Lecture 8 Absorption Probabilities and Birth-Death Markov chain

Let $S = C_T \cup C_R$. If C is an irreducible subset of C_R, then

$$P_C(x) = \begin{cases} 1 & \text{if } x \in C \\ 0 & \text{if } x \in C_R,\ x \notin C \end{cases}$$

Question: How to find the value $P_C(x)$ if x is transient?

$\boxed{\text{Observation:}}$ Let $x \in C_T$. Then in one-step, the chain may enter C or stay in C_T. It can not enter C_R but not C.

$$P_C(x) = \underbrace{\sum_{y \in C} P(x,y)}_{\text{enter } C} + \underbrace{\sum_{y \in C_T} P(x,y)\, P_C(y)}_{\text{stay in } C_T}$$

Thus $\{P_c(x) : x \in C_T\}$ is the Solution of a system of linear equations.

Two Issues

- Solutions are difficult to obtain when C_T has infinite states;
- Solutions are NOT unique.

If C_T is finite, we have the following result

Theorem 1: Let $S = C_T \cup C_R$ and C be an irreducible subset of C_R. If C_T is finite, then $\{P_C(x) : x \in C_T\}$ is the unique solution of the system of equations

$$w_x = \sum_{y \in C} P(x,y) + \sum_{y \in C_T} P(x,y)\, w_y$$

Proof: For any solution $\{w_x : x \in C_T\}$, we have

to C_T

$$w_x = \sum_{y \in C} P(x,y) + \sum_{y \in C_T} P(x,y)\, w_y$$

to C

$$= \sum_{y \in C} P(x,y) + \sum_{y \in C_T} P(x,y) \Big[\sum_{z \in C} P(y,z) + \sum_{z \in C_T} P(y,z)\, w_z \Big]$$

$$= \sum_{y \in C} P(x,y) + \sum_{y \in C_T} \sum_{z \in C} P(x,y) P(y,z) + \sum_{y \in C_T} \sum_{z \in C_T} P(x,y) P(y,z)\, w_z$$

$$= P_x(T_c = 1) + P_x(T_c = 2) + \sum_{z \in C_T} P^2(x,z)\, w_z$$

$$= \cdots$$

$$= P_x(T_c \le n) + \sum_{z \in C_T} P^n(x,z)\, w_z$$

Taking the limit of $n \to \infty$, it follows that

$$w_x = P_x(T_c < \infty) + \lim_{n \to \infty} \sum_{z \in C_T} p^n(x,z)\, w_z$$

Finite Terms

$$= P_x(T_c < \infty) + \sum_{z \in C_T} \lim_{n \to \infty} p^n(x,z)\, w_z$$

$$= P_x(T_c < \infty)$$

$$= \rho_c(x)$$

$\Rightarrow \{\rho_c(x) : x \in C_T\}$ is the unique solution.

Example 1: Consider the Markov chain $\{X_n : n = 0, 1, 2, \cdots\}$ with $S = \{0, 1, 2, 3, 4, 5\}$ and

$$\mathbb{P} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{5} & \frac{2}{5} & \frac{1}{5} & 0 & \frac{1}{5} \\ 0 & 0 & 0 & \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{4} & 0 & \frac{3}{4} \end{pmatrix}$$

Let $C = \{0\}$. Find $P_C(2)$, $P_C(3)$.

Solution: $C_T = \{1, 2\}$, $C_R = \{0\} \cup \{3, 4, 5\}$

$\Rightarrow \boxed{P_C(3) = 0}$

$$P_C(2) = P(2,0) + \sum_{y \in C_T} P(2,y) P_C(y)$$

$$= P(2,0) + P(2,1) P_C(1) + P(2,2) P_C(2)$$

$$P_C(1) = P(1,0) + \sum_{y \in C_T} P(1,y) P_C(y)$$

$$= P(1,0) + P(1,1) P_C(1) + P(1,2) P_C(2)$$

$$\Rightarrow \begin{cases} P_C(2) = \frac{1}{5} P_C(1) + \frac{2}{5} P_C(2) \\ P_C(1) = \frac{1}{4} + \frac{1}{2} P_C(1) + \frac{1}{4} P_C(2) \end{cases}$$

$$\Rightarrow \begin{cases} 3P_c(2) = P_c(1) \\ 2P_c(1) = 1 + P_c(2) \end{cases}$$

$$\Rightarrow \begin{cases} P_c(1) = \frac{3}{5} \\ \boxed{P_c(2) = \frac{1}{5}} \end{cases}$$

Birth-Death Markov Chain

$S = \{0, 1, 2, 3, \ldots d\}$ or $\{0, 1, 2, \ldots\}$

$\mathbb{P} = (P(x,y))_{x,y \in S}$

$$P(x,y) = \begin{cases} p_x & y = x+1 \\ r_x & y = x \\ q_x & y = x-1 \\ 0 & \text{else} \end{cases}$$

Satisfying

① $p_d = q_0 = 0$ if S is finite

② $p_x + r_x + q_x = 1$

③ $p_x > 0$, $q_x > 0$ for $1 < x < d$ or $x > 1$ (infinite case)

$\boxed{p_0 > 0}$ $\boxed{q_d > 0}$

Condition ③ implies that the chain is irreducible.

Question: Is the birth-death chain recurrent when $S = \{0, 1, \dots\}$?

For any $a < b$, $a, b \in S$. Define

$$u(x) = P_x(T_a < T_b), \quad a \le x \le b$$

Set $u(a) = 1$, $u(b) = 0$

By direct calculation, for any $a < y < b$

$$\begin{aligned}
u(y) &= P_y(T_a < T_b) \\
&= P_y(X_1 = y+1,\ T_a < T_b) \\
&\quad + P_y(X_1 = y,\ T_a < T_b) \\
&\quad + P_y(X_1 = y-1,\ T_a < T_b) \\
&= p_y P_{y+1}(T_a < T_b) + q_y P_{y-1}(T_a < T_b) \\
&\quad + r_y P_y(T_a < T_b)
\end{aligned}$$

$p_y + q_y + r_y = 1$

$$= p_y\, u(y+1) + q_y\, u(y-1) + r_y\, u(y)$$

$$= u(y) + p_y\big(u(y+1) - u(y)\big) + q_y\big(u(y-1) - u(y)\big)$$

$$\Rightarrow \quad p_y\big(u(y+1) - u(y)\big) = q_y\big(u(y) - u(y-1)\big)$$

$$\Rightarrow \quad u(y+1) - u(y) = \frac{q_y}{p_y}\big(u(y) - u(y-1)\big)$$

Let $\Gamma_y = \frac{q_y}{p_y} \cdots \frac{q_1}{p_1}$. then

$$u(y+1) - u(y) = \frac{\Gamma_y}{\Gamma_{y-1}}\big(u(y) - u(y-1)\big)$$

$$= \frac{\Gamma_y}{\Gamma_{y-1}} \frac{\Gamma_{y-1}}{\Gamma_{y-2}}\big(u(y-1) - u(y-2)\big)$$

$$\vdots$$

$$= \frac{\Gamma_y}{\Gamma_{y-1}} \cdots \frac{\Gamma_{a+1}}{\Gamma_a}\big(u(a+1) - u(a)\big)$$

$$= \frac{\Gamma_y}{\Gamma_a}\big(u(a+1) - u(a)\big)$$

$$\Rightarrow \quad \underbrace{u(y)-u(y+1) = \frac{\Gamma_y}{\Gamma_a}\big(u(a)-u(a+1)\big)}$$

Noting that

$$1 = u(a)-u(b) = \underbrace{u(a)-u(a+1)} + \underbrace{u(a+1)-u(a+2)} + \cdots + \underbrace{u(b-1)-u(b)}$$

$$= \big(u(a)-u(a+1)\big)\left[1+\frac{\Gamma_{a+1}}{\Gamma_a}+\cdots+\frac{\Gamma_{b-1}}{\Gamma_a}\right]$$

$$\Rightarrow \quad u(a)-u(a+1)$$

$$= \frac{\Gamma_a}{\Gamma_a+\cdots+\Gamma_{b-1}}$$

$$\Rightarrow \quad u(y)-u(y+1) = \frac{\Gamma_y}{\Gamma_a+\cdots+\Gamma_{b-1}}$$

$$\Rightarrow \quad u(y) = u(y) - 0 = u(y)-u(b)$$

$$= \underbrace{u(y)-u(y+1)} + \underbrace{u(y+1)-u(y+2)} + \cdots + \underbrace{u(b-1)-u(b)}$$

$$\Rightarrow u(y) = \frac{r_y + \cdots + r_{b-1}}{r_a + \cdots + r_{b-1}}$$