Tensor Networks

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Tensor Networks

- Framework of decomposition representation for many-body wavefunctions
- Diagrammatic language for quantum physics
- Explicit and accessible representation of entanglement between constituents in many-body
- Relevant degrees of freedom for low-energy states corresponds to TN states

Tensor Network

- **Tensor network** : product of tensors with contracted indices
- **Diagrammatic notation**

Wavefunction Decomposition

- N particles with p states : tensor product basis

$$|\Psi\rangle = \sum_{i_j=1\dots p} C_{i_1\dots i_N} |i_1\rangle \otimes \cdots |i_N\rangle .$$

  - N rank tensor; p^N (exponential) DoF
  - Not all coefficients are independent in presence of entanglement

Wavefunction Decomposition

- Decompose tensor into tensors of smaller rank
  - pN (polynomial) DoF
  - Additional DoF contribution due to dimensionality of contracted indices; **bond indices**

Wavefunction Decomposition

- Bond indices represents entanglement
- ex. Entanglement entropy between boundary and inner tensors of PEPS system

$$S(L) = -\mathrm{tr}(\rho_{\mathrm{in}} \log \rho_{\mathrm{in}}) \leq 4L \log D.$$

- For trivial TNs ($D = 1$), $S(L) = 0$; no entanglement present
- $D > 1$: area-law for entanglement entropy

Matrix Product States

- **MPS** : One-dimensional array of tensors
  - Two examples; open BC, periodic BC
  - One tensor per site in many-body system; open indices : physical DoF
- Properties
  - MPS are dense : any Hilbert state represented by increasing D; low energy states in 1D efficiently approximated with D polynomial in N
  - Area-law for entanglement entropy; ground states for large N

Matrix Product States

- Properties
  - Finitely correlated; $\langle OO' \rangle - \langle O \rangle \langle O' \rangle \sim f(r) \exp -r/\xi$

Matrix Product States

- Examples

1. GHZ State : $|\Psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle^{\otimes\infty} + |1\rangle^{\otimes\infty})$
2. 1D cluster state : simultaneous eigenket of $K^i = S^z_{i-1} S^x_i S^z_{i+1}$;
   $|\Psi\rangle = \prod_i \frac{1+K^i}{2} |0\rangle^{\otimes\infty}$
3. Ground state of 1D AKLT : $H = \sum_{\langle i,j\rangle} \boldsymbol{S}_i \cdot \boldsymbol{S}_j + \frac{1}{3}(\boldsymbol{S}_i \cdot \boldsymbol{S}_j)^2$

Density-Matrix Renormalisation Group

- 1D spin $\frac{1}{2}$ lattice; iterative method of increasing sites (sweep)
  - Split lattice into two blocks A, B + two intermediate sites; **superblock** $A \cdot \cdot B$
  - Diagonalise superblock Hamiltonian; find ground state
    $H = -J \sum_{\langle i,j \rangle} \boldsymbol{S}_i \cdot \boldsymbol{S}_j - h \sum_i S_i^z$
    $|\Psi\rangle_G = \sum_{ij} \psi_{ij} |i\rangle_{A\cdot} |j\rangle_{\cdot B}$; basis of block $A\cdot$ $|i\rangle_{A\cdot}$
  - Diagonalise reduced density operator for $A\cdot$ given ground state
    $\rho_{A\cdot} = \mathrm{Tr}_{\cdot B} |\Psi\rangle_G \langle\Psi|_G$; $(\rho_{A\cdot})_{ij} = \sum_k \psi_{ik}\psi^\star_{jk}$
    Effective basis of new block A' $|i'\rangle_{A'}$
  - Truncate for largest eigenvalues to reduce dimensionality
  - Update operators to new basis
    $\langle i'|O|j'\rangle = \sum_{ij} \langle i'|i\rangle \langle i|O|j\rangle \langle j|j'\rangle$
- Ansatz for superblock Hamltionian ground state : MPS

MPS and Machine Learning

- Feature vector constructed from tensor products; weights constructed from tensor networks (ex. MPS)

$$\mathrm{v}(x) = \begin{bmatrix} \phi_1(x_1) \\ \phi_2(x_1) \end{bmatrix} \otimes \begin{bmatrix} \phi_1(x_2) \\ \phi_2(x_2) \end{bmatrix} \otimes \cdots \otimes \begin{bmatrix} \phi_1(x_N) \\ \phi_2(x_N) \end{bmatrix}.$$

$$f(x) = W \cdot \mathrm{v}(x) = \sum_{s_i=\{1,2\}} W_{s_1 \cdots s_N} \phi_{s_1}(x_1) \cdots \phi_{s_N}(x_N).$$

MPS and Machine Learning

- ex. Unsupervised tree tensor network training (data compression) + supervised training of top layer (classification)