Homework 7

1. Suppose $B_1(t), B_2(t)$ are independent Brownian processes with variance parameters σ_1^2, σ_2^2 respectively. Define: $\forall t,\ X(t) = B_1(t) - B_2(t)$. Derive the mean and autocorrelation functions of $X(t)$.

   i.e., $\mathrm{Cov}(X(s), X(t))$

In Problems 8.1, 8.2, and 8.3, let $\{X(t),\ t \geq 0\}$ denote a Brownian motion process.

2. **8.1.** Let $Y(t) = tX(1/t)$.
   - **(a)** What is the distribution of $Y(t)$?
   - **(b)** Compute $\mathrm{Cov}(Y(s), Y(t))$.
   - **(c)** Argue that $\{Y(t),\ t \geq 0\}$ is also Brownian motion.
   - **(d)** Let

$$T = \inf\{t > 0: \quad X(t) = 0\}.$$

   Using (c) present an argument that

$$P\{T = 0\} = 1.$$

3. a) **8.2.** Let $W(t) = X(a^2 t)/a$ for $a > 0$. Verify that $W(t)$ is also Brownian motion.

   b) Verify that $Y(t) = t^{\beta} X(t^{1-2\beta})$ is not Brownian motion unless $\beta \in \{0, 1\}$.

4. For each of the following processes, compute $\mathbb{E} X_t$ and $\mathrm{Cov}(X_s, X_t)$.

   ($B(t)$ is standard Brownian motion.)

   a) $X_t = \int_0^t B(u)\,du$

   b) $X_t = \int_0^t u \cdot B(u)\,du$

   c) $X_t = \int_0^t u^2 \cdot B(u)\,du$

Note: Since B_u is a Gaussian process, so is each of these X_ts (being sum/integrals of Gaussians). Therefore they are fully determined by the means and covariances.