Lecture 2 Markov chain (cont)

Review of Previous Lecture

- Stochastic process $\begin{cases} \text{index set} \\ \text{state space} \\ \text{dependence relation} \end{cases}$

- Discrete time and continuous time

- chain

- Markov property

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- Discrete time Markov chain

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$$P(X_{n+1}=j \mid X_0=i_0, \ldots, X_{n-1}=i_{n-1}, X_n=i)$$

$$= P(X_{n+1}=j \mid X_n=i)$$

$$\underbrace{\boxed{\text{past}} \quad \boxed{\text{present}}}_{\text{History}} \quad \boxed{\text{future}}$$

Example 1. Let $\{X_n : n=0,1,2,\dots\}$ be a process such that $X_0, X_1, X_2, \dots$ are independent then $\{X_n : n=0,1,2,\dots\}$ is Markov.

Proof: For any $n \in T$, $i_0, i_1, \dots, i_{n-1}, i, j \in S$,

$$P(X_{n+1}=j \mid X_0=i_0, \dots, X_{n-1}=i_{n-1}, X_n=i)$$

$$= \frac{P(X_0=i_0, \dots, X_n=i, X_{n+1}=j)}{P(X_0=i_0, \dots, X_n=i)}$$

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$$= \frac{\underline{P(X_0=i_0, \dots, X_n=i)}\, P(X_{n+1}=j)}{\underline{P(X_0=i_0, \dots, X_n=i)}}$$

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$$= P(X_{n+1}=j) = \frac{P(X_n=i)\, P(X_{n+1}=j)}{P(X_n=i)}$$

$$= \frac{P(X_n=i, X_{n+1}=j)}{P(X_n=i)}$$

$$= P(X_{n+1}=j \mid X_n=i)$$

Independent is a special Markov case.

Definition 1 If $P(X_{n+1}=j \mid X_n=i)$ is the same for all n, then the Markov chain has stationary transition probability.

$P(X_{n+1}=j \mid X_n=i)$, denoted by P_{ij}, is called the one-step transition probability from state i to state j.

Definition 2 the matrix

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$$\mathbb{P} = (P_{ij})_{i,j\in S}$$ is called the

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one-step transition matrix

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Definition 3 For each $i \in S$, define

$$\pi_0(i) = P(X_0=i)$$ and set

$$\pi_0 = \{\pi_0(i) : i \in S\}.$$ clearly

$$\sum_{i\in S} \pi_0(i) = 1$$ and we call

π_0 the initial distribution of the Markov chain.

Proposition 1 Let $\mathbb{P} = (P_{ij})_{i,j \in S}$ be the one-step stationary transition probability matrix. Then

① $P_{ij} \geq 0, \quad i, j \in S$

② $\sum_{j \in S} P_{ij} = 1$ for all $i \in S$

Example 2: Determine which of the following are transition matrices.

(a) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$; (b) $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix}$;

(c) $\begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{8} & -\frac{1}{8} & 1 \\ \frac{1}{3} & 0 & \frac{2}{3} \end{pmatrix}$; (d) $\begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{5}{8} & \frac{3}{8} & \frac{1}{16} \\ \frac{1}{7} & \frac{2}{7} & \frac{8}{14} \end{pmatrix}$

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Given a discrete time Markov chain with stationary transition probability matrix $\mathbb{P}$ and initial distribution π_0, One would like to know

<1> $P(X_n = j)$ for all n, j

the distribution at each time n

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<2> $P(X_{n_1} = j_1, \dots, X_{n_k} = j_k)$ for all $n_1 < n_2 < \dots < n_k$, $j_1, \dots, j_k \in S$

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the joint distribution at times $n_1, \dots, n_k$

<3> $\lim_{n \to \infty} P(X_n = j)$ for all j

Asymptotic behaviour.

Proposition 2. For any $n \geq 1$, let $\mathbb{P}^n$ be the product of n $\mathbb{P}$s, i.e.,

$$\mathbb{P}^n = \mathbb{P} \times \cdots \times \mathbb{P}$$

Denote element of $\mathbb{P}^n$ as p^n_{ij}. Then for any $j \in S$

$$P(X_n = j) = \sum_{i \in S} \pi_0(i) p^n_{ij}$$

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where $$p^n_{ij} = \sum_{i_1 \cdots i_{n-1} \in S} p_{i i_1} p_{i_1 i_2} \cdots p_{i_{n-1} j}$$

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Proof: we only consider cases $n = 2$ or 3

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the proof is similar for general n.

case 1: $n = 2$

$$P(X_2 = j) = \sum_{i, i_1 \in S} P(X_2 = j, X_1 = i_1, X_0 = i)$$

Note:
$P(A \cap B)$
$= P(A) P(B|A)$

$$= \sum_{i \in S} \sum_{i_1 \in S} P(X_0 = i, X_1 = i_1, X_2 = j)$$

$$= \sum_{i \in S} \sum_{i_1 \in S} \boxed{P(X_2 = j \mid X_0 = i, X_1 = i_1) \cdot P(X_0 = i, X_1 = i_1)}$$

$$= \sum_{i \in S} \sum_{i_1 \in S} P(X_2 = j \mid X_0 = i, X_1 = i_1) \cdot P(X_1 = i_1 \mid X_0 = i) \, P(X_0 = i)$$

$$= \sum_{i \in S} \sum_{i_1 \in S} P(X_2 = j \mid X_1 = i_1) P(X_1 = i_1 \mid X_0 = i) \cdot P(X_0 = i)$$

$$= \sum_{i \in S} \sum_{i_1 \in S} \pi_0(i) \, P_{i i_1} P_{i_1 j}$$

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$$= \sum_{i \in S} \pi_0(i) \sum_{i_1 \in S} P_{i i_1} P_{i_1 j}$$

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$$= \sum_{i \in S} \pi_0(i) \, P^2_{ij}$$

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Case 3 $\quad n = 3$

$$P(X_3 = j) = \sum_{i \in S} \sum_{i_1 \in S} \sum_{i_2 \in S} P(X_0 = i, X_1 = i_1, X_2 = i_2, X_3 = j)$$

$$P(X_0 = i, X_1 = i_1, X_2 = i_2, X_3 = j)$$

$$= P(X_0 = i) P(X_1 = i_1 \mid X_0 = i) P(X_2 = i_2 \mid X_0 = i, X_1 = i_1) \cdot P(X_3 = j \mid X_0 = i, X_1 = i_1, X_2 = i_2)$$

$$= P(X_0 = i)\, P(X_1 = i_1 \mid X_0 = i)\, P(X_2 = i_2 \mid X_1 = i_1)\, P(X_3 = j \mid X_2 = i_2)$$

$$= \pi_0(i)\, P_{i i_1} P_{i_1 i_2} P_{i_2 j}$$

$$\Rightarrow \quad P(X_3 = j) = \sum_{i \in S} \pi_0(i)\, P^3_{ij}$$

$$P^3_{ij} = \sum_{i_1 \in S} \sum_{i_2 \in S} P_{i i_1} P_{i_1 i_2} P_{i_2 j}$$

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$\boxed{\text{Proposition 2}}$ For any k, n, m

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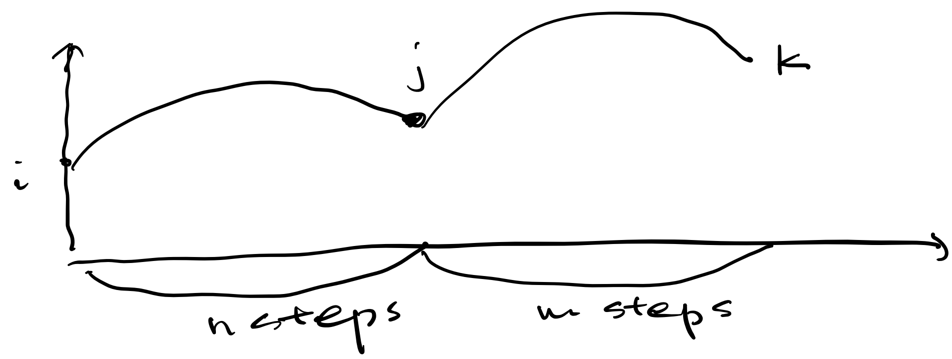
$$P(X_{m+n} = k,\ X_n = j)$$

$$= \sum_{i \in S} \sum_{i_1, \dots, i_{n-1} \in S} \sum_{i_{n+1}, \dots, i_{n+m-1} \in S} \pi_0(i)\, P_{i i_1} \cdots P_{i_{n-1} j}\, P_{j i_{n+1}} \cdots P_{i_{n+m-1}, k}$$

$$= \sum_{i \in S} \pi_0(i)\, P^n_{ij}\, P^m_{jk}$$



Theorem. The Markov chain $\{X_n : n = 0, 1, \cdots\}$ with initial distribution π_0 and one-step stationary transition probability matrix $\mathbb{P}$ is uniquely determined by $(\pi_0, \mathbb{P})$.

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$$\text{Markov chain} \Longleftrightarrow (\pi_0, \mathbb{P})$$

Notation:

$$\mathcal{S} \Leftrightarrow S, \qquad x, y \Leftrightarrow i, j$$

$$p(x,y) \Leftrightarrow P_{ij} \longleftrightarrow P(i,j)$$

$$P^n_{ij} \longleftrightarrow P^n(i,j)$$