Tensor Networks : Session 2

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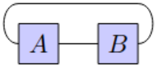
Recap

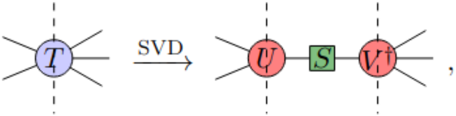
- Tensor networks are tensors (state vectors) that are made up of smaller tensors by tensor products and contraction
  - Many ways to write a tensor of a given rank N
  - Represents subspaces of Hilbert space
  - Degree of freedom polynomial in N

Examples : General TNs

- Trace

$\mathrm{Tr}(AB) =$ 

- Reduced density matrix (partial trace of bipartite state vector)
- SVD decomposition



  - Bond dimension determined by number of nonzero singular values of SVD

Recap

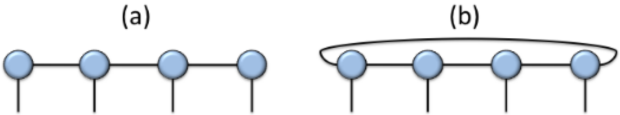
- Some comments
  - Some notation differentiates upper and lower indices by placement of 'legs'
  - Rank (p, q) tensor has $(p + q + 1)!$ reshapes generated by swapping and lowering/uppering indices
  - Do not confuse rank of tensor ex. (p, q) and rank of linear operator (dimensionality of image)
- Notation conventions
  - Tensors : either vector notation $|\psi\rangle$ or tensors A
  - Operators (MPOs) : boldface $\boldsymbol{\sigma}, \boldsymbol{H}$

Recap

- **Claimed** that the TN state subspace is physically relevant; we will see today :
    - Examples of highly entangled states/ground states as TNs
    - Bond dimension parametrises entanglement entropy; contraction characterises entanglement
    - TN states satisfy the area entropy law

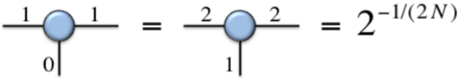
Matrix Product States

- **MPS** : 1D array of tensors
  - Two examples; open BC, periodic BC
  - One tensor per site in many-body system; open indices represents physical degrees of freedom
  - DoF : pD^2N



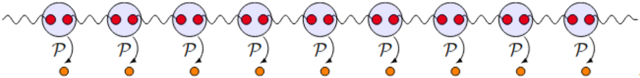
Examples : MPS

- **GHZ State** : $|\Psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle^{\otimes N} + |1\rangle^{\otimes N})$
  - Maximally entangled state : reduced density matrix is diagonal for any bipartition of system
  - MPS representation with bond dimension 2



Examples : MPS

- 1D PEPS (Projected Entangled Pair State)



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- Ground state of **1D AKLT** model : spin 1 chain with $H = \sum_{\langle i,j \rangle} \vec{S}_i \cdot \vec{S}_j + \frac{1}{3}(\vec{S}_i \cdot \vec{S}_j)^2$
  - Exact ground state is given in terms of projection of a spin $\frac{1}{2}$ chain

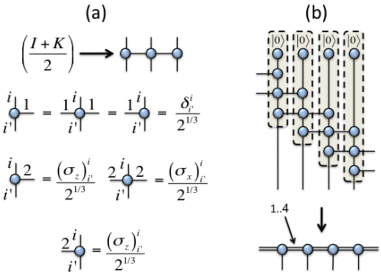
 $= \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$

 $= |+\rangle\langle\uparrow\uparrow| + |0\rangle\frac{\langle\uparrow\downarrow| + \langle\downarrow\uparrow|}{\sqrt{2}} + |-\rangle\langle\downarrow\downarrow|$

  - MPS representation with bond dimension 2

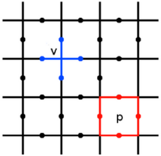
Examples : MPS

- **1D Cluster State** : simultaneous eigenket of $K^i = \sigma^x_{i-1}\sigma^z_i\sigma^x_{i+1}$; $|\Psi\rangle = \prod_i \frac{1+K^i}{2}|0\rangle^{\otimes N}$
  - Highly entangled state
  - Represent operators $\frac{1+K^i}{2}$ as TN
  - MPS representation with bond dimension 4



Examples : PEPS

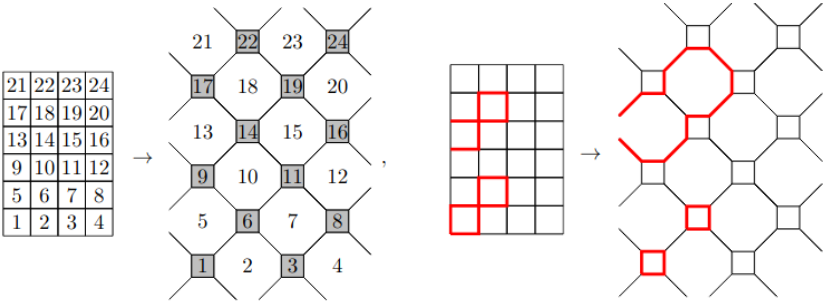
- **Toric code** eigenstates : given space of spins on edges on lattice, states where flipped spins form closed loops
  - Corresponds to trivial space of stabilisers;
    $A_v = \prod_{i \in v} \sigma_i, B_p = \prod_{i \in p} \sigma_i$



  - Used in quantum error correction

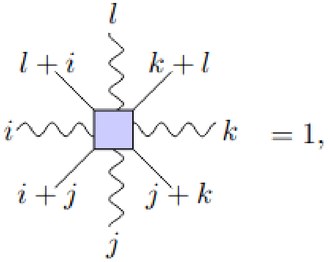
Examples : PEPS

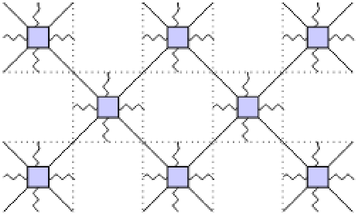
- **Toric code** eigenstate : express lattice in form below



Examples : PEPS

- **Toric code** ground state : superposition of all closed loops; admits PEPS form

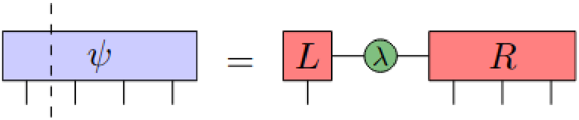




  - Assign odd numbered plaquettes to PEPS site
  - $i, j, k, l \in \mathbb{Z}_2$; straight legs indicate bond indices; wavy legs qubits on each edge of plaquette

SVD and Schmidt Decomposition

- Split wavefunction into parts; SVD gives Schmidt decomposition $|\psi\rangle = \sum_i \lambda_i |L_i\rangle \otimes |R_i\rangle$



- Recursive application decomposes state into MPS

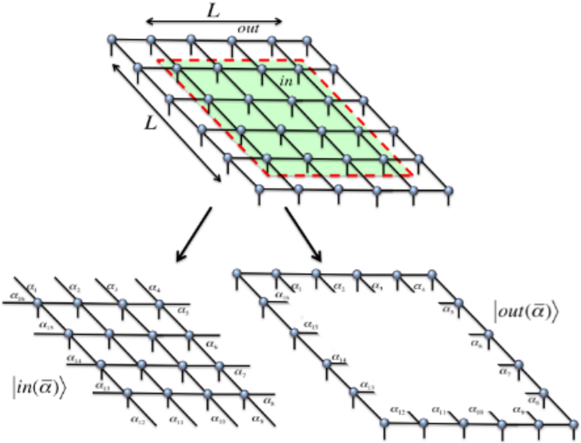
SVD and Schmidt Decomposition

- **Rank** : distinct from rank of tensor, dimensionality of image
  - Ex. $m \times n$ matrix of rank one : outer product of m vector and n vector
  - $m \times n$ matrix of rank k : total $k(m+n)$ DoF
- Number of nonzero λ_i = rank of diagonal matrix λ = bond dimension of MPS

Area Law : MPS

- Area law : constant (wrt. N) entanglement entropy
- Decomposition; $|\psi\rangle = \sum_i \lambda_i |L_i\rangle \otimes |R_i\rangle$
  - i represents the bond indices; only require one as matrix λ is diagonal
- Reduced density matrix : $\rho = \sum_i \lambda_i^2 |L_i\rangle \langle L_i|$
- Entanglement entropy : $S = -\mathrm{tr}(\rho \log \rho) = -\sum_i \lambda_i^2 \log \lambda_i^2$
- Bound : given normalisation $\sum_i \lambda_i^2 = 1$, maximum entropy at all $\lambda_i^2 = \frac{1}{D}$, $S \leq \log D$

Area Law : PEPS

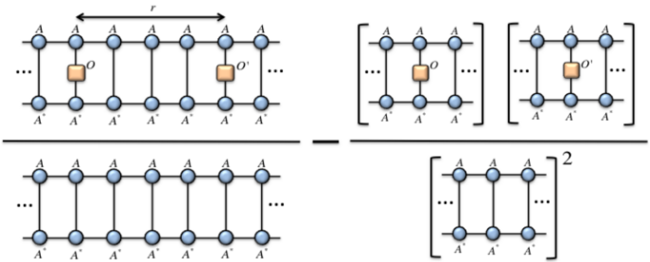


Area Law : PEPS

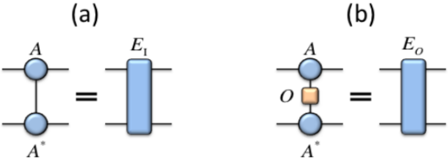
- Area law : entropy $\sim L$
- Decomposition : $|\psi\rangle = \sum_{\overline{\alpha}} |\text{in}_{\overline{\alpha}}\rangle \otimes |\text{out}_{\overline{\alpha}}\rangle$
  - $\overline{\alpha}$ represents joint bond indices; total of $D^{\partial A} = D^{4L}$ values
- Reduced density matrix : $\rho = \sum_{\overline{\alpha},\overline{\alpha}'} \langle \text{out}_{\overline{\alpha}'} | \text{out}_{\overline{\alpha}} \rangle \, |\text{in}_{\overline{\alpha}}\rangle \langle \text{in}_{\overline{\alpha}'}|$
- Similarly, maximum entropy for $S = -\text{tr}\rho \log \rho \leq 4L \log D$

Finitely Correlated : MPS

- Correlation function : $C(r) = \left\langle O_i O'_{i+r} \right\rangle - \left\langle O_i \right\rangle \left\langle O'_{i+r} \right\rangle$
  Consider infinite MPS of single tensor; 1D translational invariance with thermodynamic limit

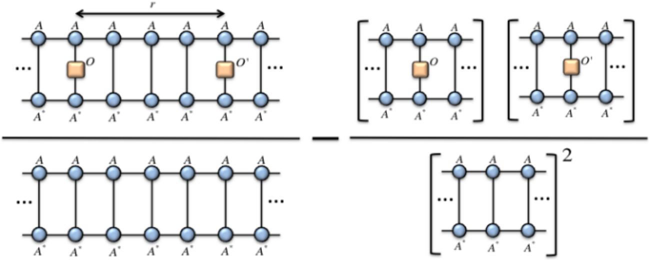


Finitely Correlated : MPS



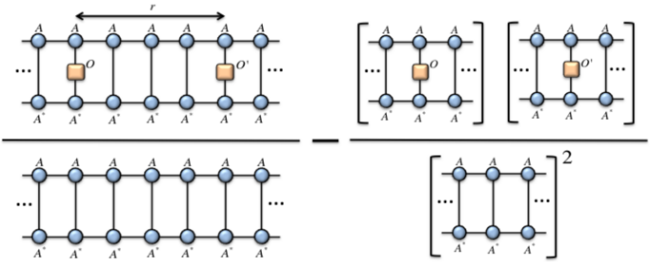
- Consider E_i as $D^2 \times D^2$ matrix; diagonalise
  $E_I = \sum_{i=1}^{D^2} \lambda_i \left|L_i\right\rangle \left\langle R_i\right|$; $E_I^r = \lambda_1^r \sum_{i=1}^{D^2} (\lambda_i/\lambda_1)^r \left|L_i\right\rangle \left\langle R_i\right|$
- Consider dominant EVs;
  $E_I^r \approx \lambda_1^r \left(\left|L_1\right\rangle \left\langle R_1\right| + (\lambda_2/\lambda_1)^r \sum \left|L_\mu\right\rangle \left\langle R_\mu\right| \right)$

Finitely Correlated : MPS



- $O_i = \mathrm{tr}(E_I^{n-1} E_O E_I^{N-n})/\mathrm{tr}(E_I^N)$
- In therm. limit, $\langle O_i \rangle \approx \langle R_1 | E_O | L_1 \rangle / \lambda_1$
- $\langle O_i \rangle \left\langle O'_{i+r} \right\rangle \approx \langle R_1 | E_O | L_1 \rangle \langle R_1 | E_{O'} | L_1 \rangle / \lambda_1^2$

Finitely Correlated : MPS



- $\langle O_i O'_{i+r} \rangle = \mathrm{tr}(E_I^{n-1} E_O E_I^{r-1} E_{O'} E_I^{N-n-r}) / \mathrm{tr}(E_I^N)$
- In therm. limit,
  $\langle O_i O'_{i+r} \rangle \approx \langle R_1 | E_O E_I^{r-1} E_{O'} | L_1 \rangle / \langle R_1 | E_I^{r+1} | L_1 \rangle$
  $\approx \langle R_1 | E_O | L_1 \rangle \langle L_1 | E_O | R_1 \rangle / \lambda_1^2 + (\lambda_2/\lambda_1)^r A$
- $C(r) \sim \exp -r/\xi$ with $\xi = -1/\log|\lambda_1/\lambda_2|$

Finitely Correlated : MPS

- Exponential decay of two-point correlation function
- Results known where such exponential decay *implies* the area law
- Correlation function mimics quantum system in lower spatial dimension

MPS Algorithms

The term MPS algorithm can refer to any of the following;

- Approximation of general tensors into MPS
  - Low rank approx.
  - MPS compression
- (Efficient) computations involving MPS
  - Inner products, expectation values etc.
- Calculating ground states
  - Variational method over TN space; DMRG
  - Imaginary time evolution

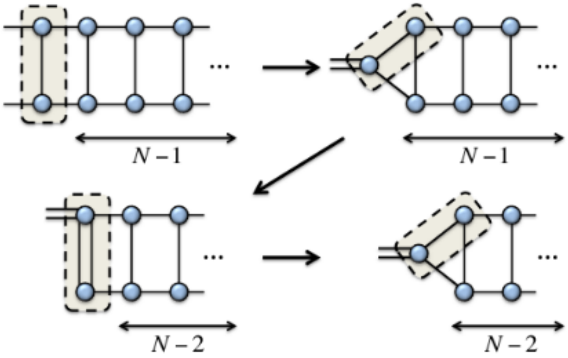
MPS Algorithms

- Can we decompose an arbitrary rank N tensor into an MPS state? Yes! By recursively applying SVD
  - For N particle system, bond dimension is bounded above by $2^{\lfloor N/2 \rfloor}$; MPS are dense
  - Expensive in terms of calculations

MPS Algorithms

- **Low rank approximation**
  - Eckart-Young theorem : optimal low rank approximation appears via SVD
  - Truncate to k singular values
- Use low rank approximation to apply cutoff to bond dimension of MPS; approximation of general state

TN Algorithms : MPS EV



TN Algorithms : PEPS EV

- Contraction of a 2D lattice is computationally complex
- Approximation required; reduce the original problem to a series of 1D problems equivalent to the ground state calculations problems above!

References and Resources

- Hand-waving and Interpretive Dance : An Introductory Course on Tensor Networks; Jacob C. Bridgeman, Christopher T. Chubb
- A Practical Introduction to Tensor Networks : Matrix Product States and Projected Entangled Pair States; Román Orús
- The density-matrix renormalization group in the age of matrix product states; Ulrich Schollwöck