MATH3075/3975
Financial Derivatives

School of Mathematics and Statistics
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Tutorial sheet 6

Background: Section 2.2 – Single-Period Market Models.

Exercise 1 Consider a single-period three-state market model $\mathcal{M} = (B, S)$ with the dates 0 and $T = 1$. We assume that there are two assets: the savings account B with the initial value $B_0 = 1$ and a risky stock with the initial price $S_0 = 4$. The risk-free simple interest rate r equals 10%. Assume that the stock price S_1 satisfies $(S_1(\omega_1), S_1(\omega_2), S_1(\omega_3)) = (8, 5, 3)$ and the real-world probability $\mathbb{P}$ satisfies $\mathbb{P}(\omega_1) = 0.3$, $\mathbb{P}(\omega_2) = 0.3$, $\mathbb{P}(\omega_3) = 0.4$.

(a) Show directly that the model $\mathcal{M} = (B, S)$ is arbitrage free, that is, no arbitrage opportunities exist in this model. Do not use here the FTAP (Theorem 2.2.1), but refer instead to Definition 2.2.3 in Course Notes.

(b) Consider the call option with the expiry date $T = 1$ and strike price $K = 4$. Examine the existence of a replicating strategy for this option.

(c) Find explicitly the class of all attainable contingent claims.

(d) Find the class $\mathbb{M}$ of all martingale measures $\mathbb{Q} = (q_1, q_2, q_3)$ on the space $\Omega = (\omega_1, \omega_2, \omega_3)$ for the model $\mathcal{M}$.

(e) Find all expected values

$$\mathbb{E}_{\mathbb{Q}}\left(\frac{(S_1 - 4)^+}{1 + r}\right)$$

where $\mathbb{Q}$ ranges over the class $\mathbb{M}$ of all risk-neutral probability measures for the model $\mathcal{M}$.

(f) (MATH3975) Find the superhedging price for X, that is, the minimal initial endowment x for which there exists a portfolio (x, ϕ) such that the inequality

$$V_1(x, \phi)(\omega) \geq (S_1(\omega) - 4)^+$$

holds for every $\omega \in \Omega$.

Exercise 2 Consider a single-period market model $\mathcal{M} = (B, S)$ on the sample space $\Omega = \{\omega_1, \omega_2, \omega_3\}$. Assume that the savings account equals $B_0 = 1$, $B_1 = 1.1$ and the stock price equals $S_0 = 5$ and

$$S_1 = (S_1(\omega_1), S_1(\omega_2), S_1(\omega_3)) = (7.7, 5.5, 4.4).$$

The real-world probability $\mathbb{P}$ is such that $\mathbb{P}(\omega_i) > 0$ for $i = 1, 2, 3$.

(a) Find the class $\mathbb{M}$ of all martingale measures for the model $\mathcal{M}$. Is this market model complete?

(b) Show that the claim $X = (X(\omega_1), X(\omega_2), X(\omega_3)) = (5.5, 3.3, 2.2)$ is attainable and compute its arbitrage price $\pi_0(X)$ using the replicating strategy for X.

(c) Consider the contingent claim $Y = (3, 1, 0)$. Show that the expected value

$$\mathbb{E}_{\mathbb{Q}}\left(\frac{Y}{B_1}\right)$$

does not depend on the choice of a martingale measure $\mathbb{Q} \in \mathbb{M}$. Is this claim attainable?

(d) Consider the contingent claim $Z = (4.4, 0, -3.3)$. Find the range of arbitrage prices

$$\pi_0(Z) = \mathbb{E}_{\mathbb{Q}}\left(\frac{Z}{B_1}\right)$$

where $\mathbb{Q} \in \mathbb{M}$. Is this claim attainable?

(e) Find the unique martingale measure $\widetilde{\mathbb{Q}}$ for the extended model $\widetilde{\mathcal{M}} = (B, S^1, S^2)$ in which $S^1 = S$ and the risky asset S^2 is defined as the claim Z traded at its initial price $\pi_0(Z) = -0.5$, that is, $S_0^2 = -0.5$ and $S_1^2 = Z$. Is the market model $\widetilde{\mathcal{M}}$ complete?

Exercise 3 (MATH3975) Let $\Omega = \{\omega_1, \omega_2\}$. We consider a single-period model $\mathcal{M} = (S^1, S^2)$ with two **risky assets** with prices S^1 and S^2 given by $S_0^1 = s_0 > 0$, $S_0^2 = z_0 > 0$ and

$$S_1^1(\omega_i) = s_i, \quad S_1^2(\omega_i) = z_i$$

for $i = 1, 2$ where $0 < s_1 < s_2$ and $0 < z_1 < z_2$. There are two traded assets, S^1 and S^2, so the wealth of a strategy ϕ equals $V_t(\phi) = \phi_t^1 S_t^1 + \phi_t^2 S_t^2$ for $t = 0, 1$.

It should be stressed that the existence of the savings account B is **not** postulated. Hence the process B should not be used at all in your solution.

(a) Under which assumptions on the (relative) values of s_0, s_1, s_2, z_0, z_1 and z_2 the model $\mathcal{M} = (S^1, S^2)$ is arbitrage-free? To answer this question in terms of some inequalities satisfied by s_0, s_1, s_2, z_0, z_1 and z_2, examine the relative wealth $\widehat{V}(\phi) = \frac{V(\phi)}{S^2}$.

(b) Assume that s_0, s_1, s_2, z_0, z_1 and z_2 are such that the model $\mathcal{M} = (S^1, S^2)$ is arbitrage-free. Check whether the model $\mathcal{M} = (S^1, S^2)$ is complete.

(c) Assume that s_0, s_1, s_2, z_0, z_1 and z_2 are such that the model $\mathcal{M} = (S^1, S^2)$ is arbitrage-free. Find the price and the replicating strategy for the contingent claim $X = (S_1^1 - S_1^2)^+$ with maturity date $T = 1$. Was it necessary to assume here that the model is complete?

(d) Find the price of the contingent claim with the payoff $Y = (S_1^2 - S_1^1)^+$ using part (c) and a suitable version of the put-call parity relationship.