MATH3075/3975
Financial Derivatives

School of Mathematics and Statistics
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Tutorial sheet 11

Background: Chapter 5 – The Black-Scholes Model.

Exercise 1 Consider the Black-Scholes model $\mathcal{M} = (B, S)$ with the initial stock price $S_0 = 9$, the continuously compounded interest rate $r = 0.01$ per annum and the stock price volatility equals $\sigma = 0.1$ per annum.

(a) Using the Black-Scholes call option pricing formula

$$C_0 = S_0 N\big(d_+(S_0, T)\big) - Ke^{-rT} N\big(d_-(S_0, T)\big)$$

compute the price C_0 of the European call option with strike price $K = 10$ and maturity $T = 5$ years.

(b) Using the Black-Scholes put option pricing formula

$$P_0 = Ke^{-rT} N\big(-d_-(S_0, T)\big) - S_0 N\big(-d_+(S_0, T)\big)$$

compute the price P_0 for the European put option with strike price $K = 10$ and maturity $T = 5$ years.

(c) Does the put-call parity relationship

$$C_0 - P_0 = S_0 - Ke^{-rT}$$

hold?

(d) Recompute the prices of call and put options for modified maturities $T = 5$ months and $T = 5$ days.

(e) Explain the observed pattern of call and put prices when the time to maturity goes to zero.

Exercise 2 Assume that the stock price S is governed under the martingale measure $\widetilde{\mathbb{P}}$ by the Black-Scholes stochastic differential equation

$$dS_t = S_t\big(r\,dt + \sigma\,dW_t\big)$$

where $\sigma > 0$ is a constant volatility and r is a constant short-term interest rate. Let $0 < L < K$ be real numbers. Consider the contingent claim with the payoff X at maturity date $T > 0$ given as $X = \min\big(|S_T - K|, L\big)$.

(a) Sketch the profile of the payoff X as the function of the stock price S_T at maturity date T and find the decomposition of the payoff X in terms of the payoffs of standard call and put options with different strikes.

(b) Compute the arbitrage price $\pi_t(X)$ at any date $t \in [0, T]$. Take for granted the Black-Scholes pricing formulae for European call and put options.

(c) Find the limits of the arbitrage price $\lim_{L\to 0}\pi_0(X)$ and $\lim_{L\to\infty}\pi_0(X)$.

(d) Find the limit of the arbitrage price $\lim_{\sigma\to\infty}\pi_0(X)$.

Exercise 3 We consider the call option pricing functions, that is, the functions $c : \mathbb{R}_+ \times [0, T] \to \mathbb{R}$ and $v : \mathbb{R}_+ \times [0, T] \to \mathbb{R}$ such that $C_t = v(S_t, t) = c(S_t, T - t)$ for all $t \in [0, T]$ where C_t is the Black-Scholes price of the call option.

(a) Show that v satisfies the terminal condition $v(s, T) = (s - K)^+$ in the sense that $\lim_{t\to T} v(s, t) = (s - K)^+$. Equivalently, the function c satisfies the initial condition $\lim_{t\to 0} c(s, t) = (s - K)^+$.

(b) (MATH3975) Show by direct computations that the pricing function v satisfies the Black-Scholes PDE. To this end, compute the partial derivatives v_s, v_{ss} and v_t (for answers, see Section 5.5 in the course notes). Write down the PDE satisfied by the function c and the initial condition.

Exercise 4 (MATH3975) Consider the stock price process S under the Black and Scholes assumption, that is,

$$S_t = S_0 \exp\left(\Big(r - \frac{1}{2}\sigma^2\Big)t + \sigma W_t\right)$$

where W is the Wiener process under the martingale measure $\widetilde{\mathbb{P}}$.

(a) Show that $\widehat{S}_t := e^{-rt} S_t$ is a martingale under $\widetilde{\mathbb{P}}$ with respect to the filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ generated by the stock price process S. Hint: Use the property that $\frac{S_t}{S_s}$ is independent of $\mathcal{F}_s$ for $0 \leq s < t$.

(b) Compute the expectation $\mathbb{E}_{\widetilde{\mathbb{P}}}(S_t)$ and the variance $\mathrm{Var}_{\widetilde{\mathbb{P}}}(S_t)$ of the stock price under the martingale measure $\widetilde{\mathbb{P}}$ using the martingale property of $\widehat{S}$ under $\widetilde{\mathbb{P}}$.