Tensor Networks : Session 1

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What is a Tensor?

- Tensor in physics : something that *transforms like a tensor*
  - Arises from a more mathematical definition
  - We are interested in a more simple definition
- Vector space V : define as rank 1 tensor
- **Tensor product** : rank $n \otimes$ rank $m =$ rank $(m+n)$ tensor
  - Components : product; $(A \otimes B)_{ij} = A_i B_j$
  - cf. Cartesian product; do not confuse rank and dimensionality!

Diagrammatic Notation

- Indices are written as legs
  - Number of legs = rank
  - Dimensionality of each index is not explicitly shown
  - Can express components of tensor by writing indices
- Product between tensors : write tensors side-by-side
- Contract over two indices (of same dimensionality) by connecting indices

What are Tensor Networks?

- Consider many-body quantum system; ex. 1D lattice of N spin-1/2 particles
  - Each site has $p = 2$ **degrees of freedom**
- General state vector : specify components for all possible configurations
- State vector is given by specifying components of a **tensor**

$$|\Psi\rangle = \sum_{\substack{s_j=\{+,-\} \\ j=1\dots N}} T_{s_1\cdots s_N} |s_1\rangle \otimes \cdots |s_N\rangle$$

  - Total p^N degrees of freedom; exponential in N

What are Tensor Networks?

- Limited class of state vectors : specify **state vectors** at each site; take **tensor product** of all vectors

$$
\begin{aligned}
|\Psi\rangle &= \left(\sum_{s_1=\{+,-\}} A_{s_1}|s_1\rangle\right) \otimes \cdots \left(\sum_{s_p=\{+,-\}} A_{s_p}|s_p\rangle\right) && (1)\\
&= \sum_{\substack{s_j=\{+,-\}\\ j=1\ldots N}} \tilde{T}_{s_1\cdots s_N}|s_1\rangle \otimes \cdots |s_N\rangle && (2)\\
& && (3)
\end{aligned}
$$

- State vector is also a **tensor**; not all components are 'independent'!
  - Total pN degrees of freedom; polynomial in N
  - Does not generate the entire Hilbert space

What are Tensor Networks?

- Limited class of state vectors : specify **tensors** at each site; **contract** indices except one
- Most general kind of tensor network
  - Dimensionality of contracted indices : **bond dimension** D
  - Typically degrees of freedom polynomial in N
- Examples
  - **Matrix Product State** : 1D array of tensors
  - **Projected Entangled Pair States** : 2D array of tensors

Why Tensor Networks?

- Using tensor networks : considering only wave functions that can be written in this form
- Why limit ourselves to tensor network states?

Why Tensor Networks?

- **Natural representation of entanglement**
  - TNs are natural representations for many-body wave functions that are visually intuitive
  - TN structure means components of wavefunction are not 'independent'
  - This dependence characterises entanglement present in states

Why Tensor Networks?

- **Hilbert space is 'too large'**
  - (Fortunately,) not all states in the Hilbert space are relevant
    - Why? Locality (states of 'neighbouring' particles are not completely independent)
    - In particular, low energy states follows the entropy area law
  - Number of states typically are polynomial in N

Some General Notes

- Complexity of contracting a TN is order dependent
  - Only relevant if bond dimensions are nonuniform
  - Expectation values of TNs is a contraction of two TNs (+ some operators)
- Given a chosen form of TN, there is still choice of bond dimension D that can be made
  - For D sufficiently large, ex. MPS can eventually cover the whole Hilbert space
  - For $D = 1$, no entanglement is present; used in mean field theory

Physical Properties : MPS

- **MPS** : 1D array of tensors
  - Two examples; open BC, periodic BC
  - One tensor per site in many-body system; open indices represents physical degrees of freedom
- MPS are dense : any Hilbert state can be represented by increasing D
- Area law : entanglement entropy is constant wrt. L