Lecture 20: Ornstein-Uhlenbeck process

Admin:

Outline:

Ornstein-Uhlenbeck process
Brownian measure and integration
Central limit principle for random functions
Applications

Reference: Taylor & Karlin, "Intro. to Stochastic Modeling" §8.5

ORNSTEIN-UHLENBECK PROCESS

https://en.wikipedia.org/wiki/Ornstein–Uhlenbeck_process

Two equivalent definitions: $B(t)$ = std. BM.

① $V(t) = v e^{-\beta t} + \frac{\sigma e^{-\beta t}}{\sqrt{2\beta}} B(e^{2\beta t} - 1)$

or

② $V(t) = v e^{-\beta t} + \sigma \int_0^t e^{-\beta(t-u)} \, dB(u)$

we'll define this integral soon

Third definition: "Langevin equation"

stochastic differential

$$dV_t = -\beta V_t dt + \sigma dB_t$$

Property: Stationary & Markovian

Claim: Given $V(s)$,

$$V(s+t) \stackrel{d}{=} V(s) e^{-\beta t} + \frac{\sigma}{\sqrt{2\beta}} e^{-\beta t} B(e^{2\beta t} - 1).$$

Proof:

$$v e^{-\beta(s+t)} + \frac{\sigma}{\sqrt{2\beta}} e^{-\beta(s+t)} B(e^{2\beta(s+t)} - 1)$$

$$= e^{-\beta t} \left(v e^{-\beta s} + \frac{\sigma}{\sqrt{2\beta}} e^{-\beta s} B(e^{2\beta(s+t)} - 1) \right)$$

$$= e^{-\beta t} \cdot V(s) + \frac{\sigma}{\sqrt{2\beta}} e^{-\beta(s+t)} \underbrace{\left(B(e^{2\beta(s+t)} - 1) - B(e^{2\beta s} - 1) \right)}_{\sim B(e^{2\beta(s+t)} - e^{2\beta s})}$$

$$\sim e^{\beta s} \cdot B\left(e^{2\beta t}-1\right) \qquad \square$$

Motivation: Modeling velocity of a particle in Brownian motion

Recall: Almost surely, B.M. is continuous, but not differentiable anywhere $\Rightarrow$ no velocity

(O-U process is not differentiable either.)

Claim: Over time $(t, t+\Delta t]$, for $\Delta V = V(t+\Delta t) - V(t)$,

$$\mathbb{E}[\Delta V \mid V(t)] = -\beta V(t) \Delta t + O((\Delta t)^2)$$

$\nwarrow$ viscosity factor (from friction)

$$\operatorname{Var}(\Delta V \mid V(t)) = \sigma^2 \Delta t + O((\Delta t)^2)$$

$\nwarrow$ random factor (from collisions)

Proof:

$$\Delta V = V(t+\Delta t) - V(t) \overset{d}{=} V(t)\left(e^{-\beta \Delta t}-1\right) + \frac{\sigma}{\sqrt{2\beta}} B\left(1-e^{-2\beta \Delta t}\right)$$

$$\mathbb{E}[\Delta V \mid V(t)=x] = \left(e^{-\beta \Delta t}-1\right)x + \frac{\sigma}{\sqrt{2\beta}} \underbrace{\mathbb{E}\left[B\left(1-e^{-2\beta \Delta t}\right)\right]}_{=0}$$

$$= -\beta(\Delta t) \cdot V(t) + o(\Delta t)$$

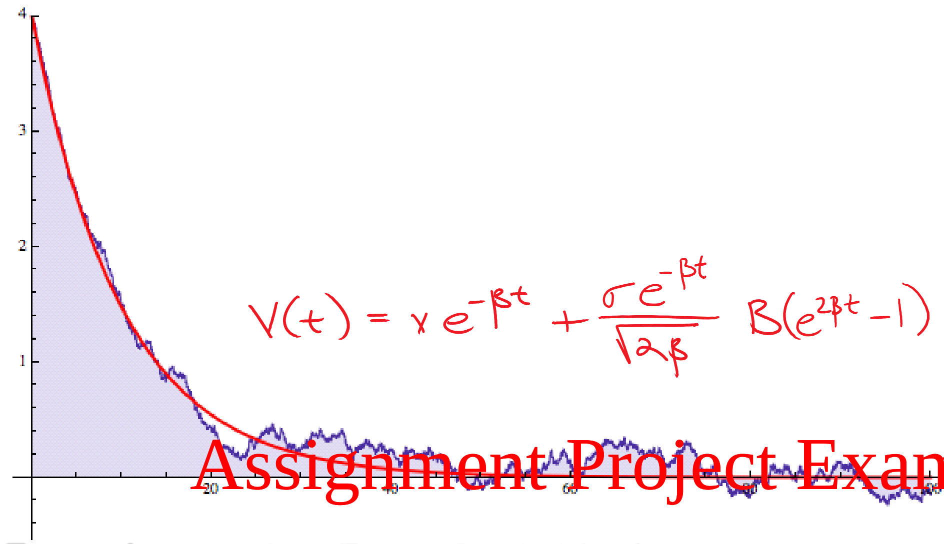
$$\operatorname{Var}(\Delta V \mid V(t)=x) = \frac{\sigma^2}{2\beta} \operatorname{Var}\left(B\left(1-e^{-2\beta \Delta t}\right)\right)$$

$$= \sigma^2 \Delta t + o(\Delta t) \qquad \square$$

```
?OrnsteinUhlenbeckProcess
```

OrnsteinUhlenbeckProcess[μ, σ, θ] represents an Ornstein-Uhlenbeck process with long-term mean μ, volatility σ, and mean reversion speed θ.
OrnsteinUhlenbeckProcess[μ, σ, θ, x_0] represents an Ornstein-Uhlenbeck process with initial condition x_0. »

```
σ = .08;
β = .1;
data = RandomFunction[OrnsteinUhlenbeckProcess[0, σ, β, 4], {0, 100, .01}];
dataplot = ListLinePlot[data, Filling → Axis, PlotRange → {-1, 4}];
expplot = Plot[4 e^(-β t), {t, 0, 100}, PlotRange → {-1, 4}, PlotStyle → {Thick, Red}];
Show[dataplot, expplot]
```



$$V(t) = x e^{-\beta t} + \frac{\sigma e^{-\beta t}}{\sqrt{2\beta}} B(e^{2\beta t} - 1)$$

Example *Tracking Error* Let $V(t)$ be the measurement error of a radar system that is attempting to track a randomly moving target. We assume $V(t)$ to be an Ornstein–Uhlenbeck process. The mean increment $E[\Delta V | V(t) = v] = -\beta v \Delta t + o(\Delta t)$ represents the controller's effort to reduce the current error, while the variance term reflects the unpredictable motion of the target. If $\beta = 0.1$, $\sigma = 2$, and the system starts on target ($v = 0$), the probability that the error is less than one at time $t = 1$ is, using (5.4),

$$\Pr\{|V(t)| < 1\} = \Phi\left(\frac{\sqrt{2\beta}}{\sigma\sqrt{1 - e^{-2\beta t}}}\right) - \Phi\left(\frac{-\sqrt{2\beta}}{\sigma\sqrt{1 - e^{-2\beta t}}}\right)$$

$$= \Phi\left(\frac{1}{\sqrt{20(1 - e^{-0.2})}}\right) - \Phi\left(\frac{-1}{\sqrt{20(1 - e^{-0.2})}}\right)$$

$$= \Phi(0.53) - \Phi(-0.53) = 0.4038.$$

As time passes, this near-target probability drops to $\Phi(1/\sqrt{20}) - \Phi(-1/\sqrt{20}) = \Phi(0.22) - \Phi(-0.22) = 0.1742$.

Finance applications

The Ornstein–Uhlenbeck process is one of several approaches used to model (with modifications) interest rates, currency exchange rates, and commodity prices stochastically. The parameter μ represents the equilibrium or mean value supported by fundamentals; σ the degree of volatility around it caused by shocks, and θ the rate by which these shocks dissipate and the variable reverts towards the mean. One application of the process is a trading strategy known as pairs trade.[11][12][13]

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Vasicek model

https://en.wikipedia.org/wiki/Vasicek_model

$$dV_t = \underbrace{\beta}_{\text{speed of reversion}}(\underbrace{\mu}_{\text{long-term mean}} - V_t)\,dt + \underbrace{\sigma}_{\text{instantaneous volatility}}\, dW_t$$

The main disadvantage is that, under Vasicek's model, it is theoretically possible for the interest rate to become negative, an undesirable feature under pre-crisis assumptions. This shortcoming was fixed in the Cox–Ingersoll–Ross model, exponential Vasicek model, Black–Derman–Toy model and Black–Karasinski model, among many others. The Vasicek model was further extended in the Hull–White model. The Vasicek model is also a canonical example of the affine term structure model, along with the Cox–Ingersoll–Ross model.

Stationary O-U process

$V(t) = v e^{-\beta t} + \frac{\sigma e^{-\beta t}}{\sqrt{2\beta}} B(e^{2\beta t}-1)$

Properties: From ①, $V(t)$ is a Gaussian process with

$$\mathbb{E}[V(t)] = V(0)\cdot e^{-\beta t}$$

$$\mathrm{Cov}(V(s),V(t)) = \frac{\sigma^2}{2\beta} e^{-\beta(t-s)}(1-e^{-2\beta s}), \quad 0<s<t$$

$$\therefore\ \mathrm{Var}(V(t)) = \frac{\sigma^2}{2\beta}(1-e^{-2\beta t}) \xrightarrow[t\to\infty]{} \frac{\sigma^2}{2\beta}. \quad \text{"long-term variance"}$$

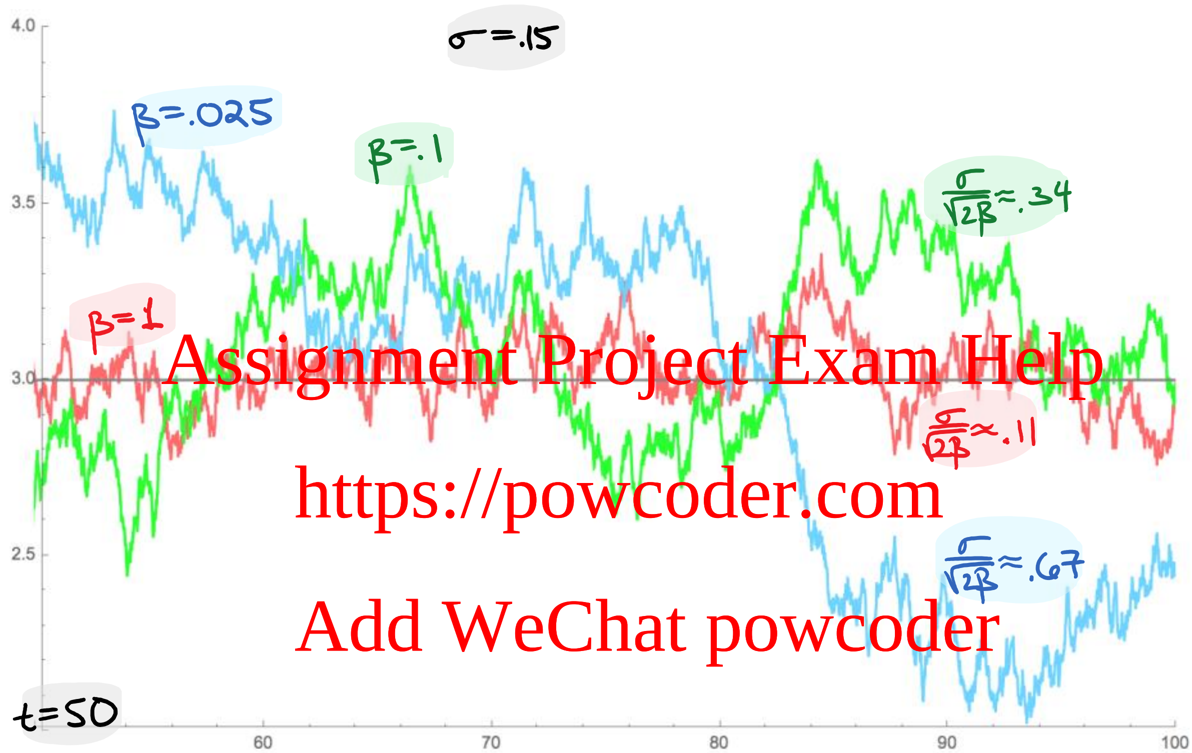
Proof:

$$\mathrm{Cov}(V(s),V(t)) = \mathbb{E}\left[\left(V(s)-ve^{-\beta s}\right)\left(V(t)-ve^{-\beta t}\right)\right]$$

$$= \frac{\sigma^2}{2\beta}e^{-\beta(s+t)}\cdot \underbrace{\mathrm{Cov}\left(B(e^{2\beta s}-1),\, B(e^{2\beta t}-1)\right)}_{= e^{2\beta s}-1}$$

(since $\mathrm{Cov}(B(a),B(b)) = \min\{a,b\}$). ✓

Example: Same plots from above, continued.



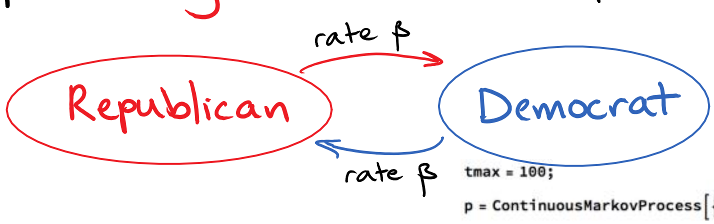
Definition: Stationary Ornstein-Uhlenbeck process

$$V^S(t) = \frac{\sigma}{\sqrt{2\beta}}e^{-\beta t}B(e^{2\beta t})$$

$$\mathbb{E}[V^S(t)]=0,\quad \mathrm{Cov}(V^S(s),V^S(t)) = \frac{\sigma^2}{2\beta}e^{-\beta|t-s|}$$

for all times s and t, not just large s and large t

Example: Many two-state Markov processes



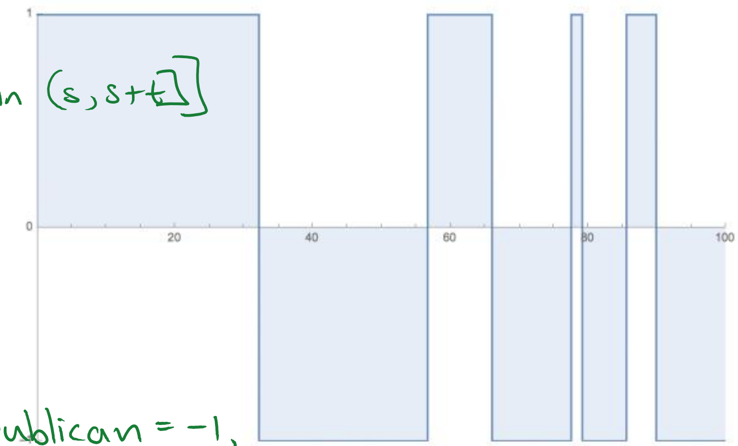
"Telegraph noise"

```
tmax = 100;
p = ContinuousMarkovProcess[{1/2, 1/2}, (-β β; β -β)];
data = 2 RandomFunction[p, {0, tmax}] - 3;
```

With one person,

rate β

```
tmax = 100;
p = ContinuousMarkovProcess[{1/2, 1/2}, ({{-β, β}, {β, -β}})];
data = 2 RandomFunction[p, {0, tmax}] - 3;
ListStepPlot[data, PlotRange → {{0, tmax}, {-1, 1}}, Filling → Axis,
 Ticks → {Automatic, {-1, 0, 1}}]
```



With one person,

$$\mathbb{P}[X(s+t)=D \mid X(s)=D]$$
$$= \mathbb{P}[\text{even \# of transitions in } (s, s+t]]$$
$$= e^{-\beta t} \sum_{j=0}^{\infty} \frac{(\beta t)^{2j}}{(2j)!}$$
$$= e^{-\beta t} \cdot \frac{1}{2}\left(e^{\beta t} - e^{-\beta t}\right)$$
$$= \frac{1}{2}\left(1 + e^{-2\beta t}\right)$$

Thus, if Democrat $= +1$, Republican $= -1$, and $\mathbb{E}[X(t)=0]$ (both parties equally likely), then

$$\mathbb{E}[X(s)X(s+t)] = \mathbb{P}[X(s+t)=X(s)] - \mathbb{P}[X(s+t)=-X(s)]$$
$$= \frac{1}{2}\left(1+e^{-2\beta t}\right) - \frac{1}{2}\left(1-e^{-2\beta t}\right)$$
$$= e^{-2\beta t}$$

Now consider N people, all moving independently.

Let $$S_N(t) = \sum_{j=1}^{N} X_j(t)$$

Central limit theorem

$$\Rightarrow V_N(t) = \frac{1}{\sqrt{N}} S_N(t)$$

is roughly normal $N(0,1)$

Central limit principle for random processes/functions:
"Sum of i.i.d. random processes should converge to a Gaussian process"

$\therefore$ $V_N(t)$ should, for large N, behave like a Gaussian process with mean 0

and $\mathrm{Cov}(V_N(s), V_N(s+t))$

$$= \frac{1}{N} \mathrm{Cov}\left(\sum_j X_j(s), \sum_j X_j(s+t)\right)$$
$$= \frac{1}{N} \sum_j \mathrm{Cov}(X_j(s), X_j(s+t)) \quad \text{by independence}$$

$$= e^{-2\beta t}$$

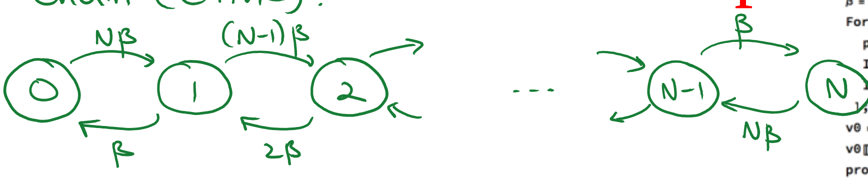
This is exactly a stationary O-U process, with $\sigma^2 = 2\beta$!

$$\therefore\ S_N(t) = \sqrt{N}\, V_N(t)$$
$$\approx \sqrt{N}\, e^{-2\beta t}\, B(e^{4\beta t})$$

```
n = 100;
p = ContinuousMarkovProcess[{1/2, 1/2}, (-β β; β -β)];
data = 1/√n Table[2 RandomFunction[p, {0, 100}] - 3, {n}];
ListStepPlot[TimeSeriesThread[Total, data], Filling → Axis, PlotRange → {-3, 3}]
```

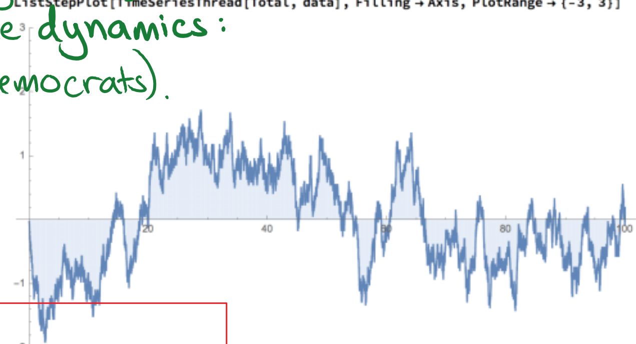
```
β = .1;
σ = √(2 β)
data = RandomFunction[OrnsteinUhlenbeckProcess[0, σ, β], {0, 100, .01}];
dataplot = ListLinePlot[data, Filling → Axis, PlotRange → {-3, 3}]
0.447214
```

Remark: This process is a birth-death continuous-time Markov chain (CTMC).



We have a closed formula for the stationary distn, but it is a mess, and tells us very little about the **dynamics**: the oscillations about ($\frac{N}{2}$ Republicans, $\frac{N}{2}$ Democrats).

```
n = 1000; (* 500 people *)
p = Table[0, {n + 1}, {n + 1}];
β = .1;
For[j = 1, j ≤ n + 1, j ++,
  p[[j, j]] = -n β;
  If[j > 1, p[[j, j - 1]] = (j - 1) β;];
  If[j ≤ n, p[[j, j + 1]] = (n - j + 1) β;];
 ];
v0 = Table[0, {n + 1}];
v0[[Round[(n + 1) / 2]]] = 1;
process = ContinuousMarkovProcess[v0, p];
data = 1/√(n/2) (RandomFunction[process, {0, 100}] - (n + 1)/2);
ListStepPlot[TimeSeriesThread[Total, data], Filling → Axis, PlotRange → {-3, 3}]
```



$$\boxed{\begin{array}{l}\text{Moral: Using the central limit principle, the O-U and stationary O-U processes can often}\\ \text{be used as simplified models for complicated combinatorial dynamics.}\end{array}}$$

be used as simplified models for complicated combinatorial dynamics.

Brownian Measure and Integration

We claimed that

$$V(t) = v e^{-\beta t} + \sigma \int_0^t e^{-\beta(t-u)} \, dB(u)$$

But what does this even mean?

Definition: For a continuous function $g(x)$, and standard Brownian motion $B(t)$, define

$$\int_0^t g(x)\, dB(x) = \lim_{n\to\infty} \sum_{k=1}^{n} g\left(\tfrac{k}{n}t\right)\left(B\left(\tfrac{k}{n}t\right) - B\left(\tfrac{k-1}{n}t\right)\right)$$

This is exactly the same as the standard (Riemann) integral.

Theorem:

1) The integral $\int_0^t g(x)\,dB(x)$ exists, i.e., the limit converges.

2) The integral is normally distributed, with

mean 0 and variance $\int_0^t g(x)^2 dx$

3) If $f(x)$ is another continuous function, then the integrals $\int_0^t f(x)\,dB(x)$ and $\int_0^t g(x)\,dB(x)$ are jointly normal, with

$$\mathrm{Cov}\left(\int_0^t f(x)\,dB(x), \int_0^t g(x)\,dB(x)\right) = \int_0^t f(x) g(x)\, dx.$$

Intuition for 2: Let $t_k = \frac{k}{n} t$.

$$\mathbb{E}\left[\sum_k g(t_k)\left(B(t_k) - B(t_{k-1})\right)\right]$$

$$= \sum_k g(t_k) \cdot \underbrace{\mathbb{E}\left[B(t_k) - B(t_{k-1})\right]}_{= 0} = 0 \quad ✓$$

$$\mathrm{Var}\left(\sum_k g(t_k) \underbrace{\left(B(t_k) - B(t_{k-1})\right)}_{\substack{\text{independent increments,} \\ \text{each with variance } t_k - t_{k-1} = \frac{1}{n}}}\right) = \sum_k g(t_k)^2 \cdot \frac{1}{n} \xrightarrow[n\to\infty]{} \int_0^t g(x)^2 dx. \quad ✓$$

Example: Integrated Brownian motion

Integration by parts:

$$g(t)\,B(t) = \int_0^t g(x)\,dB(x) + \int_0^t g'(x)\,B(x)\,dx$$

For $g(x) = t - x$, so $g(t) = 0$, $g'(x) = -1$, we get

$$\int_0^t B(x)\,dx = \int_0^t (t-x)\,dB(x)$$

is normal w/ mean 0

variance $\int_0^t (t-x)^2\,dx = \frac{1}{3}t^3$

Claim: The Ornstein-Uhlenbeck process,

$$V(t) = v\,e^{-\beta t} + \frac{\sigma e^{-\beta t}}{\sqrt{2\beta}} B(e^{2\beta t} - 1)$$

can equivalently be represented as

$$V(t) = v\,e^{-\beta t} + \sigma \int_0^t e^{-\beta(t-u)}\,dB(u)$$

Proof: The second expression for $V(t)$ describes a Gaussian process with

$$\mathbb{E}[V(t)] = v\,e^{-\beta t}$$

$$\mathrm{Cov}(V(s), V(t)) = \sigma^2 \cdot \mathrm{Cov}\left(\int_0^s e^{-\beta(s-u)}\,dB(u), \int_0^t e^{-\beta(t-u)}\,dB(u)\right)$$

$$= \sigma^2 \cdot \int_0^s e^{-\beta(s-u)} e^{-\beta(t-u)}\,du, \text{ if } s \le t$$

$$= \sigma^2 e^{-\beta(s+t)} \int_0^s e^{2\beta u}\,du$$

$$= \frac{\sigma^2}{2\beta}\left(e^{-\beta(t-s)} - e^{-\beta(t+s)}\right)$$

These are the same means and covariances that we computed earlier for $V(t)$, so they must be the same Gaussian process. $\square$

Ornstein-Uhlenbeck position process

If the velocity of a particle is $V(t)$, its position is

$$S(t) = S(0) + \int_0^t V(u)\,du.$$

Example: Stock prices can be modeled by an O-U position process. Price changes are not independent, in the short term, but have an exponentially decreasing correlation — modeling **market momentum**.

Principle: Over large time spans, the O-U position process "behaves like" Brownian motion.

Why? Assume that $\underset{\text{initial position}}{S(0)=0}$, $\underset{\text{initial velocity}}{V(0)=0}$.

$$\therefore\ S(t) = \int_0^t V(s)\,ds$$

$$= \sigma \cdot \int_0^t ds \int_0^s e^{-\beta(s-u)}\,dB(u)$$

$$= \sigma \int_0^t dB(u) \underbrace{\int_u^t ds\, e^{-\beta(s-u)}}_{e^{\beta u}\cdot\frac{1}{\beta}\left(e^{-\beta u}-e^{-\beta t}\right)} \qquad \left(\begin{matrix}\text{since}\\ u \le s\end{matrix}\right)$$

$$= \frac{\sigma}{\beta}\int_0^t dB(u)\cdot\left(1-e^{-\beta(t-u)}\right)$$

$$= \frac{\sigma}{\beta}\cdot B(t) - \frac{1}{\beta}V(t)$$

The first term is standard BM, times $\frac{\sigma}{\beta}$. Its variance is $\frac{\sigma^2}{\beta^2}t$. The second term has variance $\le \frac{1}{\beta^2}\cdot\frac{\sigma^2}{2\beta}$. This is drowned out by the first term for large t, and thus

$$S(t) \approx \frac{\sigma}{\beta}B(t). \quad \checkmark$$

Example: (Central limit theorem)

A new harddrive has a mean lifetime of one year.
For your cluster you need 100 harddrives working at all times.
Estimate the probability that you'll need over 1050 drives over the next 10 years.

Answer: Model the lifetime of a drive as $X \sim \mathrm{Exp}(\lambda = 1)$.
We want to estimate

$$\mathbb{P}[\text{total lifetime } T \text{ of } 1050 \text{ drives} < 10 \cdot 100]$$

$$= \mathbb{P}\left[\sum_{j=1}^{1050} X_j < 1000\right]$$

Now $E[T] = 1050\, E[X] = \frac{1050}{\lambda} = 1050 \doteq \mu$

$$\mathrm{Var}(T) = 1050 \cdot \mathrm{Var}(X) = \frac{1050}{\lambda^2} = 1050 \doteq \sigma^2$$

By the central limit theorem, T is roughly normally distributed.

$$\Rightarrow \mathbb{P}[T < 1000] = \mathbb{P}\left[\frac{T-\mu}{\sigma} < \frac{1000-\mu}{\sigma}\right]$$

$$\approx \Phi\left(\frac{1000-\mu}{\sigma}\right)$$

$$\approx 6.14\,\%$$


```
CDF[NormalDistribution[], (1000 - 1050)/√1050.]

0.0614113
```

Central limit principle for random functions

[Taylor & Karlin, p.482]

We have seen how the invariance principle leads to the Gaussian process called Brownian motion. Gaussian processes also arise as the limits of normalized sums of independent and identically distributed random *functions*. To sketch out this idea, let $\xi_1(t), \xi_2(t), \ldots$ be independent and identically distributed random functions, or stochastic processes. Let $\mu(t) = E[\xi(t)]$ and $\Gamma(s, t) = \mathrm{Cov}[\xi(s), \xi(t)]$ be the mean value and covariance functions, respectively. Motivated by the central limit theorem, we define

$$X_N(t) = \frac{\sum_{i=1}^{N} \{\xi_i(t) - \mu(t)\}}{\sqrt{N}}.$$

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$$X_N(t) = \frac{\sum_{i=1}^{N} \{\xi_i(t) - \mu(t)\}}{\sqrt{N}}.$$

The central limit theorem tells us that the distribution of $X_N(t)$ converges to the normal distribution for each fixed time point t. A multivariate extension of the central limit theorem asserts that for any finite set of time points $(t_1, \ldots, t_n)$, the random vector

$$(X_N(t_1), \ldots, X_N(t_n))$$

has, in the limit for large N, a multivariate normal distribution. It is not difficult to believe, then, that under ordinary circumstances, the stochastic processes $\{X_N(t);\ t \geq 0\}$ would converge, in an appropriate sense, to a Gaussian process $\{X(t);\ t \geq 0\}$ whose mean is zero and whose covariance function is $\Gamma(s, t)$. We call this the *central limit principle for random functions*. Several instances of its application appear in this chapter, the first of which is next.