MATH3411 INFORMATION, CODES & CIPHERS

Test 1 2017 S2 SOLUTIONS

Version A

Multiple choice: **b**, **b**, **c**, **c**, **a**, **e**, **e**, **c**, **c**, **b**

1. **(b)**:

2. **(b)**: Calculate the Hamming distance between the $\binom{4}{2} = 6$ pairs of codewords: 2,2,3,3,3,4. The smallest of these is 2.

3. **(c)**: There are $2^2 = 4$ codewords, there being 2 information bits.

4. **(c)**: The codeword is 10101; it has weight 3.

5. **(a)**: The two rows of this matrix form a basis for C (they are codewords, linearly independent, and there are $k = 2$ of them).

6. **(e)**: We can only correct up to $t = 3$ errors.

7. **(d)**: Rule out cases: **(a)** $100 = \mathbf{c}_1\mathbf{c}_4 = \mathbf{c}_3$; **(b)** $11 = \mathbf{c}_2 = \mathbf{c}_4\mathbf{c}_4$; **(c)** $10011 = \mathbf{c}_1\mathbf{c}_4 = \mathbf{c}_3\mathbf{c}_2$; **(d)** $111111 = \mathbf{c}_2\mathbf{c}_2\mathbf{c}_2 = \mathbf{c}_4\mathbf{c}_4$.

8. **(c)**: The Kraft-McMillan number $K = \sum \frac{1}{2^{\ell_i}}$ must be at most 1 for UD codes. Testing values of $\ell = 1, 2, 3, \ldots$ gives us that $\ell = 3$ is the minimum length that satisfies this. You can also draw a decision tree.

9. **(c)**: The codewords are 0, 100, 101, 102, 110; that last one is $\mathbf{c}_5$.

10. **(b)**: $s_2 s_3 s_3 s_1 s_1$. Start with Huff_E and recognise the codeword $\mathbf{c}_2 = 00$; therefore, next use Huff_E and so on.

11. (a) The Kraft-McMillan number is

$$K = \frac{2}{3} + \frac{3}{3^2} + \frac{1}{3^3} = \frac{28}{27} > 1$$

so there is no UD-code.

(b) Using [and drawing the steps of] the Huffman algorithm, we find that $s_1 s_1 \mapsto 0$, $s_1 s_2 \mapsto 11$, $s_2 s_1 \mapsto 100$, $s_2 s_2 \mapsto 101$.
The average length per original source symbol is
$\frac{1}{2}\left(\frac{5}{25} + \frac{9}{25} + 1\right) = \frac{39}{50}$ by Knuth's Lemma.

Version B

Multiple choice: **c**, **d**, **e**, **d**, **d**, **d**, **e**, **b**, **e**, **e**

1. **(c)**: The error lies in the 3rd row and 5th column of the block

$$\begin{matrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \end{matrix}$$

2. **(d)**: 1, 2, or 3 errors, and $\binom{12}{3} = 220$.

3. **(e)**:

4. **(d)**: $3^3 = 27$ linear combinations of the rows and thus that many codewords.

5. **(d)**: 1st row + twice 2nd row + 3rd row = 1112001

6. **(d)**: Since $d = 2t + 2$, we see that $t = 1$. The (binary) Sphere-Packing Theorem asserts that $|C| \sum_{i=0} \binom{n}{i} \leq 2^n$, which here implies that $2^k(1 + 7) \leq 2^7$, or in other words, $2^{k+3} \leq 2^7$. The largest value of k which satisfies this inequality is $k = 4$, and indeed, the code with basis $\{1000111, 0100111, 0010110, 0001101\}$ is such a code.

7. **(e)**: Trial and error. None of the four words are suitable: $101 = c_1 c_3 = c_2 c_1$.

8. **(b)**: The Kraft-McMillan number $K = \sum_i \frac{1}{\ell^i}$ must be at most 1 for UD codes. Testing values of $\ell = 1, 2, 3, \ldots$ gives us that $\ell = 3$ is the minimum length that satisfies this. You could also draw a decision tree.

9. **(e)**:

10. **(e)**: Ternary, one dummy symbol, divide by 2.

11. (a) $\mathbf{p} = M\mathbf{p}$
    (b) Draw Huffman algorithm decision tree. $L = \frac{7}{5}$
    (c) 01101100