Lecture 6 structure of state space (cont.)

For each $x \in S$, let $E_x[\cdot]$ denote the expectation of random variables with respect to $P_x(\cdot)$. In particular,

$$E_x[I_y(X_n)] = 1 \cdot P_x(I_y(X_n)=1)$$
$$= P_x(X_n = y)$$
$$= P^n(x,y)$$

$\boxed{I_y(\cdot) = \mathbb{1}_y(\cdot)}$

$$E_x[N(y)] = E_x\left[\sum_{n=1}^{\infty} I_y(X_n)\right]$$

$$= \sum_{n=1}^{\infty} E_x[I_y(X_n)]$$

$$= \sum_{n=1}^{\infty} P^n(x,y)$$
$$= P^1(x,y) + P^2(x,y) + P^3(x,y) + \cdots$$

$E_x(N(y))$ will be denoted by $G(x,y)$. It is the expected number of visit to y starting at x.

Theorem 1.

① If $y \in S$ is transient, then for all x

$$P_x(N(y) < \infty) = 1$$ and

$$G(x,y) = \frac{\rho_{xy}}{1-\rho_{yy}} < \infty$$

② If $y \in S$ is recurrent, then

$$P_x(N(y) = \infty) = \rho_{xy} \text{ for all } x \in S.$$

In particular

$$P_y(N(y) = \infty) = 1,$$

$$G(y,y) = \infty$$

Thus if $\rho_{xy} = 0$, then $G(x,y) = 0$

if $\rho_{xy} > 0$, then $G(x,y) = \infty$

Proof ① Let y be transient. Then

$\rho_{yy} < 1$. For any $x \in S$, it

follows from proposition 2 of

Lecture 5 that

$$P_x(N(y)=\infty) = \lim_{m\to\infty} P_x(N(y)\geq m)$$

$$= \lim_{m\to\infty} \rho_{xy}\,\rho_{yy}^{m-1} = 0$$

$$\Rightarrow \quad P_x(N(y)<\infty) = 1$$

$$G(x,y) = \sum_{m=1}^{\infty} m\, P_x(N(y)=m)$$

$$= \sum_{m=1}^{\infty} m\, \rho_{xy}\,\rho_{yy}^{m-1}(1-\rho_{yy})$$

$$= \rho_{xy}(1-\rho_{yy}) \sum_{m=1}^{\infty} m\,\rho_{yy}^{m-1}$$

$$= \rho_{xy}(1-\rho_{yy}) \left(\sum_{m=1}^{\infty} \rho_{yy}^{m} \right)'$$

$$= \rho_{xy}(1-\rho_{yy}) \left(\frac{\rho_{yy}}{1-\rho_{yy}} \right)'$$

$$= \rho_{xy}(1-\rho_{yy}) \frac{1}{(1-\rho_{yy})^2}$$

$$= \frac{\rho_{xy}}{1-\rho_{yy}} < \infty$$

$\uparrow$

$\rho_{yy} < 1$

③ Let $y \in S$ be recurrent. Then

$P_{yy} = 1$ and

$$P_x(N(y)=\infty) = \lim_{m\to\infty} P_x(N(y)\geq m)$$

$$= \lim_{m\to\infty} P_{xy} P_{yy}^{m-1}$$

$$= P_{xy}$$

thus

$$P_y(N(y)=\infty) = P_{yy} = 1$$

$$G(y,y) = \infty \, P_y(N(y)=\infty)$$

$$= \infty \cdot 1$$

If $P_{xy} = 0$, then

$$P_x(N(y)\geq m) = P_{xy} P_{yy}^{m-1} = 0$$

$$\Rightarrow P_x(N(y) = 0) = 1 - P_{xy} = 1$$

$$\Rightarrow G(x,y) = 0 \, P_x(N(y)=0)$$

$$= 0$$

If $P_{xy} > 0$, then

$$P_x(N(y) = \infty) = \lim_{m \to \infty} P_x(N(y) \geq m)$$

$$= \lim_{m \to \infty} P_{xy} \, P_{yy}^{m-1}$$

$$= P_{xy} > 0$$

$$\Rightarrow G(x, y) \geq \infty \cdot P_x(N(y) = \infty) = \infty$$

Remark $G(y, y) < \infty$ if and only if

y is transient.

Corollary. If $y \in S$ is transient, then

$$\lim_{n \to \infty} P_x(X_n = y) = \lim_{n \to \infty} P^n(x, y)$$

$$= 0$$

proof: y is transient $\Rightarrow$ for any $x \in S$, $G(x, y) < \infty$

$$\Rightarrow \quad G(x,y) = \sum_{n=1}^{\infty} P^n(x,y)$$

$$\Rightarrow \quad \lim_{n\to\infty} P^n(x,y) = 0$$

Definition 1. A Markov chain is called a transient chain if all its states are transient. It is called a recurrent chain if all states are recurrent.

Example 1. Let $\{X_n : n = 0,1,2\cdots\}$ be a two-state Markov chain with

$$S = \{0,1\}, \quad P(1,1) = P(0,0) = \frac{1}{2}.$$

Is the chain recurrent?

Solution: $P_0(T_0 = 1) = P(0,0) = \frac{1}{2}$

$P_0(T_0 = 2) = P(0,1)P(1,0)$
$= (\frac{1}{2})^2$

$\vdots$

$P_0(T_0 = n) = (\frac{1}{2})^n$

$$\Rightarrow \quad P_{00} = P_0(T_0 < \infty)$$

$$= \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \cdots$$

$$= \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$$

$\Rightarrow$ 0 is recurrent.

Similarly we can show that 1 is recurrent. Hence the chain is recurrent.

Example 2. Let $\{X_n : n = 0, 1, 2, \ldots\}$ be a Markov chain with state space $S = \{1, 2, 3\}$ and transition matrix

$$\mathbb{P} = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Determine whether the chain is recurrent or not.

Solution: Notting that $P(3,3)=1$, it follows that state 3 is absorbing and thus recurrent.
Since $P(2,3)=1$, $P(1,3)=\frac{1}{2}>0$, it follows that both 1 and 2 are transient.
thus the chain is NOT recurrent.

Example 3. Let $\{X_n : n=0,1,2\cdots\}$ be a Markov chain with state space S.

If S contains only finite number of states, the chain has at least one recurrent state.

Solution: Assume that all states are transient. then by the corollary

$$\lim P^n(x,y)=0 \quad \text{for all } x,y\in S$$

$$\Rightarrow \quad P_x(X_n\in S)=1 = \sum_{y\in S} P_x(X_n=y)$$

$$= \sum_{y\in S} P^n(x,y)$$

$$\Rightarrow 1 = \lim_{n\to\infty} \sum_{y\in S} P^n(x,y)$$

change of order is allowed since S is finite

$$= \sum_{y\in S} \lim_{n\to\infty} P^n(x,y)$$

$$= \sum_{y\in S} 0$$

$$= 0$$

A contradiction. Thus there is at least one recurrent state.

Example 4: Construct a transient Markov chain.

Solution: Example 3 in Lecture 3.

$S = \{0, 1, 2, \cdots\}$ $\quad P(x, x+1) = 1$

Definition 2: A state x leads to y, denoted by $x \to y$ if $P_{xy} > 0$. Two states x, y communicate if $x \to y$, $y \to x$. In this case we write $x \Leftrightarrow y$

Theorem 2. If $x \to y$, $y \to z$, then $x \to z$

Proof: $x \to y \Leftrightarrow P_{xy} > 0 \Leftrightarrow$ there exists $n \geq 1$ such that $P^n(x,y) > 0$

$y \to z \Leftrightarrow P_{yz} > 0 \Leftrightarrow$ there exists $m \geq 1$ such that $P^m(y,z) > 0$

Here "$\Rightarrow$" means "imply"
"$\Leftrightarrow$" means "equivalence"

Noting that

$P^{n+m}(x,z) \geq P^n(x,y) P^m(y,z) > 0$ it follows that $P_{xz} > 0$. Thus

$$x \to z$$