PHIL1012 Lecture 14: Semantics of MPL, Pt. 2

- $\underline{P}\,\underline{a}$ is true in M iff the referent of $\underline{a}$ in M is in the extension of $\underline{P}$ in M.
- $\forall \underline{x}\, \underline{P}\underline{x}$ is true in M iff every object in the domain of M is in the extension of $\underline{P}$.
- $\exists \underline{x}\, \underline{P}\underline{x}$ is true in M iff some object in the domain of M is in the extension of $\underline{P}$.
- Truth conditions for connectives can be read off truth tables.

Plan

(I) Truth values of general quantified propositions

(II) Analyses of logical concepts

I

Domain : { Alice , Bill , Caroline }

P : { Alice, Bill } , L : { Bill, Caroline }

D : { Alice, Bill, Caroline }

① $\exists y\,(P_y \wedge L_y)$ T

③ $\forall x\,((D_x \wedge L_x) \rightarrow P_x)$ F

② $\forall x\,(L_x \rightarrow D_x)$ T

General idea We want a rule for determining truth values of complex propositions like

$$\alpha = \forall x\,((D_x \wedge L_x) \rightarrow P_x)$$

in a model M.

- Drop quantifier from α

- Replace free x with a new name
- Check whether resulting proposition is true in every model like M that assigns a referent to the new name

Terminology

- $\alpha(\underline{x})$ is an arbitrary wff that has no free variable other than $\underline{x}$.

- $\alpha(\underline{a}/\underline{x})$ is the wff obtained from $\alpha(\underline{x})$ by replacing all free occurrences of $\underline{x}$ in $\alpha(\underline{x})$ with the name $\underline{a}$.

Examples

Suppose $\alpha(x)$ is $\overset{\alpha(x)}{\boxed{Fx \rightarrow Gx}}$. What are ... ?

- $\alpha(a/x)$: $Fa \rightarrow Ga$

$\odot\ \alpha(b/x): Fb \rightarrow Gb$

$\alpha(y): (Fy \vee Gb) \leftrightarrow Hy$

$\alpha(a/y): (Fa \vee Gb) \leftrightarrow Ha$

$\alpha(b/y): \underline{(Fb \vee Gb)} \leftrightarrow Hb$

$\alpha(z): \forall x((Gx \wedge Ha) \rightarrow Fz)$

$\alpha(a/z): \forall x((Gx \wedge Ha) \rightarrow Fa)$

$\alpha(a/z): \forall x((Gx \wedge Ha) \rightarrow Fa)$

$\alpha(r/z): \forall x((Gx \wedge Ha) \rightarrow Fr)$

How to determine whether $\forall \underline{x}\, \alpha(\underline{x})$ is true in $\mathcal{M}$?

- Start with $\alpha(\underline{x})$

- Introduce a new name $\underline{r}$, i.e. $\underline{r}$ is not assigned a referent in $\mathcal{M}$

- Consider $\alpha(\underline{r}/\underline{x})$: Is it true in every model that is exactly like $\mathcal{M}$ except that it assigns $\underline{r}$ a referent?

- $\forall \underline{x}\, \alpha(\underline{x})$ is true in $\mathcal{M}$ iff answer is "Yes."

$\mathcal{M}$

Domain: {Alice, Bill, Caroline}
P: {Alice} F: {Alice, Bill}

Is $\forall x(Px \to Fx)$ true in $\mathcal{M}$? YES!

- $\alpha(x)$: $Px \to Fx$
- New name: a
- $\alpha(a/x)$: $\boxed{Pa \to Fa}$

Domain: {Alice, Bill, Caroline} P: {Alice} F: {Alice, Bill} a: Alice	$\mathcal{M}^a_{Alice}$	$\boxed{Pa \to Fa}$ T T T
Domain: {Alice, Bill, Caroline} P: {Alice} F: {Alice, Bill} a: Bill	$\mathcal{M}^a_{Bill}$	$Pa \to Fa$ F T
Domain: {Alice, Bill, Caroline} P: {Alice} F: {Alice, Bill}	$\mathcal{M}^a_{Caroline}$	$Pa \to Fa$

a : Caroline

Caroline F T

Terminology Given a model M, an object o in the domain of M, and a name a that is not assigned a referent in M, let

$$\boxed{M^{a}_{o}}$$

be the model that is exactly like M except that it assigns a the referent o.

(Examples above)

- $\forall \underline{x}\, \alpha(\underline{x})$ is true in M iff for every object o in the domain of M, $\alpha(\underline{a}/\underline{x})$ is true in M^{a}_{o}, where $\underline{a}$ is a name that isn't assigned a referent in M.

at least one

- $\exists \underline{x}\, \alpha(\underline{x})$ is true in M iff there is some object o in the domain of M such that $\alpha(\underline{a}/\underline{x})$ is true in $M^{\underline{a}}_{o}$, where $\underline{a}$ is a name that isn't assigned a referent in M.

Examples $\exists x\,(Gx \wedge Hx \wedge \neg Fx)$

M

Domain: $\{1,2,3,4,5\}$

G: $\{1,3\}$ H: $\{3,4,5\}$

F: $\{2,5\}$

- $\alpha(x)$: $Gx \wedge Hx \wedge \neg Fx$
- New name: a

Domain: $\{1,2,3,4,5\}$

G: $\{1,\underline{3}\}$ H: $\{\underline{3},4,5\}$

F: $\{2,5\}$ a: 3

• $\alpha(a|x)$: $\underline{Ga \wedge Ha \wedge \neg}$

M^a_3 Because $\alpha(a|x)$ is true in M^a_3,

$\exists x(Gx \wedge Hx \wedge \neg Fx)$ is true in M.

M

Domain: $\{1,2,3,4,\ldots\}$

P: $\{2,4,6,\ldots\}$ a: 2

Q: $\{1,3,5,\ldots\}$

R: $\{2,4,6,\ldots\}$

II Analyses of logical concepts

- An argument is valid iff there is no model in which its premises are true and its conclusion is false. If there is such a model, it is called a counterexample and the argument is invalid.

- A proposition is a logical truth iff it is true in every model; it is a contradiction iff it is false in every model; it is satisfiable iff it is true in some model.

- Two propositions are equivalent iff they have the same truth value in every model; they are contradictory iff there is no model in which they have the same truth value; they are jointly satisfiable iff there is some model in which they are both true.

typo in recorded lecture

- A set of propositions is satisfiable iff there is some model in which every proposition in the set is true.

Example

$$\begin{array}{l} \forall x\,(Kx \to Dx) \\ Km \\ \hline Dm \end{array}$$

Let $\mathcal{M}$ make the premises true. Then, because $\forall x(Kx \to Dx)$ is true in $\mathcal{M}$, $Kb \to Db$ is true $\mathcal{M}^b_o$, where the object o is the referent of m in $\mathcal{M}$. If b and m have same referent in $\mathcal{M}^b_o$, then Dm is true in $\mathcal{M}^b_o$. But $\mathcal{M}^b_o$ is <u>exactly</u> <u>like</u> $\mathcal{M}$ with respect to m. So, Dm is true in $\mathcal{M}$.