Lecture 3 : Examples

Recall the three main issues:

- Distribution at each time
- Joint distribution at different times
- Long time distribution or asymptotic

Two factors : π_0 , $\mathbb{P}$

Example 1 Let $S = \{1, 2\}$ $T = \{0, 1, 2, \cdots\}$

$\pi_0 = (\frac{1}{2}, \frac{1}{2})$, $\mathbb{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Find $P(X_3 = 2)$, $P(X_1 = 1, X_2 = 2)$

Solution: $P(X_3 = 2) = \sum_{i \in S} \pi_0(i) P^3(i, 2)$

$$= \frac{1}{2} [P^3(1,2) + P^3(2,2)]$$

$$\mathbb{P} \times \mathbb{P} \times \mathbb{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \mathbb{P}$$

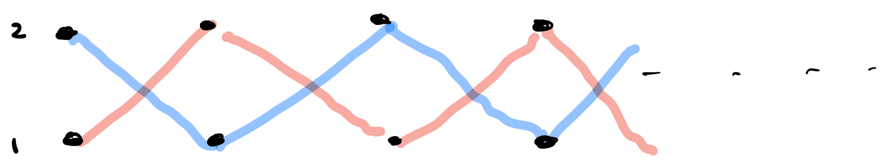
$$\Rightarrow \quad P(X_3=2) = \frac{1}{2}\left[P(1,2) + P(2,2)\right]$$
$$= \frac{1}{2}\left[1 + 0\right]$$
$$= \frac{1}{2}$$

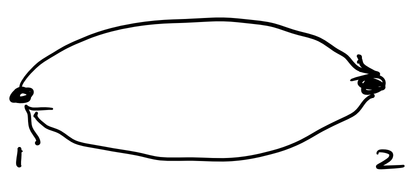
$$P(X_1=1, X_2=2) = P(X_0=1, X_1=1, X_2=2) + P(X_0=2, X_1=1, X_2=2)$$

$$= P(X_0=1)P(1,1)P(1,2) + P(X_0=2)P(2,1)P(1,2)$$

$$= \frac{1}{2}\cdot 0 \cdot 1 + \frac{1}{2}\cdot 1 \cdot 1$$

$$= \frac{1}{2}$$





Example 2 $\quad T = \{0, 1, 2, \cdots\} \qquad S = \{0, 1\}$

$$\mathbb{P} = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix} \qquad 0 \le p, q \le 1$$

Find ① $P(X_n = 0)$, $P(X_n = 1)$

② $\lim_{n\to\infty} P(X_n = 0)$

Solution: ① Let $\pi_0(0) = \alpha$, $\pi_0(1) = 1-\alpha$

Then $P(X_0 = 0) = \alpha$

$$P(X_1 = 0) = P(X_0 = 0, X_1 = 0) + P(X_0 = 1, X_1 = 0)$$

$$= P(X_0 = 0)\, P(X_1 = 0 \mid X_0 = 0) + P(X_0 = 1)\, P(X_1 = 0 \mid X_0 = 1)$$

$$= \alpha\, P(0,0) + (1-\alpha)\, P(1,0)$$

$$= \alpha(1-p) + (1-\alpha) q$$

$$= (1-p-q)\alpha + q$$

$$= \underbrace{(1-p-q)}\, \underbrace{P(X_0 = 0)} + q$$

$$P(x_2=0) = p(x_1=0, x_2=0) + P(x_1=1, x_2=0)$$

$$= p(x_1=0)\, P(x_2=0 \mid x_1=0)$$

$$+ p(x_1=1)\, P(x_2=0 \mid x_1=1)$$

$$= P(x_1=0)\, P(0,0) + (1-P(x_1=0))\, P(1,0)$$

$$= P(x_1=0)(1-p) + (1-P(x_1=0))\,\xi$$

$$= (1-p-\xi)\, P(x_1=0) + \xi$$

$$= (1-p-\xi)\left[(1-p-\xi)\, P(x_0=0) + \xi\right]$$

$$+ \xi$$

$$= (1-p-\xi)^2\, P(x_0=0)$$

$$+ (1-p-\xi)\,\xi + \xi$$

Similarly, we have

$$P(x_3=0) = (1-p-\xi)^3\, P(x_0=0)$$

$$+ (1-p-\xi)^2\,\xi + (1-p-\xi)\,\xi$$

$$+ \xi$$

By induction, we obtain

$$P(X_n=0) = (1-p-q)^n P(X_0=0) + q\left[1+(1-p-q)+(1-p-q)^2+\cdots+(1-p-q)^{n-1}\right]$$

Case 1: $p=q=0$

$$\mathbb{P} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{motionless}$$

Case 2: $p=q=1$

$$\mathbb{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{circular motion}$$

Case 3: None of above

In thise case, $0 < p+q < 2$

thus $|1-p-q| < 1$

Recall geometric sums:

$$1+a+\cdots+a^{n-1} = \frac{1-a^n}{1-a} \quad \text{if } |a|<1$$

It follows that

$$P(X_n = 0) = (1-p-q)^n \alpha + q \frac{1-(1-p-q)^n}{1-(1-p-q)}$$

$$= (1-p-q)^n \alpha + \frac{q}{p+q}\left(1-(1-p-q)^n\right)$$

$$= \frac{q}{p+q} + (1-p-q)^n \left[\alpha - \frac{q}{p+q}\right]$$

Noting that

$$\lim_{n \to \infty} (1-p-q)^n = 0$$

for $|1-p-q| < 1$, it follows that

$$\lim_{n \to \infty} P(X_n = 0) = \frac{q}{p+q}$$

Since $P(X_n = 1) = 1 - P(X_n = 0)$, it follows that

$$P(X_n = 1) = \frac{p}{p+q} + (1-p-q)^n \left[1 - \alpha - \frac{p}{p+q}\right]$$

$$\to \frac{p}{p+q}$$

For joint distribution, we have

$$P(X_0=0,\ X_1=1,\ X_2=1)$$

$$= P(X_0=0)\,P(X_1=1 \mid X_0=0)\,P(X_2=1 \mid X_1=1)$$

$$= \alpha\, P(0,1)\,P(1,1)$$

$$= \alpha\, p\,(1-\delta)$$

Suggested Exercises

Find ① $P(X_2=1,\ X_3=0)$

② $P(X_1=0,\ X_4=1)$

Example 3. $S=\{0,1,2,\cdots\}$

$$\mathbb{P} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \cdots & & & & \end{pmatrix}$$

Find $\lim_{n\to\infty} P(X_n=y)$ for any y.

Solution: For any $n \geq 1$, π_0, we have

$$P(X_n = y) = \sum_{x \in S} \pi_0(x) P^n(x, y)$$

Noting that the only transition are

$0 \to 1$, $1 \to 2$, $2 \to 3$, ...

$$\Rightarrow P(X_n = y) = P(X_n = y, X_{n-1} = y-1)$$

$$= P(X_n = y, X_{n-1} = y-1, X_{n-2} = y-2)$$

Thus X_0 must be less than y.

If $n > y$ this is impossible.

Hence

$$\lim_{n \to \infty} P(X_n = y) = 0 \quad \text{for all } y$$