PHIL1012 Lecture 9:
Functional Completeness, Pt. 1

Welcome!

Recap

① Syntax of PL

- Basic propositions A, B, C, ..
- Connectives $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$

② Semantics of PL

- Basic propositions are assigned truth

- Basic propositions are assigned truth values T, F

- Connectives are truth-functional (truth functions are represented by truth tables)

(3) Trees for proving facts about
- equivalence
- logical truth
- validity
- etc.

Question (v1)

Given any truth table, is there a formula of PL with that truth table?

Example

α	β	$\alpha \veebar \beta$	$\alpha \leftrightarrow \beta$	$\neg(\alpha \leftrightarrow \beta)$
T	T	F	T	F
T	F	T	F	T
F	T	T	F	T
F	F	F	T	F

$$\alpha \veebar \beta \equiv \neg(\alpha \leftrightarrow \beta)$$

"is logically equivalent to"

Question (v2)

Can we define all possible
①

connectives using only the connectives in the set

$$\{\neg, \vee, \wedge, \rightarrow, \leftrightarrow\} \; ?$$

Plan

1. Defining connectives
2. All possible connectives

A set of connectives that can be used to define all possible connectives is called

<u>functionally complete</u>

Question (v3)

Is

$$\{\neg, \wedge, \vee, \rightarrow, \leftrightarrow\}$$

functionally complete?

① Defining connectives

Simple example

α	β	$\alpha \rightarrow \beta$	$\neg\alpha \vee \beta$
T	T	T	T

T	F	F	F
F	T	T	T
F	F	T	T

$$\alpha \to \beta \equiv \neg\alpha \vee \beta$$

More complex example

$\alpha \to \beta \equiv \neg\alpha\vee$

$$(A \to B) \to (C \wedge D) \equiv$$

$$\underbrace{(\neg A \vee B)}_{\alpha} \to \underbrace{(C \wedge D)}_{\beta} \equiv$$

$$\neg(\neg A \vee B) \vee (C \wedge D)$$

In general, for any proposition containing $\to$, we can find an equivalent proposition that doesn't contain $\to$ and instead contains

$\neg$ and $\vee$.

So, we can define $\rightarrow$ using $\neg$ and $\vee$.

Can we define $\leftrightarrow$ using only $\neg$, $\wedge$, and $\vee$?

$$\alpha \leftrightarrow \beta \equiv \boxed{(\alpha \wedge \beta) \vee (\neg\alpha \wedge \neg\beta}$$

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② All possible connectives

How many 1-place connectives are there?

α	$*\alpha$	$\dagger\alpha$	$\neg\alpha$	$\bullet\alpha$	
T	T	T	F	F	
F	T	F	T	F	

2 rows in the truth table;

2 choices (T or F) in each row.

So, $2^2 = 4$ 1-place connectives!

How many 2-place connectives are

there?

α	β	$\alpha \textcircled{2}_1 \beta$	$\alpha \vee \beta$	$\alpha \textcircled{2}_3 \beta$	$\alpha \to \beta$	$\cdots$
T	T	T	T	T	T	
T	F	T	T	T	F	
F	T	T	T	F	T	$\cdots$
F	F	T	F	T	T	(p.12

2^2 rows in the truth table;

2 choices in each row (T or F).

So, $2^{2^2} = 2^4 = 16$ 2-place connectives

There are also 0-place connectives!

verum falsum

$\top$	$\bot$
T	F

We can also talk about 3-, 4-, and, in general, n-place connectives

For example ...

α	β	γ	$*(\alpha, \beta, \gamma)$

T	T	T	F
T	T	F	F
T	F	T	T
T	F	F	T
F	T	T	T
F	T	F	T
F	F	T	F
F	F	F	F

How many n-place connectives?

2^n rows in the truth table;

2 choices in each row (T or F).

So, 2^{2^n} n-place connectives!

Question

Can we define all possible connectives using only the connectives in the set

$$\{\neg, \wedge, \vee, \rightarrow, \leftrightarrow\} ?$$

Tomorrow we will answer this question ...