Lecture 4. Examples (cont.)

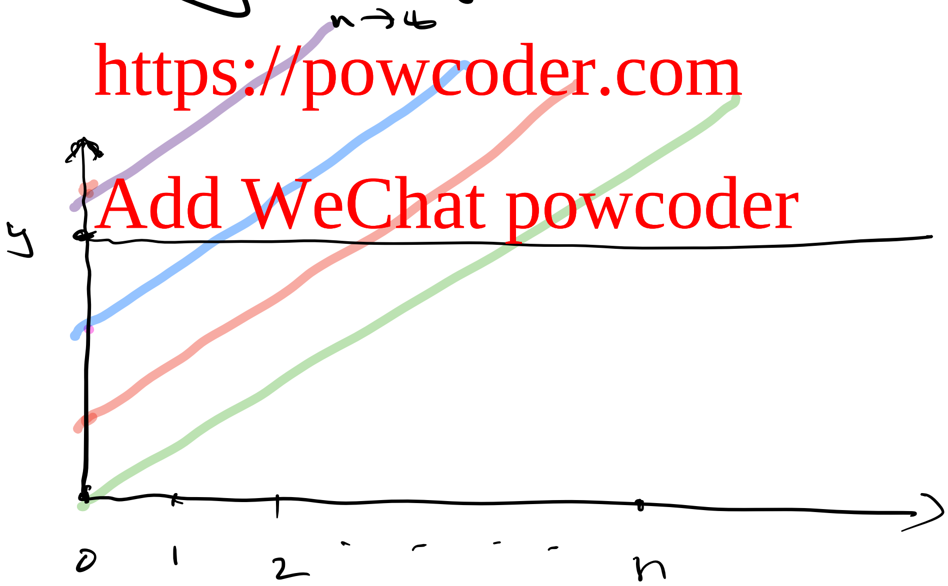
Example 3 from previous lecture:

① $\lim_{n\to\infty} P(X_n = y) = 0$ for $y \in S$

② $\sum_{y\in S} P(X_n = y) = 1$

③ $\lim_{n\to\infty} \sum_{y\in S} P(X_n = y) = 1$

④ why $\lim_{n\to\infty} \sum_{y\in S} P(X_n = y) \neq 0$?



change of order in Limit
is not allowed !!!

Example 1. Let ξ be a discrete random variable such that

$$P(\xi = k) = \alpha_k, \quad k = 0, 1, 2, \cdots$$

Let $\xi_1, \xi_2, \cdots$ be iid with the same distribution as ξ. Define

$$X_0 = 0$$
$$X_1 = \xi_1$$
$$X_2 = \xi_1 + \xi_2$$
$$\vdots$$
$$X_n = \xi_1 + \cdots + \xi_n$$

Find the π_0 and $\mathbb{P}$ for the Markov chain $\{X_n : n = 0, 1, 2, \cdots\}$

Solution: $X_0 = 0 \Rightarrow \pi_0 = (1, 0, 0, \cdots)$

i.e., $P(X_0 = k) = \pi_0(k) = 0$ for $k \neq 0$

$$\pi_0(0) = P(X_0 = 0) = 1$$

$$P(X_{n+1} = j \mid X_n = i)$$
$$= P(X_n + \xi_{n+1} = j \mid X_n = i)$$
$$= P(\xi_{n+1} = j - i)$$

$$\Rightarrow \quad \mathbb{P} = \begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \cdots \\ 0 & \alpha_0 & \alpha_1 & \alpha_2 & \cdots \\ 0 & 0 & \alpha_0 & \alpha_1 & \cdots \\ - & - & - & - & - \end{pmatrix}$$

Example 2. Ehrenfest chain

Consider two containers, labelled I & II, that contain a total number of d balls. A ball is selected at random from all balls, then the selected ball is moved to the other container. This procedure is repeated independently. For $n \geq 0$, let X_n = the number of balls in container I after the nth selection.

Find S and $\mathbb{P}$ of $\{X_n : n = 0, 1, 2, \cdots\}$

Solution: All possible values of X_n are $0, 1, 2, \ldots, d$. Hence

$$S = \{0, 1, 2, \cdots, d\}$$

For any $x, y \in S$,

$$P(X_1 = y \mid X_0 = x) = \begin{cases} \frac{x}{d} & y = x-1 \\ 1 - \frac{x}{d} & y = x+1 \\ 0 & \text{else} \end{cases}$$

Definition: A state $x \in S$ is called an absorbing state if $p(x, x) = 1$.

Example 3. Gambler's ruin chain.

A gambler starts with a certain initial capital and makes a series of \$1 bets against the house. Assume that the gambler has probability p of winning (the losing probability is thus $1-p = q$). If the total capital reaches 0, the gambler is ruined and the game stops.

Find $\mathbb{P}$

Solution: $S = \{0, 1, 2, \cdots\}$

0 is clearly an absorbing state.

$\Rightarrow P(0,0) = 1$

X_n = capital after the nth bet.

$P(x,y) = P(X_1 = y \mid X_0 = x) \qquad x \geq 1$

$$= \begin{cases} p & y = x+1 \text{ (winning)} \\ q & y = x-1 \text{ (losing)} \\ 0 & \text{else} \end{cases}$$

Example 4 Genetic Model

Consider a cell with a gene composed of d subunits among which m are mutant and d-m non-mutant. the cell duplicates before divides into two offspring cells. the gene of each offspring cell is composed of d units chosen at random from the 2m mutant units and 2(d-m) non-mutat units. Find $P(x,y)$.

Solution: X_n = number of mutant units at generation n.

$$p(x,y) = P(X_n = y \mid X_{n-1} = x)$$

2d units $\longrightarrow$ select d without replacement

2x mutant, 2(d−x) non-mutant

$$\Rightarrow p(x,y) = \frac{\binom{2x}{y}\binom{2d-2x}{d-y}}{\binom{2d}{d}}$$

name of distribution ?

$$S = \{0, 1, 2, \dots, d\}$$

Example 5. Birth-Death Markov chain

$$S = \{0, 1, 2, \dots\}.$$

$$p(x,y) = \begin{cases} p_x & y = x+1 \\ r_x & y = x \\ q_x & y = x-1 \\ 0 & \text{else} \end{cases}$$

where $p_x + r_x + q_x = 1$, $q_0 = 0$

$x \to x+1$ birth;

$x \to x-1$ death;

$x \to x$ no movement

Example 6 Queuing chain

Consider a checkout counter at a supermarket. Customers arrive at random and are served sequentially. For simplicity we assume that

① Time is measured in discrete units: 0, 1, 2, ...

② Exactly one customer that is in line at the beginning of a period is served during each unit time period. If no one is in line at the beginning of a period, then no one is served.

③ the number of arrivals during period n is η_n

④ $\eta_1, \eta_2, \ldots$ are i.i.d discrete random variables

⑤ $P(\eta_1 = k) = \alpha_k$, $k = 0, 1, 2, \ldots$

Let X_0 denote the number of customers in line initially.

For $n \geq 1$, let

X_n = number of customers in line at the end of period n

① Find the relation between X_n and $\eta_1, \eta_2, \ldots$

② Find $\mathbb{P}$

Solution: ① X_0 is given

If $X_0 = 0$, then $X_1 = \eta_1$

If $X_0 \geq 1$, then $X_1 = X_0 + \eta_1 - 1$

In general,

$$X_{n+1} = \begin{cases} \eta_{n+1} & \text{if } X_n = 0 \\ X_n + \eta_{n+1} - 1 & \text{if } X_n \geq 1 \end{cases}$$

Hence, $P(0, y) = P(\eta_1 = y) = \alpha_y$

For $x \geq 1$ we have

$$P(x, y) = P(X_{n+1} = y \mid X_n = x)$$

$$= P(X + \eta_{n+1} - 1 = y \mid X = x)$$

$$= P(\eta_{n+1} = y - x + 1 \mid X = x)$$

$$= P(\eta = y - x + 1)$$

$$= \begin{cases} \alpha_{y-x+1} & y-x+1 \geq 0 \\ 0 & \text{else} \end{cases}$$

$$\Rightarrow \mathbb{P} = \begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \cdots \\ \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \cdots \\ 0 & \alpha_0 & \alpha_1 & \alpha_2 & \cdots \\ 0 & 0 & \alpha_0 & \alpha_1 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \end{pmatrix}$$