Lecture 11 : Chapter 2 Stationary Distribution

Example 1. Let $\{X_n : n = 0, 1, 2, \cdots\}$ be a Markov chain with state space $S = \{1, 2, 3\}$ and one-step transition probability matrix

$$\mathbb{P} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

Assume that $\pi_0(2) = \pi_0(3) = \frac{2}{5}$. Find the distribution of X_1 and X_2.

Solution: Let π_1 denote the distribution of X_1, π_2 the distribution of X_2

Then

$$\pi_1(y) = \sum_{x=1}^{3} \pi_0(x)\, p(x, y)$$

$$= \begin{cases} \pi_0(3)\, p(3, 1) & y = 1 \\ \pi_0(3)\, p(3, 2) + \pi_0(1)\, p(1, 2) & y = 2 \\ \pi_0(2)\, p(2, 3) & y = 3 \end{cases}$$

$$= \begin{cases} \frac{2}{5} \cdot \frac{1}{2} & y=1 \\ \frac{2}{5} \cdot \frac{1}{2} + \frac{1}{5} \cdot 1 & y=2 \\ \frac{2}{5} \cdot 1 & y=3 \end{cases}$$

$$= \begin{cases} \frac{1}{5} & y=1 \\ \frac{2}{5} & y=2 \\ \frac{2}{5} & y=3 \end{cases}$$

$\Rightarrow \pi_0 = \pi_1$.

Similarly, we can prove that

$\pi_2 = \pi_0$. Thus the distribution of X_n is the same as X_0 for all $n \geq 1$.

The distribution of X_0 is not changing with time.

Definition 1: Let $\{X_n : n=0,1,2,\dots\}$ be a Markov chain with state space S and one-step transition probability matrix $\mathbb{P} = (p(x,y))_{x,y \in S}$.

A probability distribution π on S is called a Stationary Distribution of the chain if

① $\pi(x) \geq 0$ for all x and

$\sum_{x \in S} \pi(x) = 1$

② $\sum_{x \in S} \pi(x)\, p(x,y) = \pi(y)$ for all y

The stationary distribution π is called a Steady Stationary Distribution if

$$\lim_{n\to\infty} P^n(x,y) = \pi(y) \quad \text{for all } x,y \in S$$

Proposition 1 Let π be a stationary distribution of the chain $\{X_n\}$.

① For any $n \ge 1$, $y \in S$

$$\sum_{x \in S} \pi(x) P^n(x, y) = \pi(y)$$

② If π is steady, then for any initial distribution π_0, we have

$$P_{\pi_0}(X_n = y) \xrightarrow[n \to \infty]{} \pi(y) \quad \text{i.e.,}$$

$$\lim_{n \to \infty} \sum_{x \in S} \pi_0(x) P^n(x, y) = \pi(y)$$

Proof: ① For any $n \ge 1$, $y \in S$

$$\sum_{x \in S} \pi(x) P^n(x, y) = \sum_{x \in S} \pi(x) \sum_{z \in S} P(x, z) P^{n-1}(z, y)$$

$$= \sum_{x \in S} \sum_{z \in S} \pi(x) P(x, z) P^{n-1}(z, y)$$

$$= \sum_{z \in S} \left(\sum_{x \in S} \pi(x) P(x,z) \right) P^{n-1}(z,y)$$

$$= \sum_{z \in S} \pi(z) P^{n-1}(z,y)$$

$$= \cdots$$

$$= \sum_{u \in S} \pi(u) P(u,y)$$

$$= \pi(y)$$

② For any initial distribution π_0 and $y \in S$, we have

$$\lim_{n\to\infty} P_{\pi_0}(X_n = y) = \lim_{n\to\infty} \sum_{x \in S} \pi_0(x) P^n(x,y)$$

dominated convergence theorem

$$= \sum_{x \in S} \pi_0(x) \lim_{n\to\infty} P^n(x,y)$$

$$= \sum_{x \in S} \pi_0(x) \pi(y)$$

$$= \pi(y) \sum_{x \in S} \pi_0(x) = \pi(y)$$

A probability π on S is stationary if and only if

$$\pi \mathbb{P} = \pi$$

i.e., π is a left eigenvector of $\mathbb{P}$

Example 2 consider a Markov

chain with $S = \{1, 2, 3\}$ and

$$\mathbb{P} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{pmatrix}$$

Find the stationary distributions of the chain.

Solution: set $\pi = (\pi(1), \pi(2), \pi(3))$

$\pi \mathbb{P} = \pi$ implies

$$\left(\pi(1), \pi(2), \pi(3)\right)\begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{pmatrix} = \left(\pi(1), \pi(2), \pi(3)\right)$$

$$\Rightarrow \begin{cases} -8\pi(1) + 3\pi(2) + 2\pi(3) = 0 \\ 2\pi(1) + 2\pi(3) - 3\pi(2) = 0 \\ \pi(1) + \pi(2) + \pi(3) = 1 \end{cases}$$

$$\Rightarrow \quad \pi(1) = \frac{6}{25}, \quad \pi(2) = \frac{10}{25}, \quad \pi(3) = \frac{9}{25}$$

$\Rightarrow$ the chain has a unique stationary distribution

$$\left(\frac{6}{25}, \frac{10}{25}, \frac{9}{25}\right)$$

Example 3: Consider a Markov chain with $S = \{0, 1, 2\}$ and

$$\mathbb{P} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Find the stationary distributions.

Solution: Let $\pi = (\pi(0), \pi(1), \pi(2))$

$\pi \mathbb{P} = \pi$ implies

$$\frac{1}{2}\pi(0) + \frac{1}{6}\pi(1) = \pi(0)$$

$$\frac{1}{2}(\pi(0) + \pi(1) + \pi(2)) = \pi(1)$$

$$\frac{1}{3}\pi(1) + \frac{1}{2}\pi(2) = \pi(2)$$

$$\Rightarrow \quad \pi(1) = 3\pi(0)$$

$$\pi(1) = \frac{1}{2}$$

$$2\pi(1) = 3\pi(2)$$

$$\Rightarrow \begin{cases} \pi(0) = \frac{1}{6} \\ \pi(1) = \frac{1}{2} \\ \pi(2) = 1 - \frac{1}{6} - \frac{1}{2} = \frac{1}{3} \end{cases}$$

Example 4 Birth-Death Markov chain

with $S = \{0, 1, 2, \dots, d\}$ and

$$\mathbb{P} = \begin{pmatrix} r_0 & p_0 & 0 & 0 & \cdots & 0 \\ q_1 & r_1 & p_1 & 0 & \cdots & \vdots \\ 0 & q_2 & r_2 & p_2 & \cdots & \\ & & & & & \\ 0 & 0 & \cdots & & q_d & r_d \end{pmatrix}$$

Find the stationary distributions.

Solution Set $\pi = (\pi(0), \dots, \pi(d))$.

$\pi \mathbb{P} = \pi \Rightarrow$

$$\begin{cases} \pi(0) r_0 + \pi(1) q_1 = \pi(0) \\ \pi(0) p_0 + \pi(1) r_1 + \pi(2) q_2 = \pi(1) \\ \vdots \\ \pi(k-1) p_{k-1} + \pi(k) r_k + \pi(k+1) q_{k+1} = \pi(k) \\ \vdots \\ \pi(d-1) p_{d-1} + r_d \pi(d) = \pi(d) \end{cases}$$

Noting that $p_k + \varepsilon_k + r_k = 1$, it follows that

$$\pi(0)(1-p_0) + \pi(1)\varepsilon_1 = \pi(0)$$

$$\Rightarrow \pi(1)\varepsilon_1 = p_0\,\pi(0) \Rightarrow \boxed{\pi(1) = \frac{p_0}{\varepsilon_1}\pi(0)}$$

$$\pi(0)p_0 + (1-p_1-\varepsilon_1)\pi(1) + \pi(2)\varepsilon_2 = \pi(1)$$

$$\Rightarrow \pi(0)p_0 - \varepsilon_1\pi(1) - p_1\pi(1) + \pi(2)\varepsilon_2 = 0$$

$$\Rightarrow \pi(2)\varepsilon_2 - \pi(1)p_1 = 0$$

$$\Rightarrow \boxed{\pi(2) = \frac{p_1}{\varepsilon_2}\pi(1) = \frac{p_1}{\varepsilon_2}\cdot\frac{p_0}{\varepsilon_1}\pi(0)}$$

By induction, we obtain

$$\pi(k) = \frac{p_{k-1}}{\varepsilon_k}\cdot\frac{p_{k-2}}{\varepsilon_{k-1}}\cdots\frac{p_0}{\varepsilon_1}\pi(0)$$

Since $\pi(0) + \pi(1) + \cdots + \pi(d) = 1$, it follows that

$$\pi(0) + \frac{P_0}{\varepsilon_1}\pi(0) + \cdots + \frac{P_0 \cdots P_{d-1}}{\varepsilon_1 \cdots \varepsilon_d}\pi(0) = 1$$

$$\Rightarrow \boxed{\pi(0) = \frac{1}{1 + \frac{P_0}{\varepsilon_1} + \cdots + \frac{P_0 \cdots P_{d-1}}{\varepsilon_1 \cdots \varepsilon_d}}}$$

$$\Rightarrow \pi(k) = \frac{\frac{P_0 \cdots P_{k-1}}{\varepsilon_1 \cdots \varepsilon_k}}{1 + \frac{P_0}{\varepsilon_1} + \cdots + \frac{P_0 \cdots P_{d-1}}{\varepsilon_1 \cdots \varepsilon_d}} \qquad k = 1, \cdots d$$