Lecture 12 Stationary Distribution (cont)

Example 1. Let $\{X_n : n=0,1,2,\dots\}$ be a Markov chain with $S=\{0,1,2,3\}$ and

$$\mathbb{P} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Find the stationary distribution of the chain.

Solution: this is a birth-death Markov chain.

$$d=3 \qquad p_0 = \frac{1}{2}, \quad p_1 = \frac{1}{3}, \quad p_2 = \frac{1}{6}$$

$$\varepsilon_1 = \frac{1}{6} \quad \varepsilon_2 = \frac{1}{3}, \quad \varepsilon_3 = \frac{1}{2}$$

$$\Rightarrow \quad \pi(k) = \frac{\dfrac{p_0 \cdots p_{k-1}}{\varepsilon_1 \cdots \varepsilon_k}}{1 + \dfrac{p_0}{\varepsilon_1} + \dfrac{p_0 p_1}{\varepsilon_1 \varepsilon_2} + \dfrac{p_0 p_1 p_2}{\varepsilon_1 \varepsilon_2 \varepsilon_3}}$$

$$\Rightarrow \quad 1 + \frac{p_0}{\varepsilon_1} + \frac{p_0 p_1}{\varepsilon_1 \varepsilon_2} + \frac{p_0 p_1 p_2}{\varepsilon_1 \varepsilon_2 \varepsilon_3}$$

$$= 1 + 3 + 3 + 1 = 8$$

$$\Rightarrow \pi(0) = \frac{1}{8}, \quad \pi(1) = \frac{3}{8}, \quad \pi(2) = \frac{3}{8},$$

$$\pi(3) = \frac{1}{8}$$

Example 2. Suppose we have two boxes containing d red balls and d blue balls in total. Initially d balls are placed in box I. At each trial one ball is chosen at random from each box and then the two balls are put back in the opposite boxes. Set

X_0 = number of red balls in box I initially

X_n = number of red balls in box I after the n^{th} trial

Find ① $\mathbb{P}$;

② the stationary distribution

Solution: $S = \{0, 1, 2, \dots, d\}$

$$P(x, x) = \frac{x}{d} \cdot \frac{d-x}{d} + \frac{d-x}{d} \cdot \frac{x}{d} = 2\,\frac{x(d-x)}{d^2}$$

$$P(x, x+1) = \frac{d-x}{d} \cdot \frac{d-x}{d} = \left(\frac{d-x}{d}\right)^2$$

$$P(x, x-1) = \frac{x}{d} \cdot \frac{x}{d} = \left(\frac{x}{d}\right)^2$$

$P(x, y) = 0$ for $y \neq x, x \pm 1$

$\Rightarrow \{X_n, n = 0, 1, 2, \dots\}$ is a birth-death

Markov chain.

$$\mathbb{P} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ \left(\frac{1}{d}\right)^2 & 2\frac{d-1}{d^2} & \left(\frac{d-1}{d}\right)^2 & \cdots & & \\ & & & & & \\ 0 & 0 & \cdots & & 1 & 0 \end{pmatrix}$$

② For any $k = 1, \ldots d$,

$$\frac{p_0 \cdots p_{k-1}}{q_1 \cdots q_k} = \frac{\left(\frac{d}{d}\right)^2 \cdot \left(\frac{d-1}{d}\right)^2 \cdots \left(\frac{d-k+1}{d}\right)^2}{\left(\frac{1}{d}\right)^2 \cdots \left(\frac{k}{d}\right)^2}$$

$$= \left(\frac{d(d-1)(d-2)\cdots(d-k+1)}{1 \cdot 2 \cdots k}\right)^2$$

$$= \binom{d}{k}^2$$

$$\Rightarrow \quad \pi(k) = \frac{\binom{d}{k}^2}{1 + \binom{d}{1}^2 + \cdots + \binom{d}{d}^2}$$

$$= \frac{\binom{d}{k}^2}{\binom{2d}{d}}$$

$$1 + \binom{d}{1}^2 + \cdots + \binom{d}{d}^2 = \sum_{i=0}^{d} \binom{d}{i}\binom{d}{d-i}$$

$$\Vert$$

$$\binom{2d}{d}$$

For birth–death chain with $S = \{0, 1, 2, \cdots\}$ the stationary distribution exists if and only if

$$1 + \sum_{k=1}^{\infty} \frac{p_0 \cdots p_{k-1}}{q_1 \cdots q_k} < \infty$$

If the above holds then

$$\pi(k) = \frac{\dfrac{p_0 \cdots p_{k-1}}{q_1 \cdots q_k}}{1 + \sum_{i=1}^{\infty} \dfrac{p_0 \cdots p_{i-1}}{q_1 \cdots q_i}}$$

- Number and Average Number of visit to a state

For any $n \geq 1$, $y \in S$, set

$$N_n(y) = \sum_{k=1}^{n} \mathbb{1}_y(X_k)$$

$$G_n(x, y) = E_x[N_n(y)] = \sum_{k=1}^{n} P^k(x, y)$$

Theorem 1:

① For any transient state y

$$\lim_{n\to\infty} \frac{N_n(y)}{n} = 0 \quad \text{almost surely}$$

$$\lim_{n\to\infty} \frac{G_n(x,y)}{n} = 0$$

② For any recurrent state y

$$\lim_{n\to\infty} \frac{N_n(y)}{n} = \frac{1\{T_y<\infty\}}{m_y}$$

$$\lim_{n\to\infty} \frac{G_n(x,y)}{n} = \frac{\rho_{xy}}{m_y}$$

where $\boxed{m_y = E_y[T_y]}$.

Proof: ① By theorem 1 of Lecture 6,

$N_n(y) \to N(y) < \infty$ with probability one

$\Rightarrow \frac{N_n(y)}{n} \to 0$ with probability one

$$\frac{G_n(x,y)}{n} = \frac{E_x[N_n(y)]}{n}$$

$$= E_x\left[\frac{N_n(y)}{n}\right]$$

$$\Rightarrow \lim_{n\to\infty} \frac{G_n(x,y)}{n} = E_x\left[\lim_{n\to\infty}\frac{N_n(y)}{n}\right]$$

$$= 0$$

Dominated convergence theorem

(2) Assume that y is recurrent.

Let T_y^0 denote the hitting time of y and assume this occurs.

w_y^1 = additional number of steps for the first return to y

$\vdots$

w_y^r = additional number of steps for the rth return to y

Then $w_y^1, \dots, w_y^r$ are i.i.d with mean m_y

By the strong law of large numbers,

$$\frac{w_y^1 + \dots + w_y^r}{r} \xrightarrow[r\to\infty]{} m_y$$

with probability one.

Let T_y^k denote the time of the k^{th} visit to y. Then

$$\frac{T_y^k}{k} = \frac{T_y^0 + w_y^1 + \dots + w_y^k}{k} \xrightarrow[k\to\infty]{} m_y$$

with probability one

choosing $k = N_n(y)$, it follows that

$$T_y^{N_n(y)} \le n < T_y^{N_n(y)+1}$$

$N_n(y)^{th}$ visit

n

$(N_n(y)+1)^{th}$ visit

$$\Rightarrow \frac{T_y^{N_n(y)}}{N_n(y)} \le \frac{n}{N_n(y)} \le \frac{T_y^{N_n(y)+1}}{N_n(y)}$$

$$= \frac{N_n(y)+1}{N_n(y)} \frac{T_y^{N_n(y)+1}}{N_n(y)+1}$$

$$\Rightarrow \lim_{n\to\infty} \frac{T_y^{N_n(y)}}{N_n(y)} \le \lim_{n\to\infty} \frac{n}{N_n(y)} \le \lim \frac{T_y^{N_n(y)+1}}{N_n(y)+1}$$

$$\Rightarrow \lim_{n\to\infty} \frac{n}{N_n(y)} = m_y$$

$$\Rightarrow \lim_{n\to\infty} \frac{N_n(y)}{n} = \frac{1}{m_y}$$

If y is not accessible, then

$N_n(y) = 0$. Hence

$$\lim_{n\to\infty} \frac{N_n(y)}{n} = \frac{1_{\{T_y < \infty\}}}{m_y}$$

$$\lim_{n\to\infty} \frac{G_n(x,y)}{n} = \lim_{n\to\infty} E_x\left[\frac{N_n(y)}{n}\right]$$

$$= E\left[\lim_{n\to\infty} \frac{N_n(y)}{n}\right]$$

$$= \frac{\rho_{xy}}{m_y}$$

Null Recurrent and Positive Recurrent States

Definition: For any recurrent state $x \in S$, let $m_x = E_x[T_x]$

x is null recurrent if $m_x = \infty$

positive recurrent if $m_x < \infty$

Observation:

$$\lim_{n\to\infty} \frac{G_n(x,y)}{n} = 0 \quad \text{if}$$

y is transient or null recurrent.

Thus Null Recurrent is between Transient and Positive Recurrent.