Lecture 7 Decomposition of State space

Theorem 1. Let $x \to y$. If x is recurrent, then y is recurrent and $\rho_{yx} = \rho_{xy} = 1$. Hence

$$x \Longleftrightarrow y$$

Proof: Let x be recurrent. Then $\rho_{xx} = 1$.

If $y = x$, the result is clearly true.

Assume that $y \neq x$.

Since $x \to y$, it follows that there exists $n \geq 1$ such that $P_x(T_y = n) > 0$. Set

$$n_0 = \min\{n \geq 1 : P_x(T_y = n) > 0\}$$

For any $1 \leq m < n_0$, we have

$$P^m(x, y) = 0, \quad P^{n_0}(x, y) > 0$$

$\Rightarrow$ there exists $x_1, \cdots x_{n_0-1}$ such that

$$x_1, \cdots x_{n_0-1} \neq x, y$$

and $P(x, x_1) > 0$, $P(x_1, x_2) > 0, \cdots P(x_{n_0-1}, y) > 0$

$$\Rightarrow P_x(X_1 = x_1, \cdots X_{n_0-1} = x_{n_0-1}, X_{n_0} = y) > 0$$

If $\rho_{yx} < 1$, then with probability $1 - \rho_{yx} > 0$ the chain will not visit x starting from y.

Thus,

P_x(the chain will not return to x)

$\geq P_x(X_1 = x_1, \cdots, X_{n_0 - 1} = x_{n_0 - 1}, X_{n_0} = y,$
$\quad X_n \neq x$ for all $n > n_0)$

$= P(x, x_1) P(x_1, x_2) \cdots P(x_{n_0-1}, y)$

$\quad P_y($ not visit $x)$

$= P(x, x_1) P(x_1, x_2) \cdots P(x_{n_0-1}, y)(1 - \rho_{yx})$

> 0.

$\Rightarrow$ x is not recurrent. A contradiction.

Hence $\rho_{yx} = 1$.

This implies that there exists $n_1 \geq 1$ such that $P^{n_1}(y, x) > 0$

By direct calculation,

$$G(y,y) = \sum_{m=1}^{\infty} P^m(y,y) \geq \sum_{n=1}^{\infty} P^{n_0+n+n_1}(y,y)$$

$$\geq \sum_{n=1}^{\infty} P^{n_1}(y,x)\, P^{n}(x,x)\, P^{n_0}(x,y)$$

$$= P^{n_1}(y,x)\, P^{n_0}(x,y) \sum_{n=1}^{\infty} P^{n}(x,x)$$

$$= P^{n_1}(y,x)\, P^{n_0}(x,y)\, \underbrace{G(x,x)}_{\infty}$$

$$= \infty \qquad (\text{since } x \text{ is recurrent})$$

Thus y is recurrent.

Putting all these together, we obtain

x recurrent, $x \to y \Rightarrow P_{yx} = 1$, y recurrent

Similarly

y recurrent, $y \to x \Rightarrow P_{xy} = 1$, x recurrent

Definition 1. A subset C of S is said to be closed if for any $x \in C$, $y \notin C$, $\rho_{xy} = 0$ or equivalently $P^n(x,y) = 0$ for $n \geq 1$

Definition 2. A closed set C is said to be irreducible if for any $x, y \in C$ we have $x \leftrightarrow y$.

Theorem 2. Let $\{X_n : n = 0, 1, 2, \dots\}$ be a Markov chain with state space S.

① If C is an irreducible set of recurrent states, then for any $x, y \in C$ we have

$$\rho_{xy} = 1, \quad P_x(N(y) = \infty) = 1$$

$$G(x,y) = \infty$$

② If C is closed, irreducible, and finite, then every state in C is recurrent.

Proof: ① $P_{xy} = 1 = P_{yx}$ follows from theorem 1.

$$P_x(N(y)=\infty) = \lim_{n\to\infty} P_{xy} P_{yy}^{n-1} = 1$$

which implies that

$$G(x,y) = \infty$$

② C is finite and closed. there is at least one recurrent state in C. By theorem 1, all states in C are recurrent.

Example 1 Let $\{X_n : n = 0, 1, 2, \cdots\}$ be a Markov chain with $S = \{0, 1, 2, 3, 4, 5\}$ and

$$\mathbb{P} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{5} & \frac{2}{5} & \frac{1}{5} & 0 & \frac{1}{5} \\ 0 & 0 & 0 & \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{4} & 0 & \frac{3}{4} \end{pmatrix}$$

changed from $\frac{1}{4}$ to $\frac{1}{2}$

Determine the transient and recurrent states of the chain.

Solution: $P(0,0)=1 \Rightarrow 0$ is absorbing and thus recurrent

$3 \to 3$, $3 \to 4$, $3 \to 5$ $\qquad$ $4 \to 3$, $4 \to 5$ $\qquad$ $5 \to 3$, $5 \to 5$

$\Rightarrow \quad 3 \Leftrightarrow 4 \Leftrightarrow 5$

Set $C_1 = \{3,4,5\}$. Then for any $x \in C_1$, $y \notin C_1$, $P(x,y)=0$. Thus

C_1 is closed and irreducible $\Rightarrow$ every state in C_1 is recurrent by theorem 2.

Noting that $1 \to 0$, $2 \to 3$, it follows that 1, 2 are transient.

Final Decomposition.

recurrent

transient

$C_T = \{1,2\}$, $C_R = \{0,3,4,5\}$

$C_1 = \{3,4,5\}$, $C_2 = \{0\}$

$\Rightarrow \quad S = C_T \cup C_R = C_T \cup C_1 \cup C_2$

Theorem 3. Let $\{X_n : n=0,1,2,\dots\}$ be a Markov chain with state space S.

Let C_T = the set of all transient states

C_R = the set of all recurrent states

If $C_R \neq \phi$, there exist at most countable number of disjoint irreducible sets $C_1, C_2, \dots$ such that

$C_R = \bigcup C_i$

Proof: Assume that $C_R \neq \phi$. Then C_R contains at most countable number of states. For any $x \in C_R$, set

$$C(x) = \{ y \in C_R : x \to y \}$$

Since $x \in C_R$ is recurrent, we have $x \to x$. By theorem 1, for any $y \in C(x)$ we have $\rho_{yx} = \rho_{xy} = 1$ and

$$x \leftrightarrow y$$

Let z be any state outside $C(x)$.

Then for any $y \in C(x)$, we have $y \not\Rightarrow z$. Otherwise

$y \leftrightarrow z \Leftrightarrow x$ A contradiction.

Thus $C(x)$ is closed, irreducible.

Let D_1, D_2 be any two such different irreducible sets. If $D_1 \cap D_2 \neq \phi$

and $w \in D_1 \cap D_2$. Then for any

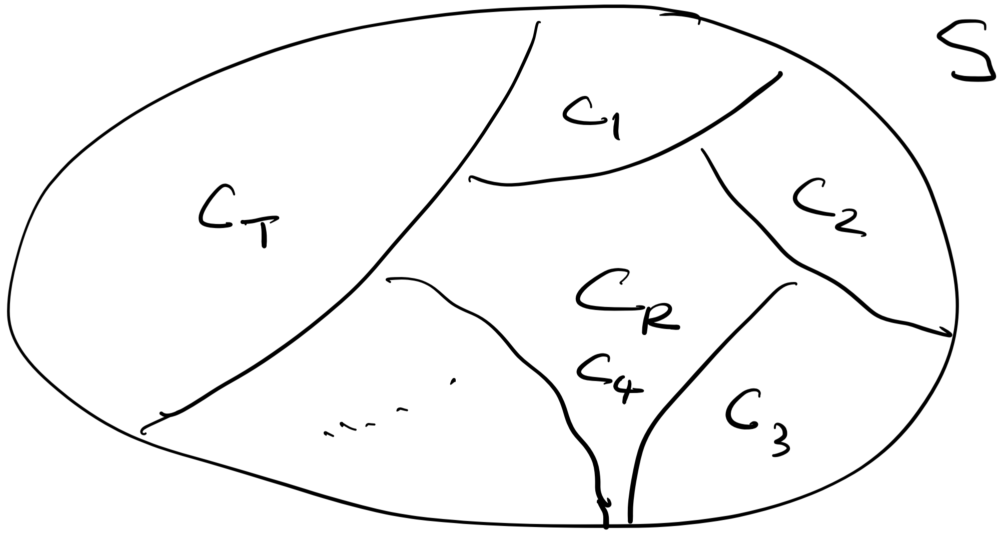
$x \in D_1$, $y \in D_2$ we have

$x \Leftrightarrow w$, $w \Leftrightarrow y$

$\Rightarrow D_1 = D_2$

Thus different D_1, D_2 are disjoint

$\Rightarrow C_R$ = the union of at most countable number of disjoint irreducible sets.



Absorption Probabilities

Definition 3: Let C be a closed, irreducible set of recurrent states.

Define

$$\rho_C(x) = P_x(T_C < \infty)$$

which is the absorption probability of state x by set C.