Lecture 10 Branching Chain & Review of Chapter 1

Example 1 Consider the following branching processes and find the corresponding extinction probability.

① $\Phi(t) = \frac{1}{4} + \frac{1}{2}t + \frac{1}{4}t^2$

② $\Phi(t) = \frac{1}{8} + \frac{7}{8}t^2$

③ $\Phi(t) = a_0 + \sum_{k=1}^{\infty} a_k t^k$ where

$a_k = \frac{\lambda^k}{k!} e^{-\lambda}$

④ the offspring number follows a binomial distribution with parameters N and p

⑤ $\Phi(t) = 0.1 + 0.1t^2 + 0.8t^{10}$

Solution:

① $\xi_1 = 0, 1$ or 2 and

$p(\xi_1 = 0) = \frac{1}{4}$, $p(\xi_1 = 1) = \frac{1}{2}$, $p(\xi_1 = 2) = \frac{1}{4}$

$\Rightarrow \mu = 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} = 1$

$\Rightarrow$ extinction probability is

$P = 1$

② $\xi_1 = 0$ or 2 and $p(\xi_1 = 0) = \frac{1}{8}$

$\Rightarrow \mu = 0 \cdot \frac{1}{8} + 2 \cdot \frac{7}{8} = \frac{7}{4} > 1$

$\Rightarrow P = t_0$ where $t_0 < 1$ and

$\frac{1}{8} + \frac{7}{8} t_0^2 = t_0$

$\Rightarrow 7t_0^2 - 8t_0 + 1 = 0$

$\Rightarrow t_0 = \frac{8 - \sqrt{64 - 28}}{14} = \frac{8-6}{14} = \frac{1}{7}$

③ $\Phi(t) = e^{-\lambda} + \sum_{k=1}^{\infty} \frac{\lambda^k}{k!} t^k e^{-\lambda}$

$$= \left(\sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} \right) e^{-\lambda}$$

$$= e^{\lambda t - \lambda} = e^{\lambda(t-1)}$$

$$\mu = E[\xi_1] = \sum_{k=1}^{\infty} k \frac{\lambda^k}{k!} e^{-\lambda} = \lambda$$

$\Rightarrow \quad \rho = 1 \quad$ if $\lambda \le 1$

$\rho = t_0 \quad$ if $\lambda > 1$ where

$$\boxed{e^{\lambda(t_0 - 1)} = t_0}$$

④ $\xi_1 \sim Bin(N, p)$

$\Rightarrow \quad E\xi_1 = Np$

$\Rightarrow \quad \rho = 1 \quad$ if $Np \le 1$

$\rho = t_0 \quad$ if $Np > 1$

$$\Phi(t) = \binom{N}{0}(1-p)^N + \sum_{k=1}^{N} \binom{N}{k} p^k (1-p)^{N-k} t^k$$

$$= (tp + 1 - p)^N = [1 + p(t-1)]^N$$

$$\Rightarrow \quad t_0 = \big(1 + p(t_0 - 1)\big)^N$$

(5) $\xi_1 = 0, 2,$ or 10

$$E[\xi_1] = 0.1 \times 2 + 0.8 \times 10$$
$$= 8.2 > 1$$

$\Rightarrow p = t_0$ where

$$\boxed{0.1 + 0.1 t_0^2 + 0.8 t_0^{10} = t_0}$$

Review of Chapter 1

1. Markov chain
2. Initial Distribution and One-step Transition Matrix
3. Three Main Issues.
   (a) Distribution at each time
   (b) Joint Distributions
   (c) Long-time behavior
4. Transient & Recurrent
5. Hitting Time & Absorption Probabilities
6. Decomposition of State Space
7. Birth-Death and Branching Chains