MATH3075/3975
Financial Derivatives

School of Mathematics and Statistics
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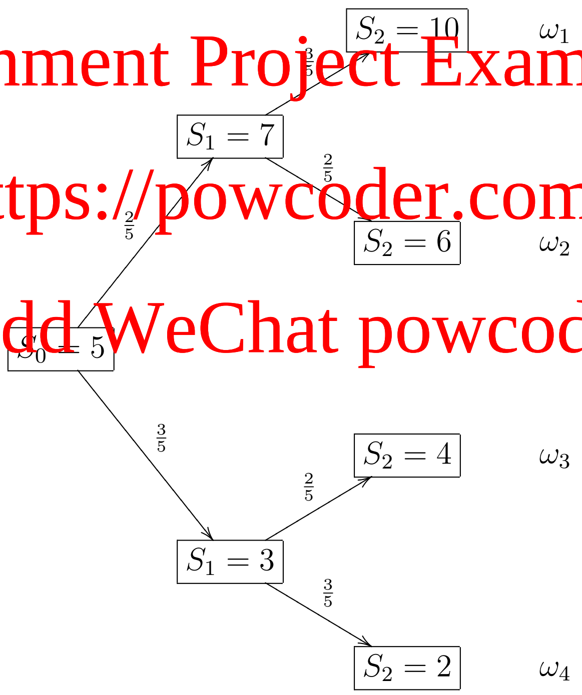
Semester 2, 2020

Tutorial sheet 8

Background: Chapter 3 – Multi-Period Market Models.

Exercise 1 We consider the two-period market model $\mathcal{M} = (B, S)$ with the savings account $B_t = (1+r)^t$ where the interest rate $r = 0.1$. The stock price process S is represented under $\mathbb{P}$ by the following diagram



(a) Find the risk-neutral probability measure $\mathbb{Q}$ for the model $\mathcal{M} = (B, S)$.

(b) Find the replicating strategy for the **digital call option** with strike $K = 8$ and maturity $T = 2$, that is, for the payoff X given by

$$X = h(S_2) = \begin{cases} 1, & \text{if } S_2 \geq 8, \\ 0, & \text{otherwise.} \end{cases}$$

Find the arbitrage price process $\pi_t(X)$ for $t = 0, 1, 2$.

(c) Compute the arbitrage price process for the *Asian option* with the payoff at maturity $T = 2$ given by the following formula

$$Y = \left(\frac{1}{3}\left(S_0 + S_1 + S_2\right) - 4\right)^+.$$

Exercise 2 Consider the CRR model with $T = 2$ and $S_0 = 80$, $S_1^u = 104$, $S_1^d = 88$. Assume that the interest rate $r = 0.2$. Consider a European contingent claim X maturing at $T = 2$ with the payoff given by the formula

$$X = (S_2 - S_1)\mathbb{1}_{\{S_2 - S_1 > 20\}} = \begin{cases} S_2 - S_1, & \text{on the event } \{S_2 - S_1 > 20\}, \\ 0, & \text{on the event } \{S_2 - S_1 \le 20\}. \end{cases}$$

(a) Show explicitly that the contingent claim X is path-dependent.

(b) Find the risk-neutral probability measure $\widetilde{\mathbb{P}}$ for the model $\mathcal{M} = (B, S)$ and compute the arbitrage price of X using the risk-neutral valuation formula

$$\pi_t(X) = B_t\,\mathbb{E}_{\widetilde{\mathbb{P}}}\left(X B_T^{-1} \mid \mathcal{F}_t\right), \quad t = 0, 1, 2.$$

(c) Find the replicating portfolio (ϕ^0, ϕ^1) for the claim X and check that the equality $V_t(\phi) = \pi_t(X)$ is satisfied for $t = 0, 1, 2$.

(d) Show that in any CRR model we have that $\mathbb{E}_{\widetilde{\mathbb{P}}}(S_2 - S_1) = r(1 + r)S_0$. Let $Y = (S_2 - S_1)\mathbb{1}_{\{S_2 - S_1 \le 20\}}$. Find the price of Y at time 0 using the additivity of arbitrage prices and the fact that $X + Y = S_2 - S_1$. Confirm your result by computing

$$\pi_0(Y) = B_0\,\mathbb{E}_{\widetilde{\mathbb{P}}}\left(Y (B_2)^{-1}\right).$$

(e) Find the unique probability measure $\widehat{\mathbb{P}}$ on $(\Omega, \mathcal{F}_2)$ such that the process $\widehat{B}_t := B_t / S_t$, $t = 0, 1, 2$ is a martingale under $\widehat{\mathbb{P}}$ with respect to the filtration $\mathbb{F} = (\mathcal{F}_t)_{t=0,1,2}$ and check that $\pi_0(Y) = S_0\,\mathbb{E}_{\widehat{\mathbb{P}}}\left(Y (S_2)^{-1}\right)$.

Exercise 3 (MATH3975) We consider a discrete-time stochastic process $X = (X_t,\, t = 0, 1, \dots)$ defined on a finite probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \ge 0}$. It is assumed throughout that a process X is adapted to the filtration $\mathbb{F}$, that is, X is $\mathbb{F}$-adapted.

(a) Assume that X has independent increments with respect to $\mathbb{F}$, meaning that for any $t = 0, 1, \dots$ the increment $X_{t+1} - X_t$ is independent of the σ-field $\mathcal{F}_t$. Show that the process Y, which is given by the following expression

$$Y_t := X_t - \mathbb{E}_{\mathbb{P}}(X_t), \quad t = 0, 1, \dots,$$

is a martingale under $\mathbb{P}$ with respect to the filtration $\mathbb{F}$.

(b) Let $A_0 = 0$ and for $t = 0, 1, \dots$

$$A_{t+1} - A_t = \mathbb{E}_{\mathbb{P}}(X_{t+1} - X_t \,|\, \mathcal{F}_t). \tag{1}$$

(b1) Verify that the process $\widetilde{Y}$ given by the equality $\widetilde{Y}_t := X_t - A_t$ for $t = 0, 1, \dots$ is a martingale under $\mathbb{P}$.

(b2) We assume that the process $\widehat{Y}_t := X_t - \widehat{A}_t$ for $t = 0, 1, \dots$ is a martingale under $\mathbb{P}$ where the process $\widehat{A}$ satisfies: $\widehat{A}_0 = 0$ and $\widehat{A}_{t+1}$ is $\mathcal{F}_t$-measurable for every $t = 0, 1, \dots$ (we then say that the process $\widehat{A}$ is $\mathbb{F}$-*predictable*). Show that $\widehat{A} = A$ where the process A is given by formula (1) with $A_0 = 0$.

Comment: In parts (b1)-(b2) we have shown that if a process X is $\mathbb{F}$-adapted, then there exists a unique $\mathbb{F}$-predictable process A with $A_0 = 0$ such that the process $\widetilde{Y} = X - A$ is a martingale under $\mathbb{P}$.

(c) Assume that a process $X = (X_t, t = 0, 1, \dots, T)$ represents a *gamble*, meaning here that if the game is played at time t then the (positive or negative) reward at time $t+1$ per one unit of the bet equals $X_{t+1} - X_t$.

The random size of the bet is given by an arbitrary $\mathbb{F}$-adapted process H called a *gambling strategy*. The profits/losses after t rounds of the game when a gambling strategy H is followed are given by the following equality (by convention, $G_0 = 0$)

$$G_t := \sum_{u=0}^{t-1} H_u (X_{u+1} - X_u).$$

Note that one does not pay any fee for the right to play the game X. By definition, we then say that the game X is *fair* if there is no gambling strategy H such that $\mathbb{E}_{\mathbb{P}}(G_t) \neq 0$ for some $t \leq T$.

(c1) Show that the game is fair if and only if X is a martingale under $\mathbb{P}$ with respect to the filtration $\mathbb{F}$.

(c2) Consider an arbitrary $\mathbb{F}$-adapted process X. Argue that the corresponding game will become a fair game if the player is required to pay at time t the fee $A_{t+1} - A_t$ per one unit of the bet where A is the unique $\mathbb{F}$-predictable process with $A_0 = 0$ that satisfies equality (1).