MATH3075/3975 Financial Mathematics

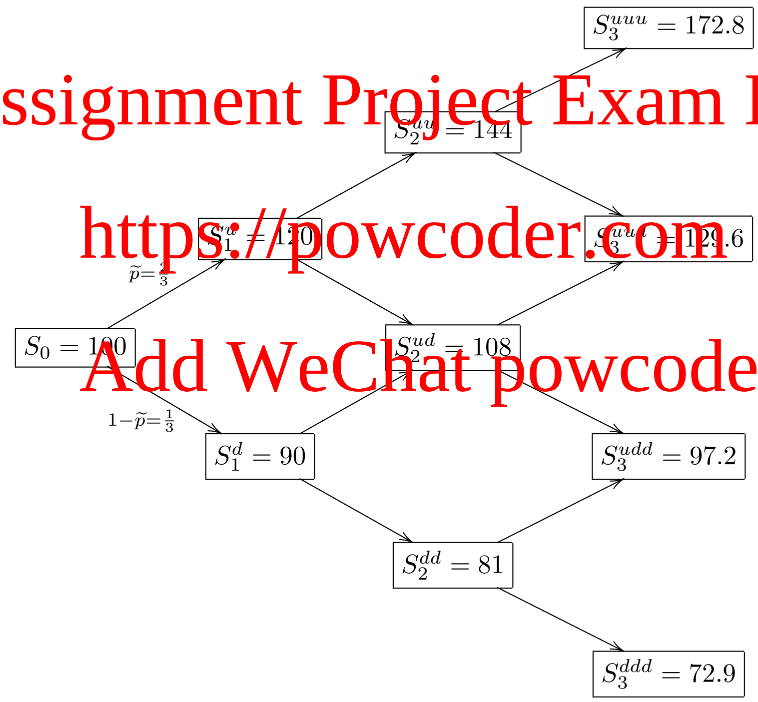
Tutorial 10: Solutions

Exercise 1 Assume the CRR model $\mathcal{M} = (B, S)$ with $T = 3$, the stock price $S_0 = 100$, $S_1^u = 120$, $S_1^d = 90$, and the risk-free interest rate $r = 0.1$. We consider the American put option on the stock S with the expiration date $T = 3$ and the constant strike price $K = 121$. The option has the reward process $g(S_t, t) = (K - S_t)^+ = (121 - S_t)^+$ for $t = 0, 1, 2, 3$,

(a) We first compute the arbitrage price P_t^a of this option for $t = 0, 1, 2, 3$. We start by noting that the unique risk-neutral probability measure $\widetilde{\mathbb{P}}$ satisfies

$$\widetilde{p} = \frac{1 + r - d}{u - d} = \frac{(1 + r)S_0 - S_1^d}{S_1^u - S_1^d} = \frac{1 + r - d}{u - d} = \frac{1.1 - 0.9}{1.2 - 0.9} = \frac{2}{3}.$$

Since $u = 1.2$ and $d = 0.9$, the stock price process S_t is given by



We first compute the terminal payoff P_T^a from the American put option at expiry date $T = 3$. We use the following convention for sample paths of the stock price: $\omega_1 = (u, u, u)$, $\omega_2 = (u, u, d)$, $\omega_3 = (u, d, u)$, $\omega_4 = (u, d, d)$, $\omega_5 = (d, u, u)$, $\omega_6 = (d, u, d)$, $\omega_7 = (d, d, u)$, $\omega_8 = (d, d, d)$. Hence the terminal payoff $P_T^a = (121 - S_T)^+$ can be represented as follows (notice that $\omega_4 = (u, d, d)$ and $\omega_5 = (d, u, u)$ so $S_T(\omega_5) = 129.6 > 97.2 = S_T(\omega_4)$)

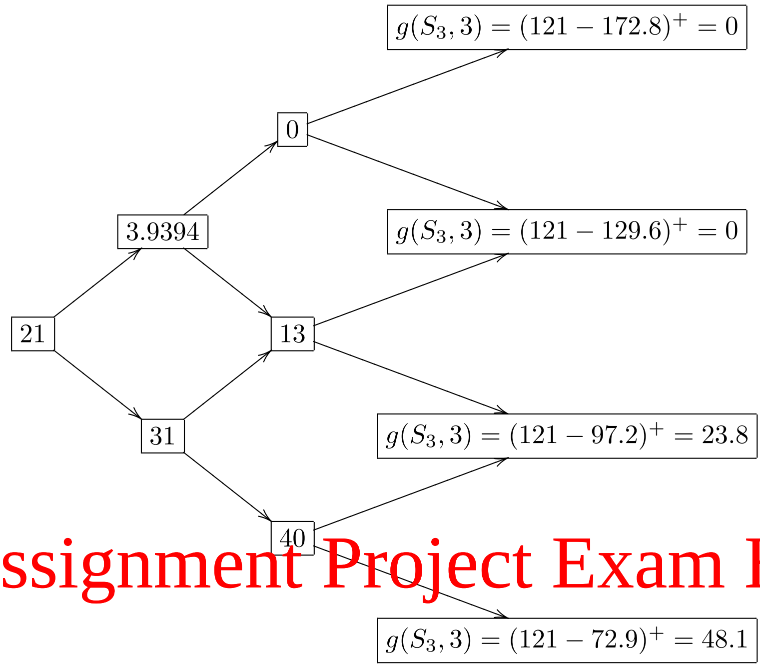
$$\big(P_T^a(\omega_1), P_T^a(\omega_2), P_T^a(\omega_3), P_T^a(\omega_4), P_T^a(\omega_5), P_T^a(\omega_6), P_T^a(\omega_7), P_T^a(\omega_8)\big) = (0, 0, 0, 23.8, 0, 23.8, 23.8, 48.1).$$

The above representation of the payoff is formally correct but not very convenient when we use the backward induction since then we start by "splitting" the model into four submodels corresponding to: $\{\omega_1, \omega_2\}$, $\{\omega_3, \omega_4\}$, $\{\omega_5, \omega_6\}$ and $\{\omega_7, \omega_8\}$, that is, to the partition generating the σ-field $\mathcal{F}_2$.

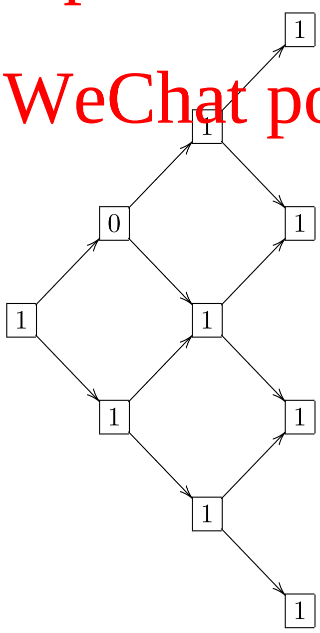
To compute the price P_t^a at times $t = 0, 1, 2$ through the risk-neutral valuation, we use the backward induction

$$P_t^a = \max\Big\{(K - S_t)^+,\ (1+r)^{-1}\,\mathbb{E}_{\widetilde{\mathbb{P}}}\big(P_{t+1}^a \,|\, \mathcal{F}_t\big)\Big\}$$

with the terminal condition $P_3^a = (K - S_3)^+ = (121 - S_3)^+$. Easy computations show that price process P_t^a can be represented by the following diagram



and the holder's rational exercise decisions are represented as follows (1 = exercise)



(b) The rational exercise times for the holder of the American put option are: $\tau_0^* = 0$,

$$\tau_1^*(\omega) = 2 \ \text{ for } \ \omega \in \{\omega_1, \omega_2, \omega_3, \omega_4\},$$
$$\tau_1^*(\omega) = 1 \ \text{ for } \ \omega \in \{\omega_5, \omega_6, \omega_7, \omega_8\},$$

$\tau_2^* = 2$ and $\tau_3^* = 3$.

(c) Suppose that the option was sold for the price $P_0^a = 21$ and it was not immediately exercised by its holder. Then the issuer may establish the replicating portfolio for the European claim $X = (3.9394, 31)$ with maturity 1 by solving the following equations

$$\begin{aligned} 1.1\,\varphi_0^0 + 120\,\varphi_0^1 &= 3.9394, \\ 1.1\,\varphi_0^0 + 90\,\varphi_0^1 &= 31. \end{aligned}$$

We find that $(\varphi_0^0, \varphi_1^0) = (101.9835, -0.90202)$ and thus the initial wealth, which is needed to establish this portfolio at time 0, equals

$$V_0(\varphi) = 101.9835 - 0.90202 \times 100 = 11.7815.$$

Hence the difference $21 - 11.7815$ is the net profit of the issuer at time 0. This argument can be extended to any date t.

Exercise 2 We consider the CRR binomial model with the risk-free rate $r = 0$ and the following values of the stock price S at times $t = 0$ and $t = 1$:

$$S_0 = 100, \quad S_1^u = 120, \quad S_1^d = 90.$$

We examine the American call option with maturity date $T = 3$ and the following reward process

$$g(S_t, t) = (S_t - K_t)^+$$

where the variable strike K_t satisfies

$$K_0 = K_1 = 100, \quad K_2 = 105, \quad K_3 = 110.$$

(a) We first compute the arbitrage price X_t^a of this option for $t = 0, 1, 2, 3$. The unique risk-neutral probability measure $\widetilde{\mathbb{P}}$ satisfies

$$\widetilde{p} = \frac{1 + r - d}{u - d} = \frac{(1 + r)S_0 - S_1^d}{S_1^u - S_1^d} = \frac{1 - d}{u - d} = \frac{1 - 0.9}{1.2 - 0.9} = \frac{1}{3}.$$

Recall that we use the following notation for sample paths: for the "upper half" of the model

$$\omega_1 = (u, u, u),\ \omega_2 = (u, u, d),\ \omega_3 = (u, d, u),\ \omega_4 = (u, d, d)$$

and for the "lower half" of the model

$$\omega_5 = (d, u, u),\ \omega_6 = (d, u, d),\ \omega_7 = (d, d, u),\ \omega_8 = (d, d, d).$$

Hence the terminal payoff C_T^a from the American call option at time $T = 3$ equals (notice that $\omega_4 = (u, d, d)$ and $\omega_5 = (d, u, u)$ so $S_T(\omega_5) = 129.6 > 97.2 = S_T(\omega_4)$)

$$\big(C_T^a(\omega_1), C_T^a(\omega_2), C_T^a(\omega_3), C_T^a(\omega_4), C_T^a(\omega_5), C_T^a(\omega_6), C_T^a(\omega_7), C_T^a(\omega_8)\big) = (62.8, 19.6, 19.6, 0, 19.6, 0, 0, 0).$$

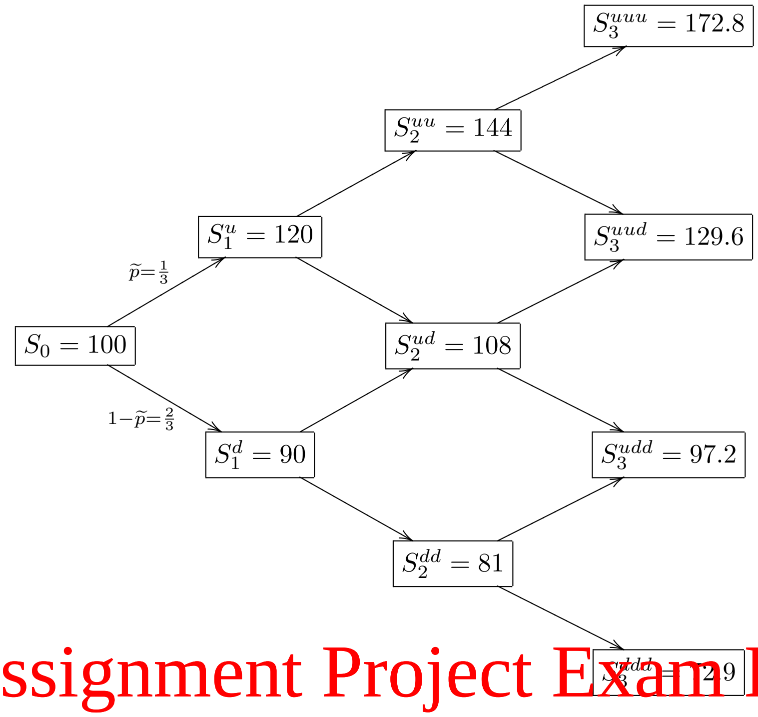
As in the previous exercise, when using the backward induction method we start by considering four submodels corresponding to: $\{\omega_1, \omega_2\}$, $\{\omega_3, \omega_4\}$, $\{\omega_5, \omega_6\}$ and $\{\omega_7, \omega_8\}$, that is, to the partition generating the σ-field $\mathcal{F}_2$.

To compute the price C_t^a at times $t = 0, 1, 2$ through the risk-neutral valuation, we use the recursive formula

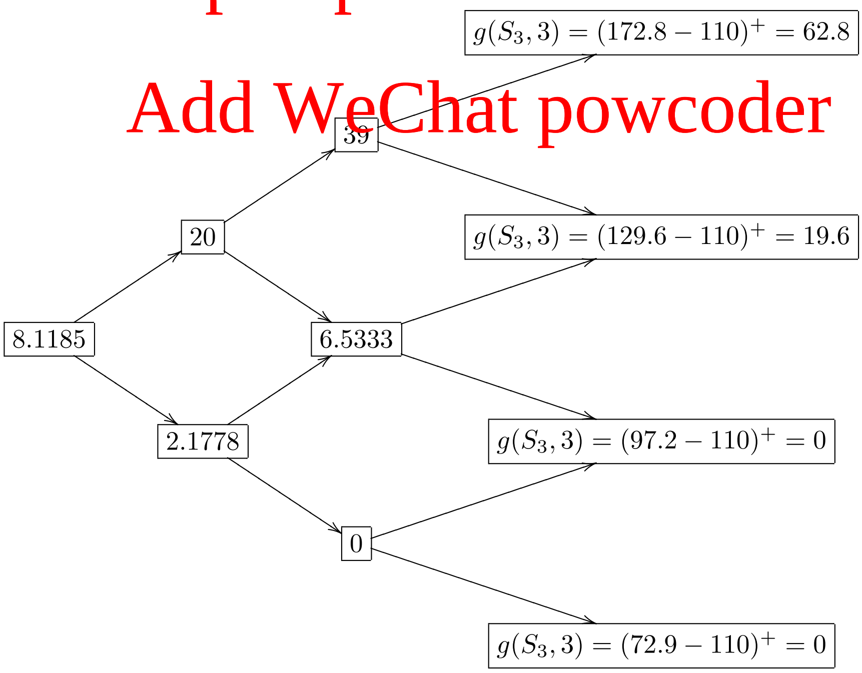
$$C_t^a = \max\Big\{(S_t - K_t)^+,\ (1 + r)^{-1}\,\mathbb{E}_{\widetilde{\mathbb{P}}}\big(C_{t+1}^a \,|\, \mathcal{F}_t\big)\Big\}$$

with the terminal condition $C_3^a = (S_3 - K_3)^+ = (S_3 - 110)^+$.

Since $u = 1.2$ and $d = 0.9$, the stock price process S_t satisfies



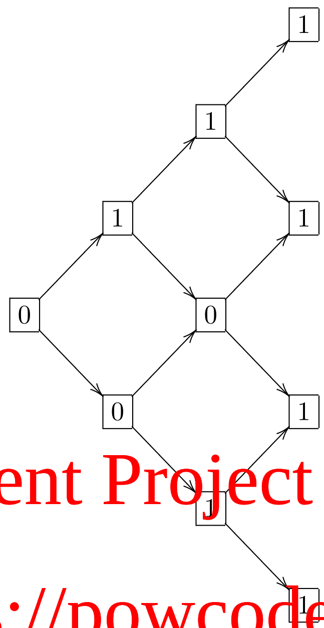
and thus the price process X_t^a of the American call option is given by



Recall that $r = 0$ and thus $B_t = 1$ for all t. It is easy to check that the process X^a is a strict supermartingale under $\widetilde{\mathbb{P}}$ since the inequality $X_t^a \geq \widetilde{p} X_{t+1}^{au} + (1-\widetilde{p}) X_{t+1}^{ad}$ is satisfied at all nodes and it is strict at some nodes. For instance, at time 1 when $S_1 = uS_0$ we obtain

$$X_1^a = 20 > (1/3)39 + (2/3)6.5333 = \widetilde{p} X_2^{au} + (1-\widetilde{p}) X_2^{ad}.$$

(b) The rational exercise decisions of the holder are given by



Hence the rational holder should exercise the American call option at time $t = 1$ whenever the stock price rises during the first period. Otherwise, he should not exercise the option till time 2. Hence the rational exercise time τ_0^* is a stopping time $\tau_0^* : \Omega \to \{0, 1, 2, 3\}$ given by

$$\begin{aligned} \tau_0^*(\omega) &= 1 \ \text{ for } \ \omega \in \{\omega_1, \omega_2, \omega_3, \omega_4\}, \\ \tau_0^*(\omega) &= 2 \ \text{ for } \ \omega \in \{\omega_7, \omega_8\}, \\ \tau_0^*(\omega) &= 3 \ \text{ for } \ \omega \in \{\omega_5, \omega_6\}. \end{aligned}$$

(c) We now take the position of the issuer of the option:

– At $t = 0$, we need to solve

$$\begin{aligned} \varphi_0^0 + 120\, \varphi_0^1 &= 20, \\ \varphi_0^0 + 90\, \varphi_0^1 &= 2.1778. \end{aligned}$$

Hence $(\varphi_0^0, \varphi_0^1) = (-51.2888, 0.5941)$ for all ωs.

– If the stock price has risen during the first period, the option is exercised by its holder. Hence we do not need to compute the strategy at time 1 for $\omega \in \{\omega_1, \omega_2\}$.

– If the stock price has fallen during the first period, we need to solve

$$\begin{aligned} \varphi_1^0 + 108\, \varphi_1^1 &= 6.5333, \\ \varphi_1^0 + 81\, \varphi_1^1 &= 0. \end{aligned}$$

Hence $(\varphi_1^0, \varphi_1^1) = (-19.599, 0.2420)$.

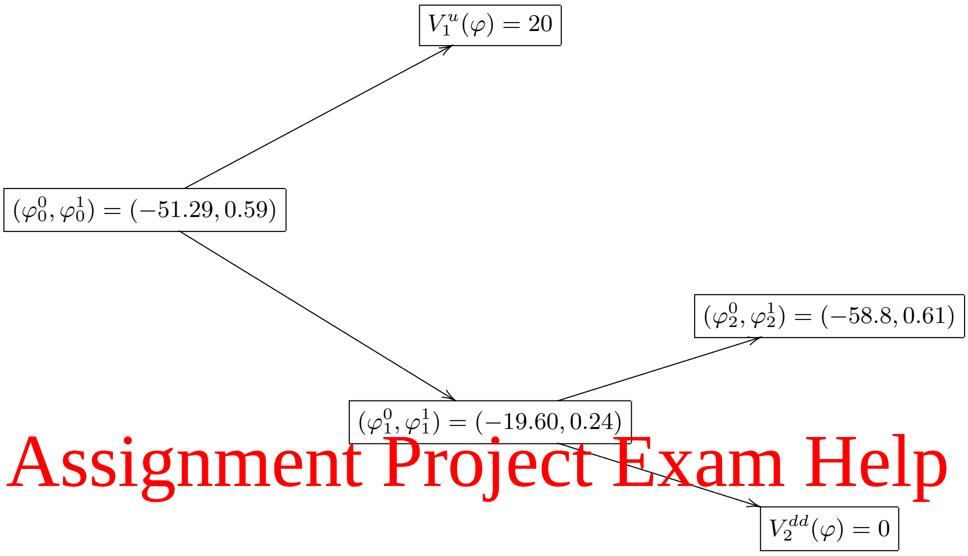
– If the stock price has fallen twice then the option is exercised and has value 0.

– If the stock price has fallen during the first period and has risen during the second period then we need to solve

$$\varphi_2^0 + 129.6\,\varphi_2^1 = 19.6,$$
$$\varphi_2^0 + 97.2\,\varphi_2^1 = 0.$$

Hence $(\varphi_1^0, \varphi_1^1) = (-58.8, 0.6050)$.

We conclude that the replicating strategy $\varphi = (\varphi^0, \varphi^1)$ is represented by the following diagram



Exercise 3 (MATH3975) We consider the European call option with strike price $K = 10$ and maturity date $T = 5$ years. We assume that the initial stock price $S_0 = 9$, the risk-free interest rate is $r = 0.01$ and the stock price volatility equals $\sigma = 0.1$ per annum.

We use the CRR parametrization for u and d with $\Delta t = 1$, that is, we set

$$u = e^{\sigma\sqrt{\Delta t}} = 1.105171, \quad d = \frac{1}{u} = 0.904837.$$

Consequently,

$$\widetilde{p} = \frac{1 + r - d}{u - d} = 0.524938.$$

Using the backward induction method, we obtain the following results.

(a) The price at time 0 of the European call option equals $C_0 = 0.5522$. The detailed computations are summarised on the next page.

(b) The price at time 0 of the European put option equals $P_0 = 1.0669$.

(c) The put-call parity at time $t = 0$ reads

$$C_0 - P_0 = 0.5522 - 1.0669 = -0.5147 = 9 - \frac{10}{(1.01)^5} = S_0 - \frac{K}{(1+r)^T}.$$

(d) The price at time 0 of the American put option equals $P_0^a = 1.2112$. The option should not be exercised at time 0, but it should be exercised by its holder at time 1 if the stock price falls during the first period. For the full description of the rational exercise decisions of the holder, see the foregoing pages.

Exercise 4 (MATH3975) Numerical results for Exercise 4 are given on the foregoing pages.

(a) Notice that the computations of the price X_t^g were done using the backward induction

$$X_t^g = \min\Big\{h(S_t,t),\, \max\big\{\ell(S_t,t),\, (1+r)^{-1}\big(\tilde{p}X_{t+1}^{gu} + (1-\tilde{p})X_{t+1}^{gd}\big)\big\}\Big\}$$

with the terminal condition $X_T^g = \pi_T(X^g) = \ell(S_T,T) = (K-S_T)^+$.

(b) Rational exercise times τ_0^* and σ_0^* can be found either using the respective expressions

$$\tau_0^* = \inf\big\{t \in \{0,1,\dots,T\} \,|\, X_t^g = \ell(S_t)\big\}$$

and

$$\sigma_0^* = \inf\big\{t \in \{0,1,\dots,T\} \,|\, X_t^g = h(S_t)\big\}$$

or when computing the price process by comparing the continuation value with respective cancellation/exercise values for the issuer and holder.

Week 10

Exercise 3

Interest	Volatility	Stock at 0	Strike price	Maturity
0.01	0.1	9	10	5

up	down	tilde p	1 - tilde p
1.105171	0.904837	0.524938	0.475062

Year	0	1	2	3	4	5
Stock price	9	9.946538	10.99262	12.14873	13.42642	14.83849
		8.143537	9	9.946538	10.99262	12.14873
			7.368577	8.143537	9	9.946538
				6.667364	7.368577	8.143537
					6.03288	6.667364
						5.458776

European call price	0.552247	0.920649	1.498351	2.357597	3.525432	4.838491
		0.156793	0.301676	0.580436	1.116781	2.148729
			0	0	0	0
				0	0	0
					0	0
						0

European put price	1.066904	0.583914	0.211627	0.011828	0	0
		1.62306	1.007577	0.436858	0.025146	0
			2.337325	1.659424	0.90099	0.053462
				3.135597	2.532413	1.856463
					3.86811	3.332636
						4.541224

Put-call parity	-0.51466

American put price	1.211179	0.650276	0.233532	0.011828	0	0
		1.856463	1.124461	0.483428	0.025146	0
			2.631423	1.856463	1	0.053462
				3.332636	2.631423	1.856463
					3.96712	3.332636
						4.541224

Exercise decision	No	No	No	No	No	Yes
		Yes	No	No	No	Yes
			Yes	Yes	Yes	Yes
				Yes	Yes	Yes
					Yes	Yes
						Yes

Week 10

Exercise 4

Interest		Stock at 0	Strike K
0.05		25	27

up	down	tilde p	1 - tilde p	alpha
1.1	0.9	0.75	0.25	0.02

Year	0	1	2	3	4	5	6
Stock price	25	27.5	30.25	33.275	36.6025	40.26275	44.28903
		22.5	24.75	27.225	29.9475	32.94225	36.23648
			20.25	22.275	24.5025	26.95275	29.64803
				18.225	20.0475	22.05225	24.25748
					16.4025	18.04275	19.84703
						14.76225	16.23848
							13.28603

Upper payoff H_t	2.02	0.02	0.02	0.02	0.02	0.02	0.02
		4.52	2.27	0.02	0.02	0.02	0.02
			6.77	4.745	2.5175	0.06725	0.02
				8.795	6.9725	4.96775	2.762525
					10.6175	8.97725	7.172975
						12.25775	10.78153
							13.73398

Lower payoff L_t	2	0	0	0	0	0	0
		4.5	2.25	0	0	0	0
			6.75	4.725	2.4975	0.04725	0
				8.775	6.9525	4.94775	2.742525
					10.5975	8.95725	7.152975
						12.23775	10.76153
							13.71398

Game	2	0.02	0.009022	0.005964	0.001683	0.000408	6.43E-05
option		4.5	2.25	0.02	0.02	0.005847	0.001519
price			6.75	4.725	2.4975	0.06725	0.02
X^g_t				8.775	6.9525	4.94775	2.742525
					10.5975	8.95725	7.152975
						12.23775	10.76153
							13.71398

Exercise	L	H	N	N	N	N	N
decision		L	L	H	H	N	N
			L	L	L	H	H
				L	L	L	L
					L	L	L
						L	L
							L

7	8	9	10	11	12
48.71793	53.58972	58.94869	64.84356	71.32792	78.46071
39.86012	43.84613	48.23075	53.05382	58.35921	64.19513
32.61283	35.87411	39.46152	43.40767	47.74844	52.52328
26.68322	29.35154	32.2867	35.51537	39.06691	42.9736
21.83173	24.0149	26.41639	29.05803	31.96383	35.16022
17.86232	19.64855	21.61341	23.77475	26.15223	28.76745
14.61463	16.07609	17.6837	19.45207	21.39728	23.537
11.95742	13.15316	14.46848	15.91533	17.50686	19.25755
	10.76168	11.83785	13.02163	14.3238	15.75618
		9.685512	10.65406	11.71947	12.89142
			8.716961	9.588657	10.54752
				7.845265	8.629791
					7.060738

0.02	0.02	0.02	0.02	0.02	0.02
0.02	0.02	0.02	0.02	0.02	0.02
0.02	0.02	0.02	0.02	0.02	0.02
0.336777	0.02	0.02	0.02	0.02	0.02
5.188272	3.0051	0.60361	0.02	0.02	0.02
9.157677	7.371445	5.40659	3.245249	0.867774	0.02
12.40537	10.94391	9.336301	7.567931	5.622724	3.482996
15.06258	13.86684	12.55152	11.10467	9.513138	7.762451
	16.25832	15.18215	13.99837	12.6962	11.26382
		17.33449	16.36594	15.30053	14.12858
			18.30304	17.43134	16.47248
				19.17474	18.39021
					19.95926

0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0.316777	0	0	0	0	0
5.168272	2.9851	0.58361	0	0	0
9.137677	7.351445	5.38659	3.225249	0.847774	0
12.38537	10.92391	9.316301	7.547931	5.602724	3.462996
15.04258	13.84684	12.53152	11.08467	9.493138	7.742451
	16.23832	15.16215	13.97837	12.6762	11.24382
		17.31449	16.34594	15.28053	14.10858
			18.28304	17.41134	16.45248
				19.15474	18.37021
					19.93926

0	0	0	0	0	0
0.00027	0	0	0	0	0
0.005572	0.001134	0	0	0	0
0.336777	0.02	0.004762	0	0	0
5.168272	2.9851	0.60361	0.02	0	0
9.137677	7.351445	5.38659	3.225249	0.847774	0
12.38537	10.92391	9.316301	7.547931	5.602724	3.462996
15.04258	13.84684	12.53152	11.08467	9.493138	7.742451
	16.23832	15.16215	13.97837	12.6762	11.24382
		17.31449	16.34594	15.28053	14.10858
			18.28304	17.41134	16.45248
				19.15474	18.37021
					19.93926

L	L	L	L	L	L
N	L	L	L	L	L
N	N	L	L	L	L
H	H	N	L	L	L
L	L	H	H	L	L
L	L	L	L	L	L
L	L	L	L	L	L
L	L	L	L	L	L
	L	L	L	L	L
		L	L	L	L
			L	L	L
				L	L
					L