# PHIL1012 Lecture 10:
# Functional Completeness, Pt. 2

## Question

- Is $\{\neg, \wedge, \vee, \rightarrow, \leftrightarrow\}$ functionally complete?
- Given any truth table, is there a formula of PL with that truth table?
- Can the connectives of PL be used to define all possible connectives?



We will start by showing that the set $\{\neg, \wedge, \vee\}$ is functionally complete.

This of course implies that the set $\{\neg, \wedge, \vee, \rightarrow, \leftrightarrow\}$ is functionally complete as well. (If we can define every connective using $\neg$, $\wedge$, and $\vee$, then we can define every connective using $\neg$, $\wedge$, $\vee$, $\rightarrow$, and $\leftrightarrow$.)



Defining 0-place connectives using only $\neg$, $\wedge$, and $\vee$.

| $A$ | $\top$ | $\bot$ | $A \vee \neg A$ | $A \wedge \neg A$ |
|---|---|---|---|---|
| T | T | F | T | F |
| F | T | F | T | F |

## Defining 1-place connectives using only $\neg$, $\wedge$, and $\vee$.

| $\alpha$ | $\textcircled{1}_1\,\alpha$ | $\textcircled{1}_2\,\alpha$ | $\neg\alpha$ | $\textcircled{1}_4\,\alpha$ |
|---|---|---|---|---|
| T | T | T | F | F |
| F | T | F | T | F |
| | $\top$ | $\alpha \vee \alpha$ | | $\bot$ |
| | $A \vee \neg A$ | $\neg\neg\alpha$ | | $A \wedge \neg A$ |
| | | $\alpha$ | | |



## Defining $\veebar$ using only $\neg$, $\wedge$, and $\vee$.



| $\alpha$ | $\beta$ | $\alpha \veebar \beta$ | row description |
|---|---|---|---|
| T | T | F | |
| T | F | T | $\alpha \wedge \neg\beta$ |
| F | T | T | $\neg\alpha \wedge \beta$ |
| F | F | F | |



| $\alpha$ | $\beta$ | $(\alpha \wedge \neg\beta) \vee (\neg\alpha \wedge \beta)$ |
|---|---|---|
| T | T | F |
| T | F | T |
| F | T | T |
| F | F | F |

## Defining *any* connective using only $\neg$, $\wedge$, and $\vee$.

| $\alpha$ | $\beta$ | $\gamma$ | $*(\alpha,\beta,\gamma)$ | row description |
|---|---|---|---|---|
| T | T | T | F | |
| T | T | F | F | |
| T | F | T | T | $\alpha \wedge \neg\beta \wedge \gamma$ ← |
| T | F | F | F | |
| F | T | T | F | |
| F | T | F | F | |
| F | F | T | T | $\neg\alpha \wedge \neg\beta \wedge \gamma$ ← |
| F | F | F | T | $\neg\alpha \wedge \neg\beta \wedge \neg\gamma$ |



$(\alpha \wedge \neg\beta \wedge \gamma) \vee (\neg\alpha \wedge \neg\beta \wedge \gamma) \vee (\neg\alpha \wedge \neg\beta \wedge \neg\gamma)$





We have shown that $\boxed{\{\neg, \wedge, \vee\}}$ is a functionally complete set of connectives!

Can we find other, smaller functionally complete sets of connectives?

$\{\neg, \wedge, \vee\}$

Defining $\vee$ using only $\neg$ and $\wedge$

Defining $\wedge$ using only $\neg$ and $\vee$

| $\alpha$ | $\beta$ | $\alpha \vee \beta$ | $\neg(\neg\alpha \wedge \neg\beta)$ | $\alpha \wedge \beta$ | $\neg(\neg\alpha \vee \neg\beta)$ |
|---|---|---|---|---|---|
| T | T | T | T | T | T |
| T | F | T | T | F | F |
| F | T | T | T | F | F |
| F | F | F | F | F | F |



So, both $\{\neg, \wedge\}$ and $\{\neg, \vee\}$ are functionally complete as well!

① $\{\neg, \wedge, \vee\}$
② $\{\neg, \wedge\}$
③ $\{\neg, \vee\}$

are functionally complet

## Important Fact

Suppose $X$ is a functionally complete set of connectives and $Y$ is another set of connectives. Then, if every connective in $X$ can be defined using only connectives in $Y$, then $Y$ is functionally complete as well.



**Example** $\boxed{X = \{\neg, \wedge, \vee\}}$, $\boxed{Y = \{\neg, \wedge\}}$



Can we find even smaller functionally comple sets? For instance ...

Is $\{\vee\}$ functionally complete?

No! But how to show it?



In general, to show that a set of connectives is not functionally comple you must find a connective that cannot be defined using only the connectives in the set.

So let's show that ¬ cannot be

So, let's show that $\neg$ cannot be defined using only $\vee$.

Start by observing that $\neg$ has the following property:

⊛ For any basic proposition $P$, $\neg P$ is false in the first row of its truth table.

| $P$ | $\neg P$ |
|---|---|
| T | F |
| F | T |



Does any formula $\alpha$ of $P$ that contain only $\vee$ have the property ⊛?

Of course not! No matter how complex $\alpha$ is, e.g.

$$\alpha = ((P \vee Q) \vee (P \vee P)) \vee R,$$

if it contains only $\vee$, it will be true in the first row of its truth table, unlike $\neg P$.

So, it is impossible to define $\neg$ using only $\vee$.

That is, $\{\vee\}$ is not functionally complete!



Is there any functionally complete set that contains only one connective?

Yes! Consider...

| $\alpha$ | $\beta$ | $\alpha \downarrow \beta$ | $\neg(\alpha \vee \beta)$ |
|---|---|---|---|
| T | T | F | F |
| T | F | F | F |
| F | T | F | F |
| F | F | T | T |



- To show that $\{\downarrow\}$ is functionally complete, we use the Important Fact from above. We define $\neg$ and $\vee$ using only $\downarrow$.





"is equivalent to"

$\alpha \downarrow \beta \equiv \neg(\alpha \vee \beta)$

Defining $\neg$ using $\downarrow$:

$\neg\alpha \equiv \neg(\alpha \vee \alpha)$

$\equiv \alpha \downarrow \alpha$

Defining $\vee$ using $\downarrow$

$\alpha \vee \beta \equiv \neg\neg(\alpha \vee \beta)$

$\equiv \neg(\alpha \downarrow \beta)$

$\equiv (\alpha \downarrow \beta) \downarrow (\alpha \downarrow \beta)$

## Summary

① We showed that $\{\neg, \wedge, \vee\}$ (and hence $\{\neg, \wedge, \vee, \rightarrow, \leftrightarrow\}$) is functionally complete

$\{\neg, \wedge, \vee, \rightarrow, \leftrightarrow\}$) is functionally complete using the "row description method".

② We showed that $\{\neg, \wedge\}$, $\{\neg, \vee\}$, and $\{\downarrow\}$ are truth functionally complete using the <u>Important Fact</u> above.

③ We showed that $\{\vee\}$ is <u>not</u> truth functionally complete.