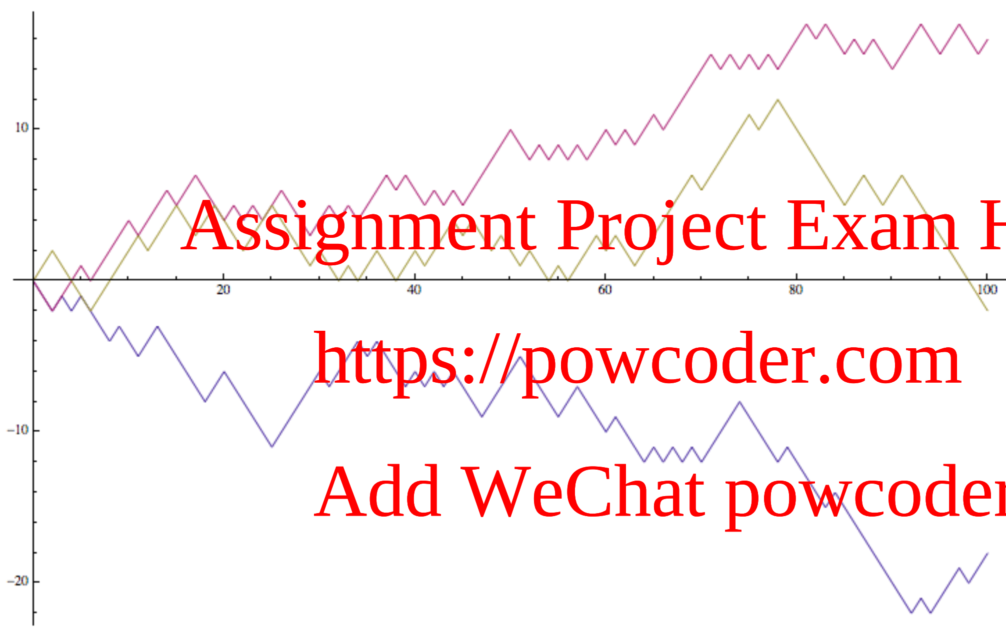
Lecture 18: Brownian motion

Admin: Reading. Ross, 8.1-8.5

Symmetric random walk

$$S_n = \underbrace{X_1 + X_2 + \cdots + X_n}_{\text{i.i.d. } \begin{cases} +1 & \text{w/ prob } 1/2 \\ -1 & \text{w/ prob } 1/2 \end{cases}}$$

```
ListLinePlot[
 Table[RandomFunction[RandomWalkProcess[1/2], {100}], {3}]
]
```



- Null-recurrent Markov chain, period 2
  - $\mathbb{P}[\text{return to } 0 \text{ from } 0] = 1$
  - $\mathbb{E}[\text{time to return to } 0] = \infty$
- Martingale
- Hitting times:
  - $\mathbb{P}[\text{go right } A \text{ before left } B] = \frac{B}{A+B}$
  - $\mathbb{E}[\text{time to go right } A \text{ or left } B] = A \cdot B$
- Limit theorems, and concentration

Recall: Central limit theorem

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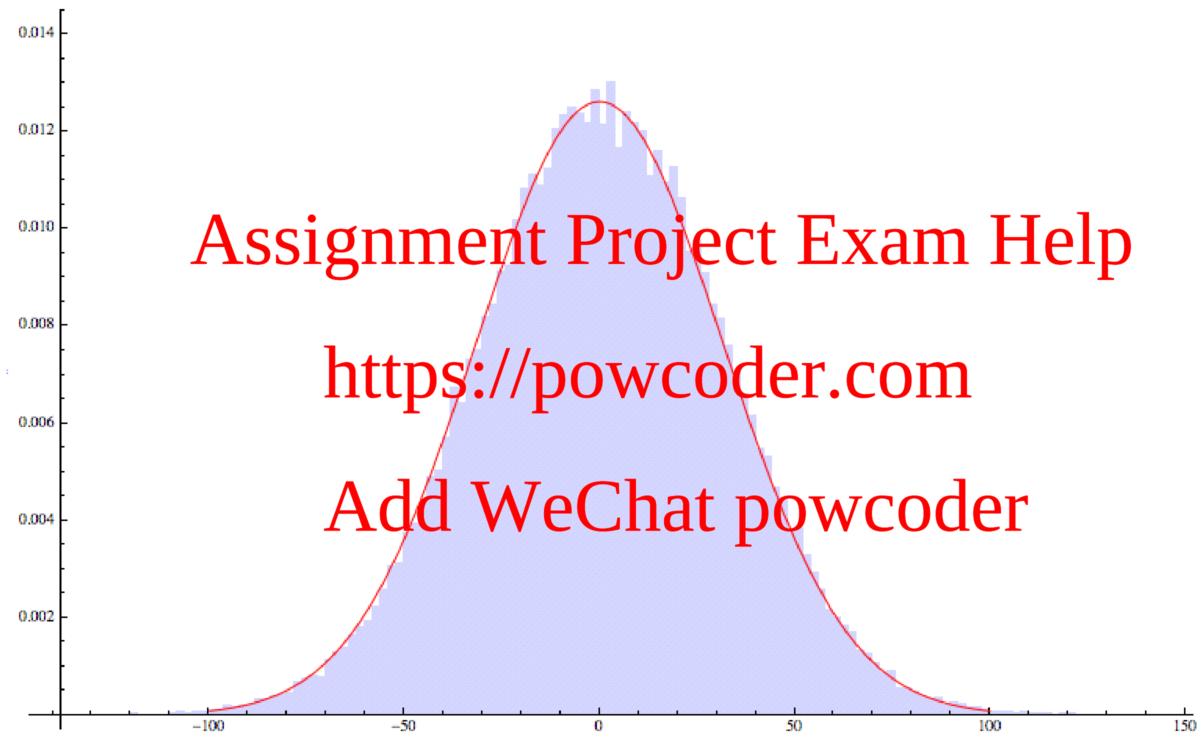
$$\lim_{n\to\infty} \mathbb{P}\left[X_1+\cdots+X_n \le \sqrt{n}\,a\right] = \underbrace{\int_{-\infty}^{a} dx \frac{1}{\sqrt{2\pi}} e^{-x^2/2}}_{\text{CDF for normal distribution}}$$

Roughly, S_n is distributed like $N(0, \sigma^2 = n)$.

```
n = 1000;
values = Table[Plus @@ Table[2 Random[Integer, {0, 1}] - 1, {n}], {100000}] // Sort;
Histogram[values, Length[Union[values]], "PDF",
 Epilog → First@Plot[PDF[NormalDistribution[0, √n], x], {x, -100, 100}, PlotStyle → Red]]
```



Observe: " $S_n \approx \sqrt{n}$ "

$$E[|S_n|] \approx \underset{N(0,n)}{E[|X|]} = 2\int_0^\infty dx \frac{x}{\sqrt{2\pi n}} e^{-x^2/2n}$$

$$= \sqrt{\frac{2}{\pi}} \cdot \sqrt{n}$$

Azuma's inequality

$$\mathbb{P}[|S_n| \geq \lambda\sqrt{n}] \le 2e^{-\lambda^2/2}$$

also, $\mathbb{P}[S_{2n} = 0] = \binom{2n}{n}\left(\frac{1}{2}\right)^{2n}$

$\approx \sqrt{\frac{2}{\pi n}}$ (Stirling: $n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$)

$\Rightarrow$ 99% of the time $\frac{1}{100}\sqrt{n} \leq |S_n| \leq 4\sqrt{n}$

Brownian Motion

INVESTIGATIONS ON THE THEORY OF THE BROWNIAN MOVEMENT

BY

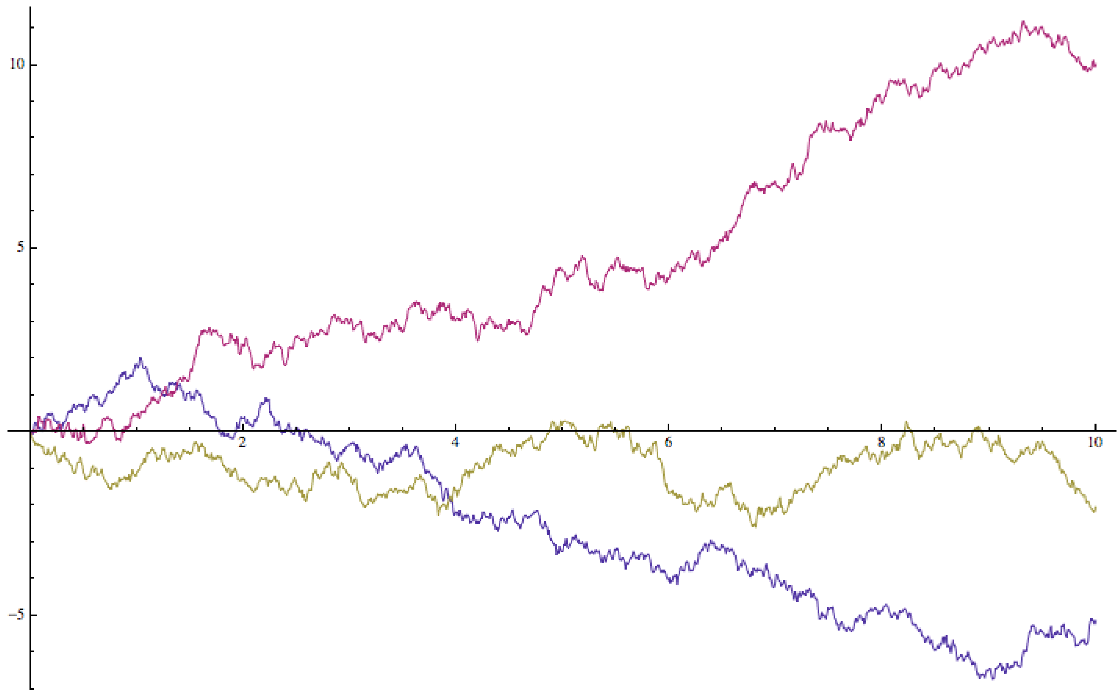
ALBERT EINSTEIN, PH.D. (1905)

I

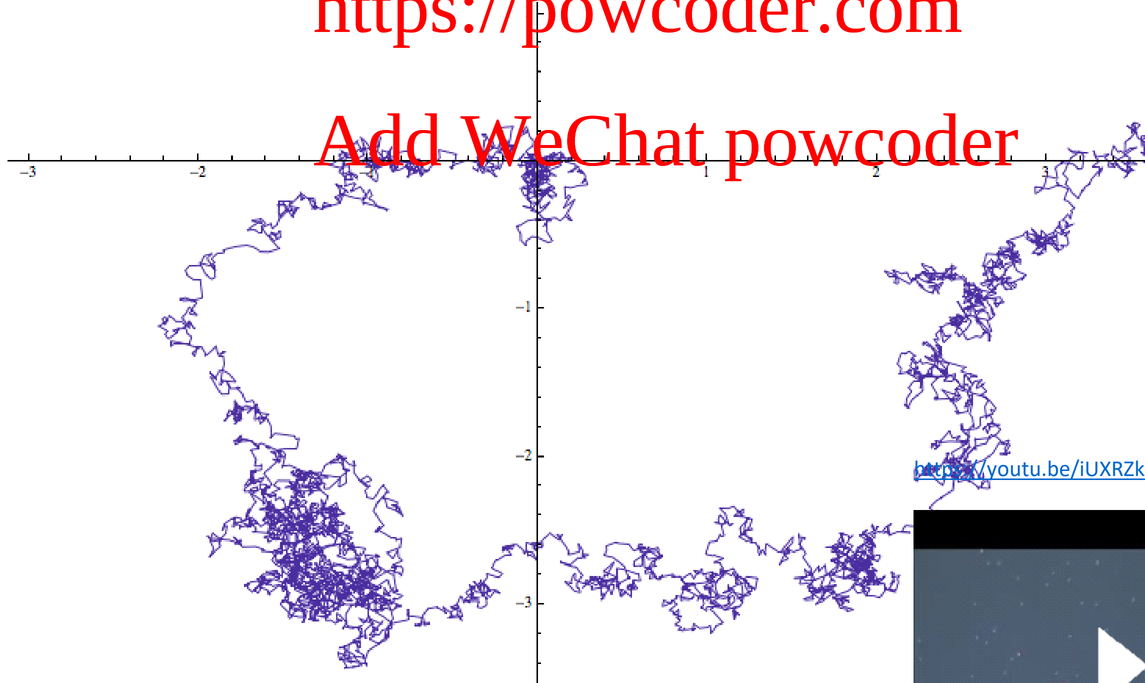
ON THE MOVEMENT OF SMALL PARTICLES SUSPENDED IN A STATIONARY LIQUID DEMANDED BY THE MOLECULAR-KINETIC THEORY OF HEAT

IN this paper it will be shown that according to the molecular-kinetic theory of heat, bodies of microscopically-visible size suspended in a liquid will perform movements of such magnitude that they can be easily observed in a microscope, on account of the molecular motions of heat. It is possible that the movements to be discussed here are identical with the so-called "Brownian molecular motion"; however, the information available to me regarding the latter is so lacking in precision, that I can form no judgment in the matter (1).

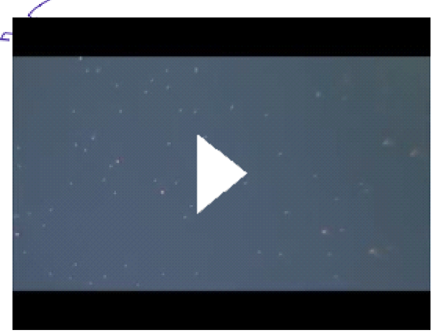
```
ListLinePlot[
 Table[RandomFunction[WienerProcess[], {0, 10, .01}], {3}]
]
```




```
data = {RandomFunction[WienerProcess[], {0, 5, .001}][[2, 1, 1]],
    RandomFunction[WienerProcess[], {0, 5, .001}][[2, 1, 1]]} // Transpose;
plot = ListLinePlot[data]
```

https://youtu.be/iUXRZkJN8OQ



"Continuous-time, symmetric random walk"

Idea: Discretize time and space

$$S_t = \Delta x \left(X_1 + X_2 + \cdots + X_{\frac{t}{\Delta t}} \right)$$

and then let $\Delta t, \Delta x \to 0$

$$S_t \sim N\left(0, (\Delta x)^2 \cdot \frac{t}{\Delta t}\right)$$

for the variance to be > 0 and $< \infty$ in the continuous limit, we must have

$$\Delta x = c\sqrt{\Delta t}$$

for a constant c. We will choose $c = 1$.

Definition:

A stochastic process $\{X(t) : t \geq 0\}$ is called standard Brownian motion or a Wiener process if:

- $X(0) = 0$
- $X(t) \sim N(0, \sigma^2 = t)$ for all $t > 0$
- It has independent stationary increments

$$X(s+t) - X(s) \sim N(0, t)$$

for $0 = t_0 < t_1 < \cdots < t_k$,

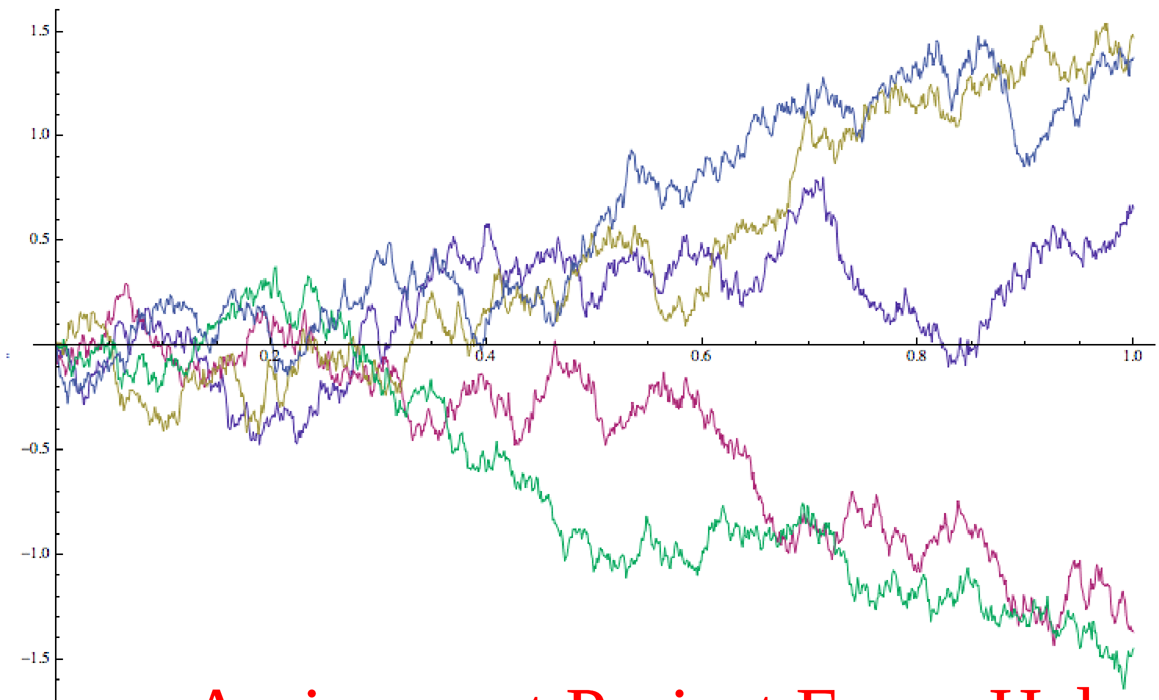
$\{X(t_1) - X(t_0), \ldots, X(t_k) - X(t_{k-1})\}$ all independent

Examples:

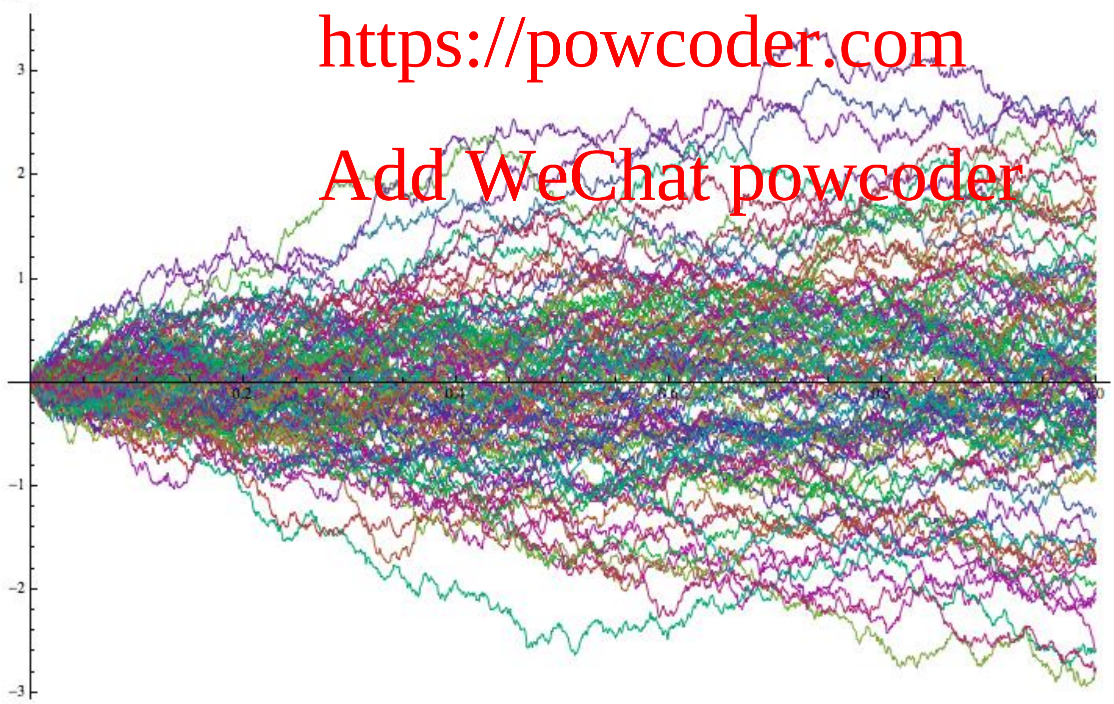
Mathematica:

```
ListLinePlot[
 Table[RandomFunction[WienerProcess[], {0, 1, .001}], {5}]
]
```

t_{min} t_{max} Δt



```
ListLinePlot[
 Table[RandomFunction[WienerProcess[], {0, 1, .001}], {100}]
]
```

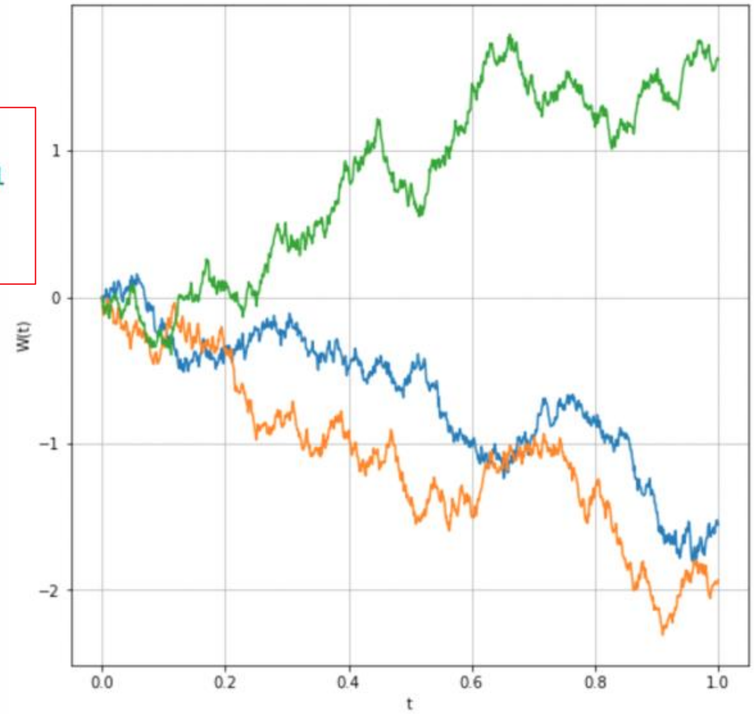



Python:

```python
import numpy as np
from math import sqrt
import matplotlib.pyplot as plt

def bm(t, n):
  r = sqrt(t/n) * np.random.normal(size=n+1
  r[0] = 0
  return np.cumsum(r)

t = 1
n = 1000

plt.figure(figsize=(8,8))
plt.xlabel('t'), plt.ylabel('W(t)')
plt.grid(True, which='major')
x = np.linspace(0, t, n+1)

for i in range(5):
  plt.plot(x, bm(t,n))

plt.show()
```

cumulative sum

$r_0, r_0+r_1, r_0+r_1+r_2, \ldots$

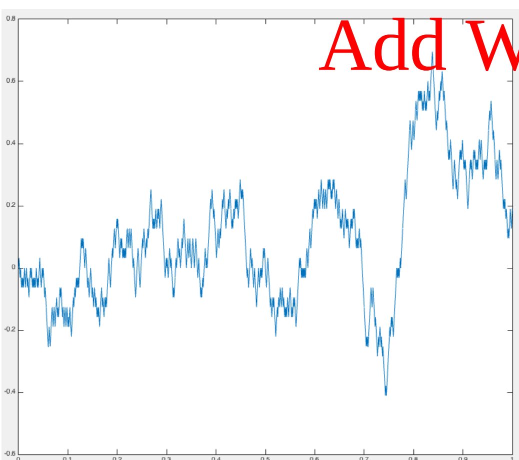


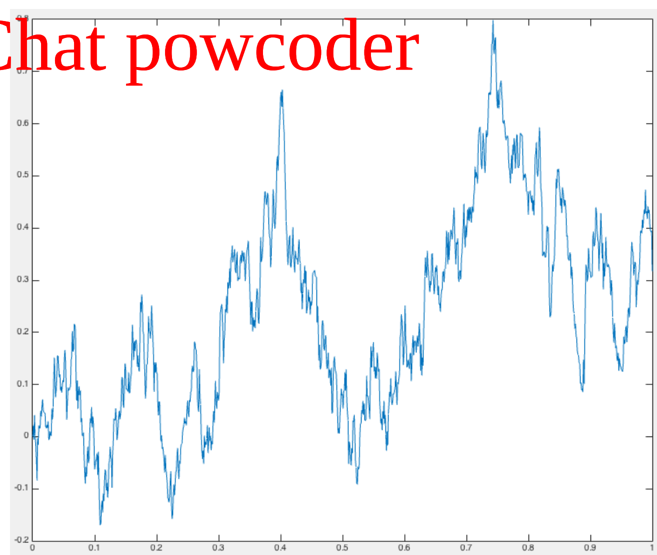
Matlab:

```matlab
>> t = 1; n = 1000; dt = t/n;
times = 0:dt:t;
steps = sqrt(dt)*(2*randi(2,1,n)-3);
plot(times, [0,cumsum(steps)])
```

generates ±1 random


```matlab
>> t = 1; n = 1000; dt = t/n;
times = 0:dt:t;
steps = sqrt(dt)*randn(1,n);
plot(times, [0,cumsum(steps)])
```



Remark: We can also use other increments, eg.,

$$S_t = \Delta x \left(\underbrace{X_1 + X_2 + \cdots + X_{\frac{t}{\Delta t}}}_{\text{i.i.d. } N(0,\sigma^2)} \right)$$

$\Rightarrow Var(S_t) = (\Delta x)^2 \cdot t \quad \sigma^2$

$$\Rightarrow Var(S_t) = (\Delta x)^2 \cdot \frac{t}{\Delta t} \cdot \sigma^2$$

$$= t \quad \text{if} \quad \sigma^2 = 1$$

Property: Brownian motion is a Gaussian process:

for all $t_1, \dots, t_n$

$X(t_1), \dots, X(t_n)$ has a multivariate Gaussian distn

determined by the means (all 0), and covariance matrix

$$Cov(X(t_i), X(t_j)) \qquad t_i \le t_j$$

$$= Cov(X(t_i), X(t_i) + (X(t_j) - X(t_i)))$$

$$= Var(X(t_i)) + Cov(X(t_i), X(t_j) - X(t_i))$$

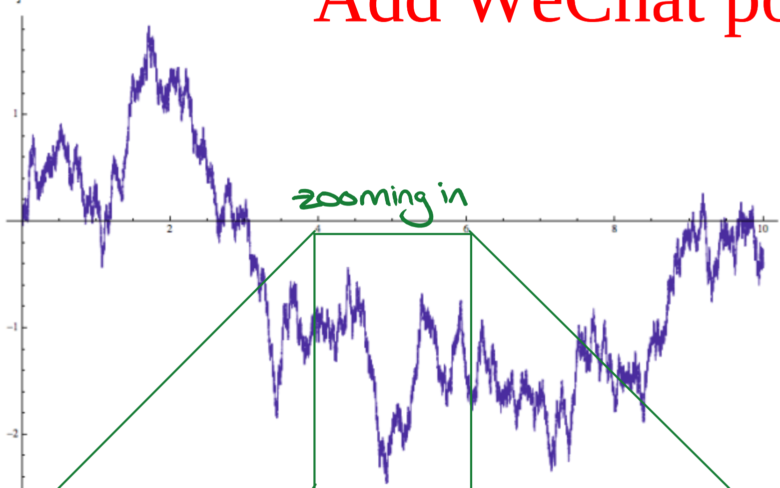
$$= t_i + 0$$

$$\boxed{Cov(X(s), X(t)) = \min(s,t)}$$

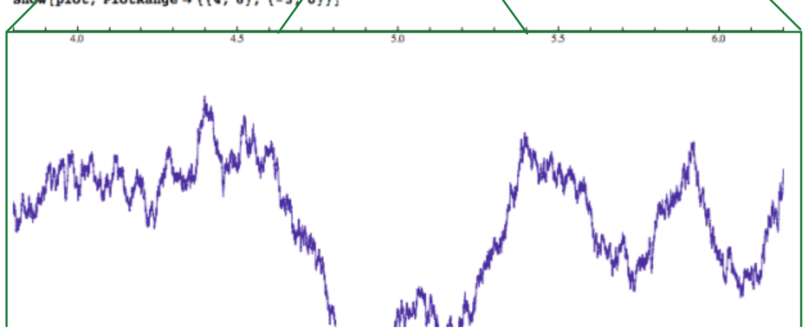
Property: Jagged paths

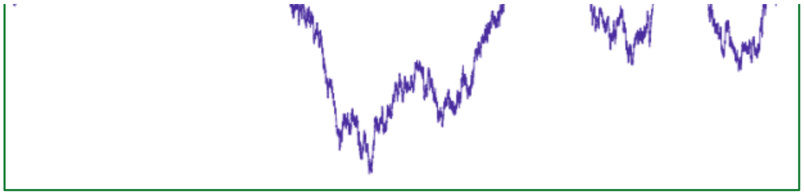
① Distributional self-similarity

```
plot = ListLinePlot[
  RandomFunction[WienerProcess[], {0, 10, .0001}]
]
```


```
Show[plot, PlotRange → {{4, 6}, {-3, 0}}]
```





$X(t) \sim N(0,t)$

$c\,X(t/c^2) \sim N(0,t)$ also

(for $c<1$, this is zooming in)

② Nowhere differentiable

Fact: With probability one, the function $X(t)$ is nowhere differentiable.

$$\frac{X(t+\Delta t)-X(t)}{\Delta t} \sim N\left(0, \frac{1}{(\Delta t)^2}\Delta t\right) = N\left(0, \frac{1}{\Delta t}\right)$$

diverges as $\Delta t \to 0$

③ Unbounded variation

Fact: For any $t>0$, with probability 1,

$$\lim_{n\to\infty} \sup_{\substack{\text{partitions} \\ 0=t_0<t_1<\cdots<t_n=t}} \sum_{j=1}^{n} |X(t_j)-X(t_{j-1})| = \infty$$

Interpretation: This gives the total distance, left and right, that the particle moves from time 0 to t.

Indeed $E\Big[\sum_{j=1}^{n} |\underbrace{X_j}_{N(0,\sigma^2=\frac{t}{n})}|\Big] = n \times \sqrt{\frac{2}{\pi}}\,\sigma = \sqrt{\frac{2}{\pi}}\sqrt{t} \times \sqrt{n} \longrightarrow \infty$ as $n \to \infty$

Compare to

Claim: Let $f: \mathbb{R} \to \mathbb{R}$ be a differentiable function with $|f'(t)| \le K$. Then for any times $t_0 < t_1 < \cdots < t_n$,

$$\sum_{j=1}^{n} |f(t_j) - f(t_{j-1})| \le K \cdot (t_n - t_0).$$

Proof:

$$|f(t_j)-f(t_{j-1})| \le \max_{t\in[t_{j-1},t_j]} |f'(t)| \cdot (t_j - t_{j-1})$$

$$\Rightarrow \sum_{j=1}^{n} |f(t_j) - f(t_{j-1})| \le K \cdot \sum_{j=1}^{n} (t_j - t_{j-1})$$
$$= K(t_n - t_0) \checkmark \qquad \square$$

Hitting times for standard Brownian motion

Proposition: If $X(t)$ is standard BM,

a) For
$$\tau = \min\{t \geq 0 : X(t) \in \{-A, B\}\},$$
$$\mathbb{P}[X(\tau) = B] = \frac{A}{A+B}$$
$$\mathbb{E}[\tau] = A \cdot B.$$

b) For $\tau = \min\{t \geq 0 : X(t) = B\}$
$$\mathbb{P}[\tau < \infty] = 1, \text{ but } \mathbb{E}[\tau] = \infty.$$

Proof idea for a):

Start with the discrete random walk
$$S_t = \Delta x (X_1 + \cdots + X_{t/\Delta t}) \text{ with } \Delta x = \sqrt{\Delta t}$$

$$\frac{A/\Delta x}{A/\Delta x + B/\Delta x} = \frac{A}{A+B} \checkmark$$

$$\left(\frac{A}{\Delta x}\right) \cdot \left(\frac{B}{\Delta x}\right) \cdot \Delta t = A \cdot B \checkmark$$

We can give a rigorous proof using martingales and the Martingale Stopping Theorem.

Proof of b)

$$\mathbb{P}[X(t) \geq B] = \mathbb{P}[X(t) \geq B \mid \tau \leq t] \cdot \mathbb{P}[\tau \leq t] + \mathbb{P}[X(t) \geq B \mid \tau > t] \cdot \mathbb{P}[\tau > t]$$
$$\mathbb{P}[X(t) \geq B \mid \tau \leq t] = 1/2, \quad \mathbb{P}[X(t) \geq B \mid \tau > t] = 0$$

$$\Rightarrow \mathbb{P}[\tau \leq t] = 2\,\mathbb{P}[X(t) \geq B]$$
$$= \frac{2}{\sqrt{2\pi t}} \int_B^\infty e^{-x^2/2t}\, dx$$
$$y = x/\sqrt{t}$$
$$= \frac{2}{\sqrt{2\pi}} \int_{B/\sqrt{t}}^\infty e^{-y^2/2}\, dy$$

$$\therefore\ \mathbb{P}[\tau<\infty] = \lim_{t\to\infty} \mathbb{P}[\tau \le t] = 1 \quad ✓$$

$$\mathbb{E}[\tau] = \int_0^\infty \mathbb{P}[\tau > t]\,dt$$

$$= \int_0^\infty \left(1 - \frac{2}{\sqrt{2\pi}} \int_{B/\sqrt{t}}^\infty e^{-y^2/2} dy\right) dt$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^\infty dt \int_0^{B/\sqrt{t}} dy\ e^{-y^2/2} \qquad y \le \frac{B}{\sqrt{t}} \Leftrightarrow t \le \frac{B^2}{y^2}$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^\infty dy \int_0^{B^2/y^2} dt\ e^{-y^2/2}$$

$$= \frac{2B^2}{\sqrt{2\pi}} \int_0^\infty dy \frac{1}{y^2} e^{-y^2/2} \qquad = \frac{2}{\sqrt{2\pi}} \int_{B/\sqrt{t}}^\infty dy\, e^{-y^2/2}$$

$$> \frac{2B^2 e^{-1/2}}{\sqrt{2\pi}} \int_0^1 dy \frac{1}{y^2} = \infty \quad ✓$$

Remark: We also get

$$\mathbb{P}[\max_{0\le s\le t} X(s) \ge B] = \mathbb{P}[\tau \le t]$$
$$= 2\,\mathbb{P}[X(t) \ge B]$$

Exercises:

1. A particle moves on a line according to a standard BM, $X(t)$.

a) What is its expected position at $t=6$?
What is the variance of its position at $t=6$?

$$\mathbb{E}[X(6)] = 0,\ \mathrm{Var}(X(6)) = 6.$$

b) If the particle is at position 1.7 at time $t=2$, what is its expected position at $t=6$?

$$\mathbb{E}[X(6)\,|\,X(2)=1.7]$$
$$= 1.7 + \mathbb{E}[X(6)-X(2)\,|\,X(2)=1.7]$$
$$= 1.7 + \mathbb{E}[X(6)-X(2)] \quad \text{(independent increments)}$$
$$\mathbb{E}[X(4)] = 0 \quad \text{(stationary increments)}$$
$$= 1.7$$

c) What is the probability that the particle hits level 10 before level -2? What is the expected time until either 10 or -2 is hit?

$A=2,\ B=10 \qquad \frac{2}{10+2}=\frac{1}{6},\quad 10\cdot 2=20.$

(2) The price of a commodity moves according to a BM,

$P(t)=\sigma X(t)+\mu t,$

with variance term $\sigma^2=4$ and drift $\mu=-5$.

Given that $P(8)=4$, what is the probability that $P(9)<1$?

$$\mathbb{P}[P(9)<1 \mid P(8)=4]$$
$$=\mathbb{P}[2X(9)-5\cdot 9<1 \mid 2X(8)-5\cdot 8=4]$$
$$=\mathbb{P}[X(9)<23 \mid X(8)=22]$$
$$=\mathbb{P}[X(1)<1]$$
$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{1} dt\, e^{-t^2/2} \quad \checkmark$$

```
CDF[NormalDistribution[], 1.]
0.841345
```

```
>> normcdf(1)
ans =
    0.8413
```


$S(t)=S_0 e^{X(t)}$

(3) A stock price per share moves according to geometric BM

Suppose that $S_0=4$. What is the probability that the stock price will reach a high of 7 before a low of 2?

$$4e^x=7 \iff x=\log\frac{7}{4}$$
$$4e^x=2 \iff x=\log\frac{1}{2}=-\log 2$$
$$\frac{\log 2}{\log 2+\log\frac{7}{4}}=\frac{\log 2}{\log\frac{7}{2}} \quad \checkmark$$

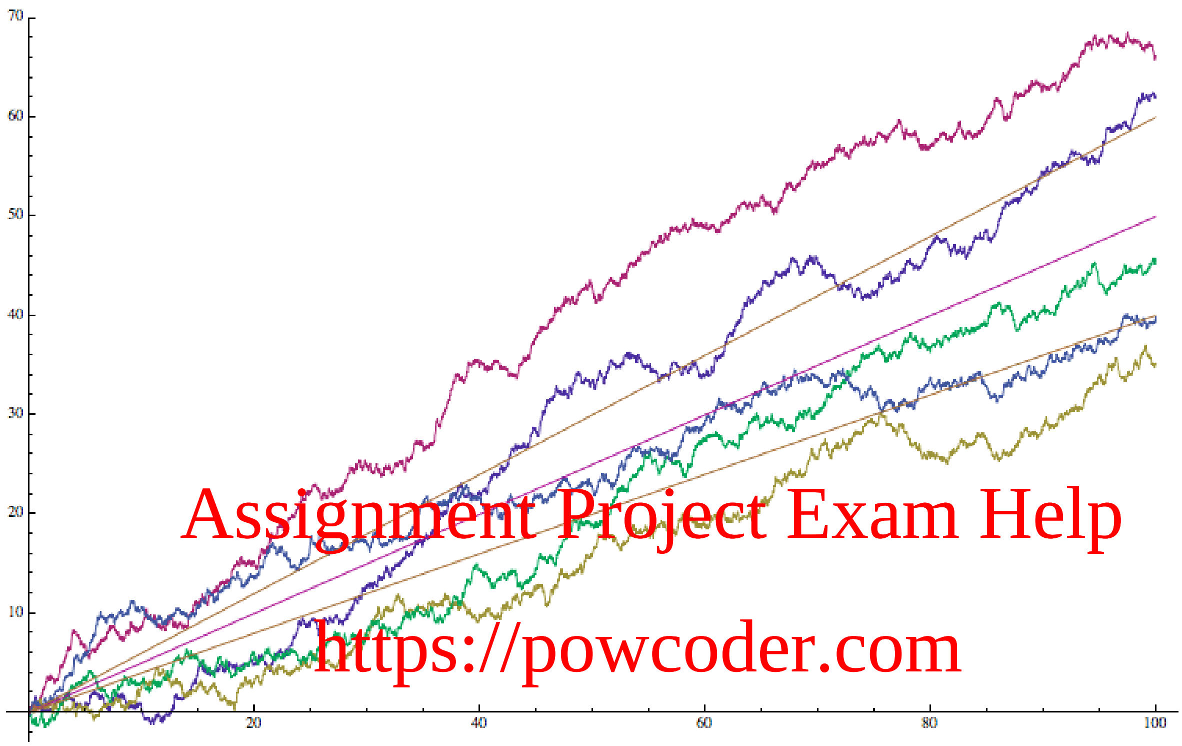
Brownian motion with drift

$$Y(t)=\underbrace{X(t)}_{\text{standard BM}}+\mu t$$

$$\mathbb{E}[Y(t)]=\mu t$$

$$Var(Y(t))=Var(X(t))=t$$

```
μ = .5;
σ2 = 1;
t = 100;
ListLinePlot[Join[
  Table[RandomFunction[WienerProcess[μ, σ2], {0, t, .01}], {5}],
  {{{0, 0}, {t, μ t}}, {{t, μ t + √t}, {0, 0}, {t, μ t - √t}}}
 ]
]
```



Equivalent definitions:

① Gaussian process with

$$\mathbb{E}[Y(t)] = \mu t, \quad \mathrm{Cov}(Y(s), Y(t)) = \min(s,t)$$

② Limit of discrete, biased random walks

$$S_t = \Delta x \cdot \underbrace{(X_1 + \cdots + X_{t/\Delta t})}_{\text{iid} \begin{cases} +1 & \text{w/ prob. } p \\ -1 & \text{w/ prob } 1-p \end{cases}} \qquad \Delta x = \sqrt{\Delta t}$$

$$\mathbb{E}[S_t] = \Delta x \cdot \frac{t}{\Delta t} \cdot (p - (1-p))$$

$$= \frac{t}{\sqrt{\Delta t}} \cdot (2p-1) = \mu t \quad \text{if} \quad p = \tfrac{1}{2}(1 + \mu\sqrt{\Delta t})$$

Hitting times for BM with drift

Just as for random walks, we can study the hitting times using martingales:

Claim: If $X(t)$ is standard BM, then

- $X(t)$
- $X(t)^2 - t$
- $e^{cX(t) - \frac{c^2}{2}t}$

are all martingales.

Proof: For $s < t$,

$$\mathbb{E}[X(t) \mid X(u), 0 \le u \le s]$$
$$\parallel$$
$$X(s) + (X(t) - X(s))$$
$$= X(s) + \mathbb{E}[X(t-s)] \quad \text{stationary, indep. increments}$$
$$= X(s) \;\checkmark$$

$$\mathbb{E}[X(t)^2 - t \mid X(u), 0 \le u \le s]$$
$$= X(s)^2 - t + X(s)\cdot\mathbb{E}[X(t-s)] + \underbrace{\mathbb{E}[X(t-s)^2]}_{= t-s}$$
$$= X(s)^2 - s \;\checkmark$$

$$\mathbb{E}[\exp(cX(t) - \tfrac{c^2}{2}t) \mid X(s)]$$
$$= e^{cX(s) - \frac{c^2}{2}t} \cdot \underbrace{\mathbb{E}[\exp(cX(t-s))]}_{= e^{(t-s)\frac{c^2}{2}}} = e^{cX(s) - \frac{c^2}{2}s} \;\checkmark$$

$\square$

Table 1.4.2

Continuous Probability Distribution	Probability Density Function, $f(x)$	Moment Generating Function, $\phi(t)$ $= \mathbb{E}[e^{tX}]$	Mean	Variance
Uniform over (a, b)	$\frac{1}{b-a}, a < x < b$	$\frac{e^{tb} - e^{ta}}{t(b-a)}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Exponential with parameter $\lambda > 0$	$\lambda e^{-\lambda x}, x \ge 0$	$\frac{\lambda}{\lambda - t}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
Gamma with parameters (n, λ), $\lambda > 0$	$\frac{\lambda e^{-\lambda x}(\lambda x)^{n-1}}{(n-1)!}, x \ge 0$	$\left(\frac{\lambda}{\lambda - t}\right)^n$	$\frac{n}{\lambda}$	$\frac{n}{\lambda^2}$
Normal with parameters (μ, σ^2)	$\frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/2\sigma^2}, -\infty < x < \infty$	$\exp\left\{\mu t + \frac{\sigma^2 t^2}{2}\right\}$	μ	σ^2
Beta with parameters $a, b, a > 0, b > 0$	$cx^{a-1}(1-x)^{b-1}, 0 < x < 1$; $c = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}$		$\frac{a}{a+b}$	$\frac{ab}{(a+b)^2(a+b+1)}$

$$Y(t) = X(t) + \mu t$$
$$T = \min\{t : Y(t) \in \{-A, +B\}\}.$$

Martingale stopping

$$\Rightarrow \quad 0 = \mathbb{E}[X(0)] = \mathbb{E}[X(T)]$$

$$= p\,\mathbb{E}[B-\mu T \mid Y(T)=B] + (1-p)\,\mathbb{E}[-A-\mu T \mid Y(T)=-A]$$

where $p = \mathbb{P}[Y(T)=B]$

$1-p = \mathbb{P}[Y(T)=-A]$

$$= -\mu\,\mathbb{E}[T] - A + p(A+B)$$

$$\Rightarrow \mathbb{E}[T] = \frac{1}{\mu}\big(p(A+B) - A\big)$$

To find p, use the third MG:

MG stopping

$$\Rightarrow 1 = \mathbb{E}\left[\exp\left(cX(0) - \tfrac{c^2}{2}0\right)\right]$$

$$= \mathbb{E}\left[\exp\left(cX(T) - \tfrac{c^2}{2}T\right)\right]$$

$$c(Y(T) - \mu T) - \tfrac{c^2}{2}T$$

$$= -2\mu Y(T) \quad \text{for } c = -2\mu$$

$$= p \cdot e^{-2\mu B} + (1-p)\,e^{+2\mu A}$$

$$\Rightarrow p = \frac{1 - e^{+2\mu A}}{e^{-2\mu B} - e^{+2\mu A}}$$

Observe: If $\mu < 0$, letting $A \to \infty$ we get

$$\mathbb{P}[Y(t) \text{ ever reaches } B] = e^{2\mu B}$$

If $\mu > 0$, $\mathbb{P}[Y(t) \text{ ever reaches } B] = 1$, and

$$\mathbb{E}[\text{time to reach } B] = \frac{1}{\mu}\big(p(A+B) - A\big) = \frac{B}{\mu}$$

Example: Value of a perpetual American call option

Suppose the price of a stock is given by

$$S(t) = S_0 \cdot \exp(\sigma \cdot X(t) - \mu t)$$

where X is standard BM and $\mu > 0$.

We are given the **option** of buying the stock at price P, at any time in the future.

When should we exercise the option?, and

What is our expected return?

Answer:

The profit from using the option is $S(t) - P$.

Obviously, we shouldn't use the option if $S(t) < P$.

But when should we use it?

Consider the policy: use the option if $S(t) = Q$.

$$E[\text{profit}] = (Q-P)\cdot \mathbb{P}[S(t) \text{ ever reaches } Q]$$

$$\Updownarrow$$

$$\sigma X(t) - \mu t = \log Q$$

$$X(t) - \frac{\mu}{\sigma} t = \frac{1}{\sigma}\log Q$$

$$\Rightarrow \mathbb{P}[S(t) \text{ ever reaches } Q]$$

$$= \exp\left(-\frac{2\mu}{\sigma}\frac{1}{\sigma}\log Q\right) = Q^{-2\mu/\sigma^2}$$

$$E[\text{profit}] = (Q-P)\cdot Q^{-2\mu/\sigma^2}$$

Now maximize over Q:

```
D[(Q - P) Q^(-2μ/σ^2), Q] // FullSimplify
Solve[% == 0, Q]
```

$$\frac{Q^{-1-\frac{2\mu}{\sigma^2}}\left(2 P \mu + Q\left(-2\mu + \sigma^2\right)\right)}{\sigma^2}$$

$$\left\{\left\{Q \to \frac{2 P \mu}{2\mu - \sigma^2}\right\}\right\} = \boxed{P \cdot \frac{1}{1 - \frac{\sigma^2}{2\mu}}}$$

observe Q increases with volatility σ^2

decreases with drift μ.

Example: Simulating geometric BM with drift

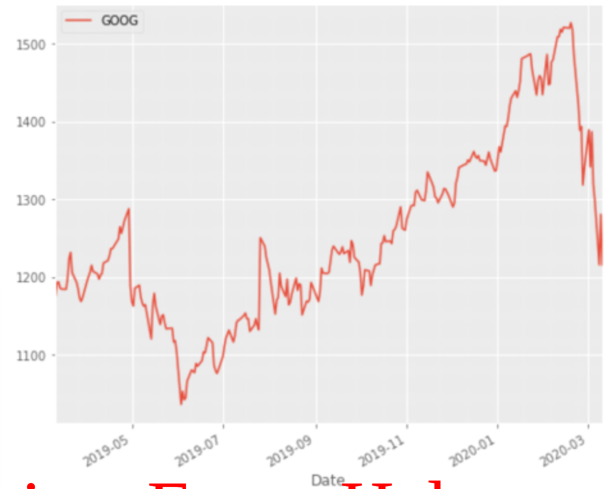
```python
import pandas as pd
import pandas_datareader.data as web
import datetime

start = datetime.datetime(2019, 3, 11)
end = datetime.datetime(2020, 3, 11)
df = web.DataReader("GOOG", 'yahoo', start, end)

from IPython.display import display

display(df.tail())

close_prices = df['Adj Close']
```

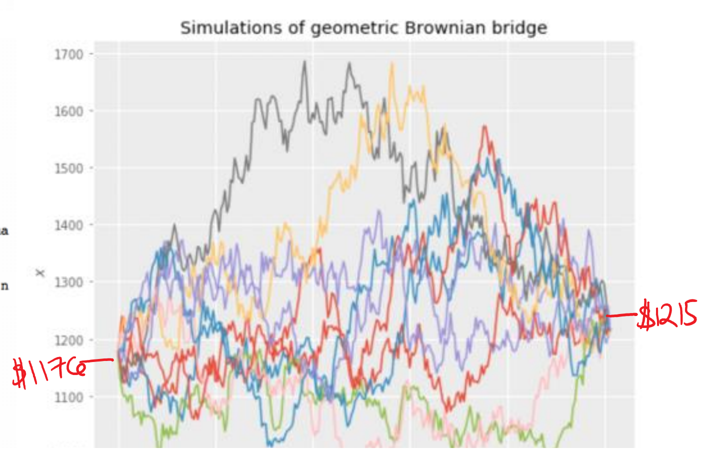
Date	High	Low	Open	Close	Volume	Adj Close
2020-03-05	1358.910034	1305.099976	1350.199951	1319.040039	2561300	1319.040039
2020-03-06	1306.219971	1261.050049	1277.060059	1298.410034	2660600	1298.410034
2020-03-09	1254.760010	1200.000000	1205.300049	1215.560059	3365400	1215.560059
2020-03-10	1281.150024	1218.770020	1260.000000	1280.390015	2609900	1280.390015
2020-03-11	1260.959961	1205.500000	1249.699951	1215.410034	2611229	1215.410034

```python
import numpy as np

returns = close_prices.pct_change()
# can also use close_px / close_px.shift(1) - 1

mu = np.mean(returns)      .0003
sigma = np.std(returns)    .02

# better expressions are: (see GBM on Wikipedia)
#   from math import log, exp
#   mu = log(mean + 1)
#   sigma2 = log(variance / (mean + 1)**2 + 1)
# but those are almost the same

# Now let us run a simulation

trials = 10

x0 = close_prices[0]
n = 253 # approx. # trading days in a year
dt = 1
np.random.seed(1)

x = np.exp(
    (mu - sigma ** 2 / 2) * dt
    + sigma * np.random.normal(0, np.sqrt(dt), size=(n, trials))
)
x = np.vstack([np.ones(trials), x])
x = x0 * x.cumprod(axis=0)

plt.plot(x)
plt.xlabel("$t$")
plt.ylabel("$x$")
plt.title("Simulations of geometric BM with drift")
plt.show()
```

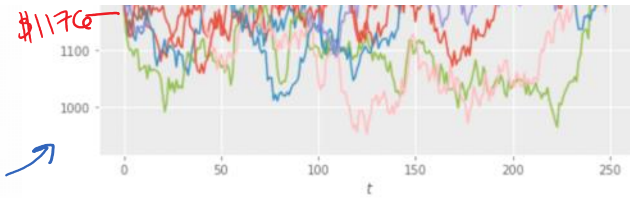
```python
# Simulating a Brownian bridge

trials = 10

x0 = close_prices[0]
xf = close_prices[-1]
n = len(close_prices) - 1
dt = 1
np.random.seed(1)

brownian_bridge_final = (np.log(xf/x0) - (mu - sigma**2 / 2) * n * dt) / sigma
brownian_steps = np.random.normal(0, np.sqrt(dt), size=(n, trials))
brownian_bridge_steps = brownian_steps \
            + brownian_bridge_final / n - np.sum(brownian_steps, axis=0) / n

x = np.exp(
    (mu - sigma ** 2 / 2) * dt
    + sigma * brownian_bridge_steps
)
x = np.vstack([np.ones(trials), x])
x = x0 * x.cumprod(axis=0)
```

```python
%matplotlib inline
import matplotlib as mpl
import matplotlib.pyplot as plt
mpl.rc('figure', figsize=(8, 7)) # adjust size
mpl.style.use('ggplot') # adjust style

close_prices.plot(label='GOOG')
plt.legend();
```





```
    + sigma * brownian_bridge_steps
)
x = np.vstack([np.ones(trials), x])
x = x0 * x.cumprod(axis=0)

plt.plot(x)
plt.xlabel("$t$")
plt.ylabel("$x$")
plt.title("Simulations of geometric BM with drift")
plt.show()

np.around((x0, xf), decimals=2)
```




```
array([1175.76, 1215.41])
```