)^{-1} \mathbb{1}_A K) - S_0 \mathbb{E}_{\hat{\mathbb{P}}}(S_T^{-1} \mathbb{1}_A S_T) \\ &= K(1+r)^{-1} \tilde{\mathbb{P}}(S_T < K) - S_0 \hat{\mathbb{P}}(S_T < K). \end{aligned}$$

Let  $C_T(K) = (S_T - K)^+$  for some  $K > 0$ . It is easy to show that the price  $C_0(K)$  satisfies

$$C_0(K) = S_0 \hat{\mathbb{P}}(S_T > K) - K(1+r)^{-1} \tilde{\mathbb{P}}(S_T > K).$$

(d) Our goal is to show that the extended model  $\mathcal{M}^e = (B, S, P(K))$  is arbitrage-free, in the sense of Definition 2.2.3 from the course notes.

- For any trading strategy  $(x, \varphi^1, \varphi^2) \in \mathbb{R}^3$ , the wealth satisfies  $V_0(x, \varphi^1, \varphi^2) = x$  and

$$V_1(x, \varphi^1, \varphi^2) = (x - \varphi^1 S_0 - \varphi^2 P_0(K))(1+r) + \varphi^1 S_1 + \varphi^2 P_1(K).$$

If  $x = 0$ , then for every  $(\varphi^1, \varphi^2) \in \mathbb{R}^2$

$$\begin{aligned} V_1(0, \varphi^1, \varphi^2) &= \varphi^1 (S_T - S_0(1+r)) + \varphi^2 (P_T(K) - P_0(K)(1+r)) \\ &= \varphi^1 (S_T - S_0(1+r)) + \varphi^2 (X - (1+r)\pi_0(X)) \end{aligned}$$

where we denote  $X = P_T(K)$  and  $\pi_0(X) = P_0(K)$ . Therefore,

$$\mathbb{E}_{\tilde{\mathbb{P}}}(V_T(0, \varphi^1, \varphi^2)) = \varphi^1 \mathbb{E}_{\tilde{\mathbb{P}}}(S_T - S_0(1+r)) + \varphi^2 \mathbb{E}_{\tilde{\mathbb{P}}}(X - (1+r)\pi_0(X)) = 0.$$

If the wealth  $V_T(0, \varphi^1, \varphi^2)$  is non-negative and has the expected value equal to zero, then necessarily  $V_T(0, \varphi^1, \varphi^2)(\omega_i) = 0$  for  $i = 1, 2$ . We conclude that arbitrage opportunities do not exist in the extended model  $\mathcal{M}^e = (B, S, P(K))$ .

(e) We take a fixed  $K$  such that  $S_0 < K < S_0 u$  and we consider the modified market model  $\mathcal{N} = (B, P(K))$ . We will show that the price of an arbitrary claim  $X$  computed in  $\mathcal{N} = (B, P(K))$  coincides with its arbitrage price computed in  $\mathcal{M} = (B, S)$ . We will also find the arbitrage price at time 0 for the claim  $X = S_T$ .

- It suffices to observe that the probability measure  $\mathbb{Q} = \tilde{\mathbb{P}}$  is the unique martingale measure for the model  $\mathcal{N} = (B, P(K))$  since

$$\mathbb{E}_{\tilde{\mathbb{P}}}(P_1(K)) = (1+r)P_0(K).$$

Hence the arbitrage price of any contingent claim  $X$  computed in  $\mathcal{N} = (B, P(K))$  and  $\mathcal{M} = (B, S)$  are identical. In particular, the arbitrage price of the claim  $X = S_T$  in the model  $\mathcal{N}$  can be computed from the risk-neutral valuation

$$\pi_0(X) = B_0 \mathbb{E}_{\mathbb{Q}}(B_T^{-1} X) = B_0 \mathbb{E}_{\tilde{\mathbb{P}}}(B_T^{-1} X) = B_0 \mathbb{E}_{\tilde{\mathbb{P}}}(B_T^{-1} S_T) = S_0$$

where the last equality is a consequence of the definition of  $\tilde{\mathbb{P}}$ . One may also show directly that the claim  $X = S_T$  can be replicated by a portfolio composed of  $B$  and  $P(K)$ .

**Exercise 2** Consider a trading strategy  $(x, \varphi) = (x, \varphi^1, \dots, \varphi^n)$ . Its wealth satisfies  $V_0(x, \varphi) = x$  and

$$V_1(x, \varphi) = \left(x - \sum_{j=1}^n \varphi^j S_0^j\right)(1+r) + \sum_{j=1}^n \varphi^j S_1^j$$

and thus

$$\begin{aligned} \hat{V}_1(x, \varphi) &:= \frac{V_1(x, \varphi)}{B_1} = \frac{V_1(x, \varphi)}{1+r} = \left(x - \sum_{j=1}^n \varphi^j S_0^j\right) + \sum_{j=1}^n \varphi^j (1+r)^{-1} S_1^j \\ &= \left(x - \sum_{j=1}^n \varphi^j S_0^j\right) + \sum_{j=1}^n \varphi^j \hat{S}_1^j \end{aligned}$$

where  $\hat{S}_1^j := (1+r)^{-1} S_1^j$ .

Consequently,

$$\begin{aligned}\widehat{G}_1(x, \varphi) &:= \widehat{V}_1(x, \varphi) - \widehat{V}_0(x, \varphi) = \widehat{V}_1(x, \varphi) - V_0(x, \varphi) = \widehat{V}_1(x, \varphi) - x \\ &= \left( x - \sum_{j=1}^n \varphi^j S_0^j \right) + \sum_{j=1}^n \varphi^j \widehat{S}_1^j - x = \sum_{j=1}^n \varphi^j (\widehat{S}_1^j - S_0^j) \\ &= \sum_{j=1}^n \varphi^j (\widehat{S}_1^j - \widehat{S}_0^j) = \sum_{j=1}^n \varphi^j \Delta \widehat{S}_1^j.\end{aligned}$$

**Exercise 3** In view of Definition 2.2.4, we need to find all probability measures  $\mathbb{Q}$  on the space  $\Omega = \{\omega_1, \omega_2, \omega_3\}$  such that  $\mathbb{Q}$  is equivalent to  $\mathbb{P}$  (that is,  $\mathbb{Q}(\omega_i) > 0$  for  $i = 1, 2, 3$ ) and

$$\mathbb{E}_{\mathbb{Q}}(\Delta \widehat{S}_1) = 0$$

where

$$\Delta \widehat{S}_1 = \widehat{S}_1 - \widehat{S}_0 = \frac{9}{10} \left( \frac{60}{9}, \frac{40}{9}, \frac{30}{9} \right) - (5, 5, 5) = (1, -1, -2).$$

Let us denote  $\mathbb{Q} = (q_1, q_2, q_3)$ . Then we search for all solutions  $(q_1, q_2, q_3)$  to the system

$$\begin{cases} 0 < q_i < 1 \text{ for } i = 1, 2, 3, \\ q_1 - q_2 = 2/3 = 0, \\ q_1 + q_2 + q_3 = 1. \end{cases}$$

We obtain

$$\begin{cases} q_1 = \frac{1}{5} q_3 + \frac{1}{2}, \\ q_2 = -\frac{3}{2} q_3 + \frac{1}{2}, \end{cases}$$

and we check that the condition  $q_i \in (0, 1)$  is satisfied for every  $i = 1, 2, 3$  whenever  $q_3 \in (0, 1/3)$  since it is clear that  $q_3$  should satisfy the following inequalities

$$\begin{cases} 0 < \frac{1}{2} q_3 + \frac{1}{2} < 1, \\ 0 < -\frac{3}{2} q_3 + \frac{1}{2} < 1, \end{cases}$$

and this holds whenever  $q_3 \in (0, 1/3)$ . Hence the set of all risk-neutral probability measures for  $\mathcal{M}$  can be represented as follows

$$\mathbb{M} = \left\{ \mathbb{Q} = \left( \frac{1}{2}, \frac{1}{2}, 0 \right) + q_3 \left( \frac{1}{2}, -\frac{3}{2}, 1 \right), \quad q_3 \in (0, \frac{1}{3}) \right\}.$$

**Exercise 4** (a) For any trading strategy  $(x, \varphi) = (x, \varphi^1, \varphi^2)$ , we have

$$V_1(x, \varphi) = \left( x - \sum_{j=1}^2 \varphi^j S_0^j \right) (1 + r) + \sum_{j=1}^2 \varphi^j S_1^j$$

or, more explicitly,

$$V_1(x, \varphi)(\omega_i) = \begin{cases} \frac{10}{9} x + \frac{10}{9} \varphi^1 + \frac{20}{9} \varphi^2, & i = 1, \\ \frac{10}{9} x + \frac{10}{9} \varphi^1 - \frac{20}{9} \varphi^2, & i = 2, \\ \frac{10}{9} x - \frac{10}{9} \varphi^1 - \frac{20}{9} \varphi^2, & i = 3, \\ \frac{10}{9} x - \frac{30}{9} \varphi^1 + \frac{20}{9} \varphi^2, & i = 4. \end{cases}$$

Since

$$G_1(x, \varphi) := V_1(x, \varphi) - V_0(x, \varphi) = V_1(x, \varphi) - x,$$

we obtain

$$G_1(x, \varphi)(\omega_i) = \begin{cases} \frac{1}{9}x + \frac{10}{9}\varphi^1 + \frac{20}{9}\varphi^2, & i = 1, \\ \frac{1}{9}x + \frac{10}{9}\varphi^1 - \frac{20}{9}\varphi^2, & i = 2, \\ \frac{1}{9}x - \frac{10}{9}\varphi^1 - \frac{20}{9}\varphi^2, & i = 3, \\ \frac{1}{9}x - \frac{30}{9}\varphi^1 + \frac{20}{9}\varphi^2, & i = 4. \end{cases}$$

Next

$$\widehat{V}_1(x, \varphi) := (1 + r)^{-1} V_1(x, \varphi) = \frac{9}{10} V_1(x, \varphi)$$

so that

$$\widehat{V}_1(x, \varphi)(\omega_i) = \begin{cases} x + \varphi^1 + 2\varphi^2, & i = 1, \\ x + \varphi^1 - 2\varphi^2, & i = 2, \\ x - \varphi^1 - 2\varphi^2, & i = 3, \\ x - 3\varphi^1 + 2\varphi^2, & i = 4. \end{cases}$$

Finally,

$$\widehat{G}_1(x, \varphi) := \widehat{V}_1(x, \varphi) - \widehat{V}_0(x, \varphi) = \widehat{V}_1(x, \varphi) - x$$

and thus

$$\widehat{G}_1(x, \varphi)(\omega_i) = \begin{cases} \varphi^1 + 2\varphi^2, & i = 1, \\ \varphi^1 - 2\varphi^2, & i = 2, \\ -\varphi^1 - 2\varphi^2, & i = 3, \\ -3\varphi^1 + 2\varphi^2, & i = 4. \end{cases}$$

We observe that

$$\widehat{G}_1(x, \varphi) = \varphi^1 \Delta \widehat{S}_1^1 + \varphi^2 \Delta \widehat{S}_1^2 = \varphi^1(1, 1, -1, -3) + \varphi^2(2, -2, -2, 2).$$

(b) It is clear that  $G_1(x, \varphi)$  depends on the initial wealth  $x$ , but  $\widehat{G}_1(x, \varphi)$  does not, so that for any  $x, y \in \mathbb{R}$  and arbitrary  $\varphi \in \mathbb{R}^2$  we have  $\widehat{G}_1(x, \varphi) = \widehat{G}_1(y, \varphi)$ . In particular,  $\widehat{G}_1(x, \varphi) = \widehat{G}_1(0, \varphi)$  for every  $x \in \mathbb{R}$  and  $\varphi \in \mathbb{R}^2$ .

**Exercise 5 (MATH3975)** (a) We have  $k = 4$  and  $n = 2$ . Recall that

$$\mathbb{W} := \{X \in \mathbb{R}^4 \mid X = \widehat{G}_1(x, \varphi) \text{ for some } (x, \varphi) \in \mathbb{R}^3\}$$

where in fact  $\widehat{G}_1(x, \varphi)$  does not depend on  $x$ . More explicitly,  $X \in \mathbb{W}$  if and only if  $X \in \mathbb{R}^4$  and  $X \in \widehat{G}_1(x, \varphi)$  that is (from the previous exercise)

$$X = \begin{cases} \varphi^1 + 2\varphi^2, & i = 1, \\ \varphi^1 - 2\varphi^2, & i = 2, \\ -\varphi^1 - 2\varphi^2, & i = 3, \\ -3\varphi^1 + 2\varphi^2, & i = 4, \end{cases}$$

for some real numbers  $\varphi^1$  and  $\varphi^2$ . This means that  $\mathbb{W}$  is a plane given by

$$\mathbb{W} = \{X \in \mathbb{R}^4 \mid X = \varphi^1(1, 1, -1, -3) + \varphi^2(2, -2, -2, 2), \varphi^1, \varphi^2 \in \mathbb{R}\}$$

or, more precisely, the two-dimensional linear subspace of  $\mathbb{R}^4$  spanned by the vectors  $(1, 1, -1, -3)$  and  $(2, -2, -2, 2)$ .

(b) Recall that the space  $\mathbb{W}^\perp$  is the orthogonal complement of  $\mathbb{W}$ , that is,

$$\mathbb{W}^\perp := \{Z \in \mathbb{R}^4 \mid \langle X, Z \rangle = 0 \text{ for all } X \in \mathbb{W}\}.$$

Hence a vector  $Z \in \mathbb{R}^4$  belongs to  $\mathbb{W}^\perp$  whenever it satisfies

$$\begin{cases} z_1 + z_2 - z_3 - 3z_4 = 0, \\ z_1 - z_2 - z_3 + z_4 = 0. \end{cases}$$

Therefore,  $z_1 = z_3 + z_4$  and  $z_2 = 2z_4$  and thus

$$\mathbb{W}^\perp = \{Z \in \mathbb{R}^4 \mid Z = z_3(1, 0, 1, 0) + z_4(1, 2, 0, 1), \ z_3, z_4 \in \mathbb{R}\}$$

which is the two-dimensional linear subspace of  $\mathbb{R}^4$  spanned by the vectors  $(1, 0, 1, 0)$  and  $(1, 2, 0, 1)$ . We conclude that  $\mathbb{W}^\perp$  is the plane orthogonal to the plane  $\mathbb{W}$ .

(c) We will show that the model  $\mathcal{M} = (B, S^1, S^2)$  is arbitrage-free using two methods.

- *First method.* In view of Remark 2.2.3, to check whether the model is arbitrage-free, it suffices to show that  $\mathbb{W} \cap \mathbb{A} = \emptyset$  where

$$\mathbb{A} = \{X \in \mathbb{R}^4 \mid X \neq 0, \ x_i \geq 0, \ i = 1, \dots, 4\}.$$

Since for all  $X \in \mathbb{W}$  we have  $x_3 = -x_1$  and for all  $X \in \mathbb{A}$  we have  $x_i \geq 0$ , it is clear that if  $X \in \mathbb{W} \cap \mathbb{A}$  then  $x_1 = x_3 = 0$ . This in turn implies that  $\varphi^1 = -2\varphi^2$  and thus  $x_2 = -4\varphi^2$  and  $x_4 = 8\varphi^2$ . Condition  $x_i \geq 0$  implies that  $\varphi^2 = 0$  and thus also  $x_1 = x_2 = x_3 = x_4 = 0$ . We conclude that  $\mathbb{W} \cap \mathbb{A} = \emptyset$  and thus the market model is arbitrage-free.

- *Second method.* In view of Remark 2.2.4, in order to confirm that the model is arbitrage-free, one may check that  $\mathbb{W}^\perp \cap \mathcal{P}^+ \neq \emptyset$  where the set  $\mathcal{P}^+$  is given by

$$\mathcal{P}^+ := \left\{Q \in \mathbb{R}^4 \mid \sum_{i=1}^4 q_i = 1, \ q_i > 0\right\}.$$

Assume that  $Z \in \mathbb{W}^\perp \cap \mathcal{P}^+$ . We observe that the condition  $q_i > 0$  for  $i = 1, \dots, 4$  implies that  $z_3 > 0$  and  $z_4 > 0$ . Thus  $z_1 = z_3 + z_4 > 0$  and  $z_2 = 2z_4 > 0$  and thus it suffices to impose the condition that  $Z$  is a probability measure so that

$$\sum_{i=1}^4 z_i = 1 \Rightarrow 2z_3 + 4z_4 = 1.$$

It is now clear that  $\mathbb{W}^\perp \cap \mathcal{P}^+ \neq \emptyset$  and thus the market model is arbitrage-free.

For instance, we may take  $z_3 = 1/4$  and  $z_4 = 1/8$ . Then we obtain the following vector

$$Z = \frac{1}{4}(1, 0, 1, 0) + \frac{1}{8}(1, 2, 0, 1) = \left(\frac{3}{8}, \frac{2}{8}, \frac{2}{8}, \frac{1}{8}\right) \in \mathbb{W}^\perp \cap \mathcal{P}^+.$$

It is easy to check that for  $Q = Z$  we have that  $\mathbb{E}_Q(\Delta \hat{S}_1^i) = 0$  for  $i = 1, 2$  since  $\Delta \hat{S}_1^1 = (1, 1, -1, -3)$  and  $\Delta \hat{S}_1^2 = (2, -2, -2, 2)$ .

(d) From Lemma 2.2.1, we know that  $\mathbb{M} = \mathbb{W}^\perp \cap \mathcal{P}^+$ . The class  $\mathbb{M}$  of all risk-neutral probabilities for this model is thus non-empty and non-uniqueness of a risk-neutral probability holds. We may represent  $\mathbb{M}$  as follows

$$\mathbb{M} = \left\{Q \in \mathbb{R}^4 \mid Q = q_3(1, 0, 1, 0) + q_4(1, 2, 0, 1), \ q_3 > 0, \ q_4 > 0, \ 2q_3 + 4q_4 = 1\right\}$$

or, equivalently,

$$\mathbb{M} = \left\{Q \in \mathbb{R}^4 \mid Q = q_3(1, 0, 1, 0) + \frac{1-2q_3}{4}(1, 2, 0, 1), \ q_3 \in (0, \frac{1}{2})\right\}.$$

**Exercise 6** (MATH3975) The proof of Proposition 2.2.1 is straightforward. It suffices to observe that  $V_1(x, \varphi)$  and  $\hat{V}_1(x, \varphi)$  have the same properties, since  $\hat{V}_1(x, \varphi) = cV_1(x, \varphi)$  for a strictly positive constant  $c$  (specifically,  $c = (1+r)^{-1}$ ). This argument shows that the first statement in Proposition 2.2.1 is valid.

Furthermore, we observe that in Definition 2.2.3 we only consider trading strategies with  $x = 0$ . Therefore, we may use the equality  $\hat{V}_1(0, \varphi) = \hat{G}_1(0, \varphi)$  and thus the second part in Proposition 2.2.1 is true as well.

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