Lecture 13 Stationary Distribution (Cont.)

Example 13.1. Let C be an irreducible set of recurrent states. Then

$$\lim_{n\to\infty} \frac{G_n(x,y)}{n} = \frac{1}{m_y} \quad \text{for any } x, y \in C$$

Solution: C is irreducible, recurrent

$\Rightarrow \rho_{xy} = 1$ for every $x, y \in C$

By theorem 1 of Lecture 12, we

obtain

$$\lim_{n\to\infty} \frac{G_n(x,y)}{n} = \frac{\rho_{xy}}{m_y} = \frac{1}{m_y}$$

Theorem 13.1: If $x \to y$ and x is positive recurrent, then y is positive recurrent.

Proof: x is positive recurrent

$\Rightarrow x$ is recurrent

$\Rightarrow$ y is recurrent and

$P_{xy} = P_{yx} = 1$

$\Rightarrow$ there exist $n_1 \geq 1$, $n_2 \geq 1$ such that $P^{n_1}(y,x) > 0$, $P^{n_2}(x,y) > 0$

Hence for $k = 1, 2, \dots, n$, we have

$$P^{n_1+k+n_2}(y,y) \geq P^{n_1}(y,x)\, P^{k}(x,x)\, P^{n_2}(x,y)$$

$$\Rightarrow \sum_{k=1}^{n} P^{n_1+n+n_2}(y,y) \geq \sum_{k=1}^{n} P^{n_1}(y,x)\, P^{k}(x,x)\, P^{n_2}(x,y)$$

$$= P^{n_1}(y,x)\, P^{n_2}(x,y) \sum_{k=1}^{n} P^{k}(x,x)$$

$$= P^{n_1}(y,x)\, P^{n_2}(x,y)\, G_n(x,x)$$

On the other hand,

$$\sum_{k=1}^{n} P^{n_1+k+n_2}(y,y) = \sum_{m=n_1+n_2+1}^{n_1+n_2+n} P^{m}(y,y)$$

$$= \sum_{m=1}^{n_1+n_2+n} P^{m}(y,y) - \sum_{m=1}^{n_1+n_2} P^{m}(y,y)$$

$$= G_{n_1+n_2+n}(y,y) - G_{n_1+n_2}(y,y)$$

Hence

$$G_{n_1+n_2+n}(y,y) - G_{n_1+n_2}(y,y)$$

$$\geq G_n(x,x)$$

$$\frac{G_{n_1+n_2+n}(y,y)}{n} - \frac{G_{n_1+n_2}(y,y)}{n}$$

$$= \frac{n_1+n_2+n}{n} \cdot \frac{G_{n_1+n_2+n}(y,y)}{n_1+n_2+n} - \frac{G_{n_1+n_2}(y,y)}{n}$$

1

$$\geq \frac{G_n(x,x)}{n}$$

Taking the limit of $n \to \infty$, it follows that

$$\frac{1}{m_y} \geq \frac{1}{m_x} > 0$$

$\Rightarrow$ $m_y < \infty$

$\Rightarrow$ y is positive recurrent.

Theorem 13.2: Let C be a finite irreducible closed subset of S. Then every state in C is positive recurrent.

Proof: For any $x \in C$, $k \geq 1$

$$1 = P_x(X_k \in C) = \sum_{y \in C} P_x(X_k = y)$$

$$= \sum_{y \in C} P^k(x,y)$$

$$\Rightarrow \sum_{k=1}^{n} \sum_{y \in C} P^k(x,y) = n$$

$$= \sum_{y \in C} \sum_{k=1}^{n} P^k(x,y) = \sum_{y \in C} G_n(x,y)$$

$$\Rightarrow \sum_{y \in C} \frac{G_n(x,y)}{n} = 1$$

$$\Rightarrow \lim_{n\to\infty} \sum_{y\in C} \frac{G_n(x,y)}{n} = \sum_{y\in C} \lim \frac{G_n(x,y)}{n}$$

$$= 1$$

$\Rightarrow$ there exists $z \in C$ such that

$$\lim_{n\to\infty} \frac{G_n(x,z)}{n} = \frac{1}{m_z} > 0$$

$\Rightarrow m_z < \infty \Rightarrow z$ positive recurrent

Irreducibility + Theorem 13.1 imply the result.

Example 13.2. Can a Markov chain having finite number of states have a null recurrent state?

Solution: $S = C_T \cup C_R$

$$C_R = C_1 \cup \cdots \cup C_m$$

By theorem 13.2, all states in C_i are positive recurrent.

Hence the answer is No!

Theorem 13.3. Let π be a stationary distribution of a Markov chain $\{X_n : n = 0, 1, 2, \cdots\}$. Then for any transient or null recurrent state y, we have

$$\pi(y) = 0$$

Proof: By proposition 1 of Lecture 11,

$$\sum_{x \in S} \pi(x) P^k(x,y) = \pi(y) \quad \text{for all } k \geq 1$$

and $y \in S$.

If y is transient or null recurrent,

then $\frac{G_n(x,y)}{n} \to 0$

$$\sum_{k=1}^{n} \sum_{x \in S} \pi(x) P^k(x,y) = n\,\pi(y)$$

$$\Rightarrow \frac{1}{n} \sum_{k=1}^{n} \sum_{x \in S} \pi(x) P^k(x,y)$$

$$= \sum_{x \in S} \pi(x) \frac{G_n(x,y)}{n} = \pi(y)$$

$$\Rightarrow \lim_{n \to \infty} \sum_{x \in S} \pi(x) \frac{G_n(x,y)}{n}$$

Dominated Convergence Theorem

$$= \sum_{x \in S} \pi(x) \lim_{n \to \infty} \frac{G_n(x,y)}{n}$$

$$= 0 = \pi(y)$$

Theorem 13.4. A Markov chain that does not have positive recurrent state has no stationary distribution.

Proof: If π is a stationary distribution, then $\pi(y)=0$ for all $y \in S$. This contradicts the fact that

$$\sum_{y \in S} \pi(y) = 1$$

Theorem 13.5. An irreducible positive recurrent Markov chain has a unique stationary distribution given by

$$\pi(y) = \frac{1}{m_y} \qquad y \in S$$

Proof: ① Uniqueness

② Existence

① Assume that π is a stationary distribution. Then for any $k \geq 1$, $y \in S$

$$\sum_{x \in S} \pi(x) p^k(x,y) = \pi(y)$$

$$\Rightarrow \sum_{k=1}^{n} \sum_{x \in S} \pi(x) p^k(x,y) = \sum_{x \in S} \pi(x) \sum_{k=1}^{n} p^k(x,y)$$

$$= \sum_{x \in S} \pi(x) G_n(x,y) = n\,\pi(y)$$

$$\Rightarrow \sum_{x \in S} \pi(x) \frac{G_n(x,y)}{n} = \pi(y)$$

Let $n \to \infty$, we obtain

$$\sum_{x \in S} \pi(x) \frac{1}{m_y} = \pi(y)$$

$$\Rightarrow \pi(y) = \frac{1}{m_y}$$ uniqueness

② Existence

claim: $\left\{ \frac{1}{m_y}, y \in S \right\}$ is a stationary distribution

By definition,

$$\sum_{y\in S} P^k(x,y) = 1 \quad \text{for all } k\ge 1,\ x\in S$$

Hence $\displaystyle\sum_{k=1}^{n} \sum_{y\in S} P^k(x,y) = \sum_{y\in S} G_n(x,y) = n$

$$\Rightarrow \sum_{y\in S} \frac{G_n(x,y)}{n} = 1$$

$$\Rightarrow \lim_{n\to\infty} \sum_{y\in D} \frac{G_n(x,y)}{n} = \sum_{y\in D} \lim_{n\to\infty} \frac{G_n(x,y)}{n} = \sum_{y\in D} \frac{1}{m_y} \le 1$$

for any finite subset D of S

Letting D approach S, it follows that

$$\boxed{\sum_{y\in S} \frac{1}{m_y} \le 1}$$

On the other hand, for any $k \geq 1$, $y, z \in S$

$$\sum_{x\in S} P^k(z,x)\,P(x,y) = P^{k+1}(z,y)$$

$$\Rightarrow \quad \sum_{x\in S} G_n(z,x)\,P(x,y) = G_{n+1}(z,y) - P(z,y)$$

$$\Rightarrow \quad \sum_{x\in S} \frac{G_n(z,x)}{n}\,P(x,y) = \frac{G_{n+1}(z,y)}{n} - \frac{P(z,y)}{n}$$

$$\Rightarrow \quad \sum_{x\in S} \left(\lim_{n\to\infty} \frac{G_n(z,x)}{n}\right) P(x,y)$$

$$= \sum_{x\in S} \frac{1}{m_x}\,P(x,y)$$

$$\leq \lim_{n\to\infty} \sum_{x\in S} \frac{G_n(z,x)}{n}\,P(x,y)$$

$$= \lim_{n\to\infty} \frac{G_{n+1}(z,y)}{n} - 0$$

$$= \frac{1}{m_y}$$

$$\Rightarrow \quad \boxed{\sum_{x\in S} \frac{1}{m_x}\,P(x,y) \leq \frac{1}{m_y}}$$

Claim: $\leq$ should be $=$

If not, then

$$\sum_{y \in S} \left(\sum_{x \in S} \frac{1}{w_x} P(x,y) \right) < \sum_{y \in S} \frac{1}{w_y}$$

But

$$\sum_{y \in S} \sum_{x \in S} \frac{1}{w_x} P(x,y)$$

$$= \sum_{x \in S} \frac{1}{w_x} \sum_{y \in S} P(x,y)$$

$$= \sum_{x \in S} \frac{1}{w_x}$$

A contradiction. Hence

$$\sum_{x \in S} \frac{1}{w_x} P(x,y) = \frac{1}{w_y}$$

Set $a = \sum_{x \in S} \frac{1}{w_x}$, $\pi(x) = \frac{1}{w_x} / a$

Then $\sum_{x \in S} \pi(x) = 1$ and

$$\sum_{x \in S} \pi(x) p(x,y) = \pi(y)$$

$\Rightarrow$ π is a stationary distribution.

By (1) $\pi(x) = \frac{1}{m_x}$

$\Rightarrow$ $a = 1$

Thus $\{ \frac{1}{m_y} : y \in S \}$ is a stationary distribution.