Lecture One : Basic concepts of stochastic processes

Review of probability

1. sample space Ω
2. σ-field $\mathcal{F}$
3. probability P : a function on $\mathcal{F}$

4. probability space $(\Omega, \mathcal{F}, P)$

5. Random variable

   $X : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B})$

   measurable map

6. conditional probability

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \quad \text{if } P(B) > 0$$

(7) Independent: $P(A \cap B) = P(A)P(B)$

Stochastic Process: A family of random variables defined on a probability space $(\Omega, \mathcal{F}, P)$ taking values in S with index set T. $\{X_t : t \in T\}$

The key features: (1) T time index

(2) S the state space

(3) Dependence relation

Example 1: $T = [0, +\infty)$ continuous time

$T = \{0, 1, 2, \cdots\}$ discrete time

$S = \mathbb{R}$ or $\{a_1, a_2, \cdots\}$

Example 2. $T = \{0, 1, 2, \cdots\}$

$X_0 = X_1 = X_2 = \cdots$

Motionless Process

If the state space S is at most countable, the stochastic process is called a stochastic chain.

without the loss of generality, we usually choose $S \subset \{0, 1, 2, \cdots\}$

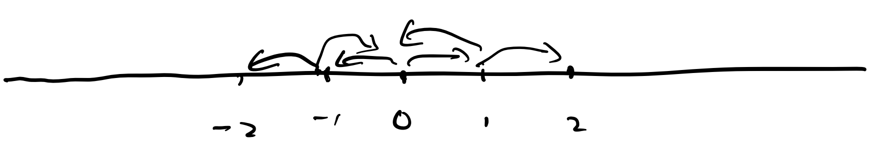
For $t \in T$, X_t represents the state of the process at time t

Example 3. Let $Y = \begin{cases} 1 & \text{with prob. } \frac{1}{2} \\ -1 & \text{-- --} \ \frac{1}{2} \end{cases}$

consider a sequence of i.i.d copies of Y, $Y_1, Y_2, \ldots$ Define

$$X_n = \begin{cases} 0 & n = 0 \\ Y_1 + \cdots + Y_n & n \geq 1 \end{cases}$$

Then $\{X_n\}$ is a stochastic chain called Random walk.



A special dependence relation:

The Markov Property

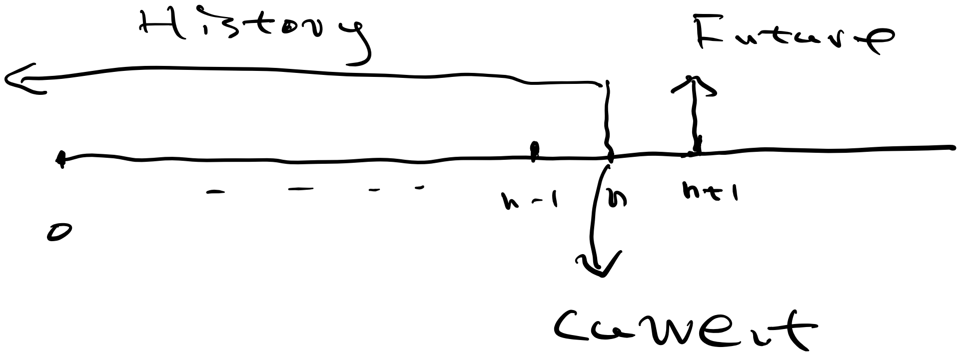
Let $T = \{0, 1, 2, \cdots\}$, $S \subset \{0, 1, 2, \cdots\}$

the stochastic chain $\{X_n : n = 0, \cdots\}$ is Markov if for any $j \in S$, $n \in T$,

$$P(X_{n+1} = j \mid X_0 = i_0, \cdots, X_{n-1} = i_{n-1}, X_n = i)$$

$$= P(X_{n+1} = j \mid X_n = i)$$

Future depends on history through current time only!



If Markov property holds, then the chain $\{X_n : n=0,1,2,\dots\}$ is called a $\boxed{\text{discrete time Markov chain}}$