Tensor Networks : Session 2

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Recap

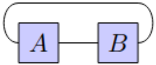
- Tensor networks are tensors (state vectors) that are made up of smaller tensors by tensor products and contraction
  - Many ways to write a tensor of a given rank N
  - Represents subspaces of Hilbert space
  - Degree of freedom polynomial in N
- Some comments
  - Some notation differentiates upper and lower indices by placement of 'legs'; rank (p, q) tensor has $(p+q+1)!$ reshapes generated by swapping and lowering/uppering indices
  - Do not confuse rank of tensor ex. (p, q) and rank of linear operator (dimensionality of image)

Recap

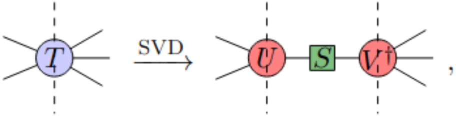
- **Claimed** that this smaller subspace is physically relevant; we will see today :
  - Examples of highly entangled states/ground states as TNs
  - Bond dimension parametrises entanglement entropy; contraction characterises entanglement
  - TN states satisfy the area entropy law

Examples : General TNs

- Trace

$\mathrm{Tr}(AB) =$ 

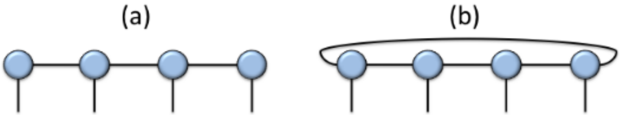
- Reduced density matrix (partial trace of bipartite state vector)
- SVD decomposition



  - Choice of index grouping, algorithm determine the bond dimension; independent of remaining discussion, set bond dimension as arbitrary value

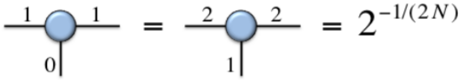
Matrix Product States

- **MPS** : 1D array of tensors
  - Two examples; open BC, periodic BC
  - One tensor per site in many-body system; open indices represents physical degrees of freedom
  - DoF : pD^2N



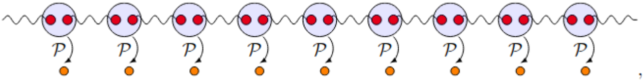
Examples : MPS

- **GHZ State** : $|\Psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle^{\otimes N} + |1\rangle^{\otimes N})$
  - Maximally entangled state : reduced density matrix is diagonal for any bipartition of system
  - MPS representation with bond dimension 2



Examples : MPS

- 1D PEPS (Projected Entangled Pair State)



- Ground state of **1D AKLT** model : spin 1 chain with $H = \sum_{\langle i,j \rangle} \boldsymbol{S}_i \cdot \boldsymbol{S}_j + \frac{1}{3}(\boldsymbol{S}_i \cdot \boldsymbol{S}_j)^2$
  - Exact ground state is given in terms of projection of a spin $\frac{1}{2}$ chain

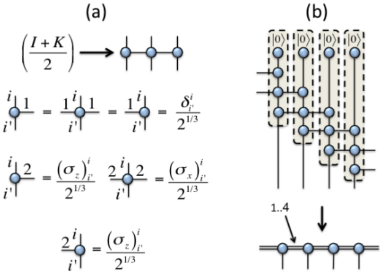
$$= \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$$

$$= |+\rangle\langle\uparrow\uparrow| + |0\rangle\frac{\langle\uparrow\downarrow| + \langle\downarrow\uparrow|}{\sqrt{2}} + |-\rangle\langle\downarrow\downarrow|$$

  - MPS representation with bond dimension 2

Examples : MPS

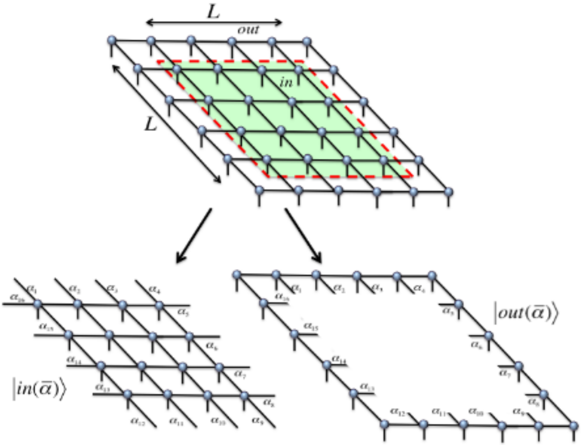
- **1D Cluster State** : simultaneous eigenket of $K^i = \sigma^x_{i-1}\sigma^z_i\sigma^x_{i+1}$; $|\Psi\rangle = \prod_i \frac{1+K^i}{2}|0\rangle^{\otimes N}$
  - Highly entangled state
  - Represent operators $\frac{1+K^i}{2}$ as TN
  - MPS representation with bond dimension 4



Physical Properties : MPS

- MPS are dense
- Area law : constant entanglement entropy
  - Split wavefunction into parts; $T_{\beta\gamma} = T^{\text{in}}_{\beta\alpha_1\alpha_2} T^{\text{out}}_{\alpha_1\alpha_2\gamma}$
    - β, γ each indices of inner and outer state
    - Combined indices $\overline{\alpha} = \{\alpha_1, \alpha_2\}$; total of D^2 values
    - Roughly $|\psi\rangle = |\text{in}\rangle \otimes |\text{out}\rangle$
  - Reduced density matrix : $\rho^{\text{in}}_{\beta\beta'} = T^{\text{out}}_{\overline{\alpha}\gamma} T^{\text{out}}_{\overline{\alpha}'\gamma} T^{\text{in}}_{\beta\overline{\alpha}} T^{\text{in}}_{\beta'\overline{\alpha}'}$
    - Inner product between T^{out}, outer product between T^{in}
    - wrt. D, $\text{tr}\rho^{\text{in}}$ scales as D^2
    - Const. wrt. subsystem size L
    - Roughly $\rho^{\text{in}} = \langle\text{out}|\text{out}\rangle\, |\text{in}\rangle\langle\text{in}|$
  - α-Rényi entropy : $S(L) = \frac{1}{1-\alpha} \log \text{tr}\rho^{\alpha} \sim \log D$

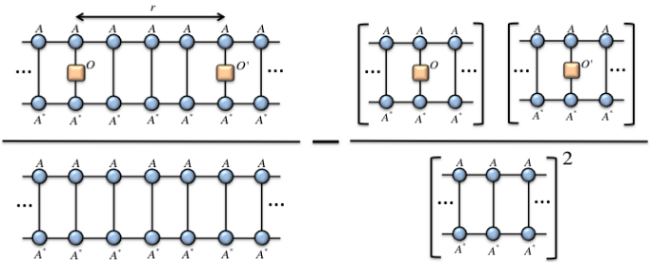
Physical Properties : PEPS



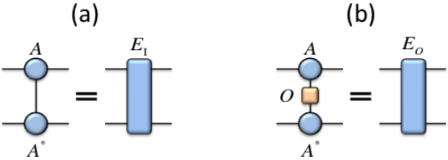
- $S(L) \sim 4L \log D$

Physical Properties - Finitely Correlated : MPS

- Correlation function : $C(r) = \left\langle O_i O'_{i+r} \right\rangle - \langle O_i \rangle \left\langle O'_{i+r} \right\rangle$
  Consider infinite MPS of single tensor; 1D translational invariance with thermodynamic limit

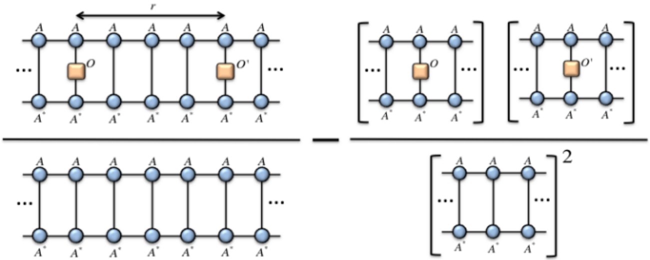


Physical Properties - Finitely Correlated : MPS



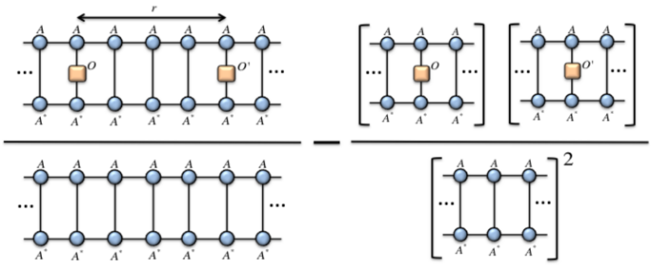
- Consider E_i as $D^2 \times D^2$ matrix; diagonalise
  $E_I = \sum_{i=1}^{D^2} \lambda_i \vec{R}_i^T \vec{L}_i$; $E_I^r = \lambda_1^r \sum_{i=1}^{D^2} (\lambda_i/\lambda_1)^r \vec{R}_i^T \vec{L}_i$
- Consider dominant EVs;
  $E_I^r \approx \lambda_1^r \left(\vec{R}_1^T \vec{L}_1 + (\lambda_2/\lambda_1)^r \sum \vec{R}_\mu^T \vec{L}_\mu \right)$

Physical Properties - Finitely Correlated : MPS



- $\langle\psi|O_i|\psi\rangle = \vec{L}_1 E_O E_I^r \vec{R}_1^T$
- In t.d. limit, $\langle O_i\rangle = \vec{L}_1 E_O E_I^r \vec{R}_1^T / E_I^{r+1} = \vec{L}_1 E_O \vec{R}_1^T / \lambda_1 + \cdots$
- $\langle O_i\rangle \left\langle O'_{i+r}\right\rangle \approx \vec{L}_1 E_O \vec{R}_1^T \vec{L}_1 E_{O'} \vec{R}_1^T / \lambda_1^2$

Physical Properties - Finitely Correlated : MPS



- $\langle\psi|O_i O'_{i+r}|\psi\rangle = \vec{L}_1 E_O E_I^{r-1} E_{O'} \vec{R}_1^T$
- In t.d. limit, $\langle O_i O'_{i+r}\rangle = \vec{L}_1 E_O E_I^{r-1} E_{O'} \vec{R}_1^T / E_I^{r+1}$
- $\langle O_i O'_{i+r}\rangle = \vec{L}_1 E_O \vec{R}_1^T \vec{L}_1 E_{O'} \vec{R}_1^T / \lambda_1^2 + (\lambda_2/\lambda_1)^r \sum \vec{R}_\mu^T \vec{L}_\mu$
- $C(r) \sim \exp -r/\xi$ with $\xi = -1/\log|\lambda_1/\lambda_2|$

Physical Properties - Finitely Correlated : MPS

- Exponential decay of two-point correlation function
- Results known where such exponential decay *implies* the area law
- Correlation function mimics quantum system in lower spatial dimension

MPS Algorithms

How do we use TNs in calculations? What are the considerations we must make?

- Approximation of general tensors into TN states
- Calculating ground states
  - Variational method over TN space; DMRG, TEBD
  - Imaginary time evolution
- (Efficient) computation of expectation values of TN states

MPS Algorithms

- Can we decompose an arbitrary rank N tensor into an MPS state? Yes! By recursively applying SVD
  - For N particle system, bond dimension is bounded above by $2^{\lfloor N/2 \rfloor}$; MPS are dense
  - Expensive in terms of calculations

MPS Algorithms

- **Rank** : distinct from rank of tensor, dimensionality of image
  - Ex. $m \times n$ matrix of rank one : outer product of m vector and n vector
  - $m \times n$ matrix of rank k : total $k(m+n)$ DoF
  - Imposing low rank conditions on TNs further compresses the data
- **Low rank approximation**
  - Eckart-Young theorem : optimal low rank approximation appears via SVD
  - Truncate to k singular values
- Use low rank approximation to apply cutoff to bond dimension of MPS; approximation of general state

MPS Algorithms

- Iteratively improve approximations by using variational methods; ex. **Density Matrix Renormalisation Group**
- Time evolve MPSs via nearest-neighbour Hamiltonians; ex. **Time-Evolving Block Decimation**

References and Resources

- Hand-waving and Interpretive Dance : An Introductory Course on Tensor Networks; Jacob C. Bridgeman, Christopher T. Chubb
- A Practical Introduction to Tensor Networks : Matrix Product States and Projected Entangled Pair States; Román Orús
- The density-matrix renormalization group in the age of matrix product states; Ulrich Schollwöck