Lecture 19: Brownian motion

Admin:

Definition: A stochastic process $\{X(t) : t \geq 0\}$ is a **Gaussian process** if for every $n \geq 1$ and all times $t_1, \ldots, t_n$, the variables

$$(X(t_1), X(t_2), \ldots, X(t_n))$$

have a multivariate Gaussian distribution.

Observe: A Gaussian process is characterized by
the means $\mathbb{E}[X(t)]$,
& the covariances $\mathrm{Cov}(X(s), X(t))$.

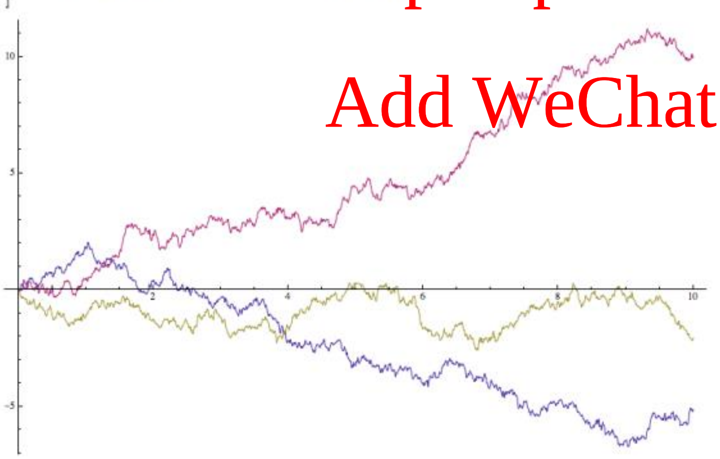
Definition: **Standard Brownian motion** is a Gaussian process with

$$\mathbb{E}[X(t)] = 0 \quad \text{and} \quad \mathrm{Cov}(X(s), X(t)) = \min\{s, t\}$$

$$\Downarrow$$

$$\mathrm{Var}(X(t)) = \mathbb{E}[X(t)^2] = t$$


```
ListLinePlot[
 Table[RandomFunction[WienerProcess[], {0, 10, .01}], {3}]
]
```



Diffusion equations [Ross §8.5]

Remark: The probability density,

$$f(X(t) = y \mid X(s) = x) = \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{(y-x)^2}{2(t-s)}}$$

satisfies

$$\frac{\partial f}{\partial t} = \frac{1}{2} \frac{\partial^2 f}{\partial y^2} \qquad \text{"forward diffusion equation"}$$

$$\frac{\partial f}{\partial t} = -\frac{1}{2} \frac{\partial^2 f}{\partial x^2} \qquad \text{"backward diffusion equation"}$$

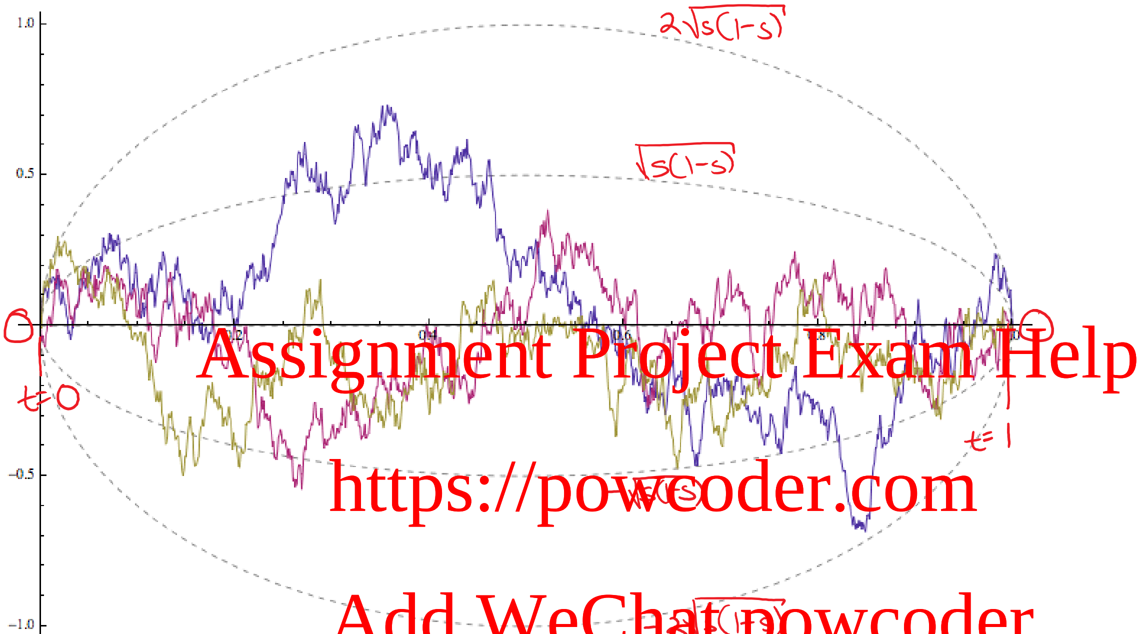
and is their unique solution.

These equations are easy to verify, or can be derived, following Einstein, by conditiong on $X(t-h)$ or $X(s+h)$, respectively, and letting $h \to 0$.

Example:

"Brownian bridge"

```
bridgeplot = ListLinePlot[
   Table[RandomFunction[BrownianBridgeProcess[], {0, 1, .001}], {3}]
  ];
stddevplot = Plot[√s (1 - s) Range[-2, 2], {s, 0, 1}, PlotStyle → {Gray, Dashed}];
Show[stddevplot, bridgeplot]
```



Problem: For $X(t)$ standard Brownian motion, what is the distribution of $X(s)$ conditioned on $X(t)=B$?

Answer:

For $s \geq t$, $X(s)$ is a Gaussian process with mean B,

$$\mathrm{Cov}(X(s_1), X(s_2)) = \min(s_1 - t, s_2 - t).$$

the same distribution as

$$B + X(s-t).$$

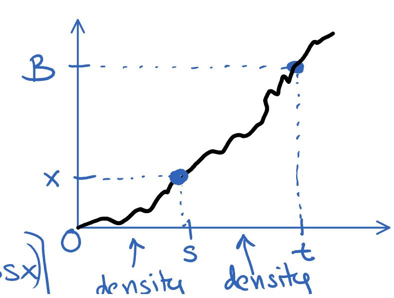
For $0 \leq s \leq t$, $X(s)$ is still a Gaussian process (starting with a multivariate Gaussian and conditioning on one of the variables, $X(t)$, still leaves a Gaussian dist$^{\underline{n}}$).

The conditional density is

$$f(X(s)=x \mid X(t)=B) \propto f_s(x) f_{t-s}(B-x)$$

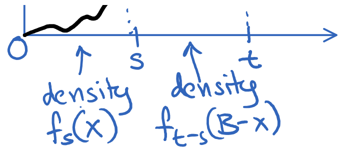
$$\propto \exp\left[\frac{-x^2}{2s} - \frac{(B-x)^2}{2(t-s)}\right]$$

$$\propto \exp\left[\frac{-1}{2s(t-s)}(tx^2 - 2Bsx)\right]$$



$$\propto \exp\left[\frac{-1}{2s(t-s)}\left(tx^2 - 2Bsx\right)\right]$$

$$\propto \exp\left[\frac{-t}{2s(t-s)}\left(x - \frac{Bs}{t}\right)^2\right]$$



$$\Rightarrow \mathbb{E}\left[X(s) \mid X(t) = B\right] = \frac{Bs}{t}$$

$$\mathrm{Var}\left(X(s) \mid X(t) = B\right) = \frac{s(t-s)}{t} = t \cdot \frac{s}{t}\left(1 - \frac{s}{t}\right)$$

$\Rightarrow \mathrm{Cov}\left(X(s_1), X(s_2) \mid X(t) = 0\right)$ for $s_1 \le s_2 \le t$

$$= \mathbb{E}\left[X(s_1)\left(X(s_1) + X(s_2) - X(s_1)\right) \mid X(t) = 0\right]$$

$$= \frac{s_1(t-s_1)}{t} + \underbrace{\mathbb{E}\left[X(s_1)\left(X(s_2) - X(s_1)\right) \mid X(t) = 0\right]}_{\mathbb{E}\left[X(s_1)\cdot\left(\frac{t-s_2}{t-s_1} - 1\right)X(s_1) \mid X(t) = 0\right]}$$

$$= \frac{s_1(t-s_1)}{t}\left[1 - \frac{s_2 - s_1}{t - s_1}\right]$$

$$= \frac{s_1}{t}\left(t - s_2\right) \quad \checkmark$$

Observe: Consider $Y(t) = X(t) - tX(1)$.

For $0 \le t \le 1$, $Y(t)$ has the same dist^n as $X(t) \mid X(1) = 0$.

Indeed,

$$\mathbb{E}[Y(t)] = 0$$

and for $0 \le s \le t \le 1$,

$$\mathrm{Cov}(Y(s), Y(t)) = \mathbb{E}\left[\left(X(s) - sX(1)\right)\left(X(t) - tX(1)\right)\right]$$

$$= \mathbb{E}\left[X(s)X(t) - sX(t)X(1) - tX(s)X(1) + st\,X(1)^2\right]$$

$$= s - s \cdot t - t \cdot s + st$$

$$= s(1-t) \quad \checkmark$$

Exercise: Verify that $Z(t) = (1-t)\,X\left(\frac{t}{1-t}\right)$, $Z(1) = 0$, has the same dist^n as a Brownian bridge on $[0,1]$.

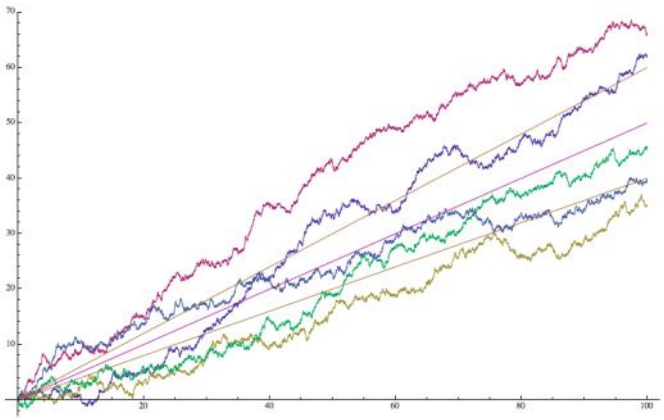
Brownian motion with drift

$$Y(t) = \underbrace{X(t)}_{\text{standard BM}} + \mu t$$

is a Gaussian process with $\mathbb{E}[Y(t)] = \mu t$

$$\mathrm{Cov}(Y(s), Y(t)) = \min(s,t).$$

```
μ = .5;
σ2 = 1;
t = 100;
ListLinePlot[Join[
  Table[RandomFunction[WienerProcess[μ, σ2], {0, t, .01}], {5}],
  {{{0, 0}, {t, μ t}}, {{t, μ t + √t}, {0, 0}, {t, μ t - √t}}}
 ]
]
```



Hitting times for BM with drift

Just as for random walks, we can study the hitting times using martingales:

Claim: If $X(t)$ is standard BM, then

- $X(t)$ • $X(t)^2 - t$ • $e^{cX(t) - \frac{c^2}{2}t}$

are all martingales.

Proof for $\exp[cX(t) - \frac{c^2}{2}t]$:

> Recall: if $X \sim N(\mu, \sigma^2)$,
> $\mathbb{E}[e^{cX}] = e^{c\mu + \frac{c^2\sigma^2}{2}}$ mgf.

$$\therefore\ \mathbb{E}\left[\exp\left(cX(t) - \tfrac{c^2}{2}t\right) \middle| X(s)\right]$$

$$= e^{cX(s) - \frac{c^2}{2}t} \cdot \underbrace{\mathbb{E}[\exp(cX(t-s))]}_{= e^{(t-s)\frac{c^2}{2}}} = e^{cX(s) - \frac{c^2}{2}s}\ \checkmark$$

$\square$

$$Y(t) = X(t) + \mu t$$

$$T = \min\{t : Y(t) \in \{-A, +B\}\}.$$

Martingale stopping

$$\Rightarrow\ 0 = \mathbb{E}[X(0)] = \mathbb{E}[X(T)]$$

$$= p\,\mathbb{E}[B - \mu T \mid Y(T) = B] + (1-p)\,\mathbb{E}[-A - \mu T \mid Y(T) = -A]$$

where $p = \mathbb{P}[Y(T) = B]$

where $p = \mathbb{P}[Y(T) = B]$

$1-p = \mathbb{P}[Y(T) = -A]$

$= -\mu \mathbb{E}[T] - A + p(A+B)$

$$\Rightarrow \mathbb{E}[T] = \frac{1}{\mu}\left(p(A+B) - A\right)$$

To find p, use the third MG:

MG stopping

$$\Rightarrow 1 = \mathbb{E}\left[\exp\left(cX(0) - \frac{c^2}{2} 0\right)\right]$$

$$= \mathbb{E}\left[\exp\left(\underbrace{cX(T) - \frac{c^2}{2}T}_{\substack{c(Y(T) - \mu T) - \frac{c^2}{2}T \\ = -2\mu Y(T) \text{ for } c = -2\mu}}\right)\right]$$

$= p \cdot e^{-2\mu B} + (1-p)e^{+2\mu A}$

$$\Rightarrow p = \frac{1 - e^{+2\mu A}}{e^{-2\mu B} - e^{+2\mu A}}$$

Observe: If $\mu < 0$, letting $A \to \infty$ we get

$\mathbb{P}[Y(t)$ ever reaches $B] = e^{2\mu B}$

If $\mu > 0$, $\mathbb{P}[Y(t)$ ever reaches $B] = 1$, and

$\mathbb{E}[\text{time to reach } B] = \frac{1}{\mu}\left(p(A+B) - A\right) = \frac{B}{\mu}$

Geometric Brownian motion

> **Recall**: If $X \sim N(\mu, \sigma^2)$,
> $\mathbb{E}[e^{cX}] = e^{c\mu + \frac{c^2\sigma^2}{2}}$ mgf.

$$Y(t) = e^{\sigma X(t)}$$

$\mathbb{E}[Y(t)] = e^{\sigma^2 t/2}$, $\mathrm{Var}(Y(t)) = \mathbb{E}(Y(t)^2) - (\mathbb{E}Y(t))^2$

$= e^{2\sigma^2 t} - e^{\sigma^2 t}$

Example 1: Value of a European call option:

Suppose a stock's price is given by

$$S(t) = S_0 \cdot e^{\sigma X(t) + \mu t}$$

At time T in the future, you have the **option** of buying the stock for price K.
What is the expected worth of the option?

$$\mathbb{E}\left[\max(S(T)-K, 0)\right]$$
$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi T}} e^{-x^2/2T} \cdot \max(0, S_0 e^{\sigma x + \mu T} - K)$$

$$\overset{\geq}{0} \quad \text{for } x \geq \frac{1}{\sigma}\left(\log\frac{K}{S_0} - \mu T\right)$$

Example 2: Value of an Asian call option

$$\max\left(0, \frac{1}{T}\sum_{n=1}^{T} S(n)\right)$$

To simulate this, use

$$S(n+1) = S(n) \cdot e^{\mu + \sigma(X(n+1) - X(n))}.$$
$$\sim N(0,1)$$

Example 3: A stock portfolio

What if you have two stocks?

$$S_1(t) = S_1(0) \cdot e^{\sigma_1 X_1(t) + \mu_1 t}$$
$$S_2(t) = S_2(0) \cdot e^{\sigma_2 X_2(t) + \mu_2 t}$$

$X_1(t)$ & $X_2(t)$ can be independent std. BM.
But what if changes in stock prices are correlated?

Observe: If $X, Y \sim N(0,1)$, $X \perp\!\!\!\perp Y$,
then $Z = \cos\theta\, X + \sin\theta\, Y \sim N(0,1)$
with $\text{cov}(X, Z) = \cos\theta$.

$\Rightarrow$ If $X(t) \perp\!\!\!\perp Y(t)$ are standard BM processes,
$Z(t) = \cos\theta\, X(t) + \sin\theta\, Y(t)$ is std BM,
with
$$\text{Cov}(X(s), Z(t)) = \cos\theta \cdot \text{Cov}(X(s), X(t))$$
$$= \cos\theta \cdot \min(s,t).$$

Example: If $\mu_1 = .01$, $\sigma_1^2 = 1$, $\mu_2 = .02$, $\sigma_2^2 = 2$,
$\text{Cov}(X_1(t), X_2(t)) = \frac{1}{\sqrt{2}} t$,
$S_1(0) = 1$, $S_2(0) = 2$,
what is $\mathbb{P}[S_1(10) + S_2(10) > 1.1(S_1(0) + S_2(0))]$?

Answer: While we can get a closed form involving a double integral, it is more practical just to simulate it:

```
t = 10;
μ1 = .01; σ1 = 1;
μ2 = .02; σ2 = Sqrt[2];
S1 = 1;
S2 = 2;

numtrials = 10^6;
wins = 0;
For[trial = 1, trial ≤ numtrials, trial++,
  {x, y} = RandomVariate[NormalDistribution[0, Sqrt[t]], 2];
  z = 1/Sqrt[2] (x + y);
  If[S1 e^(σ1 x + μ1 t) + S2 e^(σ2 z + μ2 t) > 1.1 (S1 + S2), wins++];
 ];
wins/numtrials // N
0.570815
```


Remark: To simulate standard BM at times
$0 = t_0 < t_1 < t_2 < \cdots < t_k$,
generate $Z_1, Z_2, \ldots, Z_k \sim N(0,1)$ iid.
and let
$$X(t_j) = \sum_{i=1}^{j} \sqrt{t_i - t_{i-1}}\, Z_i$$
$$= X(t_{j-1}) + \sqrt{t_i - t_{i-1}}\, Z_i \quad ✓$$

Example: Value of a perpetual American call option

Suppose the price of a stock is given by
$$S(t) = S_0 \cdot \exp(\sigma \cdot X(t) - \mu t)$$
where X is standard BM and $\mu > 0$.

Recall: If $X \sim N(\mu, \sigma^2)$, $E[e^{cX}] = e^{c\mu + \frac{c^2\sigma^2}{2}}$ mgf.

downward drift $\Rightarrow ES_t = Se^{\frac{\sigma^2 t}{2} - \mu t}$

We are given the option of buying the stock at price P, at any time in the future.

When should we exercise the option?, and

What is our expected return?

Answer:

The profit from using the option is $S(t) - P$.
Obviously, we shouldn't use the option if $S(t) < P$.
But when should we use it?

Consider the policy: use the option if $S(t) = Q$.

$$\mathbb{E}[\text{profit}] = (Q - P) \cdot \mathbb{P}[S(t) \text{ ever reaches } Q]$$

$$\Updownarrow$$

$$\sigma X(t) - \mu t = \log Q$$
$$X(t) - \frac{\mu}{\sigma} t = \frac{1}{\sigma} \log Q$$

$$\Rightarrow \mathbb{P}[S(t) \text{ ever reaches } Q]$$
$$= \exp\left(\frac{-2\mu}{\sigma} \frac{1}{\sigma} \log Q\right) = Q^{-2\mu/\sigma^2}$$

$$\mathbb{E}[\text{profit}] = (Q - P) \cdot Q^{-2\mu/\sigma^2}$$

Observe: If $\mu < 0$,
$\mathbb{P}[Y(t) \text{ ever reaches } B] = e^{2\mu B}$

Now maximize over Q:


```
D[(Q - P) Q^(-2μ/σ^2), Q] // FullSimplify
Solve[% == 0, Q]
```

$$\frac{Q^{-1-\frac{2\mu}{\sigma^2}} \left(2 P \mu + Q \left(-2\mu + \sigma^2\right)\right)}{\sigma^2}$$

$$\left\{\left\{Q \to \frac{2 P \mu}{2\mu - \sigma^2}\right\}\right\} = \boxed{P \cdot \frac{1}{1 - \frac{\sigma^2}{2\mu}}}$$

observe Q increases with volatility σ^2
decreases with drift μ.

if $\mu \le \frac{\sigma^2}{2}$, $Q = \infty$!

Brownian motion reflected at the origin

$$Z(t) = |X(t)|$$

↖ standard BM

density
$$f_{Z(t)}(z) = 2 f_{X(t)}(z) \quad \text{for } z \geq 0$$
$$= \frac{2}{\sqrt{2\pi t}} e^{-z^2/2t}$$

$$\Rightarrow \mathbb{E}[Z(t)] = \int_0^\infty z f_{Z(t)}(z)\, dz = \sqrt{\frac{2t}{\pi}}$$

$$\mathrm{Var}(Z(t)) = \left(1 - \frac{2}{\pi}\right) t.$$

Recall:

$$\mathbb{P}\left[\max_{0 \le s \le t} X(s) \geq B\right] = \mathbb{P}(T_B \le t) = 2\mathbb{P}[X(t) \geq B]$$

$$\operatorname{Var}(Z(t)) = \left(1-\frac{2}{\pi}\right)t.$$

$$P\left[\max_{0\le s\le t} X(s) \geq B\right] = P[T_B \le t] = 2P[X(t)\geq B]$$

Claim: $|X(t)|$ and $\max_{0\le s\le t} X(s)$ have the same dist$^{\underline{n}}$.

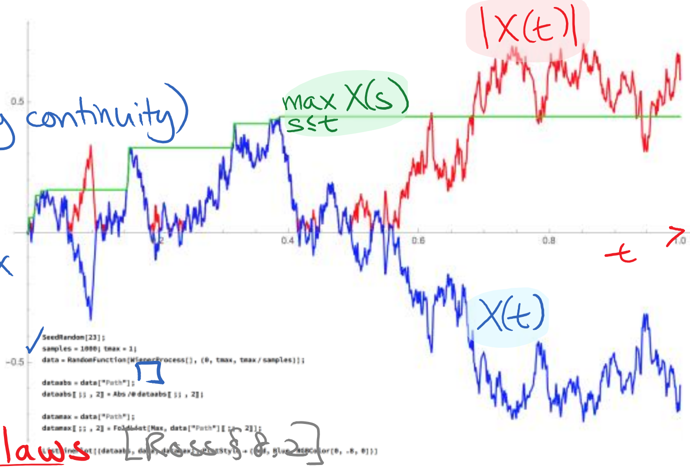
Proof:

$$P\left[\max_{0\le s\le t} X(s) \geq z\right] = P[T_z \le t] \quad \text{(by continuity)}$$

$$= 2P[X(t)\geq z]$$

$$= 2\frac{1}{\sqrt{2\pi t}}\int_z^{\infty} e^{-x^2/2t}\,dx$$

$$P\left[\max_{0\le s\le t} X(s) \le z\right] = 2\frac{1}{\sqrt{2\pi t}}\int_0^{z} e^{-x^2/2t}\,dx$$



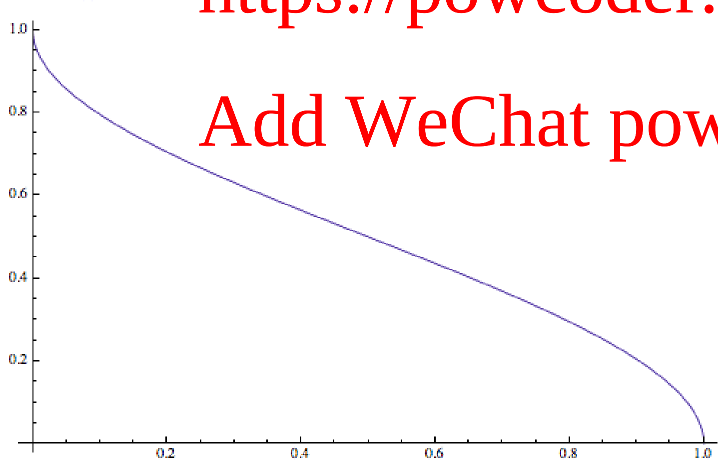
Zeros of Brownian motion, and arc sin laws [Ross §8.2]

Theorem: The probability that a standard B.M. process X has a zero in the time interval (t_0, t_1) is

$$\frac{2}{\pi}\cos^{-1}\left(\sqrt{\frac{t_0}{t_1}}\right)$$

(The probability there's no zero is $\frac{2}{\pi}\sin^{-1}\left(\sqrt{\frac{t_0}{t_1}}\right)$.)


```
Plot[2/π ArcCos[√x], {x, 0, 1}]
```



Consequences:

- $P[\text{there exists a } 0 \text{ in } (0,t)] = 1$ for all $t>0$
- $\inf\{t>0 : X(t)=0\} = 0$ almost surely.
- There are infinitely many zeros in $[0,t]$ almost surely.

harder to show

Proof of the theorem:

Let E be the event that there's a zero in (t_0, t_1).

Conditioning on $X(t_0)$

$$\mathbb{P}[E] = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi t_0}} e^{-\frac{x^2}{2t_0}} \cdot \mathbb{P}[E \mid X(t_0) = x]\, dx$$

$$\mathbb{P}[E \mid X(t_0) = x] = \mathbb{P}[T_{-x} \le t_1 - t_0]$$

$$= \mathbb{P}[T_x \le t_1 - t_0]$$

$$= 2\,\mathbb{P}[X(t_1 - t_0) \ge |x|]$$

$$= \frac{2}{\sqrt{2\pi t_0}} \int_0^{\infty} dx\, e^{\frac{-x^2}{2t_0}} \cdot \frac{2}{\sqrt{2\pi(t_1 - t_0)}} \int_x^{\infty} dz\, e^{-z^2/2(t_1 - t_0)}$$

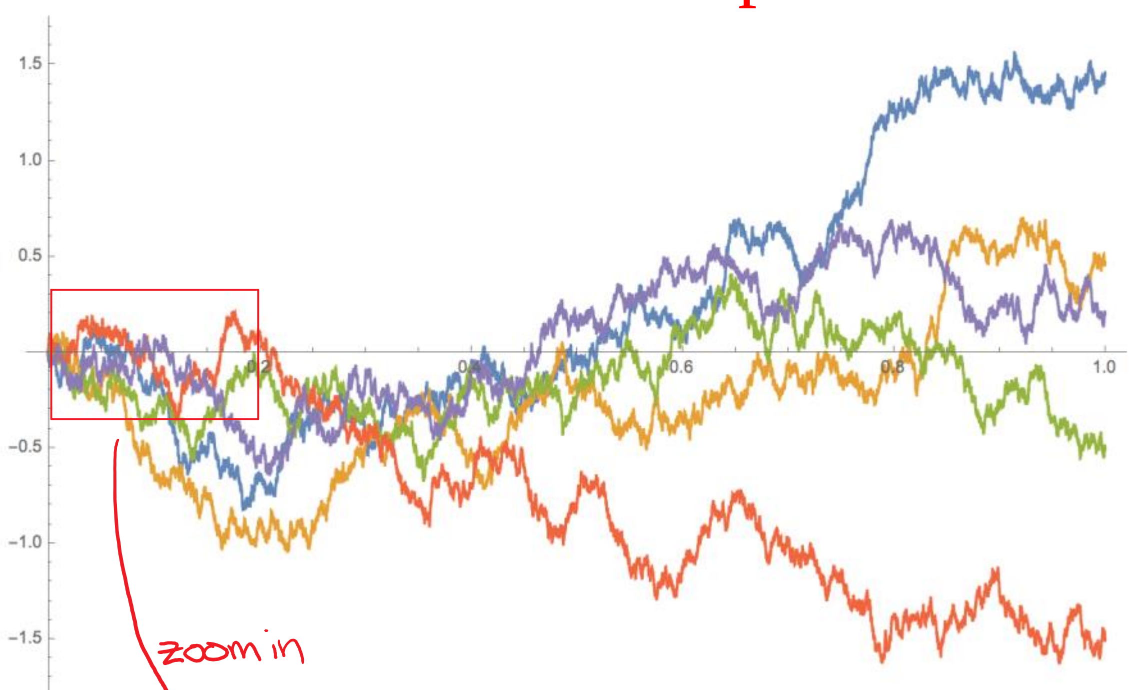
$$\frac{2}{\sqrt{2\pi t_0}} \int_0^{\infty} \left(e^{\frac{-x^2}{2 t_0}} \frac{2}{\sqrt{2\pi (t_1 - t_0)}} \int_x^{\infty} e^{\frac{-z^2}{2(t_1 - t_0)}} dz \right) dx \text{ // Simplify[\#, \{0 < } t_0 < t_1\}] \,\&$$

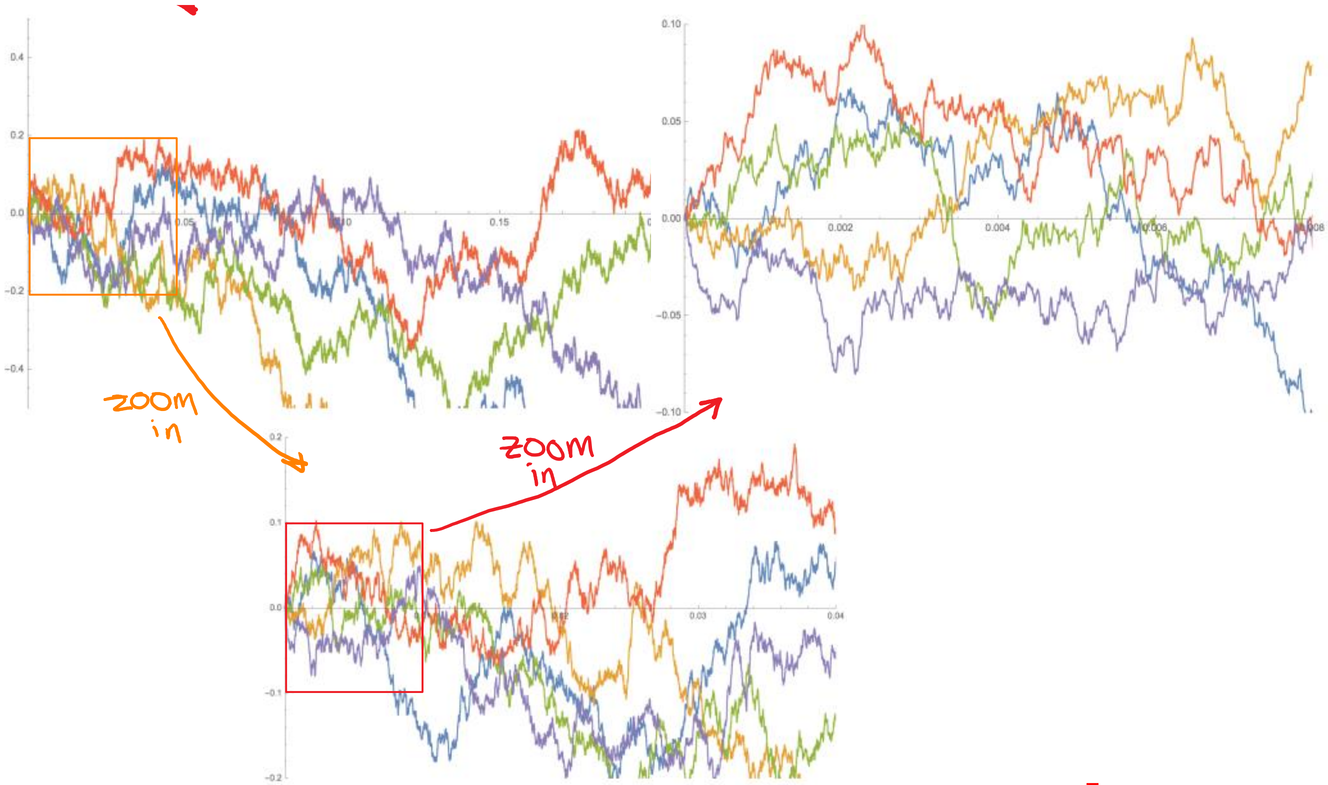
$$\frac{2\,\mathrm{ArcTan}\left[\sqrt{-1 + \frac{t_1}{t_0}}\right]}{\pi}$$

and $\tan^{-1}\sqrt{-1 + \frac{1}{x}} = \cos^{-1}\sqrt{x}$, since

$$\tan \cos^{-1}\sqrt{x} = \frac{\sin(\cos^{-1}\sqrt{x})}{\cos(\cos^{-1}\sqrt{x})} = \frac{\sqrt{1-x}}{\sqrt{x}} = \sqrt{-1 + \frac{1}{x}} \quad \square$$


```
SeedRandom[1];
samples = 100 000;
tmax = 1;
data = RandomFunction[WienerProcess[], {0, tmax, tmax/samples}, 5];
ListLinePlot[data]
```


zoom in
zoom in
0.4
0.2
0.0
-0.2
-0.4
0.05
0.10
0.15
0.1
0.03
0.04
-0.1
-0.2
0.10
0.05
0.00
0.002
0.004
-0.05
-0.10