

MATH3075/3975 Financial Mathematics

Tutorial 9: Solutions

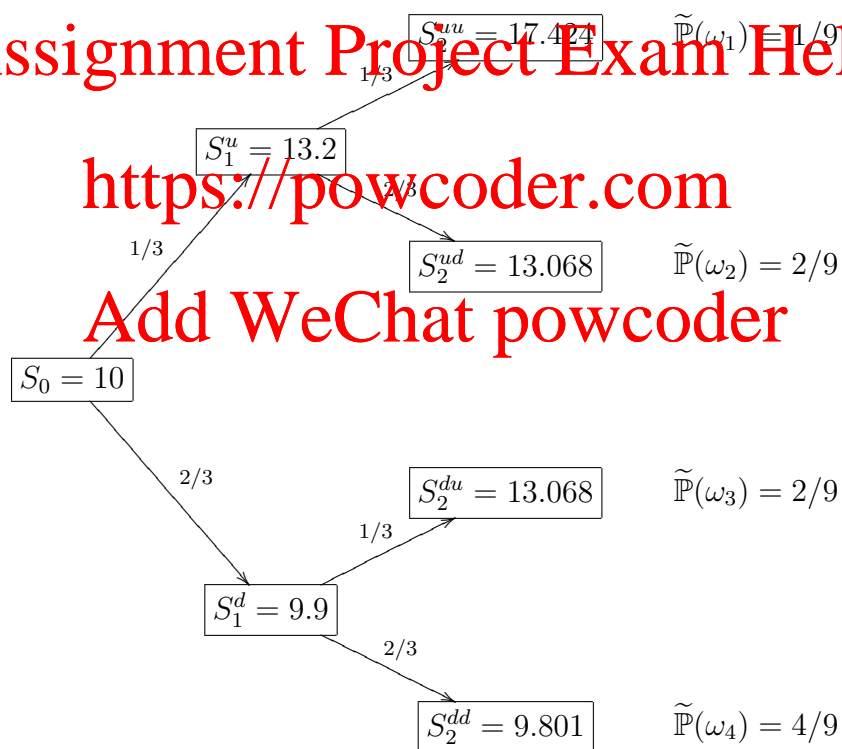
Exercise 1 Consider the CRR model $\mathcal{M} = (B, S)$ with the horizon date $T = 2$, the risk-free rate $r = 0.1$, and the following values of the stock price S at times $t = 0$ and $t = 1$:

$$S_0 = 10, \quad S_1^u = 13.2, \quad S_1^d = 9.9.$$

Let X be a European contingent claim with the maturity date $T = 2$ and the payoff

$$X = (\min(S_1, S_2) - 10)^+.$$

(a) The stock price and the martingale measure are given by the following diagram



(b) To show that the claim X is a path-dependent, it suffices to observe that

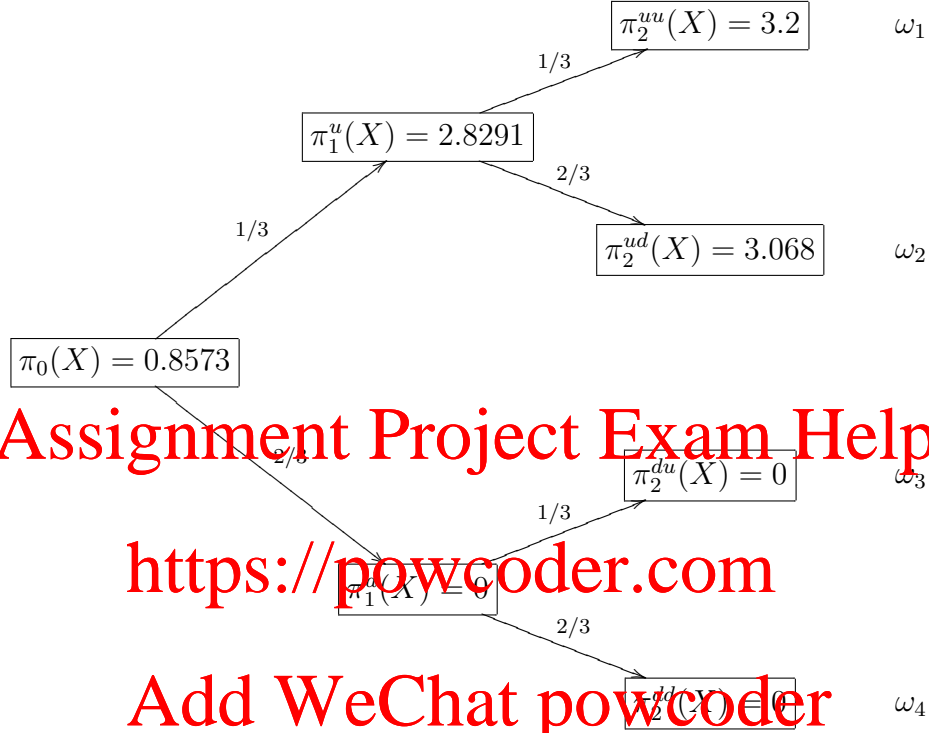
$$X = (X(\omega_1), X(\omega_2), X(\omega_3), X(\omega_4)) = (3.2, 3.068, 0, 0).$$

Although we have the same value of the stock price $S_2(\omega_2) = 13.068 = S_2(\omega_3)$, we have different values of the payoff, depending on the level of the stock price at time 1 (note that $S_1(\omega_2) = 13.2 \neq 9.9 = S_1(\omega_3)$). Hence there is no function $f: \mathbb{R} \rightarrow \mathbb{R}$ such that $X = f(S_T)$ and thus X is a path-dependent claim (a special case of the *lookback option*).

(c) Our next goal is to compute the arbitrage price of X using the risk-neutral valuation formula, for $t = 0, 1, 2$,

$$\pi_t(X) = B_t \mathbb{E}_{\tilde{\mathbb{P}}} \left(\frac{X}{B_T} \mid \mathcal{F}_t \right).$$

The price process $\pi_t(X)$ of the European claim X is given by



(d) We start by examining the replicating strategy $(\varphi_1^0, \varphi_1^1)$ at time $t = 1$.

- Assume first that the stock price has risen during the first period. Then we need to solve

$$\begin{aligned} 1.1\tilde{\varphi}_1^0 + 17.424\varphi_1^1 &= 3.2, \\ 1.1\tilde{\varphi}_1^0 + 13.068\varphi_1^1 &= 3.068. \end{aligned}$$

Hence $(\tilde{\varphi}_1^0, \varphi_1^1) = (2.4291, 0.0303)$ if the stock price has risen during the first period, that is, for $\omega \in \{\omega_1, \omega_2\}$. We check that

$$V_1^u(\varphi) = \tilde{\varphi}_1^0 + \varphi_1^1 S_1^u = 2.4291 + 0.0303 \cdot 13.2 = 2.8291 = \pi_1^u(X).$$

- Let us now assume that the stock price has fallen during the first period. Then we need to solve

$$\begin{aligned} 1.1\tilde{\varphi}_1^0 + 13.068\varphi_1^1 &= 0, \\ 1.1\tilde{\varphi}_1^0 + 9.801\varphi_1^1 &= 0. \end{aligned}$$

Hence $(\tilde{\varphi}_1^0, \varphi_1^1) = (0, 0)$ if the stock price has fallen during the first period, that is, for $\omega \in \{\omega_3, \omega_4\}$. Obviously, $V_1^d(\varphi) = 0 = \pi_1^d(X)$.

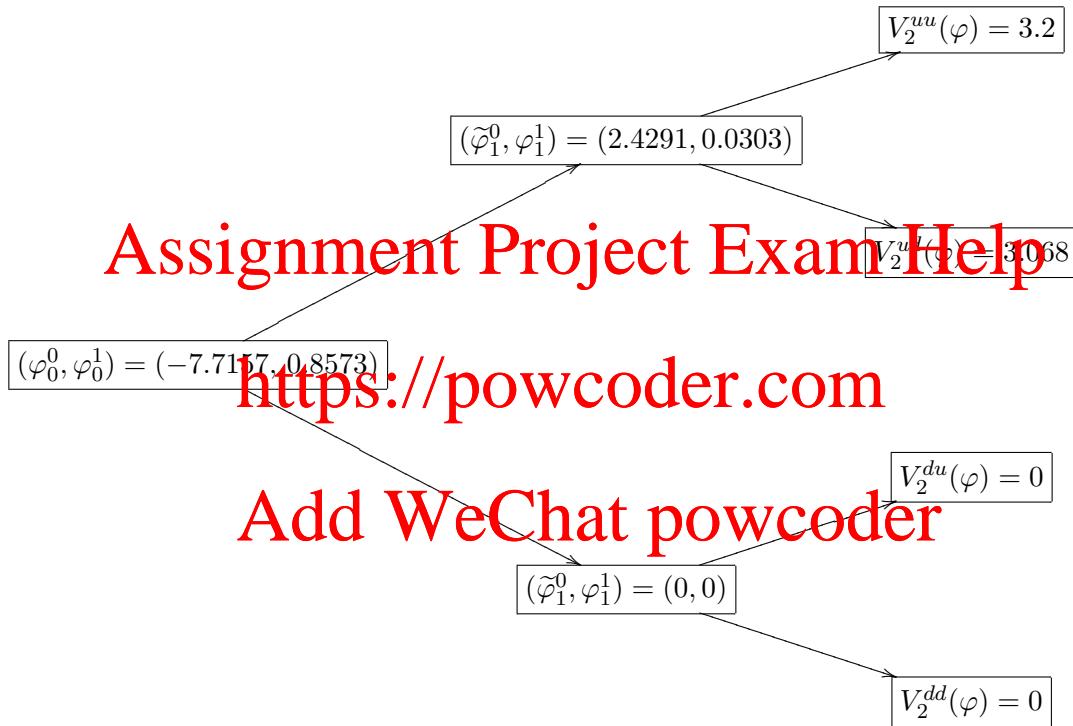
Let us find the replicating portfolio at time $t = 0$. We need to solve

$$\begin{aligned} 1.1\varphi_0^0 + 13.2\varphi_0^1 &= 2.8291, \\ 1.1\varphi_0^0 + 9.9\varphi_0^1 &= 0. \end{aligned}$$

Hence $(\varphi_0^0, \varphi_0^1) = (-7.7157, 0.8573)$ for all ω s. We check that

$$V_0(\varphi) = \varphi_0^0 + \varphi_0^1 S_0 = -7.7157 + 0.8573 \cdot 10 = 0.8573 = \pi_0(X).$$

The dynamics of the portfolio are represented by the following diagram



Exercise 2 We apply the CRR call option pricing formula

$$C_0 = S_0 \sum_{k=\hat{k}}^T \binom{T}{k} \hat{p}^k (1 - \hat{p})^{T-k} - \frac{K}{(1+r)^T} \sum_{k=\hat{k}}^T \binom{T}{k} \tilde{p}^k (1 - \tilde{p})^{T-k}$$

where \hat{k} is the smallest integer k such that

$$k > \frac{\ln\left(\frac{K}{S_0 d^T}\right)}{\ln\left(\frac{u}{d}\right)} =: \alpha_0.$$

We consider the European call option with strike price $K = 10$ and maturity date $T = 5$ years. We assume that the initial stock price $S_0 = 9$, the risk-free simple interest rate is $r = 0.01$ and the stock price volatility equals $\sigma = 0.1$ per annum. The CRR parametrization for u and d with $\Delta t = 1$ gives

$$u = e^{\sigma\sqrt{\Delta t}} = e^{\sigma} = 1.105171, \quad d = \frac{1}{u} = 0.904837.$$

Consequently,

$$\tilde{p} = \frac{1 + r - d}{u - d} = 0.524938, \quad \hat{p} = \frac{\tilde{p}u}{1 + r} = 0.574402.$$

(a) We first compute the call price at time 0. It is easy to check that $\alpha_0 = 3.02680\dots$ and thus $\hat{k} = \hat{k}(S_0, T) = 4$ and thus

$$C_0 = 9 \sum_{k=4}^5 \binom{5}{k} \tilde{p}^k (1 - \tilde{p})^{5-k} - \frac{10}{(1.01)^5} \sum_{k=4}^5 \binom{5}{k} \tilde{p}^k (1 - \tilde{p})^{5-k} = 0.552247.$$

Hence the price of the option at time 0 equals $C_0 = 0.552247$.

(b) We will now compute the prices of the option at time $t = 1$. Notice that the time to maturity is now equal to $T - 1 = 4$.

- If $S_1 = uS_0 = 9.9465$ then we obtain the new value of $\hat{k}(uS_0, T - 1) = 3$ since $\alpha_1^u = 2.02680\dots$ and thus in that case

$$C_1^u = 9.9465 \sum_{k=3}^4 \binom{4}{k} \tilde{p}^k (1 - \tilde{p})^{4-k} - \frac{10}{(1.01)^4} \sum_{k=3}^4 \binom{4}{k} \tilde{p}^k (1 - \tilde{p})^{4-k} = 0.920649.$$

- If $S_1 = dS_0 = 8.1435$ then we obtain the value of $\hat{k}(dS_0, T - 1) = 4$ since $\alpha_1^d = 3.02680\dots$ (you may check that $\alpha_1^d = \alpha_0$) and thus

$$\begin{aligned} C_1^d &= 8.1435 \sum_{k=4}^4 \binom{4}{k} \tilde{p}^k (1 - \tilde{p})^{4-k} - \frac{10}{(1.01)^4} \sum_{k=4}^4 \binom{4}{k} \tilde{p}^k (1 - \tilde{p})^{4-k} \\ &= 8.1435 (0.574402)^4 - \frac{10}{(1.01)^4} (0.524938)^4 = 0.156793. \end{aligned}$$

In Week 10, the price of the same option at any time t (in particular, for $t = 1$) will be computed using the backward induction method.

(c) The hedge ratio at time 0 equals

$$\frac{C_1^u - C_1^d}{S_1^u - S_1^d} = \frac{0.920649 - 0.156793}{9.9465 - 8.1435} = 0.423659 = \varphi_0^1$$

and thus φ_0^0 can be computed from the equality

$$C_0 = 0.552247 = \varphi_0^0 + \varphi_0^1 S_0 = \varphi_0^0 + 0.423659 \times 9.$$

We find that $\varphi_0^0 = -3.260657$, that is, we need to borrow 3.260657 units of cash at time 0.

Exercise 3 (MATH3975) (a) It is clear that $L_0 = 1$ and L is strictly positive. The martingale property of L under $\tilde{\mathbb{Q}}$ follows from the definition of $\tilde{\mathbb{Q}}$ since $L = cS/B$ where c is a constant and, by assumption, the process S/B is a martingale respect to the filtration \mathbb{F} under the probability measure $\tilde{\mathbb{Q}}$

(b) Let us denote $M = B/S$. It suffices to observe that the product $LM = L(B/S) = B_0/S_0$ is constant and thus the process LM is clearly is a martingale with respect to the filtration \mathbb{F} under the probability measure $\tilde{\mathbb{Q}}$. From part (c) in Exercise 4 in Week 7, we deduce that the process $M = B/S$ is a martingale with respect to the filtration \mathbb{F} under $\tilde{\mathbb{Q}}$.

(c) Once again, from part (b) in Exercise 4 in Week 7, we know that the equality

$$\mathbb{E}_{\tilde{\mathbb{Q}}}(Y | \mathcal{F}_t) = (L_t)^{-1} \mathbb{E}_{\tilde{\mathbb{Q}}}(Y L_s | \mathcal{F}_t). \quad (1)$$

holds for any \mathcal{F}_s -measurable random variable Y . Recall that the Radon-Nikodým density L satisfies

$$L_t = \frac{B_0}{S_0} \frac{S_t}{B_t}, \quad L_s = \frac{B_0}{S_0} \frac{S_s}{B_s}.$$

Since S_s is \mathcal{F}_s -measurable, if X is any \mathcal{F}_s -measurable random variable X , then the random variable X/S_s is \mathcal{F}_s -measurable as well.

Therefore, by applying (1) to the random variable $Y = X/S_s$, we obtain for

$$\mathbb{E}_{\tilde{\mathbb{Q}}}\left(\frac{X}{S_s} \middle| \mathcal{F}_t\right) = \frac{S_t}{B_t} \mathbb{E}_{\tilde{\mathbb{Q}}}\left(\frac{L_s X}{S_s} \middle| \mathcal{F}_t\right)$$

or, more explicitly,

$$\mathbb{E}_{\tilde{\mathbb{Q}}}\left(\frac{X}{S_s} \middle| \mathcal{F}_t\right) = \frac{S_t}{B_t} \frac{B_0}{S_0} \frac{S_s}{B_s} \mathbb{E}_{\tilde{\mathbb{Q}}}\left(\frac{B_0 S_s X}{S_0 B_s S_s} \middle| \mathcal{F}_t\right).$$

This in turn yields

$$S_t \mathbb{E}_{\tilde{\mathbb{Q}}}\left(\frac{X}{S_s} \middle| \mathcal{F}_t\right) = B_t \mathbb{E}_{\tilde{\mathbb{Q}}}\left(\frac{X}{B_s} \middle| \mathcal{F}_t\right). \quad (2)$$

as was required to show.

(d) The payoff C_T of the call option can be decomposed as follows

$$C_T = S_T \mathbf{1}_D - K \mathbf{1}_D = X_1 - X_2.$$

Since C_T is assumed to be an attainable claim, its arbitrage price can be computed using any martingale measure for the process S/B , that is, for the model \mathcal{M} . Hence we may take the martingale measure $\tilde{\mathbb{Q}} \in \mathbb{M}$ and compute the arbitrage price of the option using the risk-neutral valuation formula

$$C_t = B_t \mathbb{E}_{\tilde{\mathbb{Q}}}\left(\frac{C_T}{B_T} \middle| \mathcal{F}_t\right).$$

Using the additivity property of conditional expectation and part (c) (specifically, equation (2) applied to X_1), we obtain

$$\begin{aligned}
C_t &= B_t \mathbb{E}_{\tilde{\mathbb{Q}}} \left(\frac{C_T}{B_T} \middle| \mathcal{F}_t \right) = B_t \mathbb{E}_{\tilde{\mathbb{Q}}} \left(\frac{X_1}{B_T} \middle| \mathcal{F}_t \right) - B_t \mathbb{E}_{\tilde{\mathbb{Q}}} \left(\frac{X_2}{B_T} \middle| \mathcal{F}_t \right) \\
&\stackrel{(c)}{=} S_t \mathbb{E}_{\tilde{\mathbb{Q}}} \left(\frac{X_1}{S_T} \middle| \mathcal{F}_t \right) - B_t \mathbb{E}_{\tilde{\mathbb{Q}}} \left(\frac{X_2}{B_T} \middle| \mathcal{F}_t \right) \\
&= S_t \mathbb{E}_{\hat{\mathbb{Q}}} \left(\frac{S_T \mathbf{1}_D}{S_T} \middle| \mathcal{F}_t \right) - B_t \mathbb{E}_{\tilde{\mathbb{Q}}} \left(\frac{K \mathbf{1}_D}{B_T} \middle| \mathcal{F}_t \right) \\
&= S_t \hat{\mathbb{Q}}(D | \mathcal{F}_t) - K B_t (B_T)^{-1} \tilde{\mathbb{Q}}(D | \mathcal{F}_t) \\
&= S_t \hat{\mathbb{Q}}(D | \mathcal{F}_t) - K B(t, T) \tilde{\mathbb{Q}}(D | \mathcal{F}_t)
\end{aligned}$$

since K and B_T are deterministic and, obviously,

$$\mathbb{E}_{\hat{\mathbb{Q}}}(\mathbf{1}_D | \mathcal{F}_t) = \hat{\mathbb{Q}}(D | \mathcal{F}_t), \quad \mathbb{E}_{\tilde{\mathbb{Q}}}(\mathbf{1}_D | \mathcal{F}_t) = \tilde{\mathbb{Q}}(D | \mathcal{F}_t).$$

(e) It suffices to argue that the payoff $X_1 = 1$ at time T can be replicated by investing $\alpha = B_t/B_T$ units of cash at time t in the asset B and keeping the portfolio constant till time T . Then the wealth of the portfolio at time T will be $\alpha B_T/B_t = 1$. This means that the arbitrage price $B(t, T)$ of the unit zero-coupon bond at time t satisfies $B(t, T) = B_t/B_T$.

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