Lecture 5 Structure of State Space

Definition: For any subset A of S, we define the hitting time of A as

$$T_A = \min\{n \geq 1 : X_n \in A\}$$

If $X_n \notin A$ for all $n \geq 1$, then

$$T_A = \infty$$

If $A = \{y\}$, then we write

$$T_A = T_y$$

Proposition 1: For any $x, y \in S$, $m \geq 1$, we have

$$P^m(x,y) = \sum_{k=1}^{m} P(T_y = k \mid X_0 = x)\, P^{m-k}(y,y)$$

where $P^0(x,x) = 1$.

In the sequel, we write

$$P(\cdot \mid X_0 = x) = P_x(\cdot)$$

Proof: $P^m(x,y) = P(X_m = y \mid X_0 = x)$

$$= P(X_m = y,\ T_y \le m \mid X_0 = x)$$

$$= \sum_{k=1}^{m} P(X_m = y,\ T_y = k \mid X_0 = x)$$

$$= \sum_{k=1}^{m} \frac{P(X_0 = x,\ T_y = k,\ X_m = y)}{P(X_0 = x)}$$

$$= \sum_{k=1}^{m} \frac{P(X_0 = x,\ T_y = k)}{P(X_0 = x)} \cdot \frac{P(X_0 = x,\ T_y = k,\ X_m = y)}{P(X_0 = x,\ T_y = k)}$$

$$= \sum_{k=1}^{m} P(T_y = k \mid X_0 = x)\, P(X_m = y \mid X_0 = x,\ T_y = k)$$

$$= \sum_{k=1}^{m} P(T_y = k \mid X_0 = x) \cdot P(X_m = y \mid X_0 = x,\ X_1 \ne y,\ \dots\ X_{k-1} \ne y,\ X_k = y)$$

Markov Property

$$= \sum_{k=1}^{m} P(T_y = k \mid X_0 = x)\, P(X_m = y \mid X_k = y)$$

$$= \sum_{k=1}^{m} P(T_y = k \mid X_0 = x)\, P^{m-k}(y, y)$$

$\square$

Example 1. If y is an absorbing state,

then

$$P^n(x, y) = P_x(T_y \le n) \quad \text{for all}$$

$x \in S$ and $n \geq 1$

Solution: Since y is absorbing, we get

$$P^m(y, y) = 1 \quad \text{for all } m$$

By proposition 1, we obtain

$$P^n(x, y) = \sum_{k=1}^{n} P_x(T_y = k)\, P^{n-k}(y, y)$$

$$= \sum_{k=1}^{n} P_x(T_y = k)$$

$$= P_x(T_y \le n)$$

$$= P(T_y \le n \mid X_0 = x)$$

— Transient & Recurrent States

For any $x, y \in S$, set

$$\rho_{xy} = P_x(T_y < \infty)$$

which is the probability of reaching y in finite steps starting at x.

In particular,

ρ_{xx} is the probability of return to x in finite steps

Definition 2: A state x is recurrent if $\rho_{xx} = 1$; otherwise ($\rho_{xx} < 1$) it is called transient.

If x is recurrent, then ~~the~~ Markov chain will return to x with probability 1.

For any $x \in S$, we define

$$1_x(y) = \begin{cases} 1 & \text{if } y = x \\ 0 & \text{else} \end{cases}$$

called indicator function at x.

$$N(x) = \sum_{n=1}^{\infty} 1_x(X_n)$$

= total number of visit to x

Proposition 2:

① For any $m \geq 1$, $x, y \in S$

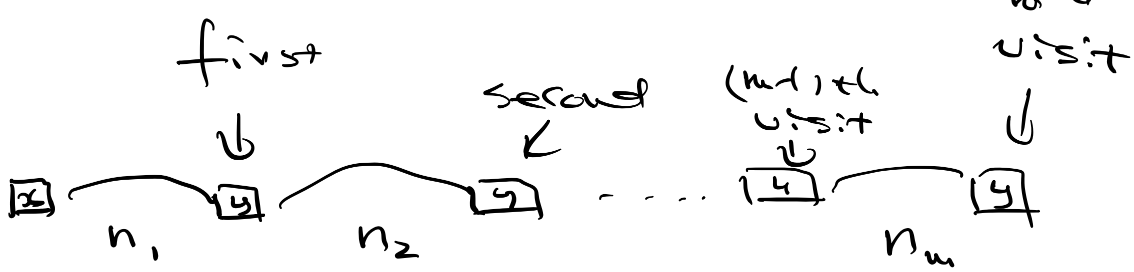
$$P_x(N(y) \geq m) = \rho_{xy}\,\rho_{yy}^{m-1}$$

② $$P_x(N(y) = m) = \rho_{xy}\,\rho_{yy}^{m-1}(1-\rho_{yy})$$

③ $$P_x(N(y) = 0) = 1 - \rho_{xy}$$

Proof: ① $P_x(N(y) \geq m)$

$= P_x($ visit y at least m times $)$



$P_x(N(y) \geq m)$

$$= \sum_{n_1 \geq 1} \cdots \sum_{n_m \geq 1} P_x\left(\text{first visit to } y \text{ occurs at step } n_1; \; \cdots\cdots; \; m^{th} \text{ visit to } y \text{ occurs at step } n_1 + \cdots + n_m\right)$$

$$= \sum_{n_1 \geq 1} \cdots \sum_{n_m \geq 1} P\left(\text{first visit to } y \text{ occurs at step } n_1 \mid X_0 = x\right)$$

$$\cdot P\left(\text{second visit to } y \text{ occurs at step } n_1 + n_2 \mid \text{first visit to } y \text{ occurs at step } n_1\right)$$

$\vdots$

$$P\left(m^{th} \text{ visit to } y \text{ occurs at step } n_1 + n_2 + \cdots + n_m \;\middle|\; (m-1)^{th} \text{ visit to } y \text{ occurs at step } n_1 + \cdots + n_{(m-1)}\right)$$

$$= \sum_{n_1 \ge 1} \cdots \sum_{n_m \ge 1} P_x(T_y = n_1)\, P_y(T_y = n_2) \cdots P_y(T_m = n_m)$$

$$= P_x(T_y < \infty)\, P_y(T_y < \infty) \cdots P_y(T_y < \infty)$$

$$= \rho_{xy}\, \rho_{yy}^{m-1}$$

(2) $P_x(N(y) = m)$

$$= P_x(N(y) \ge m) - P_x(N(y) \ge m+1)$$

$$= \rho_{xy}\, \rho_{yy}^{m-1} - \rho_{xy}\, \rho_{yy}^{m+1-1}$$

$$= \rho_{xy}\, \rho_{yy}^{m-1} (1 - \rho_{yy})$$

③ $P_x(N(y)=0)$

$= 1 - P_x(N(y) \geq 1)$

$= 1 - P_{xy} P_{yy}^{1-1} = 1 - P_{xy}$