The wave kinetic theory of three wave and four wave models

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Abstract

In this thesis, we introduce new techniques for studying the random series expansion of dispersive PDEs. We take a quadratic KdV type and a cubic Klein-Gordon type equation as examples to demonstrate the different techniques in three wave and four wave models. For three wave models, we introduce a counting argument to handle the degeneracy problems of the resonance surface and the loss of derivative problem. For four wave models, we introduce a novel renormalization argument and prove a renormalized Wick theorem. We provide a heuristic argument that this renormalization is able to remove all bad terms from the L^2 mass term, combining with an almost cancellation identity of the regular pairing and the Deng-Hani's Feynman diagram analysis [6], [9].

Acknowledgements

thank you very much

To my family and friends.

Declaration

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renormalization

In this section we introduce the new variable ψ and rewrite (NKLG) into the standard form of a dispersive equation.

Notice that in (NKLG) $\partial_{tt}u - \Delta u + u = -\lambda^2 u^3$, the operator $\partial_{tt} - \Delta + 1$ can be factorized into $(-i\partial_t + \Lambda(\nabla)) \cdot (i\partial_t + \Lambda(\nabla))$, where $\Lambda(\nabla)$ is the Fourier multiplier with symbol $\Lambda(\xi) = \sqrt{1 + |\xi|^2}$.

Therefore the equation can be simplified by introducing a new variable

$$\psi = i\partial_t u + \Lambda(\nabla)u. \tag{1.6.1}$$

Then we have

$$(-i\partial_t + \Lambda(\nabla))\psi = (-i\partial_t + \Lambda(\nabla)) \cdot (i\partial_t u + \Lambda(\nabla))u = -\lambda^2 u^3. \tag{1.6.2}$$

We can solve u in terms of ψ, $\bar{\psi}$

$$u = (2\Lambda(\nabla))^{-1}(\psi + \bar{\psi}). \tag{1.6.3}$$

We can derive the equation of ψ by substituting (1.6.3) into the right hand side of (1.6.2)

$$i\partial_t\psi - \Lambda(\nabla)\psi = \frac{\lambda^2}{8}\Big(\Lambda(\nabla)^{-1}\psi + \Lambda(\nabla)^{-1}\bar{\psi}\Big)^3. \tag{1.6.4}$$

In this section we explain the renormalization argument. Let ψ_k be the Fourier coefficient of ψ. Then in term of ψ_k equation (1.6.4) becomes

$$\begin{aligned} i\dot{\psi}_k =& \Lambda_k u_k + \frac{\lambda^2}{8L^{2d}} \sum_{(k_1,k_2,k_3)\in(\mathbb{Z}_L^d)^3} m^3_{k_1k_2k_3}\psi_{k_1}\psi_{k_2}\psi_{k_3} + m^2_{k_1k_2k_3}\psi_{k_1}\overline{\psi}_{k_2}\psi_{k_3} \\ &+ m^1_{k_1k_2k_3}\psi_{k_1}\overline{\psi}_{k_2}\overline{\psi}_{k_3} + m^0_{k_1k_2k_3}\overline{\psi}_{k_1}\overline{\psi}_{k_2}\overline{\psi}_{k_3}, \end{aligned} \tag{1.6.5}$$

where

$$\begin{cases} m^3_{k_1k_2k_3} = (\Lambda_{k_1}\Lambda_{k_2}\Lambda_{k_3})^{-1}\delta_{k_1+k_2+k_3=k} \\ m^2_{k_1k_2k_3} = (\Lambda_{k_1}\Lambda_{k_2}\Lambda_{k_3})^{-1}\delta_{k_1-k_2+k_3=k} \\ m^1_{k_1k_2k_3} = (\Lambda_{k_1}\Lambda_{k_2}\Lambda_{k_3})^{-1}\delta_{k_1-k_2-k_3=k} \\ m^0_{k_1k_2k_3} = (\Lambda_{k_1}\Lambda_{k_2}\Lambda_{k_3})^{-1}\delta_{k_1+k_2+k_3+k=0} \end{cases} \tag{1.6.6}$$

According to (3.1.1), the initial data of (1.6.5) can be written as

$$\psi_k(0) = \sqrt{n_{\text{in}}(k)/2}\,(\alpha_k + i\beta_k) \tag{1.6.7}$$

To keep the notation simpler we define

$$\begin{aligned}\eta_k(\omega) &= \frac{1}{\sqrt{2}}(\alpha_k + i\beta_k),\\ \xi_k &= \psi_k(0) = \sqrt{n_{\text{in}}(k)}\,\eta_k(\omega).\end{aligned} \tag{1.6.8}$$

Therefore $\eta_k(\omega)$ are i.i.d complex normal distributions and ξ_k are Fourier coefficients of the initial data of ψ.

As explained in Appendix **??**, the random data Cauchy problem of defocusing Klein-Gordon equation (1.6.9) is almost equivalent to above equation after change of variable $\psi = i\partial_t u + \Lambda(\nabla)u$.

$$\begin{cases}\partial_{tt}u = \Delta u - u - \lambda^2 u^3, \quad x \in \mathbb{T}^d_{L_1\cdots L_d},\\ u(0,x) = u_{\text{in}}(x).\\ \partial_t u(0,x) = u^1_{\text{in}}(x)\end{cases} \tag{1.6.9}$$

1.7 Previous research

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1.7.1 A short survey of previous papers

(1) Results about the ZK equation: ZK equation was introduced in [27] as an asymptotic model to describe the propagation of nonlinear ionic-sonic waves in a magnetized plasma. For a good reference about the physical background, see the book [5]. For rigorous results about wellposedness and derivation from the Euler-Poisson system see [19] and references therein.

(2) Previous papers about wave turbulence theory: There are numerous physics papers about the derivation of wave kinetic equation. In particular, the wave kinetic equation for the ZK equation is derived in [18]. For general references, see the books [28] and [21], and the review paper [22].

The WKE was rigorously verified for the Gibbs measure initial data by Lukkarinen and Spohn [20]. Then the basic concepts of general wave turbulence were rigorously formulated by Buckmaster, Germain, Hani, Shatah [2] and a non-trivial result that verified WKE for a short time scale was also proved by them. The WKE was proved for almost sharp time scale independently by Deng and Hani [7] and Collot and Germain [3], [4] using the ideas from the study of randomly initialized PDE. The full WKE for the sharp time was proved independently by the deep works of Deng and Hani [6] and Staffilani and Tran [26] for a four-wave problem and a three-wave problem respectively.

One key contribution of [6] and [26] was the classification of Feynman diagrams in the contexts of normal form expansion and Liouville equation respectively. WKE for the space-inhomogeneous case was derived by Ampatzoglou, Collot, and Germain [1] for almost sharp time scale. The higher order correlation functions were studied by Deng and Hani [8]. A linearized wave kinetic equation near the Rayleigh-Jeans spectrum was derived by Faou [14]. The discrete wave turbulence was studied by Dymov and Kuksin [10], [13], [11], [12].

Among the above papers, [26] is the only one working on ZK equation. [26] derived the wave kinetic equation for lattice ZK equation with random force for $t \leq T_{\text{kin}}$, while our paper is for dissipative continuous ZK equation for $t \leq L^{-\varepsilon} T_{\text{kin}}$.

(3) Previous papers about the dynamics of WKE: There are also many papers about the dynamics of WKE itself. For references, see [16], [15], [24], [25] and the reference therein.

Chapter 2

Analysis of three wave models

2.1 Introduction

2.1.1 Statement of the results

In this chapter, we study the wave turbulence theory for the following KdV type equation

$$\begin{cases} \partial_t \psi(t,x) + \Delta \partial_{x_1} \psi(t,x) - \nu \Delta \psi(t,x) = \lambda \partial_{x_1} (\psi^2(t,x)), \\ \psi(0,x) = \psi_{\text{in}}(x), \quad x \in \mathbb{T}_L^d. \end{cases} \tag{MKDV}$$

as an example of three wave system.

Here ψ is a real value function. We consider the periodic boundary condition, which implies that the spatial domain is a torus $\mathbb{T}_L^d = [0, L]^d$.

We know that the Fourier coefficients of ψ lie on the lattice $\mathbb{Z}_L^d = \{k = \frac{K}{L} : K \in \mathbb{Z}^d\}$. Let n_{in} be a known function, we assume that

$$\psi_{\text{in}}(x) = \frac{1}{L^d} \sum_{k \in \mathbb{Z}_L^d} \sqrt{n_{\text{in}}(k)} \eta_k(\omega)\, e^{2\pi i k x} \tag{2.1.1}$$

where $\eta_k(\omega)$ are mean-zero and identically distributed complex Gaussian random variables satisfying $\mathbb{E}|\eta_k|^2 = 1$. To ensure ψ_{in} to be a real value function, we assume that $n_{\text{in}}(k) = n_{\text{in}}(-k)$ and $\eta_k = \overline{\eta_{-k}}$. Finally, we assume that η_k is independent of $\{\eta_{k'}\}_{k' \neq k, -k}$.

The energy spectrum $n(t,k)$ mentioned in previous section is defined to be $\mathbb{E}|\widehat{\psi}(t,k)|^2$, where $\psi(t,k)$ are Fourier coefficients of the solution. Although the initial data is assumed to be the

Gaussian random field, it is possible to develop a theory for other types of random initial data.

We define $\Lambda(k) := k_1(k_1^2 + \cdots k_d^2)$. Under this new notations the ZK equation becomes,

$$\partial_t \psi(t,x) = i\Lambda(\nabla)\psi(t,x) + \nu\Delta\psi(t,x) + \lambda\partial_{x_1}(\psi^2(t,x)).$$

In the wave turbulence, the energy distribution $n(k)$ is supposed to evolve according to the following wave kinetic equation

$$\begin{aligned}
\partial_t n(t,k) =& \frac{1}{T_{\text{kin}}}\mathcal{K}\left(n(t,\cdot)\right),\\
\mathcal{K}(n)(k) :=& |k_x|^2 \int_{\substack{(k_1,k_2)\in\mathbb{R}^{2d}\\ k_1+k_2=k}} n(k_1)n(k_2)\delta(|k_1|^2k_{1x} + |k_2|^2k_{2x} - |k|^2k_x)\, dk_1dk_2 \\
&-2n(k)\int_{\mathbb{R}^d} k_x(k_x-k_{1x})n(k_1)\delta(|k_1|^2k_{1x} + |k_2|^2k_{2x} - |k|^2k_x)\, dk_1
\end{aligned} \tag{WKE}$$

Now we introduce the main theorem of this chapter.

Theorem 2.1.1. *Let $d \geq 3$ and L be a large number. Suppose that $n_{\text{in}} \in C_0^\infty(\mathbb{R}^d)$ is compactly supported in a domain whose diameter is bounded by D. Assume that ψ is a solution of* (MKDV) *with randomized initial data ψ_{in} given by* (2.1.1)*. Set $\alpha = \lambda L^{-\frac{d}{2}}$ to be the strength of the nonlinearity and $T_{\text{kin}} = \frac{1}{8\pi\alpha^2}$ to be the wave kinetic time. Fix a small constant $\varepsilon > 0$, set $T_{max} = L^{-\varepsilon}\alpha^{-2} = O(L^{-\varepsilon}T_{\text{kin}})$. If α satisfies*

$$\alpha^{-1} \leq L^{\frac{1}{2}} \tag{2.1.2}$$

and for some small constant c, ν satisfies

$$\nu \geq c^2 T_{max}^{-1} \tag{2.1.3}$$

then for all $L^\varepsilon \leq t \leq T_{max}$, we have the following conclusions

1. *If $\sup_k n_{\text{in}}(k) \leq C_{\text{in}}$, then $\mathbb{E}|\widehat{\psi}(t,k)|^2$ is bounded by $2C_{\text{in}}$ for $t \leq T_{max}$ and for any M, we can construct an approximation series*

$$\mathbb{E}|\widehat{\psi}(t,k)|^2 = n_{\text{in}}(k) + n^{(1)}(k) + n^{(2)}(k) + \cdots n^{(N)}(k) + O(L^{-M}) \tag{2.1.4}$$

where each terms $n^{(i)}(k)$ can be exactly calculated.

2. *Define $l_d = (\nu T_{\max})^{\frac{1}{2}} \geq c$ and $\theta = C_1\varepsilon$ (C_1 just depends on dimension d). We assume that $D \leq C_2 l_d^{-1}$. Then in the inertial range $|k| \leq \epsilon_1 l_d^{-1}$ and dissipation range $|k| \geq C_2 l_d^{-1}$, we have*

the following estimate respectively

$$n^{(1)}(k) = \begin{cases} \dfrac{t}{T_{\text{kin}}}\mathcal{K}(n_{\text{in}})(k) + O_{\ell^\infty_k}\left(L^{-\theta}\dfrac{T_{max}}{T_{\text{kin}}}\right) + \widetilde{O}_{\ell^\infty_k}\left(\epsilon_1 Err_D(k_x)\dfrac{T_{max}}{T_{\text{kin}}}\right) & \text{if } |k| \le \epsilon_1 l_d^{-1}, \\ 0, & \text{if } |k| \ge 2C_2 l_d^{-1} \end{cases} \tag{2.1.5}$$

and

$$n^{(j)}(k) = O_{\ell^\infty_k}\left(L^{-\theta}\frac{t}{T_{\text{kin}}}\right), \qquad j > 1 \tag{2.1.6}$$

where $\mathcal{K}$ is defined in (WKE)*, and $O_{\ell^\infty_k}(A)$ (resp. $\widetilde{O}_{\ell^\infty_k}(A)$) is a quantity that is bounded by A in ℓ^∞_k by some universal constant (resp. constant just depending on d). The definition of universal constant can be found in section 3.1.4. The definition of Err_D is*

$$Err_D(k_x) = \begin{cases} D^{d+1}, & \text{if } |k_x| \le D, \\ D^{d-1}(|k_x|^2 + D|k_x|), & \text{if } |k_x| \ge D. \end{cases} \tag{2.1.7}$$

3. *By 2, $n_{\text{in}}(k)+n^{(1)}(k)$, the first two terms of* (2.1.4)*, approximately equals to $n_{\text{in}}(k)+\frac{t}{T_{\text{kin}}}\mathcal{K}(n_{\text{in}})(k)$. $n_{\text{in}}(k)+\frac{t}{T_{\text{kin}}}\mathcal{K}(n_{\text{in}})(k)$ is the first two terms of the approximation series of* (WKE)*. Therefore, $\mathbb{E}|\widehat{\psi}(t,k)|^2$ is an approximate solution of the* (WKE)*.*

Remark 2.1.2. The condition $d \ge 3$ is essential. When $d = 1$, the ZK equation becomes the KdV equation which is an integrable system whose long time behavior is quasi-periodic instead of turbulent [17]. When $d = 2$, as mentioned in Remark 4.1.3, the desire number theory result is not true and a major revision to (WKE) is required to obtain a valid wave kinetic theory.

Remark 2.1.3. $\frac{t}{T_{\text{kin}}}\mathcal{K}(n_{\text{in}})(k) = O_{\ell^\infty_k}\left(\frac{t}{T_{\text{kin}}}\right) = O_{\ell^\infty_k}\left(\frac{T_{\max}}{T_{\text{kin}}}\right)$ in (2.1.5) is the largest term in (2.1.5) and (2.1.6). Compared to the second term in (2.1.5) or the first term in (2.1.6), $\frac{t}{T_{\text{kin}}}\mathcal{K}(n_{\text{in}})(k)$ is larger than them by a factor L^θ. It is also larger than the third term in (2.1.5) by a factor of ϵ_1.

Remark 2.1.4. The restriction on α^{-1} is not optimal. The optimal result is expected to be $\alpha^{-1} \le L$ for general torus and $\alpha^{-1} \le L^{d/2}$ for generic torus. Here the torus is general or generic in the sense of [7]. Except for those in the the counting results in Chapter 4, section 4.1, all the arguments in this chapter work under these stronger assumptions.

Remark 2.1.5. Due to ∂_x in the nonlinearity, there is a potential risk of loss of derivative. This causes a serious difficulty in controlling the high frequency part. To resolve this difficulty, some regularization to ZK equation is required. In [26] and this paper, grid discretization and viscosity are introduced respectively. Both of them serve as canonical high frequency truncations.

2.1.2 Ideas of the proof

The basic strategy of proving the main theorem is to construct an approximation series and use probability theory and number theory to control the size and error of this approximation.

The approximate solution

The equation of Fourier coefficients is

$$\dot{\psi}_k = i\Lambda(k)\psi_k - \nu|k|^2\psi_k + \frac{i\lambda}{L^d}\sum_{\substack{(k_1,k_2)\in(\mathbb{Z}_L^d)^2\\ k_1+k_2=k}} k_{x_1}\psi_{k_1}\psi_{k_2} \tag{2.1.8}$$

Define a new dynamical variable $\phi = e^{-it\Lambda(\nabla)}\psi$ and integrate (2.1.8) in time. Then (MKDV) with initial data (2.1.1) becomes

$$\phi_k = \xi_k + \frac{i\lambda}{L^d}\sum_{k_1+k_2=k}\int_0^t k_{x_1}\phi_{k_1}\phi_{k_2}e^{is\Omega(k_1,k_2,k)-\nu(t-s)|k|^2}ds. \tag{2.1.9}$$

Here $\Omega(k_1,k_2,k) = \Lambda(k_1)+\Lambda(k_2)-\Lambda(k)$ and ξ_k are the Fourier coefficients of the initial data of ψ defined by $\xi_k = \sqrt{n_{\text{in}}(k)}\,\eta_k(\omega)$.

Denote the second term of right hand side by $\mathcal{T}(\psi,\psi)_k$ and the right hand side by $\mathcal{F}(\psi)_k = \xi_k + \mathcal{T}(\psi,\psi)_k$. Then the equation becomes $\psi = \mathcal{F}(\psi)_k$. We can construct the approximation by iteration: $\psi = \mathcal{F}(\psi) = \mathcal{F}(\mathcal{F}(\psi)) = \mathcal{F}(\mathcal{F}(\mathcal{F}(\psi))) = \cdots$.

Define the approximate solution by $\psi_{app} = \mathcal{F}^N(\xi)$. By recursively expanding $\mathcal{F}^N$, we know that ψ_{app} is a polynomial of ξ. The expansion can be described as the following,

$$\begin{aligned}\psi_{app} =& \mathcal{F}^N(\xi) = \xi + \mathcal{T}(\mathcal{F}^{N-1}(\xi),\mathcal{F}^{N-1}(\xi))\\ =&\xi + \mathcal{T}\Big(\xi + \mathcal{T}(\mathcal{F}^{N-2}(\xi),\mathcal{F}^{N-2}(\xi)),\cdots\Big) = \xi + \mathcal{T}(\xi,\xi)+\cdots\\ =&\xi + \mathcal{T}(\xi,\xi) + \mathcal{T}(\mathcal{T}(\xi,\xi),\xi) + \mathcal{T}(\xi,\mathcal{T}(\xi,\xi))+\cdots\end{aligned}$$

In the above iteration, we recursively replace $\mathcal{F}^l(\xi)$ by $\xi + \mathcal{T}(\mathcal{F}^{l-1}(\xi),\mathcal{F}^{l-1}(\xi))$.

We need a good upper bound for each terms of ψ_{app}. To get this we introduce tree diagrams to represent terms ξ, $\mathcal{T}(\xi,\xi)$, $\mathcal{T}(\mathcal{T}(\xi,\xi),\xi)$, $\cdots$. The basic notation of tree diagrams will be introduced in section 3.2.2.

The perturbative analysis

To prove the main theorem, we need to bound the approximation error of ψ_{app} defined by $w = \psi - \psi_{app}$. To do this, we use the follow equation of w which can be derived from (2.1.9):

$$w = Err(\xi) + Lw + B(w, w) \tag{2.1.10}$$

Here $Err(\xi)$ is a polynomial of ξ whose degree $\leq N+1$ monomials vanish. Lw, $B(w,w)$ are linear, quadratic in w respectively.

We prove the smallness of w using the bootstrap method.

Define $||w||_{X^p} = \sup_k \langle k\rangle^p |w_k|$. Starting from the assumption that $\sup_t ||w||_{X^p} \leq CL^{-M}$ ($C, M \gg 1$), we need to prove that $\sup_t ||w||_{X^p} \leq (1+C/2)L^{-M} < CL^{-M}$. To prove $||w||_{X^p} \leq (1+C/2)L^{-M}$, we use (2.1.10), which gives

$$||w||_{X^p} \leq ||Err(\xi)||_{X^p} + ||Lw||_{X^p} + ||B(w,w)||_{X^p} \tag{2.1.11}$$

We just need to show that

$$||Err(\xi)||_{X^p} \leq L^{-M}, \quad ||B(w,w)||_{X^p} \leq C^2 L^{d+O(1)-2M}. \tag{2.1.12}$$

Combining with a special treatment of Lw, the above estimates imply that $||w||_{X^p} \leq (1+C/2)L^{-M}$ which closes the bootstrap.

Couple diagrams, lattice points counting and $||Err(\xi)||_{X^p}$

In this section we explain the idea of proving upper bound of $||Err(\xi)||_{X^p}$.

$(Err(\xi))_k$ is a sum of terms of the form

$$\begin{aligned}
&\mathcal{J}^0_k(\xi) = \xi_k, \quad \mathcal{J}^1_k(\xi) = \frac{i\lambda}{L^d} \sum_{k_1+k_2-k=0} H^1_{k_1k_2}\xi_{k_1}\xi_{k_2}, \quad \cdots \\
&\mathcal{J}^l_{T,k}(\xi) = \left(\frac{i\lambda}{L^d}\right)^l \sum_{k_1+k_2+\cdots+k_{l+1}-k=0} H^l_{k_1\cdots k_{l+1}}(T)\xi_{k_1}\xi_{k_2}\cdots\xi_{k_{l+1}}, \quad \cdots
\end{aligned} \tag{2.1.13}$$

According to section 3.2.2, each terms correspond to a tree diagram and their coefficients can be calculated from these diagrams. This calculation is done in section 3.3.1. As a corollary of tree diagram representation, we know that H^l is large near a surface given by $2l$ equations $S = \{S_{\mathfrak{n}_1}(T) = 0, \Omega_{\mathfrak{n}_1}(T) = 0, \cdots, S_{\mathfrak{n}_l}(T)(T) = 0, \Omega_{\mathfrak{n}_l}(T) = 0\}$.

By the large deviation principle, to obtain upper bounds of Gaussian polynomials $\mathcal{J}^l_{T,k}(\xi)$, it suffices to calculate their variance. This calculation is done in section 3.3.2 using the Wick theorem and we introduce the concept of couple diagrams to represent the final result.

As a corollary of couple diagram representation, we know that the coefficients of the variance concentrate near a surface given by n equations (n is the number of nodes in the couple) $S = \{S_{\mathfrak{n}_1}(T) = 0, \Omega_{\mathfrak{n}_1}(T) = 0, \cdots, S_{\mathfrak{n}_n}(T)(T) = 0, \Omega_{\mathfrak{n}_n}(T) = 0\}$. Then in order to estimate the variance it suffices to upper bound the number of lattice points near this surface. This is done in section 3.3.3 using the edge cutting argument to reduce the size of the couple.

The method in [7] of getting number theory estimate based on tree diagram does not work in our setting. This is because the energy conservation equation $\Lambda(k_1) + \Lambda(k_2) - \Lambda(k) = 0$ of ZK equation degenerates seriously when k_x is close to 0. In (4.1.1), the number of solutions of the diophantine equation $\Lambda(k_1) + \Lambda(k_2) = \Lambda(k) + \sigma + O(T^{-1})$ can only be bounded by $|k_x|^{-1}$ which goes to infinity when $k_x \to 0$. This difficulty is resolved by the fact that the multiplier k_x in the last term of (2.1.8) vanishes when $k_x \to 0$. Since multipliers become very complicated in higher order tree terms, we introduce the concept of norm edges to keep track of them.

In conclusion, combining the above arguments, we can show that, for any M, we can take N large enough so that $||Err(\xi)||_{X^p} \le L^{-M}$.

Upper bounds for $||B(w,w)||_{X^p}$

$||B(w,w)||_{X^p}$ is a sum of terms of the form

$$\frac{i\lambda}{L^d} \int_0^t \sum_{k_1+k_2-k=0} B_{k_1 k_2}(s) w_{k_1}(\xi) w_{k_2} \tag{2.1.14}$$

The upper bound of $||B(w,w)||_{X^p}$ can be obtained by a straight forward estimate

$$||B(w,w)||_{X^p} \le L^{O(1)} ||w||_{X^p} \le C^2 L^{O(1)-2M}, \tag{2.1.15}$$

Therefore, we get the desire upper bounds $||B(w,w)||_{X^p} \ll L^{-M}$ by taking $M > O(1)$.

A random matrix bound and Lw

To obtain a good upper bound for Lw, we need to estimate the norm of the random matrix L, following the idea in [7], [6].

In [7], they consider solutions in the Bourgain space $X^{s,b}$ and use TT^* method to get the upper bound for the operator norm of L, $||L||_{X^{s,b}\to X^{s,b}} \ll 1$. But we prefer to work in the simpler functional space X^p which is not a Hilbert space. Although the standard TT^* method is not useful in a non-Hilbert space, we can bypass it using a Neumann series argument.

Let us first explain how TT^* method works. Here we pretend that $||\cdot||_{X^p}$ is a Hilbert norm. The key idea of TT^* method is the inequality $||L||_{X^p\to X^p} = ||(LL^*)^K||^{\frac{1}{K}}_{X^p\to X^p} \leq (L^d \sup_{k,l}((LL^*)^K)_{k,l})^{1/K}$. To upper bound $||L||_{X^p\to X^p}$, we just need to estimate $(LL^*)^K)_{k,l}$ which can be calculated by couple diagrams and be estimated by the large deviation inequality. By taking K large, the loss $L^{d/K}$ could be made arbitrarily small.

Unfortunately, $||\cdot||_{X^p}$ is not a Hilbert norm. However, we can bypass the TT^* method using a Neumann series argument. Note that from (2.1.9) we have the identity

$$w - Lw = Err(\xi) + B(w,w). \tag{2.1.16}$$

We have good upper bounds for all of the three terms on the right hand side. By Neumann series argument we have

$$w = (1-L)^{-1}(RHS) = (1-L^K)^{-1}(1+L+\cdots+L^{K-1})(RHS). \tag{2.1.17}$$

By calculating $(L^K)_{k,l}$ we can show that $||L^K||_{X^p\to X^p} \ll 1$. This implies that $||(1-L^K)^{-1}||_{X^p\to X^p} \lesssim 1$. Therefore, a good upper bound of RHS gives us a good upper bound of $(1+L+\cdots+L^{K-1})(RHS)$. Combining the above arguments, we obtain the desire estimate of w. This is done in section 2.2.3 and 2.4.6.

One additional difficulty is the unboundedness of L due to the derivative ∂_x in the nonlinearity. This is controlled by the high frequency decay coming from viscosity.

Proof of the main theorem

In summary, the above arguments in this section prove that when $t \leq \alpha^{-2}$, we have $||w||_{X^p} \leq L^{-M}$ with high probability ($P(false) \lesssim e^{-CL^\theta}$).

The above inequality is equivalent to $\sup_k |\langle k\rangle^s w_k| \leq CL^{-M}$. Remember that $w := \psi - \psi_{app}$, so with high probability we have the following estimate $\sup_k \langle k\rangle^s |\psi_k - \psi_{app,k}| \leq CL^{-M}$. This implies that $\mathbb{E}|\widehat{\psi}(t,k)|^2 = \mathbb{E}|\psi_{app,k}|^2 + O(L^{-M})$. This suggests that we may get the approximation of $\mathbb{E}|\widehat{\psi}(t,k)|^2$ by calculating $\mathbb{E}|\psi_{app,k}|^2$. $\mathbb{E}|\psi_{app,k}|^2$ can be exactly calculated and the theorem can be proved by extract the main term in $\mathbb{E}|\psi_{app,k}|^2$. This is done in section 2.2.4.

2.1.3 Notations

Universal constants: In this chapter, universal constants are constants that just depend on dimension d, diameter D of the support of n_{in} and the length of the inertial range l_d^{-1}.

$O(\cdot)$, $\ll$, $\lesssim$, $\sim$: Throughout this chapter, we frequently use the notation, $O(\cdot)$, $\ll$, $\lesssim$. $A = O(B)$ or $A \lesssim B$ means that there exists C such that $A \lesssim CB$. $A \ll B$ means that there exists a small constant c such that $A \lesssim cB$. $A \sim B$ means that there exist two constant c, C such that $cB \lesssim A \lesssim CB$. Here the meaning of constant depends on the context. If they appear in conditions involving k, Λ, Ω, etc., like $|k| \lesssim 1$, $\iota_{\mathfrak{e}_1} k_{\mathfrak{e}_1} + \iota_{\mathfrak{e}_2} k_{\mathfrak{e}_2} + \iota_{\mathfrak{e}} k_{\mathfrak{e}} = 0$, then they are universal constants. If these constants appear in an estimate which gives upper bound of some quantity, like $||L^K||_{X^p \to X^p} \ll 1$ or $\sup_t \sup_k |(\mathcal{J}_T)_k| \lesssim L^{O(l(T)\theta)} \rho^{l(T)}$, then in addition to the quantities that universal constants depend, they can also depend on the quantities θ, ε, K, M, N, ϵ_1.

Order of constants: Here is the order of all constants which can appear in the exponential or superscript of L. These constants are θ, ε, K, M, N, ϵ_1.

All the constants are small compared to L in the sense they are less than L^θ for arbitrarily small $\theta > 0$.

ε can be an arbitrarily small constant less than 0.5, the reader is encouraged to assume it to be 0.01. The order of other constants can be decided by the relations $\theta \ll \varepsilon$, $K = O(\theta^{-1})$, $M \gg K$, $N \geq M/\theta$, here the constants in $\ll$, $O(\cdot)$ are universal.

$\mathbb{Z}_L^d$: $\mathbb{Z}_L^d = \{k = \frac{K}{L} : K \in \mathbb{Z}^d\}$

k_x, $k_\perp$: Given any vector k, let k_x be its first component and $k_\perp$ be the vector formed by the rest components.

$\Lambda(k)$, $\Lambda(\nabla)$: $\Lambda(k) \coloneqq k_1(k_1^2 + \cdots k_d^2)$ and $\Lambda(\nabla) = i|\nabla|^2 \partial_{x_1}$

Fourier series: The spatial Fourier series of a function $u : \mathbb{T}_L^d \to \mathbb{C}$ is defined on $\mathbb{Z}_L^d := L^{-1}\mathbb{Z}^d$ by

$$u_k = \int_{\mathbb{T}_L^d} u(x) e^{-2\pi i k\cdot x}, \qquad \text{so that} \qquad u(x) = \frac{1}{L^d} \sum_{k \in \mathbb{Z}_L^d} u_k \, e^{2\pi i k \cdot x}. \tag{2.1.18}$$

Given any function F, let F_k or $(F)_k$ be its Fourier coefficients.

Order of L: In this chapter, L is assumed to be a constant which is much larger than all the universal constants and θ, ε, K, M, N, ϵ_1.

L-certainty: If some statement S involving ω is true with probability $\geq 1 - O_\theta(e^{-L^\theta})$, then we say this statement S is L-certain.

2.2 The Perturbation Expansion

In this section, we calculate the approximation series and introduce Feynman diagrams to represent terms in this series. Then we bound the error of this approximation by the bootstrap method, assuming several propositions about the upper bounds of higher order terms. We will prove these propositions in the rest part of the chapter.

2.2.1 The approximation series and Feynman diagrams

In this section, we derive the equation for Fourier coefficients and construct the approximate solution.

The Equation of Fourier coefficients

Let ψ_k be the Fourier coefficient of ψ. Then in term of ψ_k equation (MKDV) becomes

$$\begin{cases} \dot{\psi}_k = i\Lambda(k)\psi_k - \nu|k|^2\psi_k + \frac{i\lambda}{L^d} \sum_{\substack{(k_1,k_2)\in(\mathbb{Z}_L^d)^2 \\ k_1+k_2=k}} k_x \psi_{k_1}\psi_{k_2} \\ \\ \psi_k(0) = \xi_k = \sqrt{n_{\text{in}}(k)}\, \eta_k(\omega) \end{cases} \tag{2.2.1}$$

Define the linear profile by

$$\phi_k(t) := e^{-i\Lambda(k)t}\psi_k(t) \tag{2.2.2}$$

Rewriting (2.2.1) in terms of ϕ_k gives

$$\dot{\phi}_k = -\nu|k|^2\phi_k + \frac{i\lambda}{L^d} \sum_{S(k_1,k_2,k)=0} k_x \phi_{k_1}\phi_{k_2} e^{it\Omega(k_1,k_2,k)} \tag{2.2.3}$$

where

$$\begin{aligned} &S(k_1,k_2,k) = k_1 + k_2 - k, \\ &\Omega(k_1,k_2,k) = \Lambda(k_1) + \Lambda(k_2) - \Lambda(k). \end{aligned} \tag{2.2.4}$$

We will work with (2.2.3) in the rest part of this chapter.

Integrating (2.2.3) gives

$$\phi_k = \xi_k + \underbrace{\frac{i\lambda}{L^d} \sum_{S(k_1,k_2,k)=0} \int_0^t k_x \phi_{k_1} \phi_{k_2} e^{is\Omega(k_1,k_2,k)-\nu|k|^2(t-s)} ds}_{\mathcal{T}(\phi,\phi)_k} . \tag{2.2.5}$$

Denote the second term on the right hand side by $\mathcal{T}(\phi, \phi)_k$. Denote the right hand side by $\mathcal{F}(\phi)_k = \xi_k + \mathcal{T}(\phi, \phi)_k$. With these notations, (2.2.5) becomes $\phi = \mathcal{F}(\phi)_k$.

We construct the approximation series by iteration: $\phi = \mathcal{F}(\phi) = \mathcal{F}(\mathcal{F}(\phi)) = \mathcal{F}(\mathcal{F}(\mathcal{F}(\phi))) = \cdots$. To estimate this approximation series, we need a compact graphical notation to represent the huge amount of terms generated from iteration. This is done by introducing the concept of Feynman diagrams.

Some basic definitions from graph theory

In this section, we introduce the concept of binary trees, branching nodes, leaves, subtrees, node decoration and expanding leaves.

Definition 2.2.1.

1. **Binary trees:** A binary tree T is a tree in which each node has 2 or 0 children. An example of binary trees used in this chapter is shown in Figure 3.1.

2. **Branching nodes:** A branching node in a binary tree is a node which has 2 children. The number of all branching nodes in a tree T is denoted by $l(T)$. In Figure 3.1, $l(T) = 2$.

3. **Leaves:** A leaf of a tree T is a node which has no child. In Figure 3.1, all $\star$ nodes and $\square$ nodes are leaves.

4. **Subtrees:** If any child of any node in a subset T' of a tree T is also contained in T' then T' also forms a tree, we call T' a subtree of T. If the root node of T' is $\mathfrak{n} \in T$, we say T' is the subtree rooted at $\mathfrak{n}$ or subtree of $\mathfrak{n}$ and denote it by $T_{\mathfrak{n}}$. In Figure 3.1, the tree inside the box is the subtree rooted at node $\bullet$.

5. **Node decoration:** In Figure 3.1, each node is associated with a symbol in $\{\bullet, \star, \square\}$. If a node $\mathfrak{n}$ has symbol $\bullet$ (similarly $\star$, $\square$), we say $\mathfrak{n}$ is decorated by $\bullet$ ($\star$, $\square$) or $\mathfrak{n}$ has decoration $\bullet$ ($\star$, $\square$). In what follows, we adopt the convention that leaves always have decoration $\star$ or $\square$ and nodes other than leaves always have decoration $\bullet$.

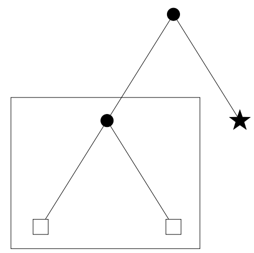


Figure 2.1: Subtrees and node decoration.

6. **Expanding and final leaves:** Leaves denoted by $\Box$ are called expanding leaves. Other leaves denoted by $\star$ are called final leaves. The notion of expanding leaves is useful in the construction of trees, in which the presence of $\Box$ means that the construction is not finishing and $\Box$ denotes leaves that may be replaced by a branching node later.

   The concept of expanding leaves and $\Box$ is only used in this section, so the readers can safely forget it after that section.

Connection between iteration and trees

In this section, we explain non-rigorously the connection between perturbation expansion and trees. Rigorous argument can be find in the next section.

This iteration process can be described as the following,

$$\begin{aligned}\phi =&\mathcal{F}(\phi) = \xi + \mathcal{T}(\phi,\phi)\\ =&\xi + \mathcal{T}\Big(\xi + \mathcal{T}(\phi,\phi),\cdots\Big) = \xi + \mathcal{T}(\xi,\xi) + \mathcal{T}\Big(\mathcal{T}(\phi,\phi),\xi\Big)\cdots\\ =&\xi + \mathcal{T}(\xi,\xi) + \mathcal{T}(\mathcal{T}(\xi,\xi),\xi) + \mathcal{T}(\xi,\mathcal{T}(\xi,\xi)) + \cdots\end{aligned}$$

In the above iteration, we recursively choose one ϕ, replace it by $\xi + \mathcal{T}(\phi,\phi)$ and use the linearity of $\mathcal{T}$ to expand into two terms.

$$\begin{aligned}&\mathcal{T}\Big(\cdots,\mathcal{T}(\mathcal{T}(\xi,\underline{\phi}),\cdots)\Big) \to \mathcal{T}\Big(\cdots,\mathcal{T}(\mathcal{T}(\xi,\underline{\xi+\mathcal{T}(\phi,\phi)}),\cdots)\Big)\\ =&\underbrace{\mathcal{T}\Big(\cdots,\mathcal{T}(\mathcal{T}(\xi,\underline{\xi}),\cdots)\Big)}_{I} + \underbrace{\mathcal{T}\Big(\cdots,\mathcal{T}(\mathcal{T}(\xi,\underline{\mathcal{T}(\phi,\phi)}),\cdots)\Big)}_{II}\end{aligned} \tag{2.2.6}$$

Here I and II are obtained by replacing ϕ by ξ and $\mathcal{T}(\phi,\phi)$ respectively.

In summary, all terms in the expansion can be generated by following steps

- **Step** 0. Add a term ϕ in the summation $\mathcal{J}$.

- **Step** i **(**$i \geq 1$**).** Assume that **Step** $i-1$ has been finished which produces a sum of terms $\mathscr{J}$, then choose a term in $\mathscr{J}$ which has least number of ξ and ϕ, remove this term from $\mathscr{J}$ and add the two terms in $\mathscr{J}$ constructed in (2.2.6).

This process is very similar to the construction of binary trees, in which we recursively replace a chosen expanding node by a leaf or branching node.

- **Step** 0**.** Start from a expanding root node $\square$.

- **Step** i **(**$i \geq 1$**).** Assume that we have finish the **Step** $i-1$ which produces a collection of trees $\mathscr{T}$, then choose a tree in $\mathscr{T}$ which has least number of expanding leaves $\square$ and final leaves $\star$, remove this tree from $\mathscr{T}$ and add two new trees in $\mathscr{T}$. In these two new trees, we replace a expanding leaf $\square$ by a final leaf $\star$ or a branching node $\bullet$ with two expanding children leaves $\square$. This construction is illustrated by Figure 3.2.

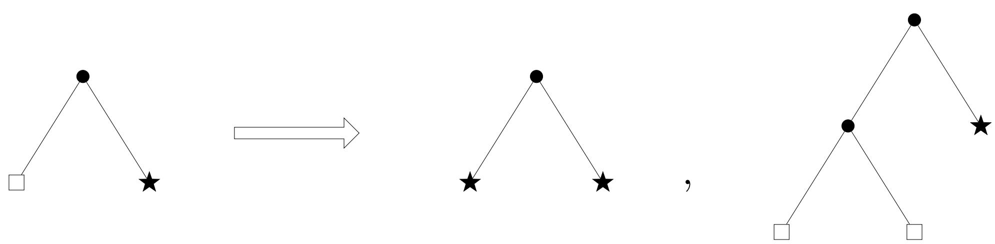


Figure 2.2: One step in the construction of binary trees

By comparing the above two process, we can make the connection between terms and trees more explicit. Each node $\bullet$ other than leaf in the tree T corresponds to a $\mathcal{T}(\cdots,\cdots)$ in a term $\mathscr{J}_T$. Each final leaf $\star$ and expanding leaf $\square$ corresponds to ξ and ϕ respectively. The **Step** i of replacing ϕ by ξ or $\mathcal{T}(\phi,\phi)$ corresponds to replacing $\square$ by $\star$ or a branching node with two children $\square$.

We have following recursive formula for calculating a term $\mathscr{J}_T$ from a binary tree T.

If T has only one node then $\mathscr{J}_T = \xi$. Otherwise let $\bullet_1$, $\bullet_2$ be two children of the root node $\bullet$, let $T_{\bullet_1}$, $T_{\bullet_2}$ be the subtrees of T rooted at the above nodes. If $\mathscr{J}_{T_{\bullet_1}}$, $\mathscr{J}_{T_{\bullet_2}}$ have been recursively calculated, then $\mathscr{J}_T$ can be calculated by

$$\mathscr{J}_T = \mathcal{T}(\mathscr{J}_{T_{\bullet_1}}, \mathscr{J}_{T_{\bullet_2}}). \tag{2.2.7}$$

The formal power series obtained by iterating $\phi = \mathcal{F}(\phi)$ can be calculated from trees by $\sum_{T\in\mathscr{T}} \mathscr{J}_T$.

Let $l(T)$ be the number of branching nodes in T, then it can be shown that $\mathcal{J}_T$ is a degree $l(T)+1$ polynomial of ξ. We define the approximation series to be a finite degree truncation of the formal power series which equals to $\sum_{l(T)\leq N} \mathcal{J}_T$.

Feynman diagrams and construction of the approximation solution

In this section we present the rigorous argument equivalent to that in the above section.

In the construction of trees, finally all $\square$ nodes will be replaced by $\bullet$, $\star$, so in what follows we only consider trees whose nodes are decorated by $\bullet$, $\star$.

Definition 2.2.2. Given a binary tree T whose nodes are decorated by $\bullet$, $\star$, we inductively define the quantity $\mathcal{J}_T$ by:

$$\mathcal{J}_T = \begin{cases} \xi, & \textit{if } T \textit{ has only one node } \star. \\ \mathcal{T}(\mathcal{J}_{T_{\mathfrak{n}_1}}, \mathcal{J}_{T_{\mathfrak{n}_2}}), & \textit{otherwise.} \end{cases} \tag{2.2.8}$$

Here $\mathfrak{n}_1$, $\mathfrak{n}_2$ are two children of the root node $\mathfrak{r}$ and $T_{\mathfrak{n}_1}$, $T_{\mathfrak{n}_2}$ are the subtrees of T rooted at the these nodes.

Definition 2.2.3. Given a large number N, define the approximate solution ϕ_{app} by

$$\phi_{app} = \sum_{l(T)\leq N} \mathcal{J}_T \tag{2.2.9}$$

Section 3.2.2 explains why the approximation series should equals to (2.2.9), a sum of many tree terms, but if we know this fact, we can directly prove it, and forget all the motivations. The lemma below proves that ϕ_{app} defined by the above expression is an approximate solution.

Lemma 2.2.4. *Define*

$$Err = \mathcal{F}(\phi_{app}) - \phi_{app}, \tag{2.2.10}$$

then we have

$$Err = \sum_{T\in\mathcal{T}^*_{>N}} \mathcal{J}_T, \tag{2.2.11}$$

where $\mathcal{T}^*_{>N}$ is defined by

$$\begin{aligned} \mathcal{T}^*_{>N} = \{T : l(T) > N,\ & l(T_{\mathfrak{n}_1}) \leq N, l(T_{\mathfrak{n}_2}) \leq N, \\ & T_{\mathfrak{n}_1}, T_{\mathfrak{n}_2} \textit{ are the subtrees defined in Definition 3.2.2}\} \end{aligned} \tag{2.2.12}$$

Remark 2.2.5. Notice that all terms in $\sum_{T\in\mathcal{T}^*_{>N}}$ are polynomials of ξ of degree $> N$. Therefore, the approximation error of ϕ_{app} is of very high order, which proves that ϕ_{app} is an appropriate approximation solution.

Proof. By (2.2.9), we get

$$\begin{aligned} Err =&\mathcal{F}(\phi_{app}) - \phi_{app} \\ =&\xi + \mathcal{T}(\phi_{app}, \phi_{app}) - \phi_{app} \\ =&\xi + \sum_{l(T_1),l(T_2)\leq N} \mathcal{T}(\mathcal{J}_{T_1}, \mathcal{J}_{T_2}) - \sum_{l(T)\leq N} \mathcal{J}_T \end{aligned} \tag{2.2.13}$$

Let T be a tree constructed by connecting the root nodes $\mathfrak{n}_1$, $\mathfrak{n}_2$ of T_1, T_2 to a new node $\mathfrak{r}$. We define $\mathfrak{r}$ to be the root node of T.

Then by (2.2.8), we have

$$\mathcal{J}_T = \mathcal{T}(\mathcal{J}_{T_1}, \mathcal{J}_{T_2}) \tag{2.2.14}$$

and

$$\sum_{l(T_1),l(T_2)\leq N} \mathcal{T}(\mathcal{J}_{T_1}, \mathcal{J}_{T_2}) = \sum_{\substack{l(T)\geq 1 \\ l(T_1),l(T_2)\leq N}} \mathcal{J}_T \tag{2.2.15}$$

By (2.2.13), we get

$$\begin{aligned} Err =&\xi + \sum_{\substack{l(T)\geq 1 \\ l(T_1),l(T_2)\leq N}} \mathcal{J}_T - \sum_{l(T)\leq N} \mathcal{J}_T \\ =& \sum_{\substack{T_1,T_2 \text{ are subtrees of } \mathfrak{r} \\ l(T_1),l(T_2)\leq N}} \mathcal{J}_T - \sum_{\substack{T_1,T_2 \text{ are subtrees of } \mathfrak{r} \\ l(T)\leq N \\ l(T_1),l(T_2)\leq N}} \mathcal{J}_T. \\ =& \sum_{T\in\mathcal{T}^*_{>N}} \mathcal{J}_T \end{aligned} \tag{2.2.16}$$

Here in the second equality, we use the fact that $\sum_{l(T)\leq N} = \sum_{\substack{l(T)\leq N \\ l(T_1),l(T_2)\leq N}}$.

Therefore, we complete the proof of this lemma. ☐

2.2.2 Estimates of the approximation solution

Estimates of tree terms

By (2.2.11), in order to control the approximation error Err, it suffices to get upper bounds of tree terms $\mathcal{J}_T$. We state the upper bound in the proposition below and delay its proof to section 2.4.5.

Let us introduce a definition before state the proposition.

Definition 2.2.6. Given a property A, we say A happens L-certainly if the probability that A happens satisfies $P(A) \geq 1 - Ce^{-L^{\theta}}$ for some $C, \theta > 0$.

Proposition 2.2.7. *We have L-certainly for any θ that*

$$\sup_t \sup_k |(\mathcal{J}_T)_k| \lesssim L^{O(l(T)\theta)} \rho^{l(T)}. \tag{2.2.17}$$

and $(\mathcal{J}_T)_k = 0$ if $|k| \gtrsim 1$. Here $(\mathcal{J}_T)_k$ is the Fourier coefficients of $\mathcal{J}_T$ and

$$\rho = \alpha T_{max}^{\frac{1}{2}}. \tag{2.2.18}$$

Linearization around the approximation solution

Let $w = \phi - \phi_{app}$ be the deviation of ϕ_{app} to the true solution ϕ_{app}. In order to estimate w, we consider the linearized equation of

$$\phi = \mathcal{F}(\phi) = \xi + \mathcal{T}(\phi, \phi). \tag{2.2.19}$$

The linearized equation is a equation of w given by

$$w = Err(\xi) + Lw + B(w, w), \tag{2.2.20}$$

where $Err(\xi)$, Lw, $B(w, w)$ are given by

$$\begin{cases} Err = \mathcal{F}(\phi_{app}) - \phi_{app}, \\ Lw = 2\mathcal{T}(\phi_{app}, w), \\ B(w, w) = \mathcal{T}(w, w). \end{cases} \tag{2.2.21}$$

As explained in section 3.1.2, to control w, we need operator norm bound $||L^K||_{X^p}$. Notice that $\phi_{app} = \sum_{l(T) \leq N} \mathcal{J}_T$, so we know that Lw is a sum of terms like $\mathcal{T}(\mathcal{J}_T, w)$.

It suffices to get following operator norm bound for $w \to \mathcal{T}(\mathcal{J}_T, w)$ in order to upper bound L^K.

Proposition 2.2.8. *Define $\rho = \alpha T_{max}^{\frac{1}{2}}$ as in Proposition 3.2.7 and $\mathcal{P}_T$ to be the linear operator*

$$w \to \mathcal{T}(\mathcal{J}_T, w). \tag{2.2.22}$$

Then for any sequence of trees $\{T_1, \cdots, T_K\}$, we have L-certainly for any θ the operator bound

$$\left\|\prod_{j=1}^{K} \mathcal{P}_{T_j}\right\|_{L_t^\infty X^p \to L_t^\infty X^p} \leq L^{O\left(1+\theta \sum_{j=1}^{K} l(T_j)\right)} \rho^{\sum_{j=1}^{K} l(T_j)}. \tag{2.2.23}$$

for any T_j with $l(T_j) \leq N$.

The the above inequality implies that

$$||L^K||_{L_t^\infty X^p \to L_t^\infty X^p} \leq L^{O(1+K\theta)} \rho^K. \tag{2.2.24}$$

The proof of this proposition can be found in section 2.4.6.

2.2.3 Bound the error of the approximation

Define $w = v - v_{app}$ to be the approximation error. In this section we prove the following theorem that gives an upper bound of w, assuming Proposition 3.2.7 and Proposition 2.2.8 in previous section.

Theorem 2.2.9. *Let $w = \phi - \phi_{app}$. Given any $M \gg 1$, there exists N such that if ϕ_{app} is the N-th order approximate solution, then $\sup_{t \leq T_{max}} ||w(t)||_{X^p} \lesssim L^{-M}$ L-certainly (with probability $\geq 1 - Ce^{-CL^\theta}$).*

Proof. If for some C sufficiently large we can show that

$$\sup_{t \leq T} ||w(t)||_{X^p} \leq CL^{-M} \text{ implies that } \sup_{t \leq T} ||w(t)||_{X^p} < CL^{-M}, \tag{2.2.25}$$

for all $T \leq T_{\max}$, then we finish the proof of this theorem.

Here is the explanation. (2.2.25) implies that if we define the set $A = \{T : \sup_{t \leq T} ||w||_{X^p} \leq CL^{-M}\}$ then the set equals to $\{T : \sup_{t \leq T} ||w||_{H^s} < CL^{-M}\}$ which is open. The original definition $A = \{T : \sup_{t \leq T} ||w||_{X^p} \leq CL^{-M}\}$ implies that this set is also closed. It is nonempty because $||w(0)||_{X^p} = 0$ implies that $0 \in A$. Therefore, A is open, closed and nonempty in $[0, T_{\max}]$, so $A = [0, T_{\max}]$ which implies that the theorem.

Now we prove (2.2.25). By (2.2.20),

$$w - Lw = Err(\xi) + B(w, w). \tag{2.2.26}$$

By Neumann series we have

$$\begin{aligned} w =&(1-L)^{-1}(Err(\xi)+B(w,w)) \\ =&(1-L^K)^{-1}(1+L+\cdots+L^{K-1})(Err(\xi)+B(w,w)). \end{aligned} \tag{2.2.27}$$

Assume that the constant in $O(1+K\theta)$ in (2.2.24) is C_{norm}. Since $T_{\max} \leq L^{-\varepsilon}\alpha^{-2}$, we know that $\rho = \alpha T_{\max}^{\frac{1}{2}} \lesssim L^{-\varepsilon}$. Take $\theta \leq C_{norm}\varepsilon/2$ and $K \gg \frac{C_{norm}}{\varepsilon}$, then $\left|\left|L^K\right|\right|_{L^\infty X^p \to L^\infty X^p} \leq L^{C_{norm}(1+K\theta)}\rho^K \lesssim L^{C_{norm}(1+K\theta)}L^{-K\varepsilon} \ll 1$, so we get $\left|\left|L^K\right|\right|_{L^\infty X^p \to L^\infty X^p} \ll 1$ and thus $\left|\left|(1-L^K)^{-1}\right|\right|_{L^\infty X^p \to L^\infty X^p} \lesssim 1$.

By (2.2.27), we get

$$||w(t)||_{X^p} \lesssim \sum_{j=1}^{K} ||L^j(Err(\xi))||_{X^p} + ||(1+L+\cdots+L^{K-1})(B(w,w))||_{X^p} \tag{2.2.28}$$

By (2.2.11), we know that Err is a sum of tree terms $\sum_{T\in\mathcal{T}^*_{>N}} \mathcal{J}_T$ of order $\geq N$. Since $L = \sum_{1\leq l(T)\leq N} \mathcal{P}_T$, we know that $L^j(Err(\xi))$ is a sum of terms like $\mathcal{P}_{T_1}\circ\cdots\circ\mathcal{P}_{T_j}(\mathcal{J}_T)$ which by (2.4.119) equals to $\mathcal{J}_{T_1\circ\cdots\circ T_j\circ T}$. By Proposition 3.2.7, we get $||\mathcal{J}_{T_1\circ\cdots\circ T_j\circ T}||_{X^p} \lesssim (L^{O(\theta)}\rho)^{l(T_1\circ\cdots\circ T_j\circ T)} \lesssim L^{O(l(T)\theta)}\rho^{l(T)}$. Since $\rho = \alpha T_{\max}^{\frac{1}{2}} \lesssim L^{-\varepsilon}$ and $l(T) > N$ in the sum of Err, we get

$$\sum_{j=1}^{K} ||L^j(Err(\xi))||_{X^p} \lesssim L^{O(N\theta)}\rho^N \lesssim L^{O(N\theta)}L^{-N\varepsilon} \ll L^{-M} \tag{2.2.29}$$

if we take $N \gg M/\varepsilon$ and $\theta \ll \varepsilon$.

Taking $K = 1$ in (2.2.23), we know that $||L||_{L^\infty X^p \to L^\infty X^p} \leq L^{O(1)}$, so $\left|\left|L^j\right|\right|_{L^\infty X^p \to L^\infty X^p} \leq L^{O(K)}$ if $j \leq K$. Taking $M \gg K$, by (2.2.25), we have $\sup_{t\leq T} ||w(t)||_{X^p} \leq CL^{-M}$. Therefore, we have

$$||(1+L+\cdots+L^{K-1})(B(w,w))||_{X^p} \lesssim L^{O(K)}L^d||w||^2_{X^p} \lesssim L^{O(K)}L^{O(1)}L^{-2M} \ll L^{-M}. \tag{2.2.30}$$

Combining (2.2.29) and (2.2.30), we prove $\sup_{t\leq T} ||w(t)||_{X^p} \ll L^{-M} < CL^{-M}$ from the assumption that $\sup_{t\leq T} ||w(t)||_{X^p} \leq CL^{-M}$. We thus complete the proof of Theorem 2.2.9. □

2.2.4 Proof of the main theorem

In this section, we prove Theorem 2.1.1.

Proof of Theorem 2.1.1. **Step 1.** ($\mathbb{E}|\widehat{\psi}(t,k)|^2$ is close to $\mathbb{E}|\psi_{app,k}|^2$) By Theorem 2.2.9, we know that when $t \le T_{\max} = L^{-\varepsilon}\alpha^{-2}$, we have $||w||_{X^p} \le L^{-M}$ with L-certainly.

The above inequality is equivalent to $\sup_k |\langle k\rangle^s w_k| \le CL^{-M}$. Remember that $w := \psi - \psi_{app}$, so L-certainly we have the following estimate

$$\sup_k \langle k\rangle^s |\psi_k - \psi_{app,k}| \le CL^{-M} \tag{2.2.31}$$

Denote by A the event that the above estimate is true, then $\mathbb{E}|\widehat{\psi}(t,k)|^2 = \mathbb{E}(|\psi_k|^2 1_A) + \mathbb{E}(|\psi_k|^2 1_{A^c})$. L-certainty implies that $\mathbb{P}(A^c) \lesssim e^{-CL^\theta}$. Since $||\psi||_{L^2}$ is conservative and $|\psi_k|^2 \le L^{d/2}||\psi||_{L^2} \le L^{d/2}$, we know that $\mathbb{E}(|\psi_k|^2 1_{A^c}) \lesssim L^{d/2}e^{-CL^\theta} = O(L^{-M})$. Therefore, $\mathbb{E}|\widehat{\psi}(t,k)|^2 = \mathbb{E}(|\psi_k|^2 1_A) + O(L^{-M})$. Since we also have $\mathbb{E}|\psi_{app,k}|^2 = \mathbb{E}(|\psi_{app,k}|^2 1_A) + O(L^{-M})$, we conclude that

$$\mathbb{E}|\widehat{\psi}(t,k)|^2 = \mathbb{E}|\psi_{app,k}|^2 + \mathbb{E}((|\psi_k|^2 - |\psi_{app,k}|^2)1_A) + O(L^{-M}) \tag{2.2.32}$$

By (2.2.31), $\mathbb{E}((|\psi_k|^2 - |\psi_{app,k}|^2)1_A) = O(L^{-M})$. We may conclude that

$$\mathbb{E}|\widehat{\psi}(t,k)|^2 = \mathbb{E}|\psi_{app,k}|^2 + O(L^{-M}). \tag{2.2.33}$$

This suggests that we may get the approximation of $\mathbb{E}|\widehat{\psi}(t,k)|^2$ by calculating $\mathbb{E}|\psi_{app,k}|^2$.

Step 2. (Expansion of $\mathbb{E}|\psi_{app,k}|^2$) By (2.2.9), we know that

$$\phi_{app} = \sum_{l(T)\le N} \mathcal{J}_T \tag{2.2.34}$$

Define

$$n^{(j)}(k) := \sum_{l(T)+l(T')=j} \mathbb{E}\mathcal{J}_{T,k}\overline{\mathcal{J}_{T',k}} \tag{2.2.35}$$

then Proposition 3.2.7 or 2.4.26 gives upper bounds of $n^{(j)}(k)$, which proves (1) and (2.1.6) of Theorem 2.1.1.

Step 3. (Asymptotics of $n^{(1)}(k)$) The only thing left in Theorem 2.1.1 is (2.1.5). This is a corollary of Proposition 2.4.38. $\square$

2.3 Lattice points counting and convergence results

2.4 Lattice points counting and convergence results

In this section, we prove Proposition 3.2.7 and 2.2.8 which gives upper bounds for tree terms $\mathcal{J}_{T,k}$ and the linearization operator $\mathcal{P}_T$. As explained before, these results are crucial in the proof of the main theorem. The proof of is divided into several steps.

In section 3.3.1, we calculate the coefficients of $\mathcal{J}_{T,k}$ as polynomials of Gaussian random variables.

In section 3.3.4, we obtain upper bounds for the coefficients of these Gaussian polynomials.

Large deviation theory suggests that an upper bound of a Gaussian polynomial can be derived from an upper bound of its expectation and variance.

In section 3.3.2, we introduce the concept of couples which is a graphical method of calculating the expectation of Gaussian polynomials.

In section 3.3.3, we use couple to establish an lattice points counting result.

In section 2.4.5 and section 2.4.6, we apply the lattice points counting result to derive upper bounds for $\mathcal{J}_{T,k}$ and $\mathcal{P}_T$ respectively. This finishes the proof of Proposition 3.2.7 and 2.2.8 and therefore the proof of the main theorem.

2.4.1 Refined expression of coefficients

From (2.2.8), it is easy to show that $\mathcal{J}_{T,k}$ are polynomials of ξ. In this section, we calculate the coefficients of $\mathcal{J}_{T,k}$ using the definition (2.2.8) of them.

Notice that all non-leaf nodes other than the root in a tree have degree 3. For convenience, we add a new edge, called leg $\mathfrak{l}$, to the root node, which makes the root also of degree 3. This process is illustrated by Figure 2.3.

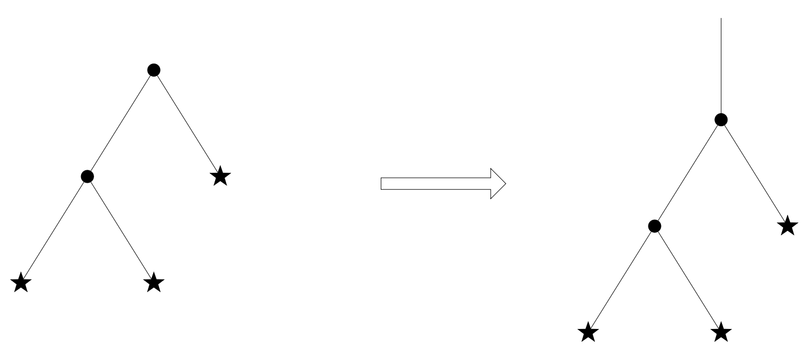


Figure 2.3: Adding a leg to a tree

We also need following concepts about trees.

Definition 2.4.1. 1. **Nodes and children of an edge:** As in Figure 2.4, let $\mathfrak{n}_u$ and $\mathfrak{n}_l$ be two endpoints of an edge $\mathfrak{e}$ and assume that $\mathfrak{n}_l$ is a children of $\mathfrak{n}_u$. We define $\mathfrak{n}_u$ (resp. $\mathfrak{n}_l$) to the upper node (resp. lower node) of $\mathfrak{e}$. Let $\mathfrak{n}_1$, $\mathfrak{n}_2$ be the two children of $\mathfrak{n}_l$ and let $\mathfrak{e}_1$(resp. $\mathfrak{e}_2$) be the edge between two nodes $\mathfrak{n}_l$ and $\mathfrak{n}_1$(resp. $\mathfrak{n}_2$). $\mathfrak{e}_1$, $\mathfrak{e}_2$ are defined to be the two children edges of $\mathfrak{e}$.

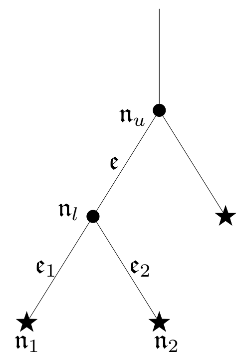


Figure 2.4: Children $\mathfrak{e}_1$, $\mathfrak{e}_2$ of an edge $\mathfrak{e}$

2. **Direction of an edge:** As in Figure 2.5, each edge $\mathfrak{e}$ is assigned with a direction. This concept is mostly used to decide the value of variables $\iota_{\mathfrak{e}} \in \{\pm\}$ that will be defined later. Although it can be shown that the final result does not depend on the choices of direction of each edge, for definiteness, we assign downward direction to each edge. The orientation in Figure 2.5 is one example of this choices.

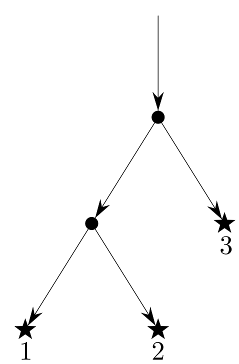


Figure 2.5: Children $\mathfrak{e}_1$, $\mathfrak{e}_2$ of an edge $\mathfrak{e}$

3. **Labelling of leaves:** As in Figure 2.5, each leaf is labelled by 1, 2, $\cdots$, $l(T)+1$ from left to right. An edge pointing to a leaf $\mathfrak{n}$ is also labelled by j if $\mathfrak{n}$ is labelled by j.

Now we calculate the coefficients of $\mathcal{J}_{T,k}$.

Lemma 2.4.2. *Given a tree T of depth $l = l(T)$, denote by T_{in} the tree formed by all non-leaf nodes $\mathfrak{n}$, then associate each node $\mathfrak{n} \in T_{in}$ and edge $\mathfrak{l} \in T$ with variables $t_{\mathfrak{n}}$ and $k_{\mathfrak{l}}$ respectively. Given a labelling of all leaves by $1, 2, \cdots, l+1$, we identify $k_{\mathfrak{e}}$ with k_j if $\mathfrak{e}$ is connected to a leaf labelled by j. Given a node $\mathfrak{n}$, let $\mathfrak{e}_1$, $\mathfrak{e}_2$ and $\mathfrak{e}$ be the three edges from or pointing to $\mathfrak{n}$ ($\mathfrak{e}$ is the parent of $\mathfrak{e}_1$ and $\mathfrak{e}_2$), $\mathfrak{n}_1$, $\mathfrak{n}_2$ be children of $\mathfrak{n}$ and $\hat{\mathfrak{n}}$ be the parent of $\mathfrak{n}$.*

Let $\mathcal{J}_T$ be terms defined in Definition 3.2.2, then their Fourier coefficients $\mathcal{J}_{T,k}$ are degree l polynomials of ξ given by the following formula

$$\mathcal{J}_{T,k} = \left(\frac{i\lambda}{L^d}\right)^l \sum_{k_1, k_2, \cdots, k_{l+1}} H^T_{k_1 \cdots k_{l+1}} \xi_{k_1} \xi_{k_2} \cdots \xi_{k_{l+1}} \tag{2.4.1}$$

where $H^T_{k_1 \cdots k_{l+1}}$ is given by

$$H^T_{k_1 \cdots k_{l+1}} = \int_{\cup_{\mathfrak{n} \in T_{in}} A_{\mathfrak{n}}} e^{\sum_{\mathfrak{n} \in T_{in}} it_{\mathfrak{n}} \Omega_{\mathfrak{n}} - \nu(t_{\hat{\mathfrak{n}}} - t_{\mathfrak{n}})|k_{\mathfrak{e}}|^2} \prod_{\mathfrak{n} \in T_{in}} dt_{\mathfrak{n}} \ \delta_{\cap_{\mathfrak{n} \in T_{in}} \{S_{\mathfrak{n}} = 0\}} \prod_{\mathfrak{e} \in T_{in}} \iota_{\mathfrak{e}} k_{\mathfrak{e},x}, \tag{2.4.2}$$

and ι, $A_{\mathfrak{n}}$, $S_{\mathfrak{n}}$, $\Omega_{\mathfrak{n}}$ are defined by

$$\iota_{\mathfrak{e}} = \begin{cases} +1 & \text{if } \mathfrak{e} \text{ pointing inwards to } \mathfrak{n} \\ -1 & \text{if } \mathfrak{e} \text{ pointing outwards from } \mathfrak{n} \end{cases} \tag{2.4.3}$$

$$A_{\mathfrak{n}} = \begin{cases} \{t_{\mathfrak{n}_1}, t_{\mathfrak{n}_2}, t_{\mathfrak{n}_3} \le t_{\mathfrak{n}}\} & \text{if } \mathfrak{n} \ne \text{the root } \mathfrak{r} \\ \{t_{\mathfrak{r}} \le t\} & \text{if } \mathfrak{n} = \mathfrak{r} \end{cases} \tag{2.4.4}$$

$$S_{\mathfrak{n}} = \iota_{\mathfrak{e}_1} k_{\mathfrak{e}_1} + \iota_{\mathfrak{e}_2} k_{\mathfrak{e}_2} + \iota_{\mathfrak{e}} k_{\mathfrak{e}} \tag{2.4.5}$$

$$\Omega_{\mathfrak{n}} = \iota_{\mathfrak{e}_1} \Lambda_{k_{\mathfrak{e}_1}} + \iota_{\mathfrak{e}_2} \Lambda_{k_{\mathfrak{e}_2}} + \iota_{\mathfrak{e}} \Lambda_{k_{\mathfrak{e}}} \tag{2.4.6}$$

For root node $\mathfrak{r}$, we impose the constrain that $k_{\mathfrak{r}} = k$ and $t_{\hat{\mathfrak{r}}} = t$ (notice that $\mathfrak{r}$ does not have a parent so $\hat{\mathfrak{r}}$ is not well defined).

Proof. We can check that $\mathcal{J}_T$ defined by (2.4.1) and (2.4.2) satisfies the recursive formula (2.2.8) by a direct substitution, so they are the unique solution of that recursive formula, and this proves Lemma 3.3.2. $\square$

2.4.2 An upper bound of coefficients in expansion series

In this section, we derive an upper bound for coefficients $H^T_{k_1 \cdots k_{l+1}}$.

Notice that in (2.4.1), $H^T_{k_1\cdots k_{l+1}}$ are integral of some oscillatory functions. An upper bound can be derived by the standard integration by parts arguments.

Associate each $\mathfrak{n} \in T_{\mathrm{in}}$ with two variables $a_{\mathfrak{n}}$, $b_{\mathfrak{n}}$. Then we define

$$F_T(t, \{a_{\mathfrak{n}}\}_{\mathfrak{n}\in T_{\mathrm{in}}}, \{b_{\mathfrak{n}}\}_{\mathfrak{n}\in T_{\mathrm{in}}}) = \int_{\cup_{\mathfrak{n}\in T_{\mathrm{in}}} A_{\mathfrak{n}}} e^{\sum_{\mathfrak{n}\in T_{\mathrm{in}}} it_{\mathfrak{n}} a_{\mathfrak{n}} - \nu(t_{\widehat{\mathfrak{n}}} - t_{\mathfrak{n}}) b_{\mathfrak{n}}} \prod_{\mathfrak{n}\in T_{\mathrm{in}}} dt_{\mathfrak{n}} \tag{2.4.7}$$

Lemma 2.4.3. *We have the following upper bound for $F_T(t, \{a_{\mathfrak{n}}\}_{\mathfrak{n}}, \{b_{\mathfrak{n}}\}_{\mathfrak{n}})$,*

$$\sup_{\{b_{\mathfrak{n}}\}_{\mathfrak{n}} \lesssim 1} |F_T(t, \{a_{\mathfrak{n}}\}_{\mathfrak{n}}, \{b_{\mathfrak{n}}\}_{\mathfrak{n}})| \lesssim \sum_{\{d_{\mathfrak{n}}\}_{\mathfrak{n}\in T_{in}} \in \{0,1\}^{l(T)}} \prod_{\mathfrak{n}\in T_{in}} \frac{1}{|q_{\mathfrak{n}}| + T^{-1}_{max}}. \tag{2.4.8}$$

Fix a sequence $\{d_{\mathfrak{n}}\}_{\mathfrak{n}\in T_{in}}$ whose elements $d_{\mathfrak{n}}$ takes boolean values $\{0,1\}$. We define the two sequences $\{q_{\mathfrak{n}}\}_{\mathfrak{n}\in T_{in}}$, $\{r_{\mathfrak{n}}\}_{\mathfrak{n}\in T_{in}}$ by following recursive formula

$$q_{\mathfrak{n}} = \begin{cases} a_{\mathfrak{r}}, & \textit{if } \mathfrak{n} = \textit{the root } \mathfrak{r}. \\ a_{\mathfrak{n}} + d_{\mathfrak{n}} q_{\mathfrak{n}'}, & \textit{if } \mathfrak{n} \neq \mathfrak{r} \textit{ and } \mathfrak{n}' \textit{ is the parent of } \mathfrak{n}. \end{cases} \tag{2.4.9}$$

$$r_{\mathfrak{n}} = \begin{cases} b_{\mathfrak{r}}, & \textit{if } \mathfrak{n} = \textit{the root } \mathfrak{r}. \\ b_{\mathfrak{n}} + d_{\mathfrak{n}} q_{\mathfrak{n}'}, & \textit{if } \mathfrak{n} \neq \mathfrak{r} \textit{ and } \mathfrak{n}' \textit{ is the parent of } \mathfrak{n}. \end{cases} \tag{2.4.10}$$

Proof. The lemma is proved by induction.

For a tree T contains only one node $\mathfrak{r}$, $F_T = 1$ and (2.4.8) is obviously true.

Assume that (2.4.8) is true for trees with $\leq n-1$ nodes. We prove the n nodes case.

For general T, let T_1, T_2 be the two subtrees and $\mathfrak{n}_1$, $\mathfrak{n}_2$ be the two children of the root $\mathfrak{r}$, then by the definition of F_T(2.4.7), we get

$$\begin{aligned} F_T(t) =& \int_{\cup_{\mathfrak{n}\in T_{\mathrm{in}}} A_{\mathfrak{n}}} e^{\sum_{\mathfrak{n}\in T_{\mathrm{in}}} it_{\mathfrak{n}} a_{\mathfrak{n}} - \nu(t_{\widehat{\mathfrak{n}}} - t_{\mathfrak{n}}) b_{\mathfrak{n}}} \prod_{\mathfrak{n}\in T_{\mathrm{in}}} dt_{\mathfrak{n}} \\ =& \int_{\cup_{\mathfrak{n}\in T_{\mathrm{in}}} A_{\mathfrak{n}}} e^{it_{\mathfrak{r}} a_{\mathfrak{r}} - \nu(t - t_{\mathfrak{r}}) b_{\mathfrak{r}}} e^{\sum_{\mathfrak{n}\in T_{\mathrm{in},1}\cup T_{\mathrm{in},2}} it_{\mathfrak{n}} a_{\mathfrak{n}} - \nu(t_{\widehat{\mathfrak{n}}} - t_{\mathfrak{n}}) b_{\mathfrak{n}}} \left(dt_{\mathfrak{r}} \prod_{j=1}^{2} \prod_{\mathfrak{n}\in T_{\mathrm{in},j}} dt_{\mathfrak{n}} \right) \\ =& \int_{\cup_{\mathfrak{n}\in T_{\mathrm{in}}} A_{\mathfrak{n}}} e^{it_{\mathfrak{r}}(a_{\mathfrak{r}} + T^{-1}_{\max} \operatorname{sgn}(a_{\mathfrak{r}})) - \nu(t - t_{\mathfrak{r}}) b_{\mathfrak{r}}} e^{-iT^{-1}_{\max} t_{\mathfrak{r}} \operatorname{sgn}(a_{\mathfrak{r}})} e^{\sum_{\mathfrak{n}\in T_{\mathrm{in},1}\cup T_{\mathrm{in},2}} it_{\mathfrak{n}} a_{\mathfrak{n}} - \nu(t_{\widehat{\mathfrak{n}}} - t_{\mathfrak{n}}) b_{\mathfrak{n}}} \left(dt_{\mathfrak{r}} \prod_{j=1}^{2} \prod_{\mathfrak{n}\in T_{\mathrm{in},j}} dt_{\mathfrak{n}} \right) \end{aligned} \tag{2.4.11}$$

We do integration by parts in the above integrals using Stokes formula. Notice that for $t_{\mathfrak{r}}$, there are three inequality constrains, $t_{\mathfrak{r}} \leq t$ and $t_{\mathfrak{r}} \geq t_{\mathfrak{n}_1}, t_{\mathfrak{n}_2}$.

$$F_T(t) = \frac{1}{ia_{\mathfrak{r}} + iT_{\max}^{-1}\mathrm{sgn}(a_{\mathfrak{r}}) + \nu b_{\mathfrak{r}}} \int_{\cup_{\mathfrak{n}\in T_{\mathrm{in}}} A_{\mathfrak{n}}} \frac{d}{dt_{\mathfrak{r}}} e^{it_{\mathfrak{r}}(a_{\mathfrak{r}}+T_{\max}^{-1}\,\mathrm{sgn}(a_{\mathfrak{r}}))-\nu(t-t_{\mathfrak{r}})b_{\mathfrak{r}}}$$

$$e^{-iT_{\max}^{-1}t_{\mathfrak{r}}\mathrm{sgn}(a_{\mathfrak{r}})} e^{\sum_{\mathfrak{n}\in T_{\mathrm{in},1}\cup T_{\mathrm{in},2}} it_{\mathfrak{n}}a_{\mathfrak{n}}-\nu(t_{\hat{\mathfrak{n}}}-t_{\mathfrak{n}})b_{\mathfrak{n}}} \left(dt_{\mathfrak{r}} \prod_{j=1}^{2} \prod_{\mathfrak{n}\in T_{\mathrm{in},j}} dt_{\mathfrak{n}} \right) \tag{2.4.12}$$

$$= \frac{1}{ia_{\mathfrak{r}} + iT_{\max}^{-1}\mathrm{sgn}(a_{\mathfrak{r}}) + \nu b_{\mathfrak{r}}} \left(\int_{\cup_{\mathfrak{n}\in T_{\mathrm{in}}} A_{\mathfrak{n}},\ t_{\mathfrak{r}}=t} - \int_{\cup_{\mathfrak{n}\in T_{\mathrm{in}}} A_{\mathfrak{n}},\ t_{\mathfrak{r}}=t_{\mathfrak{n}_1}} - \int_{\cup_{\mathfrak{n}\in T_{\mathrm{in}}} A_{\mathfrak{n}},\ t_{\mathfrak{r}}=t_{\mathfrak{n}_2}} \right)$$

$$e^{it_{\mathfrak{r}}(a_{\mathfrak{r}}+T_{\max}^{-1}\,\mathrm{sgn}(a_{\mathfrak{r}}))-\nu(t-t_{\mathfrak{r}})b_{\mathfrak{r}}} e^{-iT_{\max}^{-1}t_{\mathfrak{r}}\mathrm{sgn}(a_{\mathfrak{r}})} e^{\sum_{\mathfrak{n}\in T_{\mathrm{in},1}\cup T_{\mathrm{in},2}} it_{\mathfrak{n}}a_{\mathfrak{n}}-\nu(t_{\hat{\mathfrak{n}}}-t_{\mathfrak{n}})b_{\mathfrak{n}}} \left(dt_{\mathfrak{r}} \prod_{j=1}^{2} \prod_{\mathfrak{n}\in T_{\mathrm{in},j}} dt_{\mathfrak{n}} \right)$$

$$- \frac{1}{ia_{\mathfrak{r}} + iT_{\max}^{-1}\mathrm{sgn}(a_{\mathfrak{r}}) + \nu b_{\mathfrak{r}}} \int_{\cup_{\mathfrak{n}\in T_{\mathrm{in}}} A_{\mathfrak{n}}} e^{it_{\mathfrak{r}}(a_{\mathfrak{r}}+T_{\max}^{-1}\,\mathrm{sgn}(a_{\mathfrak{r}}))-\nu(t-t_{\mathfrak{r}})b_{\mathfrak{r}}}$$

$$\frac{d}{dt_{\mathfrak{r}}}(e^{-iT_{\max}^{-1}t_{\mathfrak{r}}\mathrm{sgn}(a_{\mathfrak{r}})}) e^{\sum_{\mathfrak{n}\in T_{\mathrm{in},1}\cup T_{\mathrm{in},2}} it_{\mathfrak{n}}a_{\mathfrak{n}}-\nu(t_{\hat{\mathfrak{n}}}-t_{\mathfrak{n}})b_{\mathfrak{n}}} \left(dt_{\mathfrak{r}} \prod_{j=1}^{2} \prod_{\mathfrak{n}\in T_{\mathrm{in},j}} dt_{\mathfrak{n}} \right)$$

$$= \frac{1}{ia_{\mathfrak{r}} + iT_{\max}^{-1}\mathrm{sgn}(a_{\mathfrak{r}}) + \nu b_{\mathfrak{r}}} (F_I - F_{T^{(1)}} - F_{T^{(2)}} - F_{II})$$

Here $T^{(j)}$, $j = 1, 2$ are trees that is obtained by deleting the root $\mathfrak{r}$, adding edges connecting $\mathfrak{n}_j$ with another node and defining $\mathfrak{n}_j$ to be the new root. For $T^{(j)}$, we can define the term $F_{T^{(j)}}$ by (2.4.7). It can be shown that $F_{T^{(j)}}$ defined in this way is the same as the $\int_{\cup_{\mathfrak{n}\in T_{\mathrm{in}}} A_{\mathfrak{n}},\ t_{\mathfrak{r}}=t_{\mathfrak{n}_j}}$ term in the second equality of (2.4.12), so the last equality of (2.4.12) is true. F_I is the $\int_{\cup_{\mathfrak{n}\in T_{\mathrm{in}}} A_{\mathfrak{n}},\ t_{\mathfrak{r}}=t}$ term and F_{II} is the last term containing $\frac{d}{dt_{\mathfrak{r}}}$.

We can apply the induction assumption to $F_{T^{(j)}}$ and show that $\frac{1}{ia_{\mathfrak{r}}+iT_{\max}^{-1}\mathrm{sgn}(a_{\mathfrak{r}})+\nu b_{\mathfrak{r}}} F_{T^{(j)}}$ can be bounded by the right hand side of (2.4.8).

A direct calculation gives that

$$F_I(t) = e^{ita_{\mathfrak{r}}} F_{T_1}(t) F_{T_2}(t). \tag{2.4.13}$$

Then the induction assumption implies that $\frac{1}{ia_{\mathfrak{r}}+iT_{\max}^{-1}\mathrm{sgn}(a_{\mathfrak{r}})+\nu b_{\mathfrak{r}}} F_I$ can be bounded by the right hand side of (2.4.8).

Another direct calculation gives that

$$F_{II}(t) = \int_0^t e^{it_{\mathfrak{r}}(a_{\mathfrak{r}}+T_{\max}^{-1}\,\mathrm{sgn}(a_{\mathfrak{r}}))-\nu(t-t_{\mathfrak{r}})b_{\mathfrak{r}}} \frac{d}{dt_{\mathfrak{r}}}(e^{-iT_{\max}^{-1}t_{\mathfrak{r}}\mathrm{sgn}(a_{\mathfrak{r}})})F_{T_1}(t_{\mathfrak{r}})F_{T_2}(t_{\mathfrak{r}})dt_{\mathfrak{r}}. \tag{2.4.14}$$

Apply the induction assumption

$$\begin{aligned} &\left|\frac{1}{ia_{\mathfrak{r}} + iT_{\max}^{-1}\mathrm{sgn}(a_{\mathfrak{r}}) + \nu b_{\mathfrak{r}}} F_{II}(t)\right| \\ \leq &\frac{1}{|q_{\mathfrak{r}}| + T_{\max}^{-1}} \prod_{j=1}^{2} \left(\sum_{\{d_{\mathfrak{n}}\}_{\mathfrak{n}\in T_{\mathrm{in},j}} \in\{0,1\}^{l(T_j)}} \prod_{\mathfrak{n}\in T_{\mathrm{in},j}} \frac{1}{|q_{\mathfrak{n}}| + T_{\max}^{-1}} \right) \\ \leq &\sum_{\{d_{\mathfrak{n}}\}_{\mathfrak{n}\in T_{\mathrm{in}}}\in\{0,1\}^{l(T)}} \prod_{\mathfrak{n}\in T_{\mathrm{in}}} \frac{1}{|q_{\mathfrak{n}}| + T_{\max}^{-1}}. \end{aligned} \tag{2.4.15}$$

Combining the bounds of F_I, $F_{T^{(1)}}$, $F_{T^{(2)}}$, F_{II}, we conclude that F_T can be bounded by the right hand side of (2.4.8) and thus complete the proof of Lemma 3.3.26. □

A straight forward application of the above lemma gives following upper bound of the coefficients $H^T_{k_1\cdots k_{l+1}}$.

Lemma 2.4.4. *We have the following upper bound for* $H^T_{k_1\cdots k_{l+1}}$*,*

$$|H^T_{k_1\cdots k_{l+1}}| \lesssim \sum_{\{d_{\mathfrak{n}}\}_{\mathfrak{n}\in T_{in}}\in\{0,1\}^{l(T)}} \prod_{\mathfrak{n}\in T_{in}} \frac{1}{|q_{\mathfrak{n}}| + T_{max}^{-1}} \prod_{\mathfrak{e}\in T_{in}} |k_{\mathfrak{e},x}|\ \delta_{\cap_{\mathfrak{n}\in T_{in}}\{S_{\mathfrak{n}}=0\}}. \tag{2.4.16}$$

Fix a sequence $\{d_{\mathfrak{n}}\}_{\mathfrak{n}\in T_{in}}$ *whose elements* $d_{\mathfrak{n}}$ *takes boolean values* $\{0,1\}$*. We define the two sequences* $\{q_{\mathfrak{n}}\}_{\mathfrak{n}\in T_{in}}$*,* $\{r_{\mathfrak{n}}\}_{\mathfrak{n}\in T_{in}}$ *by following recursive formula*

$$q_{\mathfrak{n}} = \begin{cases} \Omega_{\mathfrak{r}}, & \text{if } \mathfrak{n} = \text{the root } \mathfrak{r}. \\ \Omega_{\mathfrak{n}} + d_{\mathfrak{n}} q_{\mathfrak{n}'}, & \text{if } \mathfrak{n} \neq \mathfrak{r} \text{ and } \mathfrak{n}' \text{ is the parent of } \mathfrak{n}. \end{cases} \tag{2.4.17}$$

$$r_{\mathfrak{n}} = \begin{cases} |k_{\mathfrak{r}}|^2, & \text{if } \mathfrak{n} = \text{the root } \mathfrak{r}. \\ |k_{\mathfrak{n}}|^2 + d_{\mathfrak{n}} q_{\mathfrak{n}'}, & \text{if } \mathfrak{n} \neq \mathfrak{r} \text{ and } \mathfrak{n}' \text{ is the parent of } \mathfrak{n}. \end{cases} \tag{2.4.18}$$

Proof. This is a direct corollary of Lemma 2.4.8 if we take $a_{\mathfrak{n}} = \Omega_{\mathfrak{n}}$, $b_{\mathfrak{n}} = |k_{\mathfrak{e}}|^2$. □

Lemma 3.3.27 suggests that the coefficients are small when $|q_{\mathfrak{n}}| \gg T_{\max}^{-1}$. Therefore, in order to bound $\mathcal{J}_{T,k}$, we should count the lattice points on $|q_{\mathfrak{n}}| \lesssim T_{\max}^{-1}$

$$\{k_{\mathfrak{e}} \in \mathbb{Z}_L^d,\ |k_{\mathfrak{e}}| \lesssim 1,\ \forall \mathfrak{e} \in T : |q_{\mathfrak{n}}| \lesssim T_{\max}^{-1},\ S_{\mathfrak{n}} = 0,\ \forall \mathfrak{n} \in T.\ k_{\mathfrak{l}} = k\} \tag{2.4.19}$$

By solving (2.4.17), we know that $\Omega_{\mathfrak{n}}$ is a linear combination of $q_{\mathfrak{n}}$, so there exist constants $c_{\mathfrak{n},\mathfrak{n}'}$ such that $\Omega_{\mathfrak{n}} = \sum_{\mathfrak{n}'} c_{\mathfrak{n},\mathfrak{n}'} q_{\mathfrak{n}'}$. Therefore, $|q_{\mathfrak{n}}| \lesssim T_{\max}^{-1}$ implies that $|\Omega_{\mathfrak{n}}| \leq \sum_{\mathfrak{n}'} |c_{\mathfrak{n},\mathfrak{n}'} q_{\mathfrak{n}'}| \lesssim T_{\max}^{-1}$.

$|\Omega_{\mathfrak{n}}| \lesssim T_{\max}^{-1}$ implies that (2.4.19) is a subset of

$$\{k_{\mathfrak{e}} \in \mathbb{Z}_L^d,\ |k_{\mathfrak{e}}| \lesssim 1,\ \forall \mathfrak{e} \in T : |\Omega_{\mathfrak{n}}| \lesssim T_{\max}^{-1},\ S_{\mathfrak{n}} = 0,\ \forall \mathfrak{n} \in T.\ k_{\mathfrak{l}} = k\}. \tag{2.4.20}$$

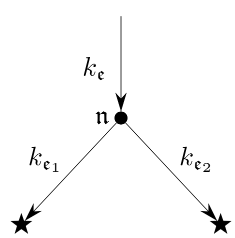
To bound the number of elements of (2.4.19), we just need to do the same thing for (2.4.20).

(2.4.20) can be read from the tree diagrams T. As in Figure 3.5, each edge corresponds to a variable $k_{\mathfrak{e}}$. The leg $\mathfrak{l}$ corresponds to equation $k_{\mathfrak{l}} = k$. Each node $\mathfrak{n}$ is connected with three edges $\mathfrak{e}_1$, $\mathfrak{e}_2$, $\mathfrak{e}$ whose corresponding variables $k_{\mathfrak{e}_1}$, $k_{\mathfrak{e}_2}$, $k_{\mathfrak{e}}$ satisfy the momentum conservation equation

$$\iota_{\mathfrak{e}_1} k_{\mathfrak{e}_1} + \iota_{\mathfrak{e}_2} k_{\mathfrak{e}_2} + \iota_{\mathfrak{e}} k_{\mathfrak{e}} = 0 \tag{2.4.21}$$

and the energy conservation equation (if the node is decorated by $\bullet$)

$$\iota_{\mathfrak{e}_1} \Lambda_{k_{\mathfrak{e}_1}} + \iota_{\mathfrak{e}_2} \Lambda_{k_{\mathfrak{e}_2}} + \iota_{\mathfrak{e}} \Lambda_{k_{\mathfrak{e}}} = O(T_{\max}^{-1}). \tag{2.4.22}$$



$$\iota_{\mathfrak{e}_1} = \iota_{\mathfrak{e}_2} = 1, \iota_{\mathfrak{e}} = -1$$
$$k_{\mathfrak{e}_1} + k_{\mathfrak{e}_2} - k_{\mathfrak{e}} = 0$$
$$\Lambda_{k_{\mathfrak{e}_1}} + \Lambda_{k_{\mathfrak{e}_2}} - \Lambda_{k_{\mathfrak{e}}} = O(T_{\max}^{-1})$$

Figure 2.6: Equations of a node $\mathfrak{n}$

The goal of the next two sections is to count the number of solutions of a modified version of the above equation (2.4.20).

2.4.3 Couples and the Wick theorem

In this section, we calculate $\mathbb{E}|\mathcal{J}_{T,k}|^2$ using Wick theorem. We also introduce another type of diagrams, the couple diagrams, to represent the result.

By the upper bound in the last section, the coefficients $H^T_{k_1\cdots k_{l+1}}$ concentrate near the surface $q_{\mathfrak{n}} = 0$, $\forall \mathfrak{n}$. But to get an upper bound of $\mathcal{J}_{T,k}$, we need upper bound of their variance $\mathbb{E}|\mathcal{J}_{T,k}|^2$. The coefficients of $\mathbb{E}|\mathcal{J}_{T,k}|^2$ also concentrate near a surface whose expression is similar to (2.4.19).

Let's derive the expression of the coefficients of $\mathbb{E}|\mathcal{J}_{T,k}|^2$ and its concentration surface.

By Lemma 3.3.2, we know that $\mathcal{J}_{T,k}$ is a polynomial of ξ which are proportional to i.i.d Gaussians. Therefore,

$$\begin{aligned}\mathbb{E}|\mathcal{J}_{T,k}|^2 =&\mathbb{E}(\mathcal{J}_{T,k}\overline{\mathcal{J}_{T,k}}) = \left(\frac{\lambda}{L^d}\right)^{2l(T)} \sum_{k_1,\, k_2,\, \cdots,\, k_{l(T)+1}} \sum_{k'_1,\, k'_2,\, \cdots,\, k'_{l(T)+1}} \\ &H^T_{k_1\cdots k_{l(T)+1}}\overline{H^T_{k'_1\cdots k'_{l(T)+1}}}\mathbb{E}\Big(\xi_{k_1}\xi_{k_2}\cdots\xi_{k_{l(T)+1}}\xi_{k'_1}\xi_{k'_2}\cdots\xi_{k'_{l(T)+1}}\Big)\end{aligned} \tag{2.4.23}$$

We just need to calculate

$$\mathbb{E}\Big(\xi_{k_1}\xi_{k_2}\cdots\xi_{k_{l(T)+1}}\xi_{k'_1}\xi_{k'_2}\cdots\xi_{k'_{l(T)+1}}\Big). \tag{2.4.24}$$

Notice that $\xi_k = \sqrt{n_{\text{in}}(k)}\,\eta_k(\omega)$ and η_k are i.i.d Gaussians. We can apply the Wick theorem to calculate the above expectations.

To introduce the Wick theorem, we need the following definition.

Definition 2.4.5. 1. **Pairing:** Suppose that we have a set $A = \{a_1, \cdots, a_{2m}\}$. A pairing is a partition of $A = \{a_{i_1}, a_{i_2}\} \cup \cdots \cup \{a_{i_{2m-1}}, a_{i_{2m}}\}$ into m disjoint subsets which have exactly two elements. Given a pairing p, elements a_{i_k}, $a_{i_{k'}}$ in the same subset of p are called paired with each other, which is denoted by $a_{i_k} \sim_p a_{i_{k'}}$.

2. $\mathcal{P}(A)$**:** Denote by $\mathcal{P}(A)$ the set of all pairings of A.

Lemma 2.4.6 (Wick theorem)**.** *Let $\{\eta_k\}_{k\in\mathbb{Z}^d_L}$ be i.i.d complex Gaussian random variables with reflection symmetry (i.e. $\eta_k = \bar{\eta}_{-k}$). Let $\mathcal{P}$ be the set of all pairings of $\{k_1, k_2, \cdots, k_{2m}\}$, then*

$$\mathbb{E}(\eta_{k_1}\cdots\eta_{k_{2m}}) = \sum_{p\in\mathcal{P}} \delta_p(k_1, \cdots, k_{2m}), \tag{2.4.25}$$

where

$$\delta_p = \begin{cases} 1 & \text{if } k_i = -k_j \text{ for all } k_i \sim_p k_j, \\ 0 & \text{otherwise.} \end{cases} \tag{2.4.26}$$

Proof. By Isserlis' theorem, for X_1, X_2, $\cdots$, X_n zero-mean i.i.d Gaussian, we have

$$\mathbb{E}[X_1 X_2 \cdots X_n] = \sum_{p \in \mathcal{P}} \prod_{i \sim_p j} \mathbb{E}[X_i X_j] \tag{2.4.27}$$

Here $\mathcal{P}$ is the set of pairings of $\{1, 2, \cdots, n\}$.

Since $\mathbb{E}[\eta_{k_i}\eta_{k_j}] = \delta_{k_i = -k_j}$, take $X_1 = \eta_{k_1}$, $\cdots$, $X_{2m} = \eta_{k_{2m}}$, then we can check that $\prod_{i \sim_p j} \mathbb{E}[X_i X_j] = \delta_p(k_1, \cdots, k_{2m})$. This finishes the proof of the Wick theorem. $\square$

Applying Wick theorem to (2.4.23), we get

$$\mathbb{E}|\mathcal{J}_{T,k}|^2 = \left(\frac{\lambda}{L^d}\right)^{2l(T)} \sum_{p \in \mathcal{P}(\{k_1, \cdots, k_{l(T)+1}, k'_1, \cdots, k'_{l(T)+1}\})} \underbrace{\sum_{k_1, k_2, \cdots, k_{l(T)+1}} \sum_{k'_1, k'_2, \cdots, k'_{l(T)+1}} H^T_{k_1 \cdots k_{l(T)+1}} \overline{H^T_{k'_1 \cdots k'_{l(T)+1}}} \delta_p(k_1, \cdots, k_{l(T)+1}, k'_1, \cdots, k'_{l(T)+1}) \sqrt{n_{\text{in}}(k_1)} \cdots}_{Term(T,p)_k}. \tag{2.4.28}$$

We see that the correlation of two tree terms is a sum of smaller expressions $Term(T, p)$. By (2.4.20), the coefficients $H^T_{k_1 \cdots k_{l(T)+1}} H^T_{k'_1 \cdots k'_{l(T)+1}}$ of $Term(T, p)$ concentrate near the subset

$$\{k_{\mathfrak{e}}, k_{\mathfrak{e}'} \in \mathbb{Z}^d_L,\ |k_{\mathfrak{e}}|, |k_{\mathfrak{e}'}| \lesssim 1,\ \forall \mathfrak{e}, \mathfrak{e}' \in T : |\Omega_{\mathfrak{n}}|, |\Omega_{\mathfrak{n}'}| \lesssim T^{-1}_{\max},\ S_{\mathfrak{n}} = S_{\mathfrak{n}'} = 0\ \forall \mathfrak{n}, \mathfrak{n}' \in T.\ k_{\mathfrak{l}} = -k'_{\mathfrak{l}} = k\}. \tag{2.4.29}$$

The pairing p in Wick theorem introduces new equations $k_i = -k'_j$ (defined in (2.4.26)) and the coefficients $H^T_{k_1 \cdots k_{l(T)+1}} H^T_{k'_1 \cdots k'_{l(T)+1}} \delta_p$ concentrate near the subset

$$\begin{aligned} &\{k_{\mathfrak{e}}, k_{\mathfrak{e}'} \in \mathbb{Z}^d_L,\ |k_{\mathfrak{e}}|, |k_{\mathfrak{e}'}| \lesssim 1,\ \forall \mathfrak{e}, \mathfrak{e}' \in T : |\Omega_{\mathfrak{n}}|, |\Omega_{\mathfrak{n}'}| \lesssim T^{-1}_{\max},\ S_{\mathfrak{n}} = S_{\mathfrak{n}'} = 0\ \forall \mathfrak{n}, \mathfrak{n}' \in T. \\ &k_{\mathfrak{l}} = -k'_{\mathfrak{l}} = k.\ k_i = -k'_j \text{ (and } k_i = -k_j,\ k'_i = -k'_j\text{) for all } k_i \sim_p k'_j \text{ (and } k_i \sim_p k_j,\ k'_i \sim_p k'_j\text{)}\}. \end{aligned} \tag{2.4.30}$$

As in the case of (2.4.20), there is a graphical representation of (2.4.30). To explain this, we need the concept of couples.

Definition 2.4.7 (Construction of couples)**.** Given two trees T and T', we flip the orientation of all edges in T' (as in the two left trees in Figure 3.7). We also label their leaves by $1, 2, \cdots, l(T)+1$ and $1, 2, \cdots, l(T')+1$ so that the corresponding variables of these leaves are $k_1, k_2, \cdots, k_{l(T)+1}$ and $k_1, k_2, \cdots, k_{l(T')+1}$. Assume that we have a pairing p of the set $\{k_1, k_2, \cdots, k_{l(T)+1}, k_1, k_2, \cdots, k_{l(T')+1}\}$, then this pairing induces a pairing between leaves (if $k_i \sim_p k_j$ then define *the i-th leaf* $\sim_p$ *the j-th leaf*). Given this pairing of leaves, we define the following procedure which glues two trees T and T' into a couple $\mathcal{C}(T, T, p)$. Some example of pairing can be find in Figure 3.7.

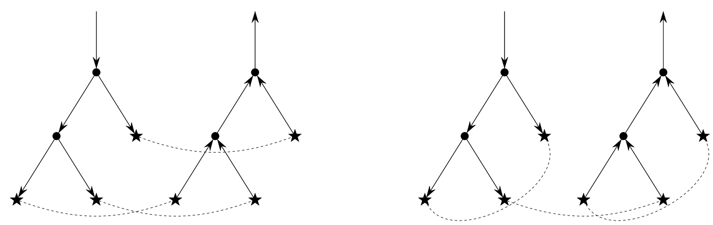


Figure 2.7: Example of pairings between trees.

1. **Merging edges connected to leaves:** Given two edges with opposite orientation connected to two paired leaves, these two edges can be merged into one edge as in Figure 3.8.

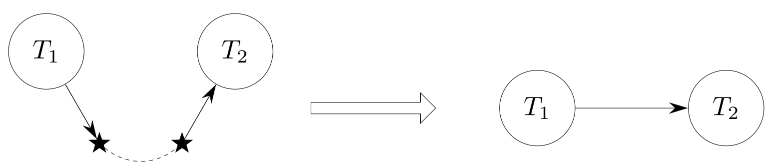


Figure 2.8: Pairing and merging of two edges

We know that two edges connected to leaves correspond to two indices k_i, k_j. Merging two such edges is a graphical interpretation that $k_i = -k_j$.

2. **Pairing of trees and couples:** Given a pairing p of the set of leaves in T, T' we merge all edges paired by p as in Figure 3.11 and the resulting combinatorial structure is called a couple.

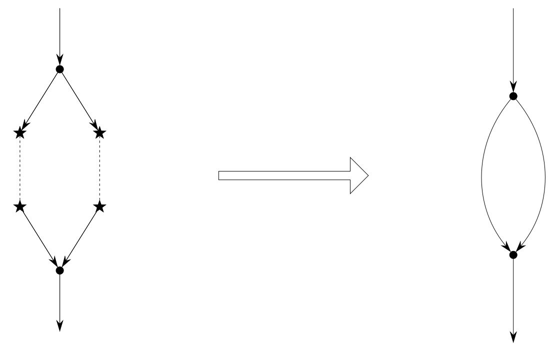


Figure 2.9: The construction of a couple

We know that each edge connected to leaf corresponds to a variable k_i. A pairing p of $\{k_1, k_2, \cdots, k_{2m}\}$ in (2.4.30) induces a pairing of edges connected to leaves. Merging paired edges corresponds to $k_i = -k'_j$ for all $k_i \sim_p k'_j$ in (2.4.30).

The following proposition introduce the graphical representation of (2.4.30).

Proposition 2.4.8. (2.4.30) *can be read from a couple diagram $\mathcal{C}(T, T, p)$. Each edge corresponds to a variable $k_{\mathfrak{e}}$. The leg $\mathfrak{l}$ corresponds to equation $k_{\mathfrak{l}} = k$. Each node corresponds to a momentum conservation equation*

$$\iota_{\mathfrak{e}_1} k_{\mathfrak{e}_1} + \iota_{\mathfrak{e}_2} k_{\mathfrak{e}_2} + \iota_{\mathfrak{e}} k_{\mathfrak{e}} = 0, \tag{2.4.31}$$

and a energy conservation equation

$$\iota_{\mathfrak{e}_1} \Lambda_{k_{\mathfrak{e}_1}} + \iota_{\mathfrak{e}_2} \Lambda_{k_{\mathfrak{e}_2}} + \iota_{\mathfrak{e}} \Lambda_{k_{\mathfrak{e}}} = O(T_{max}^{-1}). \tag{2.4.32}$$

Remark 2.4.9. In a couple diagram, we only have nodes decorated by $\bullet$. Nodes decorated by $\star$ have been removed in (1), (2) of Definition 2.4.7.

Remark 2.4.10. Through the process of (1), (2) in Definition 2.4.7, a couple diagram can automatically encode the equation $k_i = -k'_j$ for all $k_i \sim_p k'_j$. Therefore, they do not appear in Proposition 3.3.7.

Proof. This directly follows from the definition of couples. $\square$

The calculations of this section are summarized in the following proposition.

Proposition 2.4.11. *(1) Define $Term(T,p)$ in the same way as in* (2.4.28)*,*

$$Term(T,p)_k = \sum_{k_1, k_2, \cdots, k_{l(T)+1}} \sum_{k_1', k_2', \cdots, k_{l(T)+1}'} H^T_{k_1\cdots k_{l(T)+1}} \overline{H^T_{k_1'\cdots k_{l(T)+1}'}} \delta_p(k_1, \cdots, k_{l(T)+1}, k_1', \cdots, k_{l(T)+1}') \sqrt{n_{in}(k_1)} \cdots \sqrt{n_{in}(k_1')} \cdots \tag{2.4.33}$$

then $\mathbb{E}|\mathcal{J}_{T,k}|^2$ is a sum of $Term(T,p)_k$ for all $p \in \mathcal{P}$, (in (2.4.28) *the sum is over set of all possible pairing $\mathcal{P}$)*

$$\mathbb{E}|\mathcal{J}_{T,k}|^2 = \left(\frac{\lambda}{L^d}\right)^{2l(T)} \sum_{p\in\mathcal{P}(\{k_1,\cdots,k_{l(T)+1},k_1',\cdots,k_{l(T)+1}'\})} Term(T,p). \tag{2.4.34}$$

(2) $Term(T,p)$ concentrates near the subset (2.4.30) *which has a simple graphical representation given by Proposition 3.3.7.*

Proof. The proof of (1), (2) is easy and thus skipped. □

2.4.4 Counting lattice points

In this section, we use the connection between couples and concentration subsets (2.4.30) to count the number of solutions of a generalized version of (2.4.30),

$$\begin{aligned} &\{k_{\mathfrak{e}}, k_{\mathfrak{e}'} \in \mathbb{Z}^d_L,\ |k_{\mathfrak{e}}|, |k'_{\mathfrak{e}}| \lesssim 1,\ \forall \mathfrak{e}, \mathfrak{e}' \in T:\ |k_{\mathfrak{e}x}| \sim \kappa_{\mathfrak{e}}, |k'_{\mathfrak{e}x}| \sim \kappa_{\mathfrak{e}'},\ \forall \mathfrak{e}, \mathfrak{e}' \in T, \\ &|\Omega_{\mathfrak{n}} - \sigma_{\mathfrak{n}}|, |\Omega'_{\mathfrak{n}} - \sigma'_{\mathfrak{n}}| \lesssim T^{-1}_{\max},\ S_{\mathfrak{n}} = S_{\mathfrak{n}'} = 0,\ \forall \mathfrak{n}.\ k_{\mathfrak{l}} = -k'_{\mathfrak{l}} = k. \\ &k_i = -k'_j \textit{ (and } k_i = -k_j,\ k'_i = -k'_j \textit{) for all } k_i \sim_p k'_j \textit{ (and } k_i \sim_p k_j,\ k'_i \sim_p k'_j\textit{)}\}. \end{aligned} \tag{2.4.35}$$

In (2.4.35), $\kappa_{\mathfrak{e}} \in \{0\} \cup \mathcal{D}(\alpha, 1)$, where $\mathcal{D}(\alpha, 1) := \{2^{-K_{\mathfrak{e}}} : K_{\mathfrak{e}} \in \mathbb{Z} \cap [0, ln\ \alpha^{-1}]\}$. The relation $|k_{\mathfrak{e}x}| \sim \kappa_{\mathfrak{e}}$ is defined by

$$|k_{\mathfrak{e}x}| \sim \kappa_{\mathfrak{e}} \text{ if and only if } \begin{cases} \frac{1}{2}\kappa_{\mathfrak{e}} \le |k_{\mathfrak{e}x}| \le 2\kappa_{\mathfrak{e}} & \text{if } \kappa_{\mathfrak{e}} \ne 0 \\ \\ |k_{\mathfrak{e}x}| \lesssim \alpha^2,\ k_{\mathfrak{e}x} \ne 0 & \text{if } \kappa_{\mathfrak{e}} = 0 \end{cases} \tag{2.4.36}$$

(2.4.35) is obtained by replacing $\Omega_{\mathfrak{n}}$, $\Omega'_{\mathfrak{n}}$ by $\Omega_{\mathfrak{n}} - \sigma_{\mathfrak{n}}$, $\Omega'_{\mathfrak{n}} - \sigma'_{\mathfrak{n}}$ in (2.4.30) and adding conditions $|k_{\mathfrak{e}}| \sim \kappa_{\mathfrak{e}}$, where $\sigma_{\mathfrak{n}}$, $\sigma'_{\mathfrak{n}}$ and $\kappa_{\mathfrak{e}}$ are some given constants. The counterpart of Proposition 3.3.7 in this case is

Proposition 2.4.12. (2.4.35) *can be read from a couple diagram* $\mathcal{C} = \mathcal{C}(T, T, p)$*. Each edge corresponds to a variable* $k_{\mathfrak{e}}$*. The leg* $\mathfrak{l}$ *corresponds to equation* $k_{\mathfrak{l}} = k$*. Each node corresponds to a momentum conservation equation*

$$\iota_{\mathfrak{e}_1} k_{\mathfrak{e}_1} + \iota_{\mathfrak{e}_2} k_{\mathfrak{e}_2} + \iota_{\mathfrak{e}} k_{\mathfrak{e}} = 0, \tag{2.4.37}$$

and a energy conservation equation

$$\iota_{\mathfrak{e}_1} \Lambda_{k_{\mathfrak{e}_1}} + \iota_{\mathfrak{e}_2} \Lambda_{k_{\mathfrak{e}_2}} + \iota_{\mathfrak{e}} \Lambda_{k_{\mathfrak{e}}} = \sigma_{\mathfrak{n}} + O(T_{max}^{-1}). \tag{2.4.38}$$

Denote the momentum and energy conservation equations by $MC_{\mathfrak{n}}$ *and* $EC_{\mathfrak{n}}$ *respectively, then* (2.4.35) *can be rewritten as*

$$(2.4.35) = \{k_{\mathfrak{e}} \in \mathbb{Z}_L^d,\ |k_{\mathfrak{e}}| \lesssim 1\ \forall \mathfrak{e} \in \mathcal{C}:\ |k_{\mathfrak{e}x}| \sim \kappa_{\mathfrak{e}},\ \forall \mathfrak{e} \in \mathcal{C}_{norm}.\ MC_{\mathfrak{n}},\ EC_{\mathfrak{n}},\ \forall \mathfrak{n} \in \mathcal{C}.\ k_{\mathfrak{l}} = -k_{\mathfrak{l}}' = k.\} \tag{2.4.39}$$

Proof. This directly follows from Proposition 3.3.7. □

To explain the counting argument in this paper, we need the following definitions related to couples.

Definition 2.4.13.

1. **Connected couples:** A couple $\mathcal{C}$ is a connected couple if it is connected as a graph.

2. **Equations of a couple** $Eq(\mathcal{C})$**:** Given a couple $\mathcal{C}$ and constants k, $\sigma_{\mathfrak{n}}$, let $Eq(\mathcal{C}, \{\sigma_{\mathfrak{n}}\}_{\mathfrak{n}}, k)$ (or simply $Eq(\mathcal{C})$) be the system of equation (2.4.39) constructed in Proposition 3.3.11. For any system of equations Eq, let $\#(Eq)$ be its number of solutions.

3. **Normal edges and leaf edges:** Remember that any couple $\mathcal{C}$ is constructed from a pairing of two trees T, T' and therefore all edges in $\mathcal{C}$ comes from T, T'. We define edges coming from T_{in}, T_{in}' to be normal edges and edges from those connected to leaves in T, T' to be leaf edges. The set of all normal edges is denoted by $\mathcal{C}_{\text{norm}}$. A leg in $\mathcal{C}$ which is a normal edge is called a normal leg.

The main goal of this section is to prove an upper bound of $\#Eq(\mathcal{C})$. The main idea of proving this is to decompose a large couple $\mathcal{C}$ into smaller pieces and then prove this for smaller piece using induction hypothesis. To explain the idea, let us first focus on an example. Let $\mathcal{C}$ be the left couple in the following picture. (The corresponding variables of each edge are labelled near these edges.)

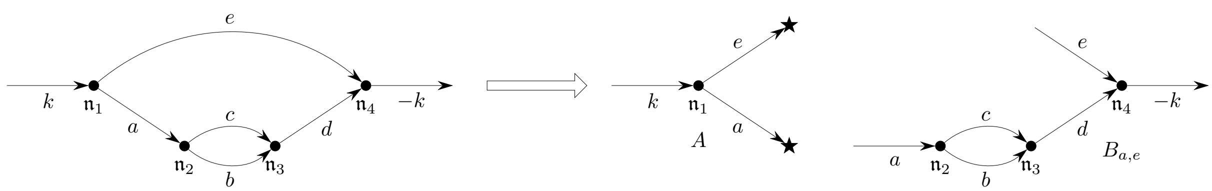


Figure 2.10: An example of decomposing a couple

By (2.4.39), we know that the couple $\mathcal{C}$ corresponds to the following equations.

$$\begin{aligned}\{a,b,c,d,e : (|a| \text{ to } |e|) \lesssim 1,\ (|a_x| \text{ to } |e_x|) \sim (\kappa_a \text{ to } \kappa_e)\\ a+e=k,\ \Lambda(a)+\Lambda(e)-\Lambda(k)=\sigma_1+O(T_{\max}^{-1})\\ a+c=b,\ \Lambda(a)+\Lambda(c)-\Lambda(b)=\sigma_2+O(T_{\max}^{-1})\\ b+c=d,\ \Lambda(b)+\Lambda(c)-\Lambda(d)=\sigma_3+O(T_{\max}^{-1})\\ d+e+k=0,\ \Lambda(d)+\Lambda(e)+\Lambda(k)=\sigma_4+O(T_{\max}^{-1})\}\end{aligned} \tag{2.4.40}$$

We know that (2.4.40) can be rewritten into the form $\bigcup_{a,e\in A} B_{a,e}$, where

$$A=\{a,e:\ |a|,|e|\lesssim 1,\ |a_x|\sim\kappa_a, |e_x|\sim\kappa_e, a+e=k,\ \Lambda(a)+\Lambda(e)-\Lambda(k)=\sigma_1+O(T_{\max}^{-1})\} \tag{2.4.41}$$

$$\begin{aligned}B_{a,e}=\{b,c,d : |b|,|c|,|d|\lesssim 1,\ |b_x|\sim\kappa_b, |c_x|\sim\kappa_c, |d_x|\sim\kappa_d\\ a+c=b,\ \Lambda(a)+\Lambda(c)-\Lambda(b)=\sigma_2+O(T_{\max}^{-1})\\ b+c=d,\ \Lambda(b)+\Lambda(c)-\Lambda(d)=\sigma_3+O(T_{\max}^{-1})\\ d+e+k=0,\ \Lambda(d)+\Lambda(e)+\Lambda(k)=\sigma_4+O(T_{\max}^{-1})\}\end{aligned} \tag{2.4.42}$$

Since an upper bound of $\#Eq(\mathcal{C})$ can be derived from upper bounds of $\#A$, $\#B_{a,e}$, we just need to consider A, $B_{a,e}$ which are systems of equations of smaller size. We can reduce the size of systems of equations in this way and prove upper bounds by induction.

One problem of applying induction argument is that A, $B_{a,e}$ cannot be represented by couple defined by Definition 2.4.7 that can contain at most two legs (an edge just connected to one node). In Definition 2.4.7, a leg is used to represent a variable which is fixed, as in the condition $k_{\mathfrak{l}}=-k'_{\mathfrak{l}}=k$ in (2.4.39). The definition of $\#B_{a,e}$ contains three fixed variables a, e, k which cannot be represented by just two legs. Therefore, we has to define a new type of couple that allows multiple legs.

Except for the lack of legs, we also have the problem of representing free variables. We know that the couple representation of A should contain one node and three edges if we insist on the rule that a node corresponds to an equation and the variables in the equation correspond to edges connected to this node. All these edges are legs, but two of three edges correspond to variable a, b which are not fixed. Therefore, we have to define a type of legs that can correspond to unfixed variables.

To solve the above problems, we introduce the following definition.

Definition 2.4.14. 1. **Couples with multiple legs:** A graph in which all nodes have degree 1 or 3 is called a couples with multiple legs. The graph A and $B_{a,e}$ in Figure 3.12 are examples of this definition.

2. **Legs:** In a couple with multiple leg, an edge connected to a degree one node is called a leg. Remember that we have encounter this concept in the second paragraph of section 3.3.1 and in what follows we call the leg defined there the root leg of a tree.

3. **Free legs and fixed legs:** In a couple with multiple leg, we use two types of node decoration for degree 1 nodes as in Figure 3.13. One is $\star$ and the other one is invisible.

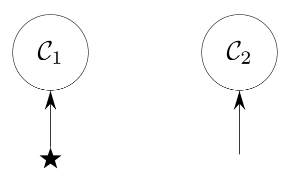


Figure 2.11: Node decoration of degree one nodes

An edge connected to a $\star$ or invisible nodes is called a free leg or fixed leg respectively.

4. **Equations of a couple** $Eq(\mathcal{C},\{c_{\mathfrak{l}}\}_{\mathfrak{l}})$**:** We define the corresponding equations for a couples with multiple legs.

$$\begin{aligned}&Eq(\mathcal{C},\{c_{\mathfrak{l}}\}_{\mathfrak{l}})\\=&\{k_{\mathfrak{e}}\in\mathbb{Z}^d_L,\ |k_{\mathfrak{e}}|\lesssim 1\ \forall\mathfrak{e}\in\mathcal{C}:\ |k_{\mathfrak{e}x}|\sim\kappa_{\mathfrak{e}},\ \forall\mathfrak{e}\in\mathcal{C}_{\text{norm}}.\ MC_{\mathfrak{n}},\ EC_{\mathfrak{n}},\ \forall\mathfrak{n}\in\mathcal{C}.\ k_{\mathfrak{l}}=c_{\mathfrak{l}},\ \forall\mathfrak{l}.\}\end{aligned} \tag{2.4.43}$$

In this representation, the corresponding variable of a fixed leg $\mathfrak{l}$ is fixed to be the constant $c_{\mathfrak{l}}$ and the corresponding variable of a free leg $\mathfrak{l}$ is not fixed.

With the above definition, it's easy to show that the couple A and $B_{a,e}$ in Figure 3.12 correspond to the system of equations (2.4.41) and (2.4.42) respectively.

Using the above argument, we can prove the following proposition which gives an upper bound of number of solutions of (2.4.35) (or (2.4.39)).

Proposition 2.4.15. *Let $\mathcal{C} = \mathcal{C}(T, T', p)$ be an connected couple with exactly one free and one fixed leg, n be the total number of nodes in $\mathcal{C}$ and $Q = L^d T_{max}^{-1}$. We fix $k \in \mathbb{R}$ for the legs $\mathfrak{l}$, $\mathfrak{l}'$ and $\sigma_{\mathfrak{n}} \in \mathbb{R}$ for each $\mathfrak{n} \in \mathcal{C}$. Assume that α satisfies* (2.1.2). *Then the number of solutions M of* (2.4.35) *(or* (2.4.39)*) is bounded by*

$$M \leq L^{O(n\theta)} Q^{\frac{n}{2}} \prod_{\mathfrak{e} \in \mathcal{C}_{norm}} \kappa_{\mathfrak{e}}^{-1}. \tag{2.4.44}$$

Proof. The proof is lengthy and therefore divided into several steps. The main idea of the proof is to use the operation of edge cutting to decompose the couple $\mathcal{C}$ into smaller ones $\mathcal{C}_1$, $\mathcal{C}_2$, then apply Lemma 3.3.16 which relates $\#Eq(\mathcal{C})$ and $\#Eq(\mathcal{C}_i)$. The desire upper bounds of $\#Eq(\mathcal{C})$ can be obtained from that of $\#Eq(\mathcal{C}_i)$ inductively.

Step 1. In this step, we explain the cutting edge argument and prove the Lemma 3.3.16 which relates $\#Eq(\mathcal{C}_1)$, $\#Eq(\mathcal{C}_2)$ and $\#Eq(\mathcal{C})$.

Here is the formal definition of cutting

Definition 2.4.16.

1. **Cutting an edge:** Given an edge $\mathfrak{e}$, we can cut it into two edges (a fixed and a free leg) as in Figure 3.14.

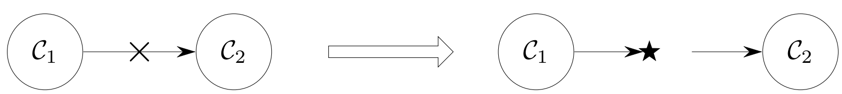


Figure 2.12: An example of cutting an edge

2. **Cut:** A <u>cut</u> c of a couple $\mathcal{C}$ is a set of edges such that $\mathcal{C}$ is disconnected after cutting all edges in c. A <u>refined cut</u> is a cut together with a map $\mathrm{rc} : c \to \{\text{left}, \text{right}\}$. For each $\mathfrak{e} \in c$, if $\mathrm{rc}(\mathfrak{e}) = \text{left}$ (resp. right), then as in Figure 3.14 the left node (resp. right node) produced by cutting $\mathfrak{e}$ is a $\star$ node (resp. invisible node). The map rc describes which one should be the free or fixed leg in the two legs produced by cutting an edge.

3. $c(\mathfrak{e})$**,** $c(\mathfrak{n})$ **and** $c(\mathfrak{l})$**:** Given an edge $\mathfrak{e}$ that is not a leg, define $c(\mathfrak{l})$ to be the cut that contains only one edge $\mathfrak{e}$. Given a node $\mathfrak{n} \in \mathcal{C}$, let $\{\mathfrak{e}_i\}$ be edges that are connected to $\mathfrak{n}$, then define $c(\mathfrak{n})$ to be the cut that consists of edges $\{\mathfrak{e}_i\}$. Given an leg $\mathfrak{l}$, let $\mathfrak{n}$ be the unique node connected to it, then define $c(\mathfrak{e})$ to be the cut $c(\mathfrak{n})$. An example of cutting $c(\mathfrak{e})$ is give by Figure 3.14. The following picture gives an example of cutting $c(\mathfrak{n})$ or $c(\mathfrak{l})$ (in this picture $\mathfrak{n} = \mathfrak{n}_1$ and $\mathfrak{l}$ is the leg labelled by k.)

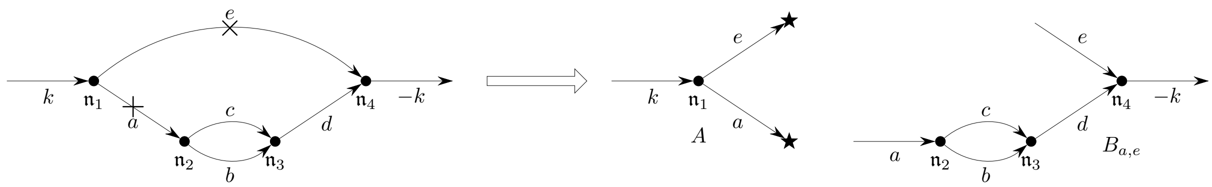


Figure 2.13: An example of cuts, $c(\mathfrak{n})$ and $c(\mathfrak{l})$

4. **Normal edges in couples with multiple legs:** In this paper, all couples with multiple legs are produced by cutting a couple defined in Definition 2.4.7. If a normal edge $\mathfrak{e}$ is cut into $\mathfrak{e}_1$ and $\mathfrak{e}_2$, then $\mathfrak{e}_1$ and $\mathfrak{e}_2$ are defined to be normal in the resulting couples with multiple legs.

Remark 2.4.17. Explicitly writing down the full definition of rc is often complicated, so in what follows, when defining rc, we will only describe which one should be the free or fixed leg in the two legs produced by cutting an edge.

The couples in Proposition 3.3.22 contains just 2 fixed legs, but after cutting, these couples may contain more fixed or free legs.

By Definition 3.3.13, for a couple $\mathcal{C}$ with multiple legs, given constants $c_{\mathfrak{l}}$ for each fixed leg $\mathfrak{l}$, the corresponding equation of $\mathcal{C}$ is denoted by $Eq(\mathcal{C}, \{c_{\mathfrak{l}}\}_{\mathfrak{l}})$. In $Eq(\mathcal{C}, \{c_{\mathfrak{l}}\}_{\mathfrak{l}})$ each edge $\mathfrak{e}$ is associated with a variable $k_{\mathfrak{e}}$ and each node $\mathfrak{n}$ is still associated with equations $MC_{\mathfrak{n}}$, $EC_{\mathfrak{n}}$. The corresponding variables of free (resp. fixed) legs are free (resp. fixed to be a constant $c_{\mathfrak{l}}$).

Let us explain how does $Eq(\mathcal{C})$ and $\#Eq(\mathcal{C})$ changes after cutting. The result is summarized in the following lemma.

Lemma 2.4.18. *Let c be a cut of $\mathcal{C}$ that consists of edges $\{\mathfrak{e}_i\}$ and $\mathcal{C}_1$, $\mathcal{C}_2$ be two components after cutting. Let $\mathfrak{e}_i^{(1)} \in \mathcal{C}_1$, $\mathfrak{e}_i^{(2)} \in \mathcal{C}_2$ be two edges obtained by cutting $\mathfrak{e}_i$. The rc map is defined by assigning $\{\mathfrak{e}_i^{(1)}\}$ to be free legs and $\{\mathfrak{e}_i^{(2)}\}$ to be fixed legs. Then we have*

$$Eq(\mathcal{C}, \{c_{\mathfrak{l}}\}_{\mathfrak{l}}) = \left\{(k_{\mathfrak{e}_1}, k_{\mathfrak{e}_2}) : \ k_{\mathfrak{e}_1} \in Eq(\mathcal{C}_1, \{c_{\mathfrak{l}_1}\}), \ k_{\mathfrak{e}_2} \in Eq\left(\mathcal{C}_2, \{c_{\mathfrak{l}_2}\}, \left\{k_{\mathfrak{e}_i^{(1)}}\right\}_i\right)\right\}. \tag{2.4.45}$$

and

$$\sup_{\{c_{\mathfrak{l}}\}_{\mathfrak{l}}} \#Eq(\mathcal{C}, \{c_{\mathfrak{l}}\}_{\mathfrak{l}}) \leq \sup_{\{c_{\mathfrak{l}_1}\}_{\mathfrak{l}_1 \in leg(\mathcal{C}_1)}} \#Eq(\mathcal{C}_1, \{c_{\mathfrak{l}_1}\}) \sup_{\{c_{\mathfrak{l}_2}\}_{\mathfrak{l}_2 \in leg(\mathcal{C}_2)}} \#Eq(\mathcal{C}_2, \{c_{\mathfrak{l}_2}\}). \tag{2.4.46}$$

Here $leg(\mathcal{C})$ is the set of fixed legs in $\mathcal{C}$ (not the set of all legs!).

Proof. By definition (2.4.43) we have

$$
\begin{aligned}
Eq(\mathcal{C},\{c_{\mathfrak{l}}\}_{\mathfrak{l}}) =&\{k_{\mathfrak{e}}\in\mathbb{Z}_L^d,\ |k_{\mathfrak{e}}|\lesssim 1:\ |k_{\mathfrak{e}x}|\sim\kappa_{\mathfrak{e}},\ \forall\mathfrak{e}\in\mathcal{C}_{\text{norm}}.\ MC_{\mathfrak{n}},\ EC_{\mathfrak{n}},\ \forall\mathfrak{n}.\ k_{\mathfrak{l}}=c_{\mathfrak{l}},\ \forall\mathfrak{l}\in\operatorname{leg}(\mathcal{C}).\}\\
=&\{(k_{\mathfrak{e}_1},k_{\mathfrak{e}_2}):\ |k_{\mathfrak{e}_1}|\lesssim 1,\ MC_{\mathfrak{n}_1},\ EC_{\mathfrak{n}_1}.\ \forall\mathfrak{e}_1,\mathfrak{n}_1\in\mathcal{C}_1.\ |k_{\mathfrak{e}x}|\sim\kappa_{\mathfrak{e}},\ \forall\mathfrak{e}\in\mathcal{C}_{\text{norm}}.\\
&\ k_{\mathfrak{l}_1}=c_{\mathfrak{l}_1},\ \forall\mathfrak{l}_1\in\operatorname{leg}(\mathcal{C})\cap\operatorname{leg}(\mathcal{C}_1)\\
&\ |k_{\mathfrak{e}_2}|\lesssim 1,\ MC_{\mathfrak{n}_2},\ EC_{\mathfrak{n}_2}.\ \forall\mathfrak{e}_2,\mathfrak{n}_2\in\mathcal{C}_2.\ |k_{\mathfrak{e}x}|\sim\kappa_{\mathfrak{e}},\ \forall\mathfrak{e}\in\mathcal{C}_{\text{norm}}.\\
&\ k_{\mathfrak{l}_2}=c_{\mathfrak{l}_2},\ \forall\mathfrak{l}_2\in\operatorname{leg}(\mathcal{C})\cap\operatorname{leg}(\mathcal{C}_2),\ k_{\mathfrak{e}_i^{(2)}}=k_{\mathfrak{e}_i^{(1)}},\ \forall\mathfrak{e}_i\in c\}\\
=&\left\{(k_{\mathfrak{e}_1},k_{\mathfrak{e}_2}):\ k_{\mathfrak{e}_1}\in Eq(\mathcal{C}_1,\{c_{\mathfrak{l}_1}\}),\ k_{\mathfrak{e}_2}\in Eq\left(\mathcal{C}_2,\{c_{\mathfrak{l}_2}\},\left\{k_{\mathfrak{e}_i^{(1)}}\right\}_i\right)\right\}
\end{aligned}
\tag{2.4.47}
$$

Here in $Eq\left(\mathcal{C}_2,\{c_{\mathfrak{l}_2}\},\left\{k_{\mathfrak{e}_i^{(1)}}\right\}_i\right)$, $k_{\mathfrak{e}_i^{(1)}}$ are view as a constant value and $k_{\mathfrak{e}_i^{(2)}}$ are fixed to be this constant value.

Therefore, we have the following identity of $Eq(\mathcal{C},\{c_{\mathfrak{l}}\}_{\mathfrak{l}})$

$$
Eq(\mathcal{C},\{c_{\mathfrak{l}}\}_{\mathfrak{l}})=\left\{(k_{\mathfrak{e}_1},k_{\mathfrak{e}_2}):\ k_{\mathfrak{e}_1}\in Eq(\mathcal{C}_1,\{c_{\mathfrak{l}_1}\}),\ k_{\mathfrak{e}_2}\in Eq\left(\mathcal{C}_2,\{c_{\mathfrak{l}_2}\},\left\{k_{\mathfrak{e}_i^{(1)}}\right\}_i\right)\right\}. \tag{2.4.48}
$$

which proves (2.4.45).

We can also find the relation between $\#Eq(\mathcal{C}_1)$, $\#Eq(\mathcal{C}_2)$ and $\#Eq(\mathcal{C})$. Applying (2.4.45),

$$
\begin{aligned}
\#Eq(\mathcal{C},\{c_{\mathfrak{l}}\}_{\mathfrak{l}}) =& \sum_{(k_{\mathfrak{e}_1},k_{\mathfrak{e}_2})\in\#Eq(\mathcal{C},\{c_{\mathfrak{l}}\}_{\mathfrak{l}})} 1\\
=& \sum_{\left\{(k_{\mathfrak{e}_1},k_{\mathfrak{e}_2}):\ k_{\mathfrak{e}_1}\in Eq(\mathcal{C}_1,\{c_{\mathfrak{l}_1}\}),\ k_{\mathfrak{e}_2}\in Eq\left(\mathcal{C}_2,\{c_{\mathfrak{l}_2}\},\left\{k_{\mathfrak{e}_i^{(1)}}\right\}_i\right)\right\}} 1\\
=& \sum_{k_{\mathfrak{e}_1}\in Eq(\mathcal{C}_1,\{c_{\mathfrak{l}_1}\})}\ \sum_{k_{\mathfrak{e}_2}\in Eq\left(\mathcal{C}_2,\{c_{\mathfrak{l}_2}\},\left\{k_{\mathfrak{e}_i^{(1)}}\right\}_i\right)} 1\\
=& \sum_{k_{\mathfrak{e}_1}\in Eq(\mathcal{C}_1,\{c_{\mathfrak{l}_1}\})} \#Eq\left(\mathcal{C}_2,\{c_{\mathfrak{l}_2}\},\left\{k_{\mathfrak{e}_i^{(1)}}\right\}_i\right)
\end{aligned}
\tag{2.4.49}
$$

Take sup in the above equation

$$\begin{aligned}
&\sup_{\{c_{\mathfrak{l}}\}_{\mathfrak{l}}} \#Eq(\mathcal{C}, \{c_{\mathfrak{l}}\}_{\mathfrak{l}}) = \sup_{\{c_{\mathfrak{l}}\}_{\mathfrak{l}}} \sum_{k_{\mathfrak{e}_1} \in Eq(\mathcal{C}_1, \{c_{\mathfrak{l}_1}\})} \#Eq\left(\mathcal{C}_2, \{c_{\mathfrak{l}_2}\}, \left\{k_{\mathfrak{e}_i^{(1)}}\right\}_i\right) \\
\leq & \sup_{\{c_{\mathfrak{l}_1}\}_{\mathfrak{l}_1 \in \mathrm{leg}(\mathcal{C}) \cap \mathrm{leg}(\mathcal{C}_1)}} \sum_{k_{\mathfrak{e}_1} \in Eq(\mathcal{C}_1, \{c_{\mathfrak{l}_1}\})} \sup_{\{c_{\mathfrak{l}_2}\}_{\mathfrak{l}_2 \in \mathrm{leg}(\mathcal{C}) \cap \mathrm{leg}(\mathcal{C}_2)}} \#Eq\left(\mathcal{C}_2, \{c_{\mathfrak{l}_2}\}, \left\{k_{\mathfrak{e}_i^{(1)}}\right\}_i\right) \\
\leq & \sup_{\{c_{\mathfrak{l}_1}\}_{\mathfrak{l}_1 \in \mathrm{leg}(\mathcal{C}_1)}} \sum_{k_{\mathfrak{e}_1} \in Eq(\mathcal{C}_1, \{c_{\mathfrak{l}_1}\})} \sup_{\{c_{\mathfrak{l}_2}\}_{\mathfrak{l}_2 \in \mathrm{leg}(\mathcal{C}_2)}} \#Eq(\mathcal{C}_2, \{c_{\mathfrak{l}_2}\}) \\
= & \sup_{\{c_{\mathfrak{l}_1}\}_{\mathfrak{l}_1 \in \mathrm{leg}(\mathcal{C}_1)}} \#Eq(\mathcal{C}_1, \{c_{\mathfrak{l}_1}\}) \sup_{\{c_{\mathfrak{l}_2}\}_{\mathfrak{l}_2 \in \mathrm{leg}(\mathcal{C}_2)}} \#Eq(\mathcal{C}_2, \{c_{\mathfrak{l}_2}\})
\end{aligned} \tag{2.4.50}$$

This proves (2.4.46). □

Step 2. In this step, we specify the cutting procedure.

Notice that Proposition 3.3.24, the multiple leg analog of Proposition 3.3.22, is only true for couples satisfying the "property P". When designing the cutting procedure, we must make sure that all couples generated during the execution of this procedure satisfy the "property P". The following proposition guarantees the existence of such procedure.

Proposition 2.4.19. *There exists a recursive algorithm that repeatedly decomposes $\mathcal{C}$ into smaller pieces and satisfies the following requirements. In the rest of this paper, we will call this algorithm "the cutting algorithm".*

(1) The input of the 0-th step of this algorithm is $\mathcal{C}(0) = \mathcal{C}$. The inputs of other steps are the outputs of previous steps of the algorithm itself.

(2) In step k, $\mathcal{C}(k)$ is decomposed into 2 or 3 connected components by cutting edges and all components with more than one node are outputted. For $\mathcal{C}(1)$, $\#Eq(\mathcal{C}(1)) = \#Eq(\mathcal{C})$ and $\mathcal{C}_{norm}(1) = \mathcal{C}_{norm}$.

(3) One of the connected components in (2) contains exactly one node $\mathfrak{n}$ and one fixed normal leg $\mathfrak{l}$. We call this component $\mathcal{C}(k)_{\mathfrak{l}}$. There are only two possibilities of $\mathcal{C}(k)_{\mathfrak{l}}$ as in Figure 2.14. We label them by $\mathcal{C}_I$, $\mathcal{C}_{II}$.

Figure 2.14: Two possibilities of $\mathcal{C}_{\mathfrak{l}}$.

(4) The cutting algorithm satisfies the requirement that the all connected components in (2)

generated in each step satisfy property P, where property P is defined below

Property P of a couple $\widetilde{\mathcal{C}}$: $\widetilde{\mathcal{C}}$ is connected and contains exactly one free leg and at least one fixed normal leg.

Remark 2.4.20. Although the definition of cut is rather general, only three special types of cuts, $c(\mathfrak{e})$, $c(\mathfrak{n})$ and $c(\mathfrak{l})$, are used in the cutting algorithm.

Remark 2.4.21. The couple $\mathcal{C}$ does not satisfy the property P because it does not have any free leg.

Proof of Proposition 2.4.19. Consider the following algorithm.

The cutting algorithm

Step 0. The input $\mathcal{C}(0)$ of this step is $\mathcal{C}$. In this step, we replace one of the two fixed legs of $\mathcal{C}$ by a free leg to obtain a new couple $\widehat{\mathcal{C}}$. By Lemma 3.3.18 (3), $\#Eq(\mathcal{C}) = \#Eq(\widehat{\mathcal{C}})$. The output $\mathcal{C}(1)$ of this step is $\widehat{\mathcal{C}}$. An example of step 0 can be found in the following picture.

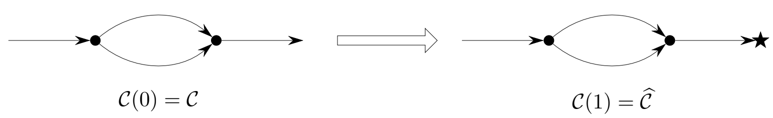


Figure 2.15: An example of step 0

Step k. Assume that the step $k-1$ have been finished. The input $\mathcal{C}(k)$ of step k is the output of step $k-1$. (If there are two output couples from step $k-1$, apply step k to these two couples separately.)

By property P, there exists a fixed normal leg in $\mathcal{C}(k)$. Choose one such leg $\mathfrak{l}$ and define $\mathcal{C}(k)_{\mathfrak{l}}$ to be the component which contains $\mathfrak{l}$ after cutting $c(\mathfrak{l})$ and define $\mathcal{C}(k)' = \mathcal{C}(k) \backslash \mathcal{C}(k)_{\mathfrak{l}}$. Check how many components does $\mathcal{C}(k)'$ have. Jump to case 1 if number of components equals to 1, otherwise jump to case 2. Examples of case 1 and case 2 can be found in the following picture.

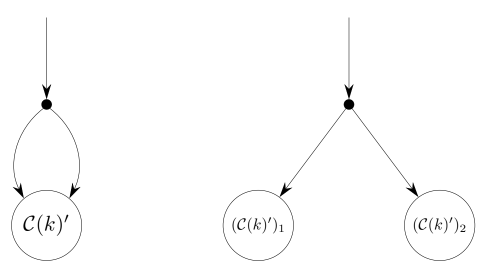


Figure 2.16: Examples of case 1 and case 2 (Left is case 1 and right is case two. $(\mathcal{C}(k)')_1$ and $(\mathcal{C}(k)')_2$ are two components of $\mathcal{C}(k)'$)

Case 1 of step k**.** In this case $\mathcal{C}(k)'$ has one components. We rename $\mathcal{C}(k)'$ to $\mathcal{C}(k)_1$. By property P, there exists a unique free leg $\mathfrak{l}_{fr}$ in $\mathcal{C}(k)$. Check if $\mathfrak{l}_{fr}$ and $\mathfrak{l}$ are connected to the same node. If yes, jump to case 1.1, otherwise jump to case 1.2.

Case 1.1. Cut edges in $c(\mathfrak{l})$ into $\{\mathfrak{e}_i^{(1)}\}_{i=1,2}$ and $\{\mathfrak{e}_i^{(2)}\}_{i=1,2}$, then $\mathcal{C}(k)$ is decomposed into $\mathcal{C}(k)_{\mathfrak{l}}$, $\mathcal{C}(k)_1 = \mathcal{C}\backslash\mathcal{C}_{\mathfrak{l}}$. As in Lemma 3.3.16, define $\{\mathfrak{e}_i^{(1)}\} \subseteq \mathcal{C}(k)_{\mathfrak{l}}$ to be free legs and $\{\mathfrak{e}_i^{(2)}\} \subseteq \mathcal{C}(k)_1$ to be fixed legs.

If $\mathcal{C}(k)_1$ satisfies the property P, define $\mathcal{C}(k+1) = \mathcal{C}(k)_1$ to be the output of step k and apply step $k+1$ to $\mathcal{C}(k+1)$.

Otherwise by Lemma 2.4.23 (2) there exist exactly one free normal leg and at least one fixed leg. By Lemma 3.3.18, we can define a new couple $\widehat{\mathcal{C}}$ such that the free normal leg becomes fixed and the fixed leg becomes free. Finally, define $\mathcal{C}(k+1) = \widehat{\mathcal{C}}$ to be the output.

Examples of cutting in case 1.1 can be found in the following picture.

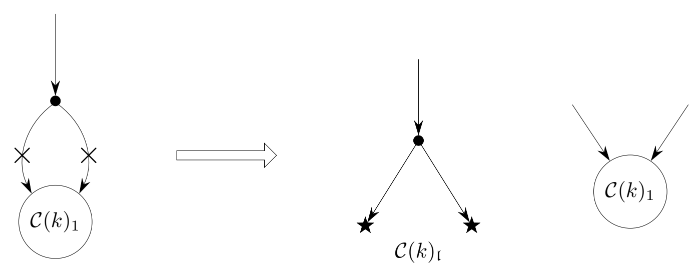


Figure 2.17: Examples of cutting in case 1.1

Case 1.2. In this case, $\mathfrak{l}_{fr}$ and $\mathfrak{l}$ are connected to the same node $\mathfrak{n}$. Let $\mathfrak{e}$ be the edge connecting $\mathfrak{n}$ and another interior node. Cut $\mathfrak{e}$ into $\{\mathfrak{e}^{(1)}, \mathfrak{e}^{(2)}\}$ and then $\mathcal{C}(k)$ is decomposed into $\mathcal{C}(k)_{\mathfrak{l}}$, $\mathcal{C}(k)_1 = \mathcal{C}(k)\backslash\mathcal{C}(k)_{\mathfrak{l}}$. Define $\mathfrak{e}^{(1)} \in \mathcal{C}(k)_{\mathfrak{l}}$ to be fixed legs and $\mathfrak{e}^{(2)} \in \mathcal{C}(k)_1$ to be free legs.

If $\mathcal{C}(k)_1$ satisfies the property P, define $\mathcal{C}(k+1) = \mathcal{C}(k)_1$.

Otherwise by Lemma 2.4.23 (2), $\mathfrak{e}^{(2)}$ is the only normal legs. Using Lemma 3.3.18 (2), we may construct a new couple $\mathcal{C}(k+1)$ by assigning $\mathfrak{e}^{(2)}$ to be fixed and another leg to be free. Finally define $\mathcal{C}(k+1)$ to be the output of step k and apply step $k+1$ to $\mathcal{C}(k+1)$.

Examples of cutting in case 1.2 can be found in the following picture.

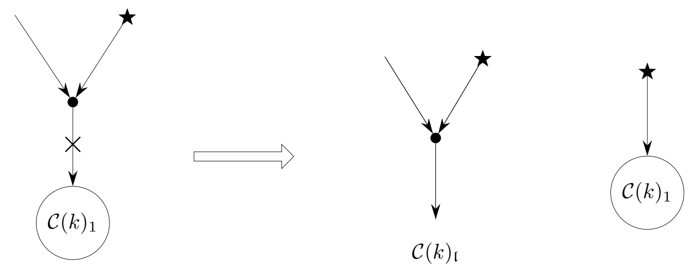


Figure 2.18: Examples of cutting in case 1.2

Case 2 of step k**.** Let the two connected components of $\mathcal{C}(k)\backslash\mathcal{C}(k)_\mathfrak{l}$ be $\mathcal{C}(k)_2$ and $\mathcal{C}(k)_3$. Let $\mathfrak{e}_2$, $\mathfrak{e}_3$ be the two edges that connect $\mathfrak{l}$ and $\mathcal{C}(k)_2$, $\mathcal{C}(k)_3$ respectively. Cut $\mathfrak{e}_2$, $\mathfrak{e}_3$ into $\{\mathfrak{e}_2^{(1)}, \mathfrak{e}_3^{(1)}\} \subseteq \mathcal{C}(k)_\mathfrak{l}$ and $\mathfrak{e}_2^{(2)} \in \mathcal{C}(k)_2$, $\mathfrak{e}_3^{(2)} \in \mathcal{C}(k)_3$. By Lemma 2.4.23, $\mathcal{C}(k)_2$, $\mathcal{C}(k)_3$ contain at least one normal leg and two legs. By symmetry, we can just consider $\mathcal{C}(k)_2$.

If $\mathcal{C}(k)_2$ contains free legs, define $\mathfrak{e}_2^{(2)} \in \mathcal{C}(k)_2$ to be fixed, otherwise define $\mathfrak{e}_2^{(2)}$ to be free.

In the case that $\mathcal{C}(k)_2$ contains free legs, define the output $\mathcal{C}(k+1)$ to be $\mathcal{C}(k)_2$ or $\mathcal{C}(k)_3$ and apply step $k+1$ to them separately.

In the case that $\mathcal{C}(k)_2$ contains no free legs, $\mathfrak{e}_2^{(2)}$ is the only normal legs and it is defined to be free. Use Lemma 3.3.18 (2) to construct a new couple $\widehat{\mathcal{C}}_2$ by assigning $\mathfrak{e}_2^{(2)}$ to be fixed and another leg to be free. Then define $\mathcal{C}(k+1)$ to be $\widehat{\mathcal{C}}_2$ or $\widehat{\mathcal{C}}_3$ and apply step $k+1$ to them separately.

Examples of cutting in case 2 can be found in the following picture.

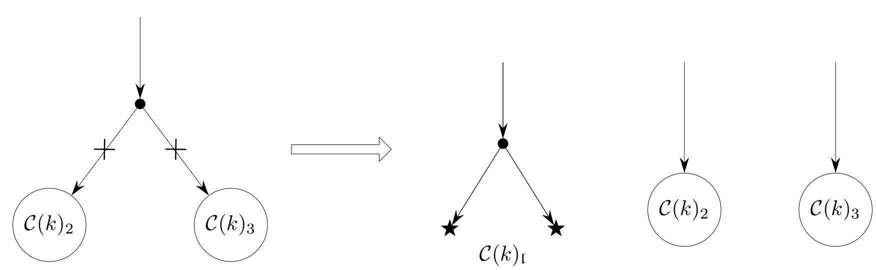


Figure 2.19: Examples of cutting in case 2

(1), (3) are true by definition. (2) is true because $\#Eq(\mathcal{C}) = \#Eq(\widehat{\mathcal{C}})$ as explained in step 0 of the algorithm. Since in step 0, we just replace a fixed leg by a free leg, $\mathcal{C}_{\text{norm}}$ should not change, so we get $\mathcal{C}_{\text{norm}}(1) = \mathcal{C}_{\text{norm}}$.

The only non-trivial part is (4), which is a corollary of Lemma 2.4.23 below.

Therefore, we complete the proof. $\square$

Lemma 2.4.22. *Given a connected couple* $\mathcal{C}$ *with multiple legs, then we have the following conclu-*

sions.

(1) Let $\{k_{\mathfrak{l}_i}\}_{i=1,\cdots,n_{leg}}$ *be the variables corresponding to legs in couple* $\mathcal{C}$*. Let* $\iota_{\mathfrak{e}}$ *be the same as* (2.4.3)*, then* $Eq(\mathcal{C})$ *implies the following momentum conservation equation*

$$\sum_{i=1}^{n_{leg}} \iota_{\mathfrak{l}_i} k_{\mathfrak{l}_i} = 0, \tag{2.4.51}$$

(2) Assume that there is exactly one free leg $\mathfrak{l}_{i_0}$ *in* $\mathcal{C}$ *and all other variables* $\{k_{\mathfrak{l}_i}\}_{i\neq i_0}$ *corresponding to fix legs are fixed to be constants* $\{c_{\mathfrak{l}_i}\}_{i\neq i_0}$*. For any* $i_1 = 1, \cdots, n_{leg}$*, we can construct a new couple* $\widehat{\mathcal{C}}$ *by replacing the* i_0 *leg by a fix leg and* i_1 *leg by a free leg. If* $i \neq i_0, i_1$*, fix* $k_{\mathfrak{l}_i}$ *to be the constant* $c_{\mathfrak{l}_i}$*, if* $i = i_0$*, fix* $k_{\mathfrak{l}_{i_0}}$ *to be the constant* $-\iota_{\mathfrak{l}_{i_0}} \sum_{i\neq i_0} \iota_{\mathfrak{l}_i} k_{\mathfrak{l}_i}$*. Under the above assumptions, we have*

$$Eq(\mathcal{C}, \{c_{\mathfrak{l}_i}\}_{i\neq i_0}) = Eq\left(\widehat{\mathcal{C}}, \{c_{\mathfrak{l}_i}\}_{i\neq i_0, i_1} \cup \{-\iota_{\mathfrak{l}_{i_0}} \sum_{i\neq i_0} \iota_{\mathfrak{l}_i} k_{\mathfrak{l}_i}\}\right). \tag{2.4.52}$$

(3) Assume that there is no free leg in $\mathcal{C}$ *and all* $\{k_{\mathfrak{l}_i}\}_{i\neq i_0}$ *are fixed to be constants* $\{c_{\mathfrak{l}_i}\}_{i\neq i_0}$*. For any* $i_1 = 1, \cdots, n_{leg}$*, we can construct a new couple* $\widehat{\mathcal{C}}$ *by replacing the* i_0 *leg by a free leg. Then we have*

$$Eq(\mathcal{C}, \{c_{\mathfrak{l}_i}\}_i) = Eq(\widehat{\mathcal{C}}, \{c_{\mathfrak{l}_i}\}_{i\neq i_0}). \tag{2.4.53}$$

(4) If the couple $\mathcal{C}$ *contains any leg, then it contains at least two legs.*

Proof. We first prove (1). Given a node $\mathfrak{n}$ and an edge $\mathfrak{e}$ connected to it, we define $\iota_{\mathfrak{e}}(\mathfrak{n})$ by the following rule

$$\iota_{\mathfrak{e}}(\mathfrak{n}) = \begin{cases} +1 & \textit{if } \mathfrak{e} \textit{ pointing towards } \mathfrak{n} \\ -1 & \textit{if } \mathfrak{e} \textit{ pointing outwards from } \mathfrak{n} \end{cases} \tag{2.4.54}$$

For a leg $\mathfrak{l}$, since it is connected to just one node, we may omit the $(\mathfrak{n})$ and just write $\iota_{\mathfrak{l}}$ as in the statement of the lemma.

For each node $\mathfrak{n}$, let $\mathfrak{e}_1(\mathfrak{n})$, $\mathfrak{e}_2(\mathfrak{n})$, $\mathfrak{e}(\mathfrak{n})$ be the three edges connected to it. For each edge $\mathfrak{e}$, let $\mathfrak{n}_1(\mathfrak{e})$, $\mathfrak{n}_2(\mathfrak{e})$ be the two nodes connected to it. Then we know that $\iota_{\mathfrak{e}}(\mathfrak{n}_1(\mathfrak{e})) + \iota_{\mathfrak{e}}(\mathfrak{n}_2(\mathfrak{e}))$, since $\mathfrak{n}_1(\mathfrak{e})$ and $\mathfrak{n}_2(\mathfrak{e})$ have the opposite direction.

Since $k_{\mathfrak{e}}$ satisfy $Eq(\mathcal{C})$, by (2.4.37), we get $\iota_{\mathfrak{e}_1(\mathfrak{n})}(\mathfrak{n}) k_{\mathfrak{e}_1(\mathfrak{n})} + \iota_{\mathfrak{e}_2(\mathfrak{n})}(\mathfrak{n}) k_{\mathfrak{e}_2(\mathfrak{n})} + \iota_{\mathfrak{e}(\mathfrak{n})}(\mathfrak{n}) k_{\mathfrak{e}(\mathfrak{n})} = 0.$

Summing over $\mathfrak{n}$ gives

$$
\begin{aligned}
0 &= \sum_{\mathfrak{n}\in\mathcal{C}} \iota_{\mathfrak{e}_1(\mathfrak{n})}(\mathfrak{n})k_{\mathfrak{e}_1(\mathfrak{n})} + \iota_{\mathfrak{e}_2(\mathfrak{n})}(\mathfrak{n})k_{\mathfrak{e}_2(\mathfrak{n})} + \iota_{\mathfrak{e}(\mathfrak{n})}(\mathfrak{n})k_{\mathfrak{e}(\mathfrak{n})} \\
&= \sum_{\mathfrak{e}\text{ is not a leg}} (\iota_{\mathfrak{e}}(\mathfrak{n}_1(\mathfrak{e})) + \iota_{\mathfrak{e}}(\mathfrak{n}_2(\mathfrak{e})))k_{\mathfrak{e}} + \sum_{\mathfrak{l}\text{ is a leg}} \iota_{\mathfrak{l}} k_{\mathfrak{l}} \\
&= \sum_{i=1}^{n_{\mathrm{leg}}} \iota_{\mathfrak{l}_i} k_{\mathfrak{l}_i}
\end{aligned}
\tag{2.4.55}
$$

This proves (2.4.51) and thus proves (1).

Now we prove (2). Since in $Eq\left(\widehat{\mathcal{C}}, \{c_{\mathfrak{l}_i}\}_{i\neq i_0,i_1} \cup \{-\iota_{\mathfrak{l}_{i_0}} \sum_{i\neq i_0} \iota_{\mathfrak{l}_i} k_{\mathfrak{l}_i}\}\right)$, $\{k_{\mathfrak{l}_i}\}_{i\neq i_0,i_1}$ are fixed to be constants $\{c_{\mathfrak{l}_i}\}_{i\neq i_0,i_1}$ and $k_{\mathfrak{l}_{i_0}}$ is fixed to be the constant $-\iota_{\mathfrak{l}_{i_0}} \sum_{i\neq i_0} \iota_{\mathfrak{l}_i} k_{\mathfrak{l}_i}$, by (2.4.51), we know that

$$
\iota_{\mathfrak{l}_{i_0}} \left(-\iota_{\mathfrak{l}_{i_0}} \sum_{i\neq i_0} \iota_{\mathfrak{l}_i} k_{\mathfrak{l}_i} \right) + \iota_{\mathfrak{l}_{i_1}} k_{\mathfrak{l}_{i_1}} + \sum_{i\neq i_0,i_1} \iota_{\mathfrak{l}_i} c_{\mathfrak{l}_i} = 0. \tag{2.4.56}
$$

This implies that $k_{\mathfrak{l}_{i_1}} = c_{\mathfrak{l}_{i_1}}$ in $Eq(\widehat{\mathcal{C}})$. Therefore, equations in $Eq(\widehat{\mathcal{C}})$ automatically imply $k_{\mathfrak{l}_{i_1}} = c_{\mathfrak{l}_{i_1}}$. Notice that whether or not containing $k_{\mathfrak{l}_{i_1}} = c_{\mathfrak{l}_{i_1}}$ is the only difference between $Eq(\mathcal{C})$ and $Eq(\widehat{\mathcal{C}})$. We conclude that $Eq(\mathcal{C}) = Eq(\widehat{\mathcal{C}})$. We thus complete the proof of (2).

The proof of (3) is similar to (2). Whether or not containing $k_{\mathfrak{l}_{i_0}} = c_{\mathfrak{l}_{i_0}}$ is the only difference between $Eq(\mathcal{C})$ and $Eq(\widehat{\mathcal{C}})$. But if $\{k_{\mathfrak{l}_i}\}_{i\neq i_0}$ are fixed to be constants $\{c_{\mathfrak{l}_i}\}_{i\neq i_0}$, by momentum conservation we know that

$$
k_{\mathfrak{l}_{i_0}} = -\iota_{\mathfrak{l}_{i_0}} \sum_{i\neq i_0} \iota_{\mathfrak{l}_i} c_{\mathfrak{l}_i}. \tag{2.4.57}
$$

Therefore, $k_{\mathfrak{l}_{i_0}}$ is fixed to be the constant $-\iota_{\mathfrak{l}_{i_0}} \sum_{i\neq i_0} \iota_{\mathfrak{l}_i} c_{\mathfrak{l}_i}$ in $Eq(\widehat{\mathcal{C}})$ and we conclude that $Eq(\mathcal{C}) = Eq(\widehat{\mathcal{C}})$. We thus complete the proof of (3).

If (4) is wrong, then $\mathcal{C}$ just has one leg $\mathfrak{l}$. By (2.4.51), $k_{\mathfrak{l}} = 0$. This contradicts with $k_{\mathfrak{l},x} \neq 0$ in (2.4.36). ☐

Lemma 2.4.23. *(1) The output $\mathcal{C}(k+1)$ of step k of the cutting algorithm satisfies the property P.*

(2) All the intermediate results $\mathcal{C}(k)_1$, $\mathcal{C}(k)_2$, $\mathcal{C}(k)_3$ satisfy the weak property P: either they satisfy the property P or they contain exactly one free normal leg and at least one fixed leg. Notice that the weak property implies that these couples contain at least one normal legs and two legs.

Proof. We prove the following stronger result by induction.

Claim. Assume that the couple $\mathcal{C} = \mathcal{C}(T, T, p)$ is the input the cutting algorithm. Then for any k, there exist a finite number of trees disjoint subtrees $T_1^{(k)}$, $T_2^{(k)}$, $\cdots$, $T_{m^{(k)}}^{(k)}$ of the two copies of T

which satisfy the following property.

(1) Let leaf(k) be the set of all leaves of these subtrees. Assume that p induce a pairing $p|_{\mathrm{leaf}_1(k)}$ of a subset $\mathrm{leaf}_1(k) \subseteq \mathrm{leaf}(k)$. Apply a similar construction to Definition 2.4.7 we can construct a couple from $T_1^{(k)}$, $T_2^{(k)}$, $\cdots$, $T_{m^{(k)}}^{(k)}$ from p and this couple equals exactly to $\mathcal{C}(k)$.

(2) The root legs of $T_1^{(k)}$, $T_2^{(k)}$, $\cdots$, $T_{m^{(k)}}^{(k)}$ are exactly the normal legs of $\mathcal{C}(k)$. The edges in $\mathrm{leaf}_2(k) = \mathrm{leaf}(k) \backslash \mathrm{leaf}_1(k)$ are exactly the leaf legs (legs that are leaf edges) of $\mathcal{C}(k)$.

(3) $\mathcal{C}(k)$ satisfies the property P.

(4) All the intermediate results $\mathcal{C}(k)_1$, $\mathcal{C}(k)_2$, $\mathcal{C}(k)_3$ satisfy the weak property P.

We show that $\mathcal{C}(0)$ satisfies (1) – (4) of the above claim.

Let $\mathcal{C}$ be the input couple of step 0 obtained by pairing two copies of T. In step 0, $\mathcal{C}(0)$ is obtained by replacing a fixed leg by a free leg. Therefore, if we define $T_1^{(0)} = T$ and $T_2^{(0)}$ to be the tree obtained by replacing the fixed root leg in T by a free leg, then $\mathcal{C}(0)$ is the couple obtained by pairing $T_1^{(0)}$ and $T_2^{(0)}$. Here the pairing is p and $\mathrm{leaf}_1(0) = \mathrm{leaf}(0)$ and $\mathrm{leaf}_2(0) = \emptyset$. Therefore, (1) is true for $\mathcal{C}(0)$.

The two fixed legs in $\mathcal{C}$ are all normal edges, because they come from the two root legs which belong to T_{in}. Therefore, the two legs in $\mathcal{C}(1)$ are also normal. Since the two legs in $\mathcal{C}(1)$ come from the root legs in $T_1^{(0)}$ and $T_2^{(0)}$, (2) is also true.

Since in step 0, we replace a fixed leg by a free leg, $\mathcal{C}(0)$ contains exactly one free and one fixed leg which are all normal. Therefore, the output $\mathcal{C}(0)$ of step 0 satisfies property P and (3) is proved.

In step 0, (4) does not need any proof.

Assume that $\mathcal{C}(k)$ satisfies (1) – (4) of the above claim, then we prove the same for $\mathcal{C}(k+1)$.

Remember that in step k of the algorithm, the input is $\mathcal{C}(k)$. The induction assumption implies that $\mathcal{C}(k)$ is obtained by pairing $T_1^{(k)}$, $T_2^{(k)}$, $\cdots$, $T_{m^{(k)}}^{(k)}$. There are several different cases in step k and we treat them separately.

In case 1.1 of the cutting algorithm, we cut $c(\mathfrak{l})$ into $\{\mathfrak{e}_i^{(1)}\}_{i=1,2}$ and $\{\mathfrak{e}_i^{(2)}\}_{i=1,2}$. By (2) $\mathfrak{l}$ is the root leg of some subtree $T_{j_0}^{(k)}$. Assume that in $T_{j_0}^{(k)}$ the two subtrees of the root are $T_{j_0,1}^{(k)}$, $T_{j_0,2}^{(k)}$. After cutting $c(\mathfrak{l})$, $T_{j_0}^{(k)}$ becomes two trees $T_{j_0,1}^{(k)}$, $T_{j_0,2}^{(k)}$ and $T_j^{(k)}$ ($j \neq j_0$) do not change. We treat three different case separately.

Case 1.1 (i). (Both $T_{j_0,1}^{(k)}$ and $T_{j_0,2}^{(k)}$ are one node trees.) In this case all edges in $\{\mathfrak{e}_1^{(1)}, \mathfrak{e}_2^{(1)}, \mathfrak{e}_1^{(2)}, \mathfrak{e}_2^{(2)}\}$ are leaf edges of some trees in $\{T_j^{(k)}\}$. Define $T_j^{(k+1)} = T_j^{(k)}$ for $j < j_0$ and $T_j^{(k+1)} = T_{j+1}^{(k)}$ for $j > j_0$, then $\mathcal{C}(k)_1$ can be constructed from these trees with $\mathrm{leaf}_1(k+1) = \mathrm{leaf}_1(k) \backslash \{\mathfrak{e}_1^{(1)}, \mathfrak{e}_2^{(1)}, \mathfrak{e}_1^{(2)}, \mathfrak{e}_2^{(2)}\}$. Remember that in the case 1.1 of the algorithm, the final result $\mathcal{C}(k+1)$ is obtained by replace (or not) some free normal leg by fixed normal leg in $\mathcal{C}(k)_1$. Therefore, if we change some free root leg of $\{T_j^{(k)}\}$ to be fixed, then $\mathcal{C}(k+1)$ can also be obtained from pairing these trees. Therefore, (1) is true for $\mathcal{C}(k+1)$ with $\mathrm{leaf}_1(k+1) = \mathrm{leaf}_1(k)\backslash\{\mathfrak{e}_1^{(1)}, \mathfrak{e}_2^{(1)}, \mathfrak{e}_1^{(2)}, \mathfrak{e}_2^{(2)}\}$.

Since both the sets of root legs and normal legs are deleted by one element $\mathfrak{l}$ and in step $k-1$ they are the same, they continue to be the same in step k. In case 1.1 (i), $\mathrm{leaf}_2(k+1) = \mathrm{leaf}_2(k)\cup\{\mathfrak{e}_i^{(2)}\}_{i=1,2}$ and the set of leaf legs in $\mathcal{C}(k)$ also changes in this way, so they continue to be the same in step k. Therefore, (2) is true.

We prove (4) by contradiction. The weak property P is equivalent to that $\mathcal{C}(k)_i$ contains exactly one free leg and at least one normal leg. In case 1.1 (i), $i = 1$, assume that the weak property P is not true, then either $\mathcal{C}(k)_1$ does not any contain normal leg or the number of free leg is not 1. If $\mathcal{C}(k)_1 \neq \emptyset$, then the number of $\{T_j^{(k)}\}$ is not zero. Because the root of $\{T_j^{(k)}\}$ are normal legs, there exists at least one normal leg in $\mathcal{C}(k)_1$. By our hypothesis that weak property P is wrong, the number of free leg is not 1. Since $\mathcal{C}(k)_1$ is obtained from $\mathcal{C}(k)$ by cutting a fixed leg, the number of free legs in $\mathcal{C}(k)_1$ equals to $\mathcal{C}(k)$ which is one. Therefore, we find a contradiction.

Notice that in the case 1.1 of the algorithm we change the free normal leg in $\mathcal{C}(k)_1$ to be fixed if $\mathcal{C}(k)_1$ does not satisfy the property P. The result $\mathcal{C}(k+1)$ must satisfy the property P, so we proves (3).

Case 1.1 (ii). (One of $T_{j_0,1}^{(k)}$ or $T_{j_0,2}^{(k)}$ is an one node tree.) Without loss of generality assume that $T_{j_0,2}^{(k)}$ is an one node tree. In this case, $\mathfrak{e}_2^{(1)}$, $\mathfrak{e}_2^{(2)}$ are leaf edges of some trees in $\{T_j^{(k)}\}$. Define $T_j^{(k+1)} = T_j^{(k)}$ for $j \neq j_0$ and $T_{j_0}^{(k+1)} = T_{j_0,1}^{(k)}$, then $\mathcal{C}(k)_1$ can be constructed from these trees with $\mathrm{leaf}_1(k+1) = \mathrm{leaf}_1(k)\backslash\{\mathfrak{e}_2^{(1)}, \mathfrak{e}_2^{(2)}\}$. By the same reason as in case 1.1 (i), (1) is true for $\mathcal{C}(k+1)$ with $\mathrm{leaf}_1(k+1) = \mathrm{leaf}_1(k)\backslash\{\mathfrak{e}_2^{(1)}, \mathfrak{e}_2^{(2)}\}$.

By the same reason as in case 1.1 (i), the set of root legs and the set of normal legs continue to be the same in step k. In case 1.1 (ii), $\mathrm{leaf}_2(k+1) = \mathrm{leaf}_2(k) \cup \{\mathfrak{e}_2^{(2)}\}$ and the set of leaf legs also changes in this way, so they continue to be the same in step k. Therefore, (2) is true.

The proof of (3), (4) in case 1.1 (ii) is the same as that in case 1.1 (i).

Case 1.1 (iii). (Neither $T_{j_0,1}^{(k)}$ or $T_{j_0,2}^{(k)}$ is an one node tree.) In this case, no edge in $\{\mathfrak{e}_1^{(1)}, \mathfrak{e}_2^{(1)}, \mathfrak{e}_1^{(2)}, \mathfrak{e}_2^{(2)}\}$ is leaf edge of $T_j^{(k)}$. Define $T_j^{(k+1)} = T_j^{(k)}$ for $j \leq j_0$, $T_j^{(k+1)} = T_{j-1}^{(k)}$ for $j \geq j_0 + 2$, $T_{j_0}^{(k+1)} = T_{j_0,1}^{(k)}$ and $T_{j_0+1}^{(k+1)} = T_{j_0,2}^{(k)}$, then $\mathcal{C}(k)_1$ can be constructed from these trees with $\mathrm{leaf}_1(k+1) = \mathrm{leaf}_1(k)$. By the same reason as in case 1.1 (i), (1) is true for $\mathcal{C}(k+1)$ with $\mathrm{leaf}_1(k+1) = \mathrm{leaf}_1(k)$.

By the same reason as in case 1.1 (i), the set of root legs and the set of normal legs continue to be the same in step k. In case 1.1 (iii), $\mathrm{leaf}_2(k+1) = \mathrm{leaf}_2(k)$ and the set of leaf legs also changes in this way, so they continue to be the same in step k. Therefore, (2) is true.

The proof of (3), (4) in case 1.1 (iii) is the same as that in case 1.1 (i).

In case 1.2 of the cutting algorithm, we cut $\mathfrak{e}$ into $\{\mathfrak{e}^{(1)}, \mathfrak{e}^{(2)}\}$. By (2), $\mathfrak{l}$ and $\mathfrak{l}_{fr}$ are the root leg and leaf of some subtrees $T_{j_0}^{(k)}$. Assume that in $T_{j_0}^{(k)}$ one subtree of the root is $T_{j_0,1}^{(k)}$ and the other one is a one node tree with edge $\mathfrak{l}_{fr}$. After cutting $c(\mathfrak{l})$, $T_{j_0}^{(k)}$ becomes $T_{j_0,1}^{(k)}$ and $T_j^{(k)}$ $(j \neq j_0)$ do not change. We treat two different case separately.

Case 1.2 (i). ($T_{j_0,1}^{(k)}$ is an one node trees.) In this case $\mathfrak{e}^{(1)}$, $\mathfrak{e}^{(2)}$ are leaf edges of some trees in $\{T_j^{(k)}\}$. Define $T_j^{(k+1)} = T_j^{(k)}$ for $j < j_0$ and $T_j^{(k+1)} = T_{j+1}^{(k)}$ for $j > j_0$, then $\mathcal{C}(k)_1$ can be constructed from these trees with $\mathrm{leaf}_1(k+1) = \mathrm{leaf}_1(k) \backslash \{\mathfrak{e}^{(1)}, \mathfrak{e}^{(2)}\}$. By the same reason as in case 1.1 (i), (1) is true for $\mathcal{C}(k+1)$ with $\mathrm{leaf}_1(k+1) = \mathrm{leaf}_1(k) \backslash \{\mathfrak{e}^{(1)}, \mathfrak{e}^{(2)}\}$.

By the same reason as in case 1.1 (i), the set of root legs and the set of normal legs continue to be the same in step k. In case 1.2 (i), $\mathrm{leaf}_2(k+1) = \mathrm{leaf}_2(k) \cup \{\mathfrak{e}^{(2)}\}$ and the set of leaf legs also changes in this way, so they continue to be the same in step k. Therefore, (2) is true.

The proof of (3), (4) in case 1.2 (i) is the same as that in case 1.1 (i).

Case 1.2 (ii). ($T_{j_0,1}^{(k)}$ is not an one node trees.) In this case $\mathfrak{e}^{(1)}$, $\mathfrak{e}^{(2)}$ are not leaf edges of any trees in $\{T_j^{(k)}\}$. Define $T_j^{(k+1)} = T_j^{(k)}$ for $j \neq j_0$ and $T_{j_0}^{(k+1)} = T_{j_0,1}^{(k)}$, then $\mathcal{C}(k)_1$ can be constructed from these trees with $\mathrm{leaf}_1(k+1) = \mathrm{leaf}_1(k)$. By the same reason as in case 1.1 (i), (1) is true for $\mathcal{C}(k+1)$ with $\mathrm{leaf}_1(k+1) = \mathrm{leaf}_1(k)$.

By the same reason as in case 1.1 (i), the set of root legs and the set of normal legs continue to be the same in step k. In case 1.2 (ii), $\mathrm{leaf}_2(k+1) = \mathrm{leaf}_2(k)$ and the set of leaf legs also changes in this way, so they continue to be the same in step k. Therefore, (2) is true.

The proof of (3), (4) in case 1.2 (ii) is the same as that in case 1.1 (i).

In case 2 of the cutting algorithm, we cut $c(\mathfrak{l}) = \{\mathfrak{e}_2, \mathfrak{e}_3\}$ into $\{\mathfrak{e}_2^{(1)}, \mathfrak{e}_3^{(1)}\}$ and $\{\mathfrak{e}_2^{(2)}, \mathfrak{e}_3^{(2)}\}$. By (2), $\mathfrak{l}$ is the root leg of some subtree $T_{j_0}^{(k)}$. Assume that in $T_{j_0}^{(k)}$ the two subtrees of the root are $T_{j_0,1}^{(k)}$, $T_{j_0,2}^{(k)}$. After cutting $c(\mathfrak{l})$, $T_{j_0}^{(k)}$ becomes two trees $T_{j_0,1}^{(k)}$, $T_{j_0,2}^{(k)}$ and $T_j^{(k)}$ $(j \neq j_0)$ do not change. Since $\mathcal{C}(k)_2$ and $\mathcal{C}(k)_3$ are disjoint, for each $j \neq j_0$, $T_j^{(k)}$ should be a subset of one of these couples. Without loss of generality we assume that $j_0 = 1$ and $T_2^{(k)}, \cdots, T_{m^{(k)\prime}}^{(k)} \subseteq \mathcal{C}(k)_2$ and $T_{m^{(k)\prime}+1}^{(k)}, \cdots, T_{m^{(k)}}^{(k)} \subseteq \mathcal{C}(k)_3$. Since the output $\mathcal{C}(k+1)$ either comes from $\mathcal{C}(k)_2$ or $\mathcal{C}(k)_3$, without loss of generality we assume that the output comes from $\mathcal{C}(k)_2$. We treat two different case separately.

Case 2 (i). ($T_{j_0,1}^{(k)}$ is an one node trees.) In this case, all edges in $\{\mathfrak{e}_2^{(1)}, \mathfrak{e}_2^{(2)}\}$ are leaf edges of some trees in $\{T_j^{(k)}\}$. Define $T_j^{(k+1)} = T_{j+1}^{(k)}$, then $\mathcal{C}(k)_2$ can be constructed from $T_2^{(k)}, \cdots, T_{m^{(k)\prime}}^{(k)}$ with $\mathrm{leaf}_1(k+1) = \{\text{all leaves in } T_2^{(k)}, \cdots, T_{m^{(k)\prime}}^{(k)}\}$. Remember that in the case 2 of the algorithm, the final result $\mathcal{C}(k+1)$ is obtained by replace (or not) some free normal leg by fixed normal leg in $\mathcal{C}(k)_2$, so by the same argument as in case 1.1 (i), (1) is true for $\mathcal{C}(k+1)$.

By the same reason as in case 1.1 (i), the set of root legs and the set of normal legs continue to be the same in step k. In case 2 (i), in the output $\mathcal{C}(k+1)$ coming from $\mathcal{C}(k)_2$, $\mathrm{leaf}_2(k+1) = (\mathrm{leaf}_1(k+1) \cap \mathcal{C}(k)_2) \cup \{\mathfrak{e}_2^{(2)}\}$ and the set of leaf legs also changes in this way, so they continue to be the same in step k. Therefore, (2) is true.

The proof of (3), (4) in case 2 (i) is the same as that in case 1.1 (i).

Case 2 (ii). ($T_{j_0,1}^{(k)}$ is not an one node trees.) In this case $\{\mathfrak{e}_2^{(1)}, \mathfrak{e}_2^{(2)}\}$ are not leaf edges of any trees in $\{T_j^{(k)}\}$. Define $T_j^{(k+1)} = T_j^{(k)}$ for $j > 1$ and $T_1^{(k+1)} = T_{j_0,1}^{(k)}$, then $\mathcal{C}(k)_2$ can be constructed from $T_1^{(k+1)}, \cdots, T_{m^{(k)\prime}}^{(k+1)}$ with $\mathrm{leaf}_1(k+1) = \{$all leaves in $T_1^{(k+1)}, \cdots, T_{m^{(k)\prime}}^{(k+1)}\}$. By the same reason as in case 1.1 (i), (1) is true for $\mathcal{C}(k+1)$.

By the same reason as in case 1.1 (i), the set of root legs and the set of normal legs continue to be the same in step k. In case 2 (ii), in the output $\mathcal{C}(k+1)$ coming from $\mathcal{C}(k)_2$, $\mathrm{leaf}_2(k+1) = (\mathrm{leaf}_1(k+1) \cap \mathcal{C}(k)_2)$ and the set of leaf legs also changes in this way, so they continue to be the same in step k. Therefore, (2) is true.

The proof of (3), (4) in case 2 (i) is the same as that in case 1.1 (i). □

Step 3. In this step, we state Proposition 3.3.24 which is a stronger version of Proposition 3.3.22 and derive Proposition 3.3.22 from it. In the end, we prove part (1), (2) and the 1 node case of part (3) of Proposition 3.3.24.

Proposition 2.4.24. *Let $\mathcal{C}(k)$ be a couple which is the output of step k of the cutting algorithm Proposition 2.4.19. For any couple $\mathcal{C}$, let $n(\mathcal{C})$ be the total number of nodes in $\mathcal{C}$ and $n_e(\mathcal{C})$ (resp. $n_{fx}(\mathcal{C})$, $n_{fr}(\mathcal{C})$) be the total number of non-leg edges (resp. fixed legs, free legs). We fix $\sigma_{\mathfrak{n}} \in \mathbb{R}$ for each $\mathfrak{n} \in \mathcal{C}(k)$ and $c_{\mathfrak{l}} \in \mathbb{R}$ for each fixed leg $\mathfrak{l}$. Assume that α satisfies* (2.1.2)*. Then we have*

(1) We have following relation of $n(\mathcal{C})$, $n_e(\mathcal{C})$, $n_{fx}(\mathcal{C})$ and $n_{fr}(\mathcal{C})$

$$2n_e(\mathcal{C}) + n_{fx}(\mathcal{C}) + n_{fr}(\mathcal{C}) = 3n(\mathcal{C}) \tag{2.4.58}$$

(2) For any couple $\mathcal{C}$, define $\chi(\mathcal{C}) = n_e(\mathcal{C}) + n_{fr}(\mathcal{C}) - n(\mathcal{C})$. Let c, $\mathcal{C}_1$, $\mathcal{C}_2$ be the same as in Lemma 3.3.16 and also assume that $\{\mathfrak{e}_i^{(1)}\}$ are free legs and $\{\mathfrak{e}_i^{(2)}\}$ are fixed legs. Then

$$\chi(\mathcal{C}) = \chi(\mathcal{C}_1) + \chi(\mathcal{C}_2). \tag{2.4.59}$$

(3) If we assume that $\mathcal{C}(k)$ satisfies the property P, then (recall that $Q = L^d T_{max}^{-1}$)

$$\sup_{\{c_{\mathfrak{l}}\}_{\mathfrak{l}}} \#Eq(\mathcal{C}(k), \{c_{\mathfrak{l}}\}_{\mathfrak{l}}) \le L^{O(\chi(\mathcal{C}(k))\theta)} Q^{\chi(\mathcal{C}(k))} \prod_{\mathfrak{e}\in\mathcal{C}_{norm}(k)} \kappa_{\mathfrak{e}}^{-1}. \tag{2.4.60}$$

Derivation of Proposition 3.3.22 from Proposition 3.3.24: By Proposition 2.4.19 (2), $\mathcal{C}_{\text{norm}}(1) = \mathcal{C}_{\text{norm}}$ and $\#Eq(\mathcal{C}) = \#Eq(\mathcal{C}(1))$. Then by Proposition 3.3.24 (3),

$$\#Eq(\mathcal{C}) = \#Eq(\mathcal{C}(1)) \le L^{O(\chi(\mathcal{C}(1))\theta)} Q^{\chi(\mathcal{C}(1))} \prod_{\mathfrak{e}\in\mathcal{C}_{\text{norm}}(1)} \kappa_{\mathfrak{e}}^{-1} \tag{2.4.61}$$

Since in $\mathcal{C}(1)$, $n_{fx}(\mathcal{C}(1)) = n_{fr}(\mathcal{C}(1)) = 1$, by Proposition 3.3.24 (1), $2n_e(\mathcal{C}(1)) + 2 = 3n(\mathcal{C}(1))$. Using this fact and the definition of χ, we get $\chi(\mathcal{C}(1)) = n_e(\mathcal{C}(1)) + 1 - n(\mathcal{C}(1)) = n(\mathcal{C}(1))/2$. Substituting this expression of $\chi(\mathcal{C}(1))$ into (2.4.61) proves the conclusion of Proposition 3.3.22.

The proof of Proposition 3.3.24 is the main goal of the rest of the proof.

Proof Proposition 3.3.24 (1): Consider the set $\mathcal{S} = \{(\mathfrak{n}, \mathfrak{e}) \in \mathcal{C} : \mathfrak{n} \textit{ is an end point of } \mathfrak{e}\}$, then

$$\#\mathcal{S} = \sum_{\substack{(\mathfrak{n},\mathfrak{e})\in\mathcal{C} \\ \mathfrak{n} \textit{ is an end point of } \mathfrak{e}}} 1 \tag{2.4.62}$$

First sum over $\mathfrak{e}$ and then over $\mathfrak{n}$, we get

$$\#\mathcal{S} = \sum_{\mathfrak{n}} \sum_{\substack{\mathfrak{e}\in\mathcal{C} \\ \mathfrak{n} \textit{ is an end point of } \mathfrak{e}}} 1 = \sum_{\mathfrak{n}} 3 = 3n(\mathcal{C}). \tag{2.4.63}$$

In the second equality, $\sum_{\mathfrak{e}\in\mathcal{C}:\mathfrak{n} \textit{ is an end point of } \mathfrak{e}} 1 = 3$ because for each node $\mathfrak{n}$ there are 3 edges connected to it.

Switch the order of summation, we get

$$\begin{aligned} \#\mathcal{S} =& \sum_{\mathfrak{e} \textit{ is a non-leg edges}} \sum_{\substack{\mathfrak{n}\in\mathcal{C} \\ \mathfrak{n} \textit{ is an end point of } \mathfrak{e}}} 1 + \sum_{\mathfrak{e} \textit{ is a leg}} \sum_{\substack{\mathfrak{n}\in\mathcal{C} \\ \mathfrak{n} \textit{ is an end point of } \mathfrak{e}}} 1 \\ =& \sum_{\mathfrak{e} \textit{ is a non-leg edges}} 2 + \sum_{\mathfrak{e} \textit{ is a leg}} 1 \\ =& 2n_e(\mathcal{C}) + n_{fx}(\mathcal{C}) + n_{fr}(\mathcal{C}) \end{aligned} \tag{2.4.64}$$

In the second equality, $\sum_{\substack{\mathfrak{n}\in\mathcal{C} \\ \mathfrak{n} \textit{ is an end point of } \mathfrak{e}}} 1$ equals to 1 or 2 because for each non-leg edge (resp. leg) there are 2 (resp. 1) nodes connected to it.

Because the value of $\#\mathcal{S}$ does not depend on the order of summation, we conclude that $2n_e(\mathcal{C}) +$

$n_{fx}(\mathcal{C}) + n_{fr}(\mathcal{C}) = 3n(\mathcal{C})$, which proves Proposition 3.3.24 (1).

Proof Proposition 3.3.24 (2): Since cutting does change the number of nodes, we have $n(\mathcal{C}) = n(\mathcal{C}_1) + n(\mathcal{C}_2)$. Let c be the cut that consists of edges $\{\mathfrak{e}_i\}$ and $n(c)$ be the number of edges in c. When cutting $\mathcal{C}$ into $\mathcal{C}_1$ and $\mathcal{C}_2$, $n(c)$ non-leg edges are cut into pairs of free and fixed legs, so $n_e(\mathcal{C}) = n(\mathcal{C}_1) + n(\mathcal{C}_2) + n(c)$. Because we have $n(c)$ additional free legs after cutting, so $n_{fr}(\mathcal{C}) = n_{fr}(\mathcal{C}_1) + n_{fr}(\mathcal{C}_2) - n(c)$. Therefore, we get

$$\begin{aligned}
\chi(\mathcal{C}) =& n_e(\mathcal{C}) + n_{fr}(\mathcal{C}) - n(\mathcal{C}) \\
=& n(\mathcal{C}_1) + n(\mathcal{C}_2) + n(c) + n_{fr}(\mathcal{C}_1) + n_{fr}(\mathcal{C}_2) - n(c) - (n(\mathcal{C}_1) + n(\mathcal{C}_2)) \\
=& (n(\mathcal{C}_1) + n_{fr}(\mathcal{C}_1) - n(\mathcal{C}_1)) + (n(\mathcal{C}_2) + n_{fr}(\mathcal{C}_2) - n(\mathcal{C}_2)) \\
=& \chi(\mathcal{C}_1) + \chi(\mathcal{C}_2).
\end{aligned} \tag{2.4.65}$$

We complete the proof of (2) of Proposition 3.3.24.

Proof of 1 node case of Proposition 3.3.24 (3): We prove the following Lemma 3.3.25 which is the 1 node case of Proposition 3.3.24 (3). Recall that $\mathcal{C}_I$ and $\mathcal{C}_{II}$ are the two possibilities of the 1 node couple $\mathcal{C}_{\mathfrak{l}}$ in Proposition 2.4.19 (3).

Lemma 2.4.25. *$\mathcal{C}_I$, $\mathcal{C}_{II}$ satisfy the bound* (2.4.60) *in Proposition 3.3.24. In other words, fix c_1 (resp. c_1, c_2) for the fixed legs of $\mathcal{C}_I$ (resp. $\mathcal{C}_{II}$), then we have*

$$\#Eq(\mathcal{C}_I) \leq L^\theta Q \kappa_{\mathfrak{e}}^{-1}, \qquad \#Eq(\mathcal{C}_{II}) \leq Q^0 = 1. \tag{2.4.66}$$

Proof. Given c_1, c_2, the equation of $\mathcal{C}_{II}$ is

$$\begin{cases} k_{\mathfrak{e}_1} + k_{\mathfrak{e}_2} - k_{\mathfrak{e}} = 0,\ k_{\mathfrak{e}_1} = c_1,\ k_{\mathfrak{e}} = c_2,\ |k_{\mathfrak{e}x}| \sim \kappa_{\mathfrak{e}} \\ \Lambda_{k_{\mathfrak{e}_1}} + \Lambda_{k_{\mathfrak{e}_2}} - \Lambda_{k_{\mathfrak{e}}} = \sigma_{\mathfrak{n}} + O(T_{\max}^{-1}) \end{cases} \tag{2.4.67}$$

It's obvious that there is at most one solution to this system of equations.

Given c_1, the equation of $\mathcal{C}_I$ is

$$\begin{cases} k_{\mathfrak{e}_1} + k_{\mathfrak{e}_2} - k_{\mathfrak{e}} = 0,\ k_{\mathfrak{e}} = c_1,\ |k_{\mathfrak{e}x}| \sim \kappa_{\mathfrak{e}} \\ \Lambda_{k_{\mathfrak{e}_1}} + \Lambda_{k_{\mathfrak{e}_2}} - \Lambda_{k_{\mathfrak{e}}} = \sigma_{\mathfrak{n}} + O(T_{\max}^{-1}) \end{cases} \tag{2.4.68}$$

By Theorem 4.3.1, the number of solutions of the above system of equations can be bounded by $L^\theta L^d T_{\max}^{-1} |k_{\mathfrak{e}x}|^{-1} \lesssim L^\theta Q \kappa_{\mathfrak{e}}^{-1}$

Therefore, we complete the proof of this lemma. $\square$

Step 4. In this step, we apply the edge cutting algorithm Proposition 2.4.19 to prove Proposition 3.3.24 (3) by induction.

If $\mathcal{C}$ has only one node ($n = 1$), then $\mathcal{C}$ equals to $\mathcal{C}_I$ or $\mathcal{C}_{II}$ and Proposition 3.3.24 (3) in this case follows from Lemma 3.3.25.

Suppose that Proposition 3.3.24 (3) holds true for couples with number of nodes $\leq n-1$. We prove it for couples with number of nodes n.

Given the couple $\mathcal{C}(k)$ which is the output of the $k-1$-th step, apply the cutting algorithm to $\mathcal{C}(k)$. Then according to Proposition 2.4.19, $\mathcal{C}(k)$ (2) is decomposed into 2 or 3 components.

In the first case, denote by $\mathcal{C}(k)_{\mathfrak{l}}$ and $\mathcal{C}(k)_1$ the two components after cutting. In the second case, denote by $\mathcal{C}(k)_{\mathfrak{l}}$, $\mathcal{C}(k)_2$ and $\mathcal{C}(k)_3$ the three components after cutting. By Proposition 2.4.19 (4), $\mathcal{C}(k)_1$, $\mathcal{C}(k)_2$, $\mathcal{C}(k)_3$ are couples of nodes $\leq n-1$ that satisfy the property P and thus satisfy the assumption of Proposition 3.3.24 (3). Therefore, the induction assumption is applicable and Proposition 3.3.24 (3) is true for these three couples.

Case 1. Assume that there are two components after cutting. One example of this case is the left couple in Figure 2.16.

Then applying Lemma 3.3.16 gives

$$\begin{aligned}
\sup_{\{c_{\mathfrak{l}}\}_{\mathfrak{l}}} \#Eq(\mathcal{C}(k), \{c_{\mathfrak{l}}\}_{\mathfrak{l}}) \leq & \sup_{\{c_{\mathfrak{l}_1}\}_{\mathfrak{l}_1 \in \mathrm{leg}(\mathcal{C}(k)_{\mathfrak{l}})}} \#Eq(\mathcal{C}(k)_{\mathfrak{l}}, \{c_{\mathfrak{l}_1}\}) \sup_{\{c_{\mathfrak{l}_2}\}_{\mathfrak{l}_2 \in \mathrm{leg}(\mathcal{C}(k)_1)}} \#Eq(\mathcal{C}(k)_1, \{c_{\mathfrak{l}_2}\}) \\
\lesssim & L^{\theta} Q^{\chi(\mathcal{C}_1)} L^{O(\chi(\mathcal{C}_2)\theta)} Q^{\chi(\mathcal{C}_2)} = L^{O((\chi(\mathcal{C}_1)+\chi(\mathcal{C}_2))\theta)} Q^{\chi(\mathcal{C}_1)+\chi(\mathcal{C}_2)} \\
= & L^{O(\chi(\mathcal{C})\theta)} Q^{\chi(\mathcal{C})}.
\end{aligned} \tag{2.4.69}$$

Here the second inequality follows from induction assumption and Lemma 3.3.25. The last equality follows from Proposition 3.3.24 (2).

Case 2. Assume that there are three components after cutting. One example of this case is the right couple in Figure 2.16.

Then applying Lemma 3.3.16 gives

$$\begin{aligned}\sup_{\{c_{\mathfrak{l}}\}_{\mathfrak{l}}} \#Eq(\mathcal{C}(k),\{c_{\mathfrak{l}}\}_{\mathfrak{l}}) \le & \sup_{\{c_{\mathfrak{l}_1}\}_{\mathfrak{l}_1\in \mathrm{leg}(\mathcal{C}(k)\backslash\mathcal{C}(k)_2)}} \#Eq(\mathcal{C}(k)\backslash\mathcal{C}(k)_2,\{c_{\mathfrak{l}_1}\}) \sup_{\{c_{\mathfrak{l}_2}\}_{\mathfrak{l}_2\in \mathrm{leg}(\mathcal{C}(k)_2)}} \#Eq(\mathcal{C}(k)_2,\{c_{\mathfrak{l}_2}\}) \\ \lesssim & L^{O(\chi(\mathcal{C}(k)_2)\theta)} Q^{\chi(\mathcal{C}(k)_2)} \sup_{\{c_{\mathfrak{l}_1}\}_{\mathfrak{l}_1\in \mathrm{leg}(\mathcal{C}(k)\backslash\mathcal{C}(k)_2)}} \#Eq(\mathcal{C}(k)\backslash\mathcal{C}(k)_2,\{c_{\mathfrak{l}_1}\}).\end{aligned} \tag{2.4.70}$$

Here in the second inequality, we apply the induction assumption to $\mathcal{C}(k)_2$.

Applying Lemma 3.3.16 and (2.4.70) gives

$$\begin{aligned}&\sup_{\{c_{\mathfrak{l}}\}_{\mathfrak{l}}} \#Eq(\mathcal{C}(k),\{c_{\mathfrak{l}}\}_{\mathfrak{l}}) \lesssim L^{O(\mathcal{C}(k)_2\theta)} Q^{\chi(\mathcal{C}(k)_2)} \sup_{\{c_{\mathfrak{l}_1}\}_{\mathfrak{l}_1\in \mathrm{leg}(\mathcal{C}(k)\backslash\mathcal{C}(k)_2)}} \#Eq(\mathcal{C}(k)\backslash\mathcal{C}(k)_2,\{c_{\mathfrak{l}_1}\}) \\ \lesssim & L^{O(\chi(\mathcal{C}(k)_2)\theta)} Q^{\chi(\mathcal{C}(k)_2)} \sup_{\{c_{\mathfrak{l}_1}\}_{\mathfrak{l}_1\in \mathrm{leg}(\mathcal{C}(k)_{\mathfrak{l}})}} \#Eq(\mathcal{C}(k)_{\mathfrak{l}},\{c_{\mathfrak{l}_1}\}) \sup_{\{c_{\mathfrak{l}_3}\}_{\mathfrak{l}_3\in \mathrm{leg}(\mathcal{C}(k)_3)}} \#Eq(\mathcal{C}(k)_3,\{c_{\mathfrak{l}_3}\}) \\ \lesssim & L^{O((\chi(\mathcal{C}(k)_2)+\chi(\mathcal{C}(k)_3))\theta)} Q^{\chi(\mathcal{C}(k)_2)+\chi(\mathcal{C}(k)_3)} \sup_{\{c_{\mathfrak{l}_1}\}_{\mathfrak{l}_1\in \mathrm{leg}(\mathcal{C}(k)_{\mathfrak{l}})}} \#Eq(\mathcal{C}(k)_{\mathfrak{l}},\{c_{\mathfrak{l}_1}\}) \\ \lesssim & (L^{O(\theta)}Q)^{\chi(\mathcal{C}(k)_2)+\chi(\mathcal{C}(k)_3)+\chi(\mathcal{C}(k)_{\mathfrak{l}})} = L^{O(\chi(\mathcal{C})\theta)} Q^{\chi(\mathcal{C})}.\end{aligned} \tag{2.4.71}$$

Here in the third inequality, we apply the induction assumption to $\mathcal{C}(k)_3$. The last equality follows from Proposition 3.3.24 (2).

Therefore, we complete the proof of Proposition 3.3.24 and thus the proof of Proposition 3.3.22.

□

2.4.5 An upper bound of tree terms

In this section, we first prove the following Proposition which gives an upper bound of the variance of $\mathcal{J}_{T,k}$ and then prove Proposition 3.2.7 as its corollary.

Proposition 2.4.26. *Assume that α satisfies (2.1.2) and $\rho = \alpha T_{max}^{\frac{1}{2}}$. For any $\theta > 0$, we have*

$$\sup_k \mathbb{E}|(\mathcal{J}_T)_k|^2 \lesssim L^{O(l(T)\theta)} \rho^{2l(T)}. \tag{2.4.72}$$

and $\mathbb{E}|(\mathcal{J}_T)_k|^2 = 0$ if $|k| \gtrsim 1$.

Proof. By (2.4.33) and (2.4.34), we know that $\mathcal{J}_{T,k}$ is a linear combination of $Term(T,p)_k$.

$$\mathbb{E}|\mathcal{J}_{T,k}|^2 = \left(\frac{\lambda}{L^d}\right)^{2l(T)} \sum_{p\in\mathcal{P}(\{k_1,\cdots,k_{l(T)+1},k_1',\cdots,k_{l(T)+1}'\})} Term(T,p)_k. \tag{2.4.73}$$

Since $\alpha = \frac{\lambda}{L^{\frac{d}{2}}}$, $\frac{\lambda}{L^d} = \alpha L^{-\frac{d}{2}}$. Since the number of elements in $\mathcal{P}$ can be bounded by a constant, by Lemma 2.4.27 proved below, we get

$$\mathbb{E}|\mathcal{J}_{T,k}|^2 \lesssim (\alpha L^{-\frac{d}{2}})^{2l(T)} L^{O(n\theta)} Q^{\frac{n}{2}} T_{\max}^n. \tag{2.4.74}$$

By definition, n is the total number of $\bullet$ nodes in the couple constructed from tree T and pairing p. Therefore, n equals to $2l(T)$. Replacing n by $2l(T)$ and Q by $L^d T_{\max}^{-1}$ in (2.4.74), we get

$$\begin{aligned}\mathbb{E}|\mathcal{J}_{T,k}|^2 \lesssim &(\alpha L^{-\frac{d}{2}})^{2l(T)} L^{O(l(T)\theta)} (L^d T_{\max}^{-1})^{l(T)} T_{\max}^{2l(T)} \\ =&L^{O(l(T)\theta)} (\alpha^2 T_{\max})^{l(T)} \\ =&L^{O(l(T)\theta)} \rho^{2l(T)}.\end{aligned} \tag{2.4.75}$$

By Lemma 2.4.27 below $Term(T,p)_k = 0$ if $|k| \gtrsim 1$, we know that the same is true for $\mathbb{E}|(\mathcal{J}_T)_k|^2 = 0$.

Therefore, we complete the proof of this proposition. □

Lemma 2.4.27. *Let n and $Q = L^d T_{max}^{-1}$ be the same as in Proposition 3.3.22. Assume that α satisfies* (2.1.2) *and $n_{\text{in}} \in C_0^\infty(\mathbb{R}^d)$ is compactly supported. Let $\mathcal{C}$ be the couple constructed from tree T and pairing p. Then for any $\theta > 0$, we have*

$$\sup_k |Term(T,p)_k| \leq L^{O(n\theta)} Q^{\frac{n}{2}} T_{max}^n. \tag{2.4.76}$$

and $Term(T,p)_k = 0$ if $|k| \gtrsim 1$.

Proof. By (2.4.33), we get

$$\begin{aligned}Term(T,p)_k = &\sum_{k_1,\, k_2,\, \cdots,\, k_{l(T)+1}} \sum_{k_1',\, k_2',\, \cdots,\, k_{l(T)+1}'} H^T_{k_1 \cdots k_{l(T)+1}} H^T_{k_1' \cdots k_{l(T)+1}'} \\ &\delta_p(k_1, \cdots, k_{l(T)+1}, k_1', \cdots, k_{l(T)+1}') \sqrt{n_{\text{in}}(k_1)} \cdots \sqrt{n_{\text{in}}(k_1')} \cdots\end{aligned} \tag{2.4.77}$$

Since n_{in} are compactly supported and there are bounded many of them in $Term(T,p)_k$, by $k_1 + k_2 + \cdots + k_{l(T)+1} = k$, we know that $Term(T,p)_k = 0$ if $|k| \gtrsim 1$.

By (2.4.16), we get

$$|H^T_{k_1 \cdots k_{l+1}}| \lesssim \sum_{\{d_\mathfrak{n}\}_{\mathfrak{n} \in T_{\text{in}}} \in \{0,1\}^{l(T)}} \prod_{\mathfrak{n} \in T_{\text{in}}} \frac{1}{|q_\mathfrak{n}| + T_{\max}^{-1}} \prod_{\mathfrak{e} \in T_{\text{in}}} |k_{\mathfrak{e},x}| \; \delta_{\cap_{\mathfrak{n} \in T_{\text{in}}} \{S_\mathfrak{n} = 0\}}. \tag{2.4.78}$$

By (2.4.17), $q_{\mathfrak{n}}$ is a linear combination of $\Omega_{\mathfrak{n}}$, so there exist constants $c_{\mathfrak{n},\widetilde{\mathfrak{n}}}$ such that $q_{\mathfrak{n}} = \sum_{\widetilde{\mathfrak{n}}} c_{\mathfrak{n},\widetilde{\mathfrak{n}}}\Omega_{\widetilde{\mathfrak{n}}}$. Let c be the matrix $[c_{\mathfrak{n},\widetilde{\mathfrak{n}}}]$ and $\mathscr{M}(T)$ be the set of all possible such matrices, then the number of elements in $\mathscr{M}(T)$ can be bounded by a constant. Let $c(\Omega)$ be the vector $\{\sum_{\widetilde{\mathfrak{n}}} c_{\mathfrak{n},\widetilde{\mathfrak{n}}}\Omega_{\widetilde{\mathfrak{n}}}\}_{\mathfrak{n}}$ and $c(\Omega)_{\mathfrak{n}} = \sum_{\widetilde{\mathfrak{n}}} c_{\mathfrak{n},\widetilde{\mathfrak{n}}}\Omega_{\widetilde{\mathfrak{n}}}$ be the components of $c(\Omega)$.

With this notation, we know that $q_{\mathfrak{n}} = c(\Omega)_{\mathfrak{n}}$ and the right hand side of (2.4.78) becomes $\sum_{c\in\mathscr{M}(T)}\prod_{\mathfrak{n}\in T_{\text{in}}}\frac{1}{|c(\Omega)_{\mathfrak{n}}|+T_{\max}^{-1}}$. Therefore, using the fact that n_{in} are compactly supported, we have

$$\begin{aligned}|Term(T,p)_k| \lesssim & \sum_{\substack{k_1,\cdots,k_{l(T)+1},k'_1,\cdots,k'_{l(T)+1}\\ |k_j|,|k'_j|\lesssim 1,\forall j}} \sum_{c\in\mathscr{M}(T)}\prod_{\mathfrak{n}\in T_{\text{in}}}\frac{1}{|c(\Omega)_{\mathfrak{n}}|+T_{\max}^{-1}}\prod_{\mathfrak{e}\in T_{\text{in}}}|k_{\mathfrak{e},x}|\ \delta_{\cap_{\mathfrak{n}\in T_{\text{in}}}\{S_{\mathfrak{n}}=0\}}\\ & \sum_{c'\in\mathscr{M}(T)}\prod_{\mathfrak{n}'\in T_{\text{in}}}\frac{1}{|c'(\Omega)_{\mathfrak{n}'}|+T_{\max}^{-1}}\prod_{\mathfrak{e}'\in T_{\text{in}}}|k_{\mathfrak{e}',x}|\ \delta_{\cap_{\mathfrak{n}'\in T_{\text{in}}}\{S_{\mathfrak{n}'}=0\}}\delta_p(k_1,\cdots,k_{l(T)+1},k'_1,\cdots,k'_{l(T)+1})\end{aligned} \tag{2.4.79}$$

Switch the order of summations and products in (2.4.79), then we get

$$\begin{aligned}|Term(T,p)_k| \lesssim & \sum_{\substack{k_1,\cdots,k_{l(T)+1},k'_1,\cdots,k'_{l(T)+1}\\ |k_j|,|k'_j|\lesssim 1,\forall j}} \sum_{c,c'\in\mathscr{M}(T)}\prod_{\mathfrak{n},\mathfrak{n}'\in T_{\text{in}}}\frac{1}{|c(\Omega)_{\mathfrak{n}}|+T_{\max}^{-1}}\\ & \frac{1}{|c'(\Omega)_{\mathfrak{n}'}|+T_{\max}^{-1}}\prod_{\mathfrak{e},\mathfrak{e}'\in T_{\text{in}}}(|k_{\mathfrak{e},x}||k_{\mathfrak{e}',x}|)\ \delta_{\cap_{\mathfrak{n},\mathfrak{n}'\in T_{\text{in}}}\{S_{\mathfrak{n}}=0,S_{\mathfrak{n}'}=0\}}\delta_p(k_1,\cdots,k_{l(T)+1},k'_1,\cdots,k'_{l(T)+1}).\end{aligned} \tag{2.4.80}$$

Given a tree T and pairing p, we can construct a couple $\mathcal{C}$. We show that

$$\sum_{c,c'\in\mathscr{M}(T)} = \sum_{c\in\mathscr{M}(\mathcal{C})}, \qquad \prod_{\mathfrak{n},\mathfrak{n}'\in T_{\text{in}}} = \prod_{\mathfrak{n}\in\mathcal{C}}, \qquad \prod_{\mathfrak{e},\mathfrak{e}'\in T_{\text{in}}} = \prod_{\mathfrak{e}\in\mathcal{C}_{\text{norm}}}, \qquad \cap_{\mathfrak{n},\mathfrak{n}'\in T_{\text{in}}} = \cap_{\mathfrak{n}\in\mathcal{C}}. \tag{2.4.81}$$

Remember that $\mathcal{C}$ is constructed by glueing two copies of T by p. In (2.4.80), $\mathfrak{n}$, $\mathfrak{e}$, $\mathfrak{n}'$, $\mathfrak{e}'$ denote nodes and edges in the first or second copy respectively. Since all nodes in $\mathcal{C}$ come from the two copies of T, we get $\prod_{\mathfrak{n},\mathfrak{n}'\in T_{\text{in}}} = \prod_{\mathfrak{n}\in\mathcal{C}}$ and $\cap_{\mathfrak{n},\mathfrak{n}'\in T_{\text{in}}} = \cap_{\mathfrak{n}\in\mathcal{C}}$. Remember that T_{in} is the tree formed by all non-leaf nodes, so edges of $\mathcal{C}_{\text{norm}}$ all come from the two copies of T_{in}. Therefore $\prod_{\mathfrak{e},\mathfrak{e}'\in T_{\text{in}}} = \prod_{\mathfrak{e}\in\mathcal{C}_{\text{norm}}}$. Given two matrix c, c', we can construct a new matrix $c\oplus c'$ as the following. Consider two vectors $\Omega = \{\Omega_{\mathfrak{n}}\}_{\mathfrak{n}'\in T}$, $\Omega' = \{\Omega_{\mathfrak{n}'}\}_{\mathfrak{n}'\in T}$, then define $\Omega\oplus\Omega' := \{\Omega_{\mathfrak{n}},\Omega_{\mathfrak{n}'}\}_{\mathfrak{n},\mathfrak{n}'\in T}$. We know that $c(\Omega) = \sum_{\widetilde{\mathfrak{n}}} c_{\mathfrak{n},\widetilde{\mathfrak{n}}}\Omega_{\widetilde{\mathfrak{n}}}$ and $c(\Omega') = \sum_{\widetilde{\mathfrak{n}}'} c_{\mathfrak{n}',\widetilde{\mathfrak{n}}'}\Omega_{\widetilde{\mathfrak{n}}'}$. Define $c\oplus c'$ to be the linear map whose domain is all vector of the form $\Omega\oplus\Omega'$ and whose action is $c\oplus c'(\Omega\oplus\Omega') = \{\sum_{\widetilde{\mathfrak{n}}} c_{\mathfrak{n},\widetilde{\mathfrak{n}}}\Omega_{\widetilde{\mathfrak{n}}},\sum_{\widetilde{\mathfrak{n}}'} c_{\mathfrak{n}',\widetilde{\mathfrak{n}}'}\Omega_{\widetilde{\mathfrak{n}}'}\}_{\mathfrak{n},\mathfrak{n}'\in T}$. Define $\mathscr{M}(\mathcal{C}) = \{c\oplus c' : c, c'\in\mathscr{M}(T)\}$, then we get $\sum_{c,c'\in\mathscr{M}(T)} = \sum_{c\in\mathscr{M}(\mathcal{C})}$.

By (2.4.81) and the fact that leaves corresponding to k_j, k'_j are merged in $\mathcal{C}$, (2.4.80) is equivalent to

$$|Term(T,p)_k| \lesssim \sum_{\substack{k_1,\cdots,k_{l(T)+1},k'_1,\cdots,k'_{l(T)+1}\\ |k_j|,|k'_j|\lesssim 1,\forall j}} \sum_{c\in\mathscr{M}(\mathcal{C})} \prod_{\mathfrak{n}\in\mathcal{C}} \frac{1}{|c(\Omega)_{\mathfrak{n}}|+T_{\max}^{-1}} \prod_{\mathfrak{e}\in\mathcal{C}_{\text{norm}}} |k_{\mathfrak{e},x}|\ \delta_{\cap_{\mathfrak{n}\in\mathcal{C}}\{S_{\mathfrak{n}}=0\}} \tag{2.4.82}$$

Assigning a number $\sigma_{\mathfrak{n}} \in \mathbb{Z}_{T_{\max}}$ for each node $\mathfrak{n}\in\mathcal{C}$, a number $\kappa_{\mathfrak{e}} \in \mathcal{D}(\alpha,1) \coloneqq \{2^{-K_{\mathfrak{e}}} : K_{\mathfrak{e}} \in \mathbb{Z}\cap[0, ln\ \alpha^{-1}]\}$ for each edge $\mathfrak{e}$ and a number $k\in\mathbb{Z}_L^d$ for each fixed leg, we can define the associated equation $Eq(\mathcal{C},\{\sigma_{\mathfrak{n}}\}_{\mathfrak{n}},\{\kappa_{\mathfrak{e}}\}_{\mathfrak{e}},k) = Eq(\mathcal{C})$ as in (2.4.39). Then we have

$$\sum_{\substack{k_1,\cdots,k_{l(T)+1},k'_1,\cdots,k'_{l(T)+1}\\ \cap_{\mathfrak{n}\in\mathcal{C}}\{S_{\mathfrak{n}}=0\}}} = \sum_{\kappa_{\mathfrak{e}}\in\mathcal{D}(\alpha,1)} \sum_{\sigma_{\mathfrak{n}}\in\mathbb{Z}_{T_{\max}}} \sum_{Eq(\mathcal{C},\{\sigma_{\mathfrak{n}}\}_{\mathfrak{n}},\{\kappa_{\mathfrak{e}}\}_{\mathfrak{e}},k)}, \tag{2.4.83}$$

which implies that

$$|Term(T,p)_k| \lesssim \sum_{\kappa_{\mathfrak{e}}\in\mathcal{D}(\alpha,1)} \sum_{\sigma_{\mathfrak{n}}\in\mathbb{Z}_{T_{\max}}} \sum_{Eq(\mathcal{C},\{\sigma_{\mathfrak{n}}\}_{\mathfrak{n}},\{\kappa_{\mathfrak{e}}\}_{\mathfrak{e}},k)} \sum_{c\in\mathscr{M}(\mathcal{C})} \prod_{\mathfrak{n}\in\mathcal{C}} \frac{1}{|c(\Omega)_{\mathfrak{n}}|+T_{\max}^{-1}} \prod_{\mathfrak{e}\in\mathcal{C}_{\text{norm}}} |k_{\mathfrak{e},x}| \tag{2.4.84}$$

Remember in the definition of $Eq(\mathcal{C})$, we have conditions that $|k_{\mathfrak{e}}| \lesssim 1$. These conditions come from the fact that on the support of $\delta_{\cap_{\mathfrak{n}\in T_{\text{in}}}\{S_{\mathfrak{n}}=0\}}$, if $k_1,\cdots,k_{l(T)+1}$ are bounded, then $|k_{\mathfrak{e}}|$ are also bounded. Recall that by (2.4.5), $S_{\mathfrak{n}} = \iota_{\mathfrak{e}_1}k_{\mathfrak{e}_1} + \iota_{\mathfrak{e}_2}k_{\mathfrak{e}_2} + \iota_{\mathfrak{e}}k_{\mathfrak{e}}$. The conditions that $S_{\mathfrak{n}} = 0$ imply that the variables $k_{\mathfrak{e}}$ of the parent edge $\mathfrak{e}$ are linear combinations of the variables $k_{\mathfrak{e}_1}$, $k_{\mathfrak{e}_2}$ of children edges $\mathfrak{e}_1$, $\mathfrak{e}_2$. The variables of children edges $\mathfrak{e}_j$ are again a linear combinations of the variables of their children. We may iterate this argument to show that all variables $k_{\mathfrak{e}}$ are linear combinations of variables of leaves $k_1,\cdots,k_{l(T)+1}$. Because $k_1,\cdots,k_{l(T)+1}$ are bounded, then their linear combinations $k_{\mathfrak{e}}$ are also bounded.

In the definition of $Eq(\mathcal{C},\{\sigma_{\mathfrak{n}}\}_{\mathfrak{n}},\{\kappa_{\mathfrak{e}}\}_{\mathfrak{e}},k)$, $|\Omega_{\mathfrak{n}}-\sigma_{\mathfrak{n}}| = O(T_{\max}^{-1})$ and $|k_{\mathfrak{e}x}| \sim \kappa_{\mathfrak{e}}$. Denote the constant in $O(T_{\max}^{-1})$ by δ and then we have $|\Omega_{\mathfrak{n}}-\sigma_{\mathfrak{n}}| \le \delta T_{\max}^{-1}$. Therefore, we have $|c(\Omega)_{\mathfrak{n}} - c(\{\sigma_{\mathfrak{n}}\})_{\mathfrak{n}}| \lesssim \delta T_{\max}^{-1}$. We have the freedom of choosing δ in the definition and we take it sufficiently small so that $|c(\Omega)_{\mathfrak{n}} - c(\{\sigma_{\mathfrak{n}}\})_{\mathfrak{n}}| \le \frac{1}{2}T_{\max}^{-1}$. This implies that $|c(\Omega)_{\mathfrak{n}}| + T_{\max}^{-1} \gtrsim |c(\{\sigma_{\mathfrak{n}}\})_{\mathfrak{n}}| + T_{\max}^{-1}$.

Since we also have $|k_{\mathfrak{e}x}| \sim \kappa_{\mathfrak{e}}$, by (2.4.84) we get

$$\begin{aligned}
&Term(T,p)_k \\
\lesssim &\sum_{\kappa_{\mathfrak{e}} \in \mathcal{D}(\alpha,1)} \sum_{\sigma_{\mathfrak{n}} \in \mathbb{Z}_{T_{\max}}} \sum_{Eq(\mathcal{C},\{\sigma_{\mathfrak{n}}\}_{\mathfrak{n}},\{\kappa_{\mathfrak{e}}\}_{\mathfrak{e}},k)} \sum_{c\in\mathscr{M}(\mathcal{C})} \prod_{\mathfrak{n}\in\mathcal{C}} \frac{1}{|c(\Omega)_{\mathfrak{n}}| + T_{\max}^{-1}} \prod_{\mathfrak{e}\in\mathcal{C}_{\text{norm}}} |k_{\mathfrak{e},x}| \\
\lesssim &\sum_{c\in\mathscr{M}(\mathcal{C})} \sum_{\kappa_{\mathfrak{e}} \in \mathcal{D}(\alpha,1)} \sum_{\substack{\sigma_{\mathfrak{n}} \in \mathbb{Z}_{T_{\max}} \\ |\sigma_{\mathfrak{n}}|\lesssim 1}} \prod_{\mathfrak{n}\in\mathcal{C}} \frac{1}{|c(\{\sigma_{\mathfrak{n}}\})_{\mathfrak{n}}| + T_{\max}^{-1}} \sum_{Eq(\mathcal{C},\{\sigma_{\mathfrak{n}}\}_{\mathfrak{n}},\{\kappa_{\mathfrak{e}}\}_{\mathfrak{e}},k)} \prod_{\mathfrak{e}\in\mathcal{C}_{\text{norm}}} |k_{\mathfrak{e},x}| \\
\lesssim &\sum_{c\in\mathscr{M}(\mathcal{C})} \sum_{\kappa_{\mathfrak{e}} \in \mathcal{D}(\alpha,1)} \sum_{\substack{\sigma_{\mathfrak{n}} \in \mathbb{Z}_{T_{\max}} \\ |\sigma_{\mathfrak{n}}|\lesssim 1}} \prod_{\mathfrak{n}\in\mathcal{C}} \frac{1}{|c(\{\sigma_{\mathfrak{n}}\})_{\mathfrak{n}}| + T_{\max}^{-1}} \left(\prod_{\mathfrak{e}\in\mathcal{C}_{\text{norm}}} \kappa_{\mathfrak{e}}\right) \# Eq(\mathcal{C},\{\sigma_{\mathfrak{n}}\}_{\mathfrak{n}},\{\kappa_{\mathfrak{e}}\}_{\mathfrak{e}},k) \\
\lesssim &\sum_{c\in\mathscr{M}(\mathcal{C})} \sum_{\kappa_{\mathfrak{e}} \in \mathcal{D}(\alpha,1)} \sum_{\substack{\sigma_{\mathfrak{n}} \in \mathbb{Z}_{T_{\max}} \\ |\sigma_{\mathfrak{n}}|\lesssim 1}} \prod_{\mathfrak{n}\in\mathcal{C}} \frac{1}{|c(\{\sigma_{\mathfrak{n}}\})_{\mathfrak{n}}| + T_{\max}^{-1}} \left(\prod_{\mathfrak{e}\in\mathcal{C}_{\text{norm}}} \kappa_{\mathfrak{e}}\right) L^{O(n\theta)} Q^{\frac{n}{2}} \prod_{\mathfrak{e}\in\mathcal{C}_{\text{norm}}} \kappa_{\mathfrak{e}}^{-1}
\end{aligned} \tag{2.4.85}$$

Here in the last inequality we applied (2.4.44) in Proposition 3.3.22.

After simplification, (2.4.85) gives us

$$\begin{aligned}
|Term(T,p)_k| \lesssim &L^{O(n\theta)} Q^{\frac{n}{2}} \sum_{c\in\mathscr{M}(\mathcal{C})} \sum_{\kappa_{\mathfrak{e}} \in \mathcal{D}(\alpha,1)} \sum_{\substack{\sigma_{\mathfrak{n}} \in \mathbb{Z}_{T_{\max}} \\ |\sigma_{\mathfrak{n}}|\lesssim 1}} \prod_{\mathfrak{n}\in\mathcal{C}} \frac{1}{|c(\{\sigma_{\mathfrak{n}}\})_{\mathfrak{n}}| + T_{\max}^{-1}} \\
\lesssim &L^{O(n\theta)} Q^{\frac{n}{2}} \left(\sum_{\kappa_{\mathfrak{e}} \in \mathcal{D}(\alpha,1)} 1\right) \sum_{c\in\mathscr{M}(\mathcal{C})} \sum_{\substack{\sigma_{\mathfrak{n}} \in \mathbb{Z}_{T_{\max}} \\ |\sigma_{\mathfrak{n}}|\lesssim 1}} \prod_{\mathfrak{n}\in\mathcal{C}} \frac{1}{|c(\{\sigma_{\mathfrak{n}}\})_{\mathfrak{n}}| + T_{\max}^{-1}} \\
\lesssim &L^{O(n\theta)} Q^{\frac{n}{2}} \sum_{c\in\mathscr{M}(\mathcal{C})} \sum_{\substack{\sigma_{\mathfrak{n}} \in \mathbb{Z}_{T_{\max}} \\ |\sigma_{\mathfrak{n}}|\lesssim 1}} \prod_{\mathfrak{n}\in\mathcal{C}} \frac{1}{|c(\{\sigma_{\mathfrak{n}}\})_{\mathfrak{n}}| + T_{\max}^{-1}}
\end{aligned} \tag{2.4.86}$$

Here in the last step we use the fact that $\#\mathcal{D}(\alpha, 1) \lesssim ln(\alpha^{-1})$. $|\sigma_{\mathfrak{n}}| \lesssim 1$ in the sum of the first two inequalities comes from the fact that $\#Eq(\mathcal{C}, \{\sigma_{\mathfrak{n}}\}_{\mathfrak{n}}, k) = 0$ if some $|\sigma_{\mathfrak{n}}| \gtrsim 1$. This fact is true because in $Eq(\mathcal{C}, \{\sigma_{\mathfrak{n}}\}_{\mathfrak{n}}, k)$, all $|k_{\mathfrak{e}}| \lesssim 1$, which implies that $|\Omega_{\mathfrak{n}}| \lesssim 1$ and therefore $|\sigma_{\mathfrak{n}}| \gtrsim 1$ (notice that $|\Omega_{\mathfrak{n}} - \sigma_{\mathfrak{n}}| = O(T_{\max}^{-1})$).

We claim that

$$\sup_c \sum_{\substack{\sigma_{\mathfrak{n}} \in \mathbb{Z}_{T_{\max}} \\ |\sigma_{\mathfrak{n}}|\lesssim 1}} \prod_{\mathfrak{n}\in\mathcal{C}} \frac{1}{|c(\{\sigma_{\mathfrak{n}}\})_{\mathfrak{n}}| + T_{\max}^{-1}} \lesssim L^{O(n\theta)} T_{\max}^{n} \tag{2.4.87}$$

Since there are only bounded many matrices in $\mathscr{M}(\mathcal{C})$. Given the above claim, we know that

$$|Term(T,p)_k| \lesssim L^{O(n\theta)} Q^{\frac{n}{2}} T_{\max}^{n}, \tag{2.4.88}$$

which proves the lemma.

Now prove the claim. Notice that there are n nodes in $\mathcal{C}$. We label these nodes by $h = 1, \cdots, n$ and denote $\sigma_{\mathfrak{n}}$ by σ_h if $\mathfrak{n}$ is labelled by h. Since $\sigma_h \in \mathbb{Z}_{T_{\max}}$, there exists $m_h \in \mathbb{Z}$ such that $\sigma_h = \frac{m_h}{T_{\max}}$. (2.4.87) is thus equivalent to

$$\sum_{\substack{m_h \in \mathbb{Z} \\ |m_h| \lesssim T_{\max}}} \prod_{h=1}^{n} \frac{T_{\max}}{|c(\{m_h\})_h| + 1} \lesssim L^{O(n\theta)} T_{\max}^n \tag{2.4.89}$$

Before proving (2.4.89), let's first look at one of its special case. If $c = Id$, then $c(\{m_h\})_h = m_h$. The right hand side of (2.4.89) becomes

$$T_{\max}^n \sum_{\substack{m_h \in \mathbb{Z} \\ |m_h| \lesssim T_{\max}}} \prod_{h=1}^{n} \frac{1}{|m_h| + 1} = T_{\max}^n \prod_{h=1}^{n} \left(\sum_{\substack{m_h \in \mathbb{Z} \\ |m_h| \lesssim T_{\max}}} \frac{1}{|m_h| + 1} \right) \lesssim T_{\max}^n (ln(\alpha^{-1}))^n = L^{O(n\theta)} T_{\max}^n. \tag{2.4.90}$$

Here we use the fact that $\sum_{\substack{j \in \mathbb{Z} \\ |j| \lesssim T_{\max}}} \frac{1}{|j|+1} = O(ln(\alpha^{-1}))$.

Now we prove (2.4.89). We just need to show that

$$\sum_{\substack{m_h \in \mathbb{Z} \\ |m_h| \lesssim T_{\max}}} \prod_{h=1}^{n} \frac{1}{|c(\{m_h\})_h| + 1} \lesssim L^{O(n\theta)} \tag{2.4.91}$$

By Euler-Maclaurin formula (4.2.1) and change of variable formula, we get

$$\begin{aligned} \sum_{\substack{m_h \in \mathbb{Z} \\ |m_h| \lesssim T_{\max}}} \prod_{h=1}^{n} \frac{1}{|c(\{m_h\})_h| + 1} &\leq \int_{|m_h| \lesssim T_{\max}} \prod_{h=1}^{n} \frac{1}{|c(\{m_h\})_h| + 1} \prod_{h=1}^{n} dm_h \\ &= \int_{c(\{|m_h| \lesssim T_{\max}\})} \prod_{h=1}^{n} \frac{1}{|m_h| + 1} |\det c| \prod_{h=1}^{n} dm_h \\ &\lesssim \prod_{h=1}^{n} \int_{|m_h| \lesssim T_{\max}} \frac{1}{|m_h| + 1} dm_h \\ &\lesssim (ln(\alpha^{-1}))^n \lesssim L^{O(n\theta)} \end{aligned} \tag{2.4.92}$$

We complete the proof of the claim and thus the proof of the lemma. □

Now we prove Proposition 3.2.7. To start with, recall the large deviation estimate for Gaussian polynomial.

Lemma 2.4.28 (Large deviation for Gaussian polynomial)**.** *Let $\{\eta_k(\omega)\}$ be i.i.d. complex Gaussian variables with mean 0 and variance 1. Let $F = F(\omega)$ be an degree n polynomial of $\{\eta_k(\omega)\}$ defined* *by*

$$F(\omega) = \sum_{k_1,\cdots,k_n} a_{k_1\cdots k_n} \prod_{j=1}^{n} \eta_{k_j}^{\iota_j}, \tag{2.4.93}$$

where $a_{k_1\cdots k_n}$ are constants, then we have

$$\mathbb{P}\left(|F(\omega)| \geq A \cdot \left(\mathbb{E}|F(\omega)|^2\right)^{\frac{1}{2}}\right) \leq Ce^{-cA^{\frac{2}{n}}} \tag{2.4.94}$$

Proof. This is a corollary of the hypercontractivity of Ornstein-Uhlenbeck semigroup. A good reference of this topic is [23]. (2.4.94) is equivalent to (B.9) in [23]. $\square$

Proposition 3.2.7 is a corollary of the above large deviation estimate and Proposition 2.4.26.

Proof of Proposition 3.2.7. Because $(\mathcal{J}_T)_k$ are Gaussian polynomials, we can take $F(\omega) = (\mathcal{J}_T)_k$ in Lemma 2.4.28. Then we obtained

$$|(\mathcal{J}_T)_k(t)| \lesssim L^{\frac{n}{2}\theta}\sqrt{\mathbb{E}|(\mathcal{J}_T)_k|^2} \tag{2.4.95}$$

with probability less than $e^{-c(L^{\frac{n}{2}\theta})^{\frac{2}{n}}} = e^{-cL^{\theta}}$. By definition 2.2.6, the above inequality holds true L-certainly.

Since by Proposition 2.4.26, $\mathbb{E}|(\mathcal{J}_T)_k|^2 \lesssim L^{O(l(T)\theta)}\rho^{2l(T)}$, then we get

$$|(\mathcal{J}_T)_k(t)| \lesssim L^{O(l(T)\theta)}\rho^{l(T)}, \qquad L\textit{-certainly} \tag{2.4.96}$$

(2.4.96) is very similar to the final goal (2.2.17), except for the $\sup_t$ and $\sup_k$ in front. In what follows we apply the standard epsilon net and union bound method to remove these two sup.

Assume that $|t-t'| \lesssim \rho^{l(T)}L^{-M}$, it's not hard to show that $|(\mathcal{J}_T)_k(t)-(\mathcal{J}_T)_k(t')| \lesssim \rho^{l(T)}$. Therefore, if $\sup_i \sup_k |(\mathcal{J}_T)_k(i\rho^{l(T)}L^{-M})| \lesssim L^{O(l(T)\theta)}\rho^{l(T)}$, then $\sup_t \sup_k |(\mathcal{J}_T)_k(t)| \lesssim L^{O(l(T)\theta)}\rho^{l(T)}$ and the proof is completed.

Notice that

$$
\begin{aligned}
&\mathbb{P}\left(\sup_{i\in\mathbb{Z}}\sup_{k}|(\mathcal{J}_T)_k(i\rho^{l(T)}L^{-M})|\gtrsim L^{O(l(T)\theta)}\rho^{l(T)}\right)\\
=&\mathbb{P}\left(\bigcup_{i\in\mathbb{Z}\cap[0,T_{\max}\rho^{-l(T)}L^{M}]}\bigcup_{k\in\mathbb{Z}_L\cap[0,1]}\{|(\mathcal{J}_T)_k(i\rho^{l(T)}L^{-M})|\gtrsim L^{O(l(T)\theta)}\rho^{l(T)}\}\right)\\
\lesssim&\sum_{i\in\mathbb{Z}\cap[0,T_{\max}\rho^{-l(T)}L^{M}]}\sum_{k\in\mathbb{Z}_L\cap[0,1]}\mathbb{P}\left(|(\mathcal{J}_T)_k(i\rho^{l(T)}L^{-M})|\gtrsim L^{O(l(T)\theta)}\rho^{l(T)}\right)\\
\lesssim&\sum_{i\in\mathbb{Z}\cap[0,T_{\max}\rho^{-l(T)}L^{M}]}\sum_{k\in\mathbb{Z}_L\cap[0,1]}e^{-O(L^{\theta})}=L^{2M}e^{-O(L^{\theta})}=e^{-O(L^{\theta})}
\end{aligned}
\tag{2.4.97}
$$

Here in the second inequality the two ranges $[0, T_{\max}\rho^{-l(T)}L^M]$ and $[0,1]$ of i and k come from the fact that $t = i\rho^{l(T)}L^{-M} \lesssim T_{\max}$ and $(\mathcal{J}_T)_k = 0$ for $|k| \gtrsim 1$. In the last line we can replace the probability by $e^{-O(L^\theta)}$ because the estimate $|(\mathcal{J}_T)_k(t)| \lesssim L^{O(l(T)\theta)}\rho^{l(T)}$ holds true L-certainly.

Now we complete the proof of Proposition 3.2.7. □

2.4.6 Norm estimate of random matrices

In this section, we prove Proposition 2.2.8.

Remember that by definition of $\mathcal{P}_T$ and $\mathcal{T}$,

$$\mathcal{P}_T(w) = \mathcal{T}(\mathcal{J}_T, w) = \frac{i\lambda}{L^d}\sum_{S(k_1,k_2,k)=0}\int_0^t k_x\mathcal{J}_{T,k_1}w_{k_2}e^{is\Omega(k_1,k_2,k)-\nu|k|^2(t-s)}ds. \tag{2.4.98}$$

Dyadic decomposition of $\mathcal{P}_T$

Remember that we have the dyadic decomposition $[0,1] = \bigcup_{\tau=0}^{\infty}[2^{-\tau}, 2^{-\tau-1}]$.

We can then construct a dyadic decomposition $\mathcal{P}_T = \sum_{l=0}^{\infty}\mathcal{P}_T^l$. Here $\mathcal{P}_T^l$ is defined by the following formula.

$$\mathcal{P}_T^l(w) = \frac{i\lambda}{L^d}\sum_{S(k_1,k_2,k)=0}\int_{(t-s)/T_{\max}\in[2^{-\tau},2^{-\tau-1}]}k_x\mathcal{J}_{T,k_1}w_{k_2}e^{is\Omega(k_1,k_2,k)-\nu|k|^2(t-s)}ds \tag{2.4.99}$$

We also introduce the bilinear operator $\mathcal{T}^l(\phi,\phi)_k$

$$\mathcal{T}^l(\phi,\phi)_k = \frac{i\lambda}{L^d}\sum_{S(k_1,k_2,k)=0}\int_{(t-s)/T_{\max}\in[2^{-\tau},2^{-\tau-1}]}k_x\phi_{k_1}\phi_{k_2}e^{is\Omega(k_1,k_2,k)-\nu|k|^2(t-s)}ds \tag{2.4.100}$$

Proposition 2.2.8 is a corollary of the following proposition.

Proposition 2.4.29. *Let $\rho = \alpha T_{max}^{\frac{1}{2}}$ and $\mathcal{P}_T^l$ be the operator defined above. Define the space $X_{L^{2M}}^p = \{w \in X^p : w_k = 0$ if $|k| \gtrsim L^{2M}\}$ and the norm $||w||_{X_{L^{2M}}^p} = \sup_{|k|\lesssim L^{2M}} \langle k\rangle^p |w_k|$. Then for any sequence of trees and numbers $\{T_1, \cdots, T_K\}$ and $\{\tau_1, \cdots, \tau_K\}$, we have L-certainly the operator bound*

$$\left|\left|\sum_{\tau_1,\cdots,\tau_K}\prod_{j=1}^{K}\mathcal{P}_{T_j}^{\tau_j}\right|\right|_{L_t^\infty X_{L^{2M}}^p \to L_t^\infty X^p} \leq L^{O\left(1+\theta\sum_{j=1}^K l(T_j)\right)}\rho^{\sum_{j=1}^K l(T_j)}. \tag{2.4.101}$$

for any T_j with $l(T_j) \leq N$.

Proof of Proposition 2.2.8: By Proposition 2.4.29, we have

$$\left|\left|\prod_{j=1}^{K}\mathcal{P}_{T_j}\right|\right|_{L_t^\infty X_{L^{2M}}^p \to L_t^\infty X^p} = \left|\left|\sum_{\tau_1,\cdots,\tau_K}\prod_{j=1}^{K}\mathcal{P}_{T}^{\tau_j}\right|\right|_{L_t^\infty X_{L^{2M}}^p \to L_t^\infty X^p} \leq L^{O\left(1+\theta\sum_{j=1}^K l(T_j)\right)}\rho^{\sum_{j=1}^K l(T_j)}. \tag{2.4.102}$$

Define $\left(X_{L^{2M}}^p\right)^\perp = \{w \in X^p : w_k = 0 \text{ if } |k| \lesssim L^{2M}\}$. To prove Proposition 2.2.8, it suffices to show that for all $w \in \left(X_{L^{2M}}^p\right)^\perp$,

$$\left|\left|\prod_{j=1}^{K}\mathcal{P}_{T_j}w\right|\right|_{L_t^\infty X^p} \lesssim L^{O\left(1+\theta\sum_{j=1}^K l(T_j)\right)}\rho^{\sum_{j=1}^K l(T_j)}\,||w||_{L_t^\infty X^p}\,. \tag{2.4.103}$$

By (2.4.98), if $w \in \left(X_{L^{2M}}^p\right)^\perp$, then $\mathcal{P}_{T_j}w \in \left(X_{L^{2M}}^p\right)^\perp$ if we enlarge the constant in $|k| \lesssim L^{2M}$ in the definition of $\left(X_{L^{2M}}^p\right)^\perp$. Therefore, to prove (2.4.103), it suffices to prove

$$||\mathcal{P}_{T_j}w||_{L_t^\infty X^p} \lesssim L^{O(1+l(T_j)\theta)}\rho^{l(T_j)}\,||w||_{L_t^\infty X^p}\,, \tag{2.4.104}$$

for all $w \in \left(X_{L^{2M}}^p\right)^\perp$.

By (2.4.98),

$$\begin{aligned}
|\mathcal{P}_{T_j}(w)_k| \leq& \frac{\lambda}{L^d}\sum_{S(k_1,k_2,k)=0}\int_0^t |k_x||\mathcal{J}_{T_j,k_1}||w_{k_2}|e^{-\nu|k|^2(t-s)}ds \\
\lesssim& L^{O(l(T_j)\theta)}\rho^{l(T_j)}\frac{\lambda}{L^d}|k_x|\int_0^t e^{-\nu|k|^2(t-s)}ds\sum_{k_2:|k_2-k|\lesssim 1}\langle k_2\rangle^{-p} \\
\lesssim& L^{O(l(T_j)\theta)}\rho^{l(T_j)}\frac{\lambda}{L^d}|k_x|\nu^{-1}\langle k\rangle^{-2}\sum_{k_2:|k_2-k|\lesssim 1}\langle k_2\rangle^{-p} \\
\lesssim& L^{O(l(T_j)\theta)}\rho^{l(T_j)}\frac{\lambda}{L^d}\nu^{-1}\langle k\rangle^{-1}\langle k\rangle^{-p} \lesssim L^{O(l(T_j)\theta)}\rho^{l(T_j)}\frac{\lambda}{L^d}\nu^{-1}L^{-2M}\langle k\rangle^{-p} \\
\lesssim& L^{-M}\rho^{l(T_j)}\langle k\rangle^{-p}
\end{aligned} \tag{2.4.105}$$

Here in the second inequality, we apply Proposition 3.2.7. In the third line we use the fact that $\int_0^t e^{-\nu|k|^2(t-s)}ds \leq \nu^{-1}\langle k\rangle^{-2}$. In the fourth line we use the fact that $\sum_{k_2:|k_2-k|\lesssim 1}\langle k_2\rangle^{-p} \leq \langle k\rangle^{-p}$ and $|k| \gtrsim L^{2M}$ (since if $|k| \lesssim L^{2M}$ then $|\mathcal{P}_{T_j}(w)_k|$ vanishes and there is nothing to prove).

(2.4.105) implies that $\left|\left|\mathcal{P}_{T_j}w\right|\right|_{L_t^\infty X^p} \lesssim L^{-M}\rho^{l(T_j)}\left|\left|w\right|\right|_{L_t^\infty X^p} \lesssim L^{O(1+l(T_j)\theta)}\rho^{l(T_j)}\left|\left|w\right|\right|_{L_t^\infty X^p}$, so we have proved (2.4.104) and thus (2.2.23).

Now we prove (2.2.24). Because $L = \sum_{1\leq l(T)\leq N}\mathcal{P}_T$ and $L^K = \sum_{1\leq l(T_1),\cdots l(T_K)\leq N}\mathcal{P}_{T_1}\cdots\mathcal{P}_{T_K}$, by (2.2.23)

$$\left|\left|L^K\right|\right|_{L_t^\infty X^p\to L_t^\infty X^p} \lesssim L^{O(1)}\sum_{1\leq l(T_1),\cdots l(T_K)\leq N}(L^{O(\theta)}\rho)^{\sum_{j=1}^K l(T_j)} \leq L^{O\left(1+\theta\sum_{j=1}^K l(T_j)\right)}\rho^K \tag{2.4.106}$$

Here in the last step we use the fact that $l(T_j)\geq 1$ for all j and there are bounded many trees satisfy $1\leq l(T_1),\cdots l(T_K)\leq N$.

Therefore, we complete the proof of Proposition 2.2.8. □

Formulas for product of random matrices $\mathcal{P}_{T_j}^l$

In order to prove Proposition 2.4.29, it is very helpful to find a good formula of $\prod_{j=1}^K\mathcal{P}_{T_j}^{\tau_j}$, which is the main goal of this section.

$\mathcal{P}_T^l$ is almost the same as $\mathcal{P}_T$ except for the limits in the time integral, so in the rest part of this section we do not stress their difference. Now let's find a tree representation for $\mathcal{P}_T^l$ or $\mathcal{P}_T$.

By (2.2.8), we know that $\mathcal{J}_T = \mathcal{T}(\mathcal{J}_{T_{\mathfrak{n}_1}}, \mathcal{J}_{T_{\mathfrak{n}_2}})$ corresponds to the tree T in which the two subtrees of the root nodes are $T_{\mathfrak{n}_1}$ and $T_{\mathfrak{n}_2}$. Taking $T_{\mathfrak{n}_2}$ to be an one node tree, as the left tree in Figure 2.20, then this graph represents a term $\mathcal{T}(\mathcal{J}_{T_{\mathfrak{n}_1}}, \xi)$. The right tree in Figure 2.20 represents the term $\mathcal{T}(\mathcal{J}_{T_{\mathfrak{n}_1}}, w)$, because as in section 3.2.2, the □ node represents a function w (in section 3.2.2 the function is ϕ).

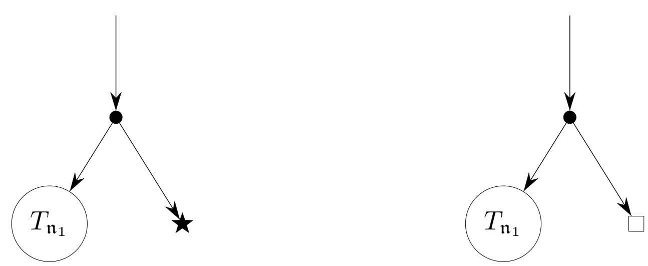


Figure 2.20: Graphical representations of $\mathcal{T}(\mathcal{J}_{T_{\mathfrak{n}_1}}, \xi)$ and $\mathcal{T}(\mathcal{J}_{T_{\mathfrak{n}_1}}, w)$.

Let's find the tree representation for $\prod_{j=1}^K\mathcal{P}_{T_j}^{\tau_j}$ or $\prod_{j=1}^K\mathcal{P}_{T_j}$. Recall the expansion process in section 3.2.2: the replacement of □ by a branching node indicates the substitution of ϕ by $\mathcal{T}(\xi,\xi)$.

The action of composition $\mathcal{P}_{T_1} \circ \mathcal{P}_{T_2}(w) = \mathcal{T}(\mathcal{J}_{T_1}, \mathcal{T}(\mathcal{J}_{T_2}, w))$ is the substitution of $\cdot$ by $\mathcal{T}(\mathcal{J}_{T_2}, w)$ in $\mathcal{T}(\mathcal{J}_{T_1}, \cdot)$. As in Figure 2.21, if $\mathcal{P}_{T_1} = \mathcal{T}(\mathcal{J}_{T_1}, \cdot)$ is represented by the left tree, then as an analog that a $\square$ node is replaced by a branching node, the substitution of $\cdot$ by $\mathcal{T}(\mathcal{J}_{T_2}, w)$ should correspond to the operation that the $\square$ node in the left tree is replaced by the middle tree corresponding to $\mathcal{T}(\mathcal{J}_{T_2}, w)$, and the finally resulting right tree should correspond to $\mathcal{P}_{T_1} \circ \mathcal{P}_{T_2}(w)$.

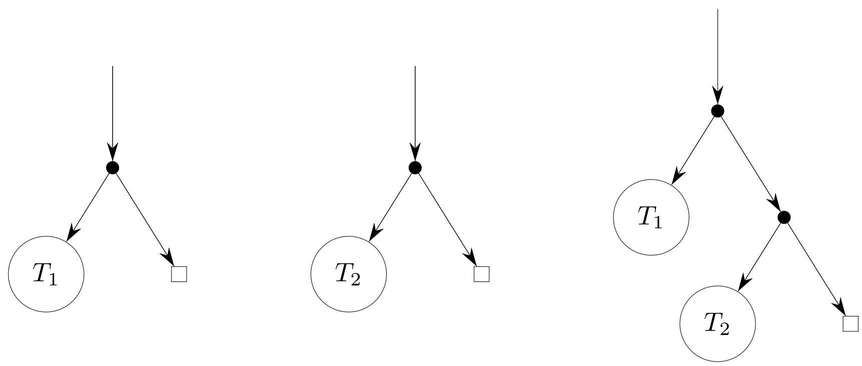


Figure 2.21: The substitution process.

In conclusion, $\mathcal{P}_{T_1} \circ \mathcal{P}_{T_2}(w)$ corresponds to the right tree in Figure 2.21 and more generally, $\prod_{j=1}^{K} \mathcal{P}_{T_j}(w)$ (or $\prod_{j=1}^{K} \mathcal{P}_{T_j}^{\tau_j}(w)$) corresponds to the tree in in Figure 2.22.

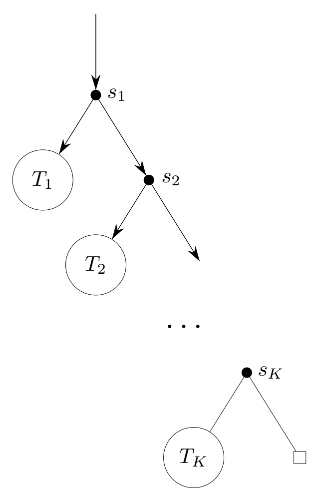


Figure 2.22: Picture of $T_1 \circ \cdots \circ T_K$

We introduce the following definition.

Definition 2.4.30. 1. **Definition of $T_1 \circ \cdots \circ T_K$:** We define $T_1 \circ \cdots \circ T_K$ to be the tree in Figure 2.22.

2. **Substitution nodes:** A node in a tree T with $\square$ nodes is defined to be a <u>substitution node</u> if

it is an ancestor of some $\square$ nodes. For a substitution node, we assign a number τ to it, called its index. In Figure 2.22, $s_1, \cdots, s_K$ are all the substitution nodes in $T_1 \circ \cdots \circ T_K$ and we assign index $\tau_1, \cdots, \tau_K$ to them. Notice that s_1 is the root $\mathfrak{r}$.

Because the tree in this section contains $\square$ nodes and substitution nodes, we propose the following generalization of Definition 3.2.2 of tree terms.

Definition 2.4.31. Given a binary tree T with $\square$ nodes and substitution nodes, we inductively define the quantity $\mathcal{J}_T$ by:

$$\mathcal{J}_T = \begin{cases} \xi, & \textit{if } T \textit{ has only one node } \star. \\ w, & \textit{if } T \textit{ has only one node } \square. \\ \mathcal{T}^l(\mathcal{J}_{T_{\mathfrak{n}_1}}, \mathcal{J}_{T_{\mathfrak{n}_2}}), & \textit{if the root of } T \textit{ is a substitution node with index } \tau. \\ \mathcal{T}(\mathcal{J}_{T_{\mathfrak{n}_1}}, \mathcal{J}_{T_{\mathfrak{n}_2}}), & \textit{otherwise.} \end{cases} \tag{2.4.107}$$

Here $\mathfrak{n}_1$, $\mathfrak{n}_2$ are two children of the root node $\mathfrak{r}$ and $T_{\mathfrak{n}_1}$, $T_{\mathfrak{n}_2}$ are the subtrees of T rooted at above nodes.

Then we have the following formula of $\prod_{j=1}^K \mathcal{P}_{T_j}^{\tau_j}(w)$

Lemma 2.4.32. *With above definition of $T_1 \circ \cdots \circ T_K$ and $\mathcal{J}_T$, we have*

$$\prod_{j=1}^K \mathcal{P}_{T_j}^{\tau_j}(w) = \mathcal{J}_{T_1 \circ \cdots \circ T_K} \tag{2.4.108}$$

Proof. This lemma follows from the above explanation . $\square$

To get a good upper bound, we need a better formula of $\prod_{j=1}^K \mathcal{P}_{T_j}^{\tau_j}(w)$ which is an analog of Lemma 3.3.2.

Lemma 2.4.33. *(1) Using the same notation as Lemma 3.3.2. Given a tree T with $\square$ nodes and substitution nodes. Assume that the root $\mathfrak{r}$ is a substitution nodes of index τ_1. Let $\mathcal{J}_T$ be terms defined in Definition 2.4.31, then their Fourier coefficients $\mathcal{J}_{T,k}$ are degree l polynomials of ξ and w given by the following formula*

$$\mathcal{J}_{T,k} = \left(\frac{i\lambda}{L^d}\right)^l \sum_{k_1,\, k_2,\, \cdots,\, k_{l+1}} \int_{\cup_{\mathfrak{n}\in T_{in}} A_{\mathfrak{n}}} e^{\sum_{\mathfrak{n}\in T_{in}} it_{\mathfrak{n}}\Omega_{\mathfrak{n}} - \nu(t_{\widehat{\mathfrak{n}}} - t_{\mathfrak{n}})|k_{\mathfrak{e}}|^2} \prod_{j=1}^{l+1} (\xi|w)_{k_j} \prod_{\mathfrak{n}\in T_{in}} dt_{\mathfrak{n}}\, \delta_{\cap_{\mathfrak{n}\in T_{in}}\{S_{\mathfrak{n}}=0\}} \prod_{\mathfrak{e}\in T_{in}} \iota_{\mathfrak{e}} k_{\mathfrak{e},x} \tag{2.4.109}$$

Here ι, $(\xi|w)_{k_j}$, $A_{\mathfrak{n}}$, $S_{\mathfrak{n}}$, $\Omega_{\mathfrak{n}}$ *are defined by*

$$\iota_{\mathfrak{e}} = \begin{cases} +1 & \text{if } \mathfrak{e} \text{ pointing inwards to } \mathfrak{n} \\ -1 & \text{if } \mathfrak{e} \text{ pointing outwards from } \mathfrak{n} \end{cases} \tag{2.4.110}$$

$$(\xi|w)_{k_j} = \begin{cases} \xi & \text{if } j\text{-th leaf node is a } \star \text{ node} \\ w & \text{if } j\text{-th leaf node is a } \Box \text{ node} \end{cases} \tag{2.4.111}$$

$$A_{\mathfrak{n}} = \begin{cases} \{t_{\mathfrak{r}} \le t, (t - t_{\mathfrak{r}})/T_{max} \in [2^{-\tau_1}, 2^{-\tau_1 - 1}]\} & \text{if } \mathfrak{n} \text{ is the root } \mathfrak{r} \\ \{t_{\mathfrak{n}_1}, t_{\mathfrak{n}_2}, t_{\mathfrak{n}_3} \le t_{\mathfrak{n}}\} & \text{if } \mathfrak{n} \neq \mathfrak{r} \text{ and is not a substitution node} \\ \{t_{\mathfrak{n}_1}, t_{\mathfrak{n}_2}, t_{\mathfrak{n}_3} \le t_{\mathfrak{n}}, (t_{\widehat{\mathfrak{n}}} - t_{\mathfrak{n}})/T_{max} \in [2^{-\tau}, 2^{-\tau-1}]\} & \text{if } \mathfrak{n} \neq \mathfrak{r} \text{ is an index } \tau \text{ substitution node} \end{cases} \tag{2.4.112}$$

$$S_{\mathfrak{n}} = \iota_{\mathfrak{e}_1} k_{\mathfrak{e}_1} + \iota_{\mathfrak{e}_2} k_{\mathfrak{e}_2} + \iota_{\mathfrak{e}} k_{\mathfrak{e}} \tag{2.4.113}$$

$$\Omega_{\mathfrak{n}} = \iota_{\mathfrak{e}_1} \Lambda_{k_{\mathfrak{e}_1}} + \iota_{\mathfrak{e}_2} \Lambda_{k_{\mathfrak{e}_2}} + \iota_{\mathfrak{e}} \Lambda_{k_{\mathfrak{e}}} \tag{2.4.114}$$

For root node $\mathfrak{r}$*, we impose the constrain that* $k_{\mathfrak{r}} = k$*. We also define* $t_{\widehat{\mathfrak{r}}} = t$ *to fix the problem that* $\widehat{\mathfrak{r}}$ *is not well defined because* $\mathfrak{r}$ *does not have a parent.*

(2) Define $T = T_1 \circ \cdots \circ T_K$ *and by definition there are only one* $\Box$ *leaf. Assume that there are* $l+1$ *leaves in* T *and label all* $\star$ *leaves by* $1, \cdots, l$*, then* $l = \sum_{j=1}^{K} l(T_j)$*. As a corollary of (1), we have the following formula for Fourier coefficients of* $\prod_{j=1}^{K} \mathcal{P}_{T_j}^{\tau_j}$*.*

$$\left(\prod_{j=1}^{K} \mathcal{P}_{T_j}^{\tau_j}(w)\right)_k (t) = \sum_{k'} \int_0^t H_{Tkk'}^{\tau_1 \cdots \tau_K}(t,s) w_{k'}(s) ds \tag{2.4.115}$$

and the kernel $H_{Tkk'}^{\tau_1 \cdots \tau_K}$ *is given by*

$$H_{Tkk'}^{\tau_1 \cdots \tau_K}(t,s) = \left(\frac{i\lambda}{L^d}\right)^l \sum_{k_1, k_2, \cdots, k_l} H_{Tk_1 \cdots k_l kk'}^{\tau_1 \cdots \tau_K} \xi_{k_1} \cdots \xi_{k_l} \tag{2.4.116}$$

and the coefficients $H_{Tk_1 \cdots k_l kk'}^{\tau_1 \cdots \tau_K}$ *of the kernel is given by*

$$H_{Tk_1 \cdots k_l kk'}^{\tau_1 \cdots \tau_K}(t,s) = \int_{\cup_{\mathfrak{n} \in T_{in}} B_{\mathfrak{n}}} e^{\sum_{\mathfrak{n} \in T_{in}} it_{\mathfrak{n}} \Omega_{\mathfrak{n}} - \nu(t_{\widehat{\mathfrak{n}}} - t_{\mathfrak{n}})|k_{\mathfrak{e}}|^2} \prod_{\mathfrak{n} \in T_{in}} dt_{\mathfrak{n}}\ \delta_{\cap_{\mathfrak{n} \in T_{in}} \{S_{\mathfrak{n}} = 0\}} \prod_{\mathfrak{e} \in T_{in}} \iota_{\mathfrak{e}} k_{\mathfrak{e},x} \tag{2.4.117}$$

Here $k_1, \cdots, k_l$ are all variables corresponding to $\star$ leaves, k' is the variable corresponding to the $\square$ node and $B_{\mathfrak{n}}$ is defined by

$$B_{\mathfrak{n}} = \begin{cases} \{t_{\mathfrak{r}} \leq t, (t-t_{\mathfrak{r}})/T_{max} \in [2^{-\tau_1}, 2^{-\tau_1 - 1}]\} & \text{if } \mathfrak{n} \text{ is the root } \mathfrak{r} \\ \{t_{\mathfrak{n}} \geq s\} & \text{if } \mathfrak{n} \text{ is a parent of the } \square \text{ nodes} \\ & \text{and is not a substitution nodes} \\ \{t_{\mathfrak{n}} \geq s, (t_{\widehat{\mathfrak{n}}} - t_{\mathfrak{n}})/T_{max} \in [2^{-\tau}, 2^{-\tau-1}]\} & \text{if } \mathfrak{n} \text{ is a parent of the } \square \text{ nodes} \\ & \text{and is a substitution nodes of index } \tau \\ \{t_{\mathfrak{n}_1}, t_{\mathfrak{n}_2}, t_{\mathfrak{n}_3} \leq t_{\mathfrak{n}}, (t_{\widehat{\mathfrak{n}}} - t_{\mathfrak{n}})/T_{max} \in [2^{-\tau}, 2^{-\tau-1}]\} & \text{if not in the first three cases} \\ & \text{and } \mathfrak{n} \text{ is a substitution node of index } \tau \\ \{t_{\mathfrak{n}_1}, t_{\mathfrak{n}_2}, t_{\mathfrak{n}_3} \leq t_{\mathfrak{n}}\} & \text{otherwise} \end{cases} \tag{2.4.118}$$

Proof. Lemma 2.4.33 (1) can be proved by the same method as Lemma 3.3.2. We can check that $\mathcal{J}_T$ defined by (2.4.1) and (2.4.2) satisfies the recursive formula (2.2.8) by a direct substitution, so they are the unique solution of that recursive formula, and this proves (1).

(2) is a corollary of (1). $\square$

We can also calculate $\prod_{j=1}^{K} \mathcal{P}_{T_j}^{\tau_j}(\mathcal{J}_T)$, replacing w by $\mathcal{J}_T$ corresponds to replacing a $\square$ node by a tree T, so $\prod_{j=1}^{K} \mathcal{P}_{T_j}^{\tau_j}(\mathcal{J}_T) = \mathcal{J}_{T_1 \circ T_2 \circ \cdots \circ T_K \circ T}$. Since T does not contain $\square$ node, so does $T_1 \circ T_2 \circ \cdots \circ T_K \circ T$ and $\mathcal{J}_{T_1 \circ T_2 \circ \cdots \circ T_K \circ T}$ is a polynomial of Gaussian variables ξ_k. Then we have the following lemma

Lemma 2.4.34. *We have*

$$\prod_{j=1}^{K} \mathcal{P}_{T_j}^{\tau_j}(\mathcal{J}_T) = \mathcal{J}_{T_1 \circ T_2 \circ \cdots \circ T_K \circ T} \tag{2.4.119}$$

Proof. This lemma follows from the above explanation . $\square$

The upper bound for coefficients

In this section, we prove the Lemma 2.4.35 which gives an upper bound for $H^{\tau_1 \cdots \tau_K}_{T k_1 \cdots k_l k k'}$. This lemma is an analog of Lemma 3.3.27.

Lemma 2.4.35. *Assume that $|k_1|, \cdots, |k_l| \lesssim 1$, then for $t \leq T_{max}$, we have the following upper* *bound for* $H^{\tau_1\cdots\tau_K}_{Tk_1\cdots k_lkk'}$,

$$|H^{\tau_1\cdots\tau_K}_{Tk_1\cdots k_lkk'}| \lesssim \sum_{\{d_\mathfrak{n}\}_{\mathfrak{n}\in T_{in}}\in\{0,1\}^{l(T)}} \prod_{\mathfrak{n}\in T_{in}} \frac{2^{-\frac{\tau_\mathfrak{n}}{2}}}{|q_\mathfrak{n}|+T_{max}^{-1}}\ \delta_{\cap_{\mathfrak{n}\in T_{in}}\{S_\mathfrak{n}=0\}} \prod_{\mathfrak{e}\in T_{in}} p_\mathfrak{e}. \tag{2.4.120}$$

Here $\tau_\mathfrak{n}$ *is defined by*

$$\tau_\mathfrak{n} = \begin{cases} 0 & \textit{if } \mathfrak{n} \textit{ is not a substitution node} \\ \tau_j & \textit{if } \mathfrak{n} \textit{ is the } j\textit{-th substitution node} \end{cases} \tag{2.4.121}$$

Fix a sequence $\{d_\mathfrak{n}\}_{\mathfrak{n}\in T_{in}}$ *whose elements* $d_\mathfrak{n}$ *takes boolean values* $\{0,1\}$. *We define the sequences* $\{p_\mathfrak{n}\}_{\mathfrak{n}\in T_{in}}$, $\{q_\mathfrak{n}\}_{\mathfrak{n}\in T_{in}}$, $\{r_\mathfrak{n}\}_{\mathfrak{n}\in T_{in}}$ *by following formulas*

$$p_\mathfrak{e} = \frac{|k_{\mathfrak{e},x}|}{|k_{\mathfrak{e},x}|+1} \tag{2.4.122}$$

$$q_\mathfrak{n} = \begin{cases} \Omega_\mathfrak{r}, & \textit{if } \mathfrak{n} = \textit{the root } \mathfrak{r}. \\ \Omega_\mathfrak{n} + d_\mathfrak{n} q_{\mathfrak{n}'}, & \textit{if } \mathfrak{n}\neq\mathfrak{r} \textit{ and } \mathfrak{n}' \textit{ is the parent of } \mathfrak{n}. \end{cases} \tag{2.4.123}$$

$$r_\mathfrak{n} = \begin{cases} |k_\mathfrak{r}|^2, & \textit{if } \mathfrak{n} = \textit{the root } \mathfrak{r}. \\ |k_\mathfrak{n}|^2 + d_\mathfrak{n} q_{\mathfrak{n}'}, & \textit{if } \mathfrak{n}\neq\mathfrak{r} \textit{ and } \mathfrak{n}' \textit{ is the parent of } \mathfrak{n}. \end{cases} \tag{2.4.124}$$

Proof. By definition

$$H^{\tau_1\cdots\tau_K}_{Tk_1\cdots k_lkk'}(t,s) = \int_{\cup_{\mathfrak{n}\in T_{\text{in}}} B_\mathfrak{n}} e^{\sum_{\mathfrak{n}\in T_{\text{in}}} it_\mathfrak{n}\Omega_\mathfrak{n} - \nu(t_{\widehat{\mathfrak{n}}}-t_\mathfrak{n})|k_\mathfrak{e}|^2} \prod_{\mathfrak{n}\in T_{\text{in}}} dt_\mathfrak{n}\ \delta_{\cap_{\mathfrak{n}\in T_{\text{in}}}\{S_\mathfrak{n}=0\}} \prod_{\mathfrak{e}\in T_{\text{in}}} \iota_\mathfrak{e} k_{\mathfrak{e},x} \tag{2.4.125}$$

For any edge $\mathfrak{e}$, assume that the two end points of $\mathfrak{e}$ are $\mathfrak{n}_1$ and $\mathfrak{n}_2$. If neither $\mathfrak{n}_1$ and $\mathfrak{n}_2$ is a substitution node, then we claim that $|k_\mathfrak{e}| \lesssim 1$.

This is because if no end point of $\mathfrak{e}$ is a substitution node, then the subtree $T_\mathfrak{e}$ rooted at the upper end points of $\mathfrak{e}$ does not contain the $\square$ node as its leaf. Therefore, the momentum conservation (Lemma 3.3.18) is applicable which implies that $k_\mathfrak{e}$ is a linear combination of $k_1,\cdots,k_l$. By $|k_1|,\cdots,|k_l| \lesssim 1$, we get $|k_\mathfrak{e}| \lesssim 1$.

Since $|k_\mathfrak{e}| \lesssim 1$, we get $\left|\prod_{\mathfrak{e}\in T_{\text{in}}} \iota_\mathfrak{e} k_{\mathfrak{e},x}\right| \lesssim \prod_{j=1}^K |k_{s_j,x}|$. Here $s_1,\cdots,s_K$ are all substitution nodes and k_{s_j} are corresponding variables of edges pointing towards s_j.

As in section 3.3.4, we define

$$F_T(t,\{a_\mathfrak{n}\}_{\mathfrak{n}\in T_{\text{in}}},\{b_\mathfrak{n}\}_{\mathfrak{n}\in T_{\text{in}}}) = \int_{\cup_{\mathfrak{n}\in T_{\text{in}}} B_\mathfrak{n}} e^{\sum_{\mathfrak{n}\in T_{\text{in}}} it_\mathfrak{n} a_\mathfrak{n} - \nu(t_{\widehat{\mathfrak{n}}}-t_\mathfrak{n}) b_\mathfrak{n}} \prod_{\mathfrak{n}\in T_{\text{in}}} dt_\mathfrak{n} \prod_{j=1}^K |k_{s_j,x}| \tag{2.4.126}$$

Keeping notation the same as Lemma 3.3.26, if we can show that

$$|F_T(t, \{a_{\mathfrak{n}}\}_{\mathfrak{n}\in T_{\text{in}}}, \{b_{\mathfrak{n}}\}_{\mathfrak{n}\in T_{\text{in}}})| \lesssim \sum_{\{d_{\mathfrak{n}}\}_{\mathfrak{n}\in T_{\text{in}}}\in\{0,1\}^{l(T)}} \prod_{\mathfrak{n}\in T_{\text{in}}} \frac{2^{-\frac{\tau_{\mathfrak{n}}}{2}}}{|q_{\mathfrak{n}}| + T_{\max}^{-1}} \prod_{\mathfrak{e}\in T_{\text{in}}} p_{\mathfrak{e}}, \tag{2.4.127}$$

then this lemma can be proved by taking $a_{\mathfrak{n}} = \Omega_{\mathfrak{n}}$, $b_{\mathfrak{n}} = |k_{\mathfrak{e}}|^2$ in (2.4.127).

Therefore, it suffices to prove (2.4.127).

We run a similar inductive integration by parts argument of Lemma 3.3.26. If the roots $\mathfrak{r}$ is not a substitution node, then the same argument of Lemma 3.3.26 works. Therefore, we just consider the case that the roots $\mathfrak{r}$ is a substitution node.

Using the same calculation as (2.4.11), we get

$$\begin{aligned} F_T(t) = \int_{\cup_{\mathfrak{n}\in T_{\text{in}}} B_{\mathfrak{n}}} & e^{it_{\mathfrak{r}}(a_{\mathfrak{r}}+T_{\max}^{-1}\operatorname{sgn}(a_{\mathfrak{r}}))-\nu(t-t_{\mathfrak{r}})b_{\mathfrak{r}}} e^{-iT_{\max}^{-1}t_{\mathfrak{r}}\operatorname{sgn}(a_{\mathfrak{r}})} \\ & e^{\sum_{\mathfrak{n}\in T_{\text{in},1}\cup T_{\text{in},2}} it_{\mathfrak{n}}a_{\mathfrak{n}}-\nu(t_{\widehat{\mathfrak{n}}}-t_{\mathfrak{n}})b_{\mathfrak{n}}} \left(dt_{\mathfrak{r}} \prod_{j=1}^{2} \prod_{\mathfrak{n}\in T_{\text{in},j}} dt_{\mathfrak{n}}\right) |k_{s_1,x}| \prod_{j=2}^{K} |k_{s_j,x}| \end{aligned} \tag{2.4.128}$$

We do integration by parts in above integrals using Stokes formula. Notice that for $t_{\mathfrak{r}}$, there are four inequality constrains, $t_{\mathfrak{r}} \le t - 2^{-\tau_1}T_{\max}$, $t_{\mathfrak{r}} \ge t - 2^{-\tau_1-1}T_{\max}$ and $t_{\mathfrak{r}} \ge t_{\mathfrak{n}_1}, t_{\mathfrak{n}_2}$. Notice that the first two come from $(t-t_{\mathfrak{r}})/T_{\max} \in [2^{-\tau_1}, 2^{-\tau_1-1}]$.

$$\begin{aligned} F_T(t) = & \frac{|k_{s_1,x}|}{ia_{\mathfrak{r}} + iT_{\max}^{-1}\operatorname{sgn}(a_{\mathfrak{r}}) + \nu b_{\mathfrak{r}}} \int_{\cup_{\mathfrak{n}\in T_{\text{in}}} B_{\mathfrak{n}}} \frac{d}{dt_{\mathfrak{r}}} e^{it_{\mathfrak{r}}(a_{\mathfrak{r}}+T_{\max}^{-1}\operatorname{sgn}(a_{\mathfrak{r}}))-\nu(t-t_{\mathfrak{r}})b_{\mathfrak{r}}} \\ & e^{-iT_{\max}^{-1}t_{\mathfrak{r}}\operatorname{sgn}(a_{\mathfrak{r}})} e^{\sum_{\mathfrak{n}\in T_{\text{in},1}\cup T_{\text{in},2}} it_{\mathfrak{n}}a_{\mathfrak{n}}-\nu(t_{\widehat{\mathfrak{n}}}-t_{\mathfrak{n}})b_{\mathfrak{n}}} \left(dt_{\mathfrak{r}} \prod_{j=1}^{2} \prod_{\mathfrak{n}\in T_{\text{in},j}} dt_{\mathfrak{n}}\right) \prod_{j=2}^{K} |k_{s_j,x}| \end{aligned} \tag{2.4.129}$$

$$\begin{aligned} = & \frac{|k_{s_1,x}|}{ia_{\mathfrak{r}} + iT_{\max}^{-1}\operatorname{sgn}(a_{\mathfrak{r}}) + \nu b_{\mathfrak{r}}} \\ & \left(\int_{\cup_{\mathfrak{n}\in T_{\text{in}}} B_{\mathfrak{n}},\ t_{\mathfrak{r}}=t-2^{-\tau_1}T_{\max}} - \int_{\cup_{\mathfrak{n}\in T_{\text{in}}} B_{\mathfrak{n}},\ t_{\mathfrak{r}}=t-2^{-\tau_1-1}T_{\max}} - \int_{\cup_{\mathfrak{n}\in T_{\text{in}}} B_{\mathfrak{n}},\ t_{\mathfrak{r}}=t_{\mathfrak{n}_1}} - \int_{\cup_{\mathfrak{n}\in T_{\text{in}}} B_{\mathfrak{n}},\ t_{\mathfrak{r}}=t_{\mathfrak{n}_2}}\right) \\ & e^{it_{\mathfrak{r}}(a_{\mathfrak{r}}+T_{\max}^{-1}\operatorname{sgn}(a_{\mathfrak{r}}))-\nu(t-t_{\mathfrak{r}})b_{\mathfrak{r}}} e^{-iT_{\max}^{-1}t_{\mathfrak{r}}\operatorname{sgn}(a_{\mathfrak{r}})} e^{\sum_{\mathfrak{n}\in T_{\text{in},1}\cup T_{\text{in},2}} it_{\mathfrak{n}}a_{\mathfrak{n}}-\nu(t_{\widehat{\mathfrak{n}}}-t_{\mathfrak{n}})b_{\mathfrak{n}}} \left(dt_{\mathfrak{r}} \prod_{j=1}^{2} \prod_{\mathfrak{n}\in T_{\text{in},j}} dt_{\mathfrak{n}}\right) \prod_{j=2}^{K} |k_{s_j,x}| \end{aligned}$$

$$-\frac{|k_{s_1,x}|}{ia_{\mathfrak{r}}+iT_{\max}^{-1}\mathrm{sgn}(a_{\mathfrak{r}})+\nu b_{\mathfrak{r}}}\int_{\cup_{\mathfrak{n}\in T_{\mathrm{in}}}B_{\mathfrak{n}}} e^{it_{\mathfrak{r}}(a_{\mathfrak{r}}+T_{\max}^{-1}\,\mathrm{sgn}(a_{\mathfrak{r}}))-\nu(t-t_{\mathfrak{r}})b_{\mathfrak{r}}}$$
$$\frac{d}{dt_{\mathfrak{r}}}(e^{-iT_{\max}^{-1}t_{\mathfrak{r}}\mathrm{sgn}(a_{\mathfrak{r}})})e^{\sum_{\mathfrak{n}\in T_{\mathrm{in},1}\cup T_{\mathrm{in},2}}it_{\mathfrak{n}}a_{\mathfrak{n}}-\nu(t_{\widehat{\mathfrak{n}}}-t_{\mathfrak{n}})b_{\mathfrak{n}}}\left(dt_{\mathfrak{r}}\prod_{j=1}^{2}\prod_{\mathfrak{n}\in T_{\mathrm{in},j}}dt_{\mathfrak{n}}\right)\prod_{j=2}^{K}|k_{s_j,x}|$$

$$=\frac{|k_{s_1,x}|}{ia_{\mathfrak{r}}+iT_{\max}^{-1}\mathrm{sgn}(a_{\mathfrak{r}})+\nu b_{\mathfrak{r}}}(F_I-F_{I'}-\widetilde{F}_{T^{(1)}}-\widetilde{F}_{T^{(2)}}-F_{II})$$

We now derive upper bounds for F_I, $F_{I'}$, $\widetilde{F}_{T^{(1)}}$, $\widetilde{F}_{T^{(2)}}$, F_{II}.

The argument of F_I and $F_{I'}$ is very similar, so we just consider F_I. By a direct calculation, we know that $\frac{|k_{s_1,x}|}{ia_{\mathfrak{r}}+iT_{\max}^{-1}\mathrm{sgn}(a_{\mathfrak{r}})+\nu b_{\mathfrak{r}}}F_I(t)$ equals to

$$\begin{aligned}
&\frac{|k_{s_1,x}|}{ia_{\mathfrak{r}}+iT_{\max}^{-1}\mathrm{sgn}(a_{\mathfrak{r}})+\nu b_{\mathfrak{r}}}\int_{\cup_{\mathfrak{n}\in T_{\mathrm{in}}}B_{\mathfrak{n}},\ t_{\mathfrak{r}}=t-2^{-\tau_1}T_{\max}} e^{it_{\mathfrak{r}}(a_{\mathfrak{r}}+T_{\max}^{-1}\,\mathrm{sgn}(a_{\mathfrak{r}}))-\nu(t-t_{\mathfrak{r}})b_{\mathfrak{r}}}e^{-iT_{\max}^{-1}t_{\mathfrak{r}}\mathrm{sgn}(a_{\mathfrak{r}})}\\
&e^{\sum_{\mathfrak{n}\in T_{\mathrm{in},1}\cup T_{\mathrm{in},2}}it_{\mathfrak{n}}a_{\mathfrak{n}}-\nu(t_{\widehat{\mathfrak{n}}}-t_{\mathfrak{n}})b_{\mathfrak{n}}}\left(dt_{\mathfrak{r}}\prod_{j=1}^{2}\prod_{\mathfrak{n}\in T_{\mathrm{in},j}}dt_{\mathfrak{n}}\right)\prod_{j=2}^{K}|k_{s_j,x}|\\
=&\frac{|k_{s_1,x}|}{ia_{\mathfrak{r}}+iT_{\max}^{-1}\mathrm{sgn}(a_{\mathfrak{r}})+\nu b_{\mathfrak{r}}}e^{i(t-2^{-\tau_1}T_{\max})a_{\mathfrak{r}}-\nu T_{\max}2^{-\tau_1}b_{\mathfrak{r}}}\int_{\cup_{\mathfrak{n}\in T_{\mathrm{in},1}\cup T_{\mathrm{in},2}}B_{\mathfrak{n}},\ t_{\mathfrak{r}_1},t_{\mathfrak{r}_2}\lesssim t-2^{-\tau_1}T_{\max}}\\
&e^{\sum_{\mathfrak{n}\in T_{\mathrm{in},1}\cup T_{\mathrm{in},2}}it_{\mathfrak{n}}a_{\mathfrak{n}}-\nu(t_{\widehat{\mathfrak{n}}}-t_{\mathfrak{n}})b_{\mathfrak{n}}}\left(dt_{\mathfrak{r}}\prod_{j=1}^{2}\prod_{\mathfrak{n}\in T_{\mathrm{in},j}}dt_{\mathfrak{n}}\right)\prod_{j=2}^{K}|k_{s_j,x}|\\
=&O\left(\left|\frac{|k_{s_1,x}|}{ia_{\mathfrak{r}}+iT_{\max}^{-1}\mathrm{sgn}(a_{\mathfrak{r}})+\nu b_{\mathfrak{r}}}e^{i(t-2^{-\tau_1}T_{\max})a_{\mathfrak{r}}-\nu T_{\max}2^{-\tau_1}b_{\mathfrak{r}}}\right||F_{T_1}(t)||F_{T_2}(t)|\right)\\
=&O\left(\frac{(|k_{s_1,x}|+1)e^{-2^{-\tau_1}|k_{s_1}|^2}}{|q_{\mathfrak{r}}|+T_{\max}^{-1}}|F_{T_1}(t)||F_{T_2}(t)|\frac{|k_{s_1,x}|}{|k_{s_1,x}|+1}\right)=O\left(\frac{2^{-\frac{\tau_1}{2}}|F_{T_1}(t)||F_{T_2}(t)|}{|q_{\mathfrak{r}}|+T_{\max}^{-1}}p_{\mathfrak{e}}\right)
\end{aligned} \tag{2.4.130}$$

Here in the last line we use the fact that $b_{\mathfrak{r}}=|k_{s_1}|^2$, $\nu T_{\max}\gtrsim 1$ (2.1.3) and $(|k_{s_1,x}|+1)e^{-2^{-\tau_1}|k_{s_1}|^2}\lesssim 2^{-\frac{\tau_1}{2}}$. By above equation and the induction assumption, we know that $\frac{|k_{s_1,x}|}{ia_{\mathfrak{r}}+iT_{\max}^{-1}\mathrm{sgn}(a_{\mathfrak{r}})+\nu b_{\mathfrak{r}}}F_I(t)$ can be bounded by the right hand side of (2.4.120).

Now we find upper bounds of $\widetilde{F}_{T^{(1)}}$ and $\widetilde{F}_{T^{(2)}}$. Let $T^{(j)}$, $j=1,2$ are trees that is obtained by deleting the root $\mathfrak{r}$, adding edges connecting $\mathfrak{n}_j$ with another node and defining $\mathfrak{n}_j$ to be the new root. For $T^{(j)}$, we can define the term $F_{T^{(j)}}$ by (2.4.126). By a direct calculation, we know that

$\frac{|k_{s_1,x}|}{ia_{\mathfrak{r}}+iT_{\max}^{-1}\mathrm{sgn}(a_{\mathfrak{r}})+\nu b_{\mathfrak{r}}}\widetilde{F}_{T^{(1)}}(t)$ equals to

$$\begin{aligned}
&\frac{|k_{s_1,x}|}{ia_{\mathfrak{r}}+iT_{\max}^{-1}\mathrm{sgn}(a_{\mathfrak{r}})+\nu b_{\mathfrak{r}}}\int_{\cup_{\mathfrak{n}\in T_{\mathrm{in}}}B_{\mathfrak{n}},\ t_{\mathfrak{r}}=t_{\mathfrak{n}_j}} e^{it_{\mathfrak{r}}(a_{\mathfrak{r}}+T_{\max}^{-1}\mathrm{sgn}(a_{\mathfrak{r}}))-\nu(t-t_{\mathfrak{r}})b_{\mathfrak{r}}}e^{-iT_{\max}^{-1}t_{\mathfrak{r}}\mathrm{sgn}(a_{\mathfrak{r}})}\\
&e^{\sum_{\mathfrak{n}\in T_{\mathrm{in},1}\cup T_{\mathrm{in},2}} it_{\mathfrak{n}}a_{\mathfrak{n}}-\nu(t_{\hat{\mathfrak{n}}}-t_{\mathfrak{n}})b_{\mathfrak{n}}}\left(dt_{\mathfrak{r}}\prod_{j=1}^{2}\prod_{\mathfrak{n}\in T_{\mathrm{in},j}}dt_{\mathfrak{n}}\right)\prod_{j=2}^{K}|k_{s_j,x}|\\
=&\frac{|k_{s_1,x}|}{ia_{\mathfrak{r}}+iT_{\max}^{-1}\mathrm{sgn}(a_{\mathfrak{r}})+\nu b_{\mathfrak{r}}}e^{-\nu T_{\max}2^{-\tau_1}b_{\mathfrak{r}}}\int_{\cup_{\mathfrak{n}\in T_{\mathrm{in}}}B_{\mathfrak{n}}} e^{it_{\mathfrak{n}_j}(a_{\mathfrak{r}}+T_{\max}^{-1}\mathrm{sgn}(a_{\mathfrak{r}}))-\nu((t-T_{\max}2^{-\tau_1})-t_{\mathfrak{n}_j})b_{\mathfrak{r}}}\\
&e^{\sum_{\mathfrak{n}\in T_{\mathrm{in},1}\cup T_{\mathrm{in},2}} it_{\mathfrak{n}}a_{\mathfrak{n}}-\nu(t_{\hat{\mathfrak{n}}}-t_{\mathfrak{n}})b_{\mathfrak{n}}}\left(dt_{\mathfrak{r}}\prod_{j=1}^{2}\prod_{\mathfrak{n}\in T_{\mathrm{in},j}}dt_{\mathfrak{n}}\right)\prod_{j=2}^{K}|k_{s_j,x}|\\
=&O\left(\frac{|k_{s_1,x}|e^{-2^{-\tau_1}|k_{s_1}|^2}}{|q_{\mathfrak{r}}|+T_{\max}^{-1}}|F_{T^{(j)}}(t)|\right)=O\left(\frac{2^{-\frac{\tau_1}{2}}|F_{T^{(j)}}(t)|}{|q_{\mathfrak{r}}|+T_{\max}^{-1}}p_{\mathfrak{e}}\right)
\end{aligned}\tag{2.4.131}$$

We can apply the induction assumption to $F_{T^{(j)}}$ and show that $\frac{|k_{s_1,x}|}{ia_{\mathfrak{r}}+iT_{\max}^{-1}\mathrm{sgn}(a_{\mathfrak{r}})+\nu b_{\mathfrak{r}}}F_{T^{(j)}}$ can be bounded by the right hand side of (2.4.120).

Another direct calculation gives that

$$F_{II}(t)=\int_{(t-t_{\mathfrak{r}})/T_{\max}\in[2^{-\tau_1},2^{-\tau_1-1}]} e^{it_{\mathfrak{r}}(a_{\mathfrak{r}}+T_{\max}^{-1}\mathrm{sgn}(a_{\mathfrak{r}}))-\nu(t-t_{\mathfrak{r}})b_{\mathfrak{r}}}\frac{d}{dt_{\mathfrak{r}}}(e^{-iT_{\max}^{-1}t_{\mathfrak{r}}\mathrm{sgn}(a_{\mathfrak{r}})})F_{T_1}(t_{\mathfrak{r}})F_{T_2}(t_{\mathfrak{r}})dt_{\mathfrak{r}}.\tag{2.4.132}$$

Apply the induction assumption

$$\begin{aligned}
&\left|\frac{|k_{s_1,x}|}{ia_{\mathfrak{r}}+iT_{\max}^{-1}\mathrm{sgn}(a_{\mathfrak{r}})+\nu b_{\mathfrak{r}}}F_{II}(t)\right|\\
\le&\frac{T_{\max}^{-1}|k_{s_1,x}|}{|q_{\mathfrak{r}}|+T_{\max}^{-1}}\int_{(t-t_{\mathfrak{r}})/T_{\max}\in[2^{-\tau_1},2^{-\tau_1-1}]} e^{-\nu(t-t_{\mathfrak{r}})b_{\mathfrak{r}}}|F_{T_1}(t_{\mathfrak{r}})||F_{T_2}(t_{\mathfrak{r}})|dt_{\mathfrak{r}}\\
\le&\frac{T_{\max}^{-1}|k_{s_1,x}|e^{-\nu T_{\max}2^{-\tau_1}|k_{s_1}|^2}}{|q_{\mathfrak{r}}|+T_{\max}^{-1}}\prod_{j=1}^{2}\left(\sum_{\{d_{\mathfrak{n}}\}_{\mathfrak{n}\in T_{\mathrm{in},j}}\in\{0,1\}^{l(T_j)}}\prod_{\mathfrak{n}\in T_{\mathrm{in},j}}\frac{2^{-\frac{\tau_{\mathfrak{n}}}{2}}}{|q_{\mathfrak{n}}|+T_{\max}^{-1}}\prod_{\mathfrak{e}\in T_{\mathrm{in},j}}p_{\mathfrak{e}}\right)\\
\le&\sum_{\{d_{\mathfrak{n}}\}_{\mathfrak{n}\in T_{\mathrm{in}}}\in\{0,1\}^{l(T)}}\prod_{\mathfrak{n}\in T_{\mathrm{in}}}\frac{2^{-\frac{\tau_{\mathfrak{n}}}{2}}}{|q_{\mathfrak{n}}|+T_{\max}^{-1}}\prod_{\mathfrak{e}\in T_{\mathrm{in}}}p_{\mathfrak{e}}.
\end{aligned}\tag{2.4.133}$$

Therefore, we get an upper bound for F_{II}.

Combining the bounds of F_I, $F_{I'}$, $\widetilde{F}_{T^{(1)}}$, $\widetilde{F}_{T^{(2)}}$, F_{II}, we conclude that F_T can be bounded by the right hand side of (2.4.120) and thus complete the proof of Lemma 3.3.26. □

The upper bound for expectation of entries

In this section, we prove the Proposition 2.4.36 which gives an upper bound for $\mathbb{E}|H^{\tau_1\cdots\tau_K}_{Tkk'}|^2$. This lemma is an analog of Proposition 2.4.26.

Proposition 2.4.36. *Assume that α satisfies* (2.1.2). *For any $\theta > 0$, we have*

$$\sup_k \mathbb{E}|H^{\tau_1\cdots\tau_K}_{Tkk'}|^2 \lesssim L^{O(l(T)\theta)} 2^{-\frac{1}{2}\sum_{j=1}^K \tau_j} \rho^{2l(T)}. \tag{2.4.134}$$

and $\mathbb{E}|H^{\tau_1\cdots\tau_K}_{Tkk'}|^2 = 0$ if $|k-k'| \gtrsim 1$.

Proof. We first find a formula of $\mathbb{E}|H^{\tau_1\cdots\tau_K}_{Tkk'}|^2$ which is similar to (2.4.33) and (2.4.34).

A direct calculation gives

$$\begin{aligned}\mathbb{E}|H^{\tau_1\cdots\tau_K}_{Tkk'}|^2 =& \mathbb{E}(H^{\tau_1\cdots\tau_K}_{Tkk'}\overline{H^{\tau_1\cdots\tau_K}_{Tkk'}}) = \left(\frac{\lambda}{L^d}\right)^{2l(T)} \sum_{k_1,\,k_2,\,\cdots,\,k_{l(T)}} \sum_{k'_1,\,k'_2,\,\cdots,\,k'_{l(T)}} \\ & H^{\tau_1\cdots\tau_K}_{Tk_1\cdots k_{l(T)}kk'}\overline{H^{\tau_1\cdots\tau_K}_{Tk'_1\cdots k'_{l(T)}kk'}}\mathbb{E}\Big(\xi_{k_1}\xi_{k_2}\cdots\xi_{k_{l(T)}}\xi_{k'_1}\xi_{k'_2}\cdots\xi_{k'_{l(T)}}\Big)\end{aligned} \tag{2.4.135}$$

Applying Wick theorem (Lemma 3.3.4) to (2.4.135), we get

$$\mathbb{E}|H^{\tau_1\cdots\tau_K}_{Tkk'}|^2 = \left(\frac{\lambda}{L^d}\right)^{2l(T)} \sum_{p\in\mathcal{P}(\{k_1,\cdots,k_{l(T)},k'_1,\cdots,k'_{l(T)}\})} Term(T,p)_{op,k,k'}. \tag{2.4.136}$$

and

$$\begin{aligned}&Term(T,p)_{op,k,k'} \\ =& \sum_{k_1,\,k_2,\,\cdots,\,k_{l(T)}} \sum_{k'_1,\,k'_2,\,\cdots,\,k'_{l(T)}} H^{\tau_1\cdots\tau_K}_{Tk_1\cdots k_{l(T)}kk'}\overline{H^{\tau_1\cdots\tau_K}_{Tk'_1\cdots k'_{l(T)}kk'}}\delta_p(k_1,\cdots,k_{l(T)},k'_1,\cdots,k'_{l(T)})\sqrt{n_{\text{in}}(k_1)}\cdots.\end{aligned} \tag{2.4.137}$$

Since $\alpha = \frac{\lambda}{L^{\frac{d}{2}}}$, $\frac{\lambda}{L^d} = \alpha L^{-\frac{d}{2}}$. Since the number of elements in $\mathcal{P}$ can be bounded by a constant, by Lemma 2.4.37 proved below, we get

$$\begin{aligned}\mathbb{E}|H^{\tau_1\cdots\tau_K}_{Tkk'}|^2 \lesssim& (\alpha L^{-\frac{d}{2}})^{2l(T)} L^{O(l(T)\theta)} 2^{-\frac{1}{2}\sum_{j=1}^K\tau_j} (L^d T^{-1}_{\max})^{l(T)} T^{2l(T)}_{\max} \\ =& L^{O(l(T)\theta)} 2^{-\frac{1}{2}\sum_{j=1}^K\tau_j} \rho^{2l(T)}\end{aligned} \tag{2.4.138}$$

By Lemma 2.4.27 below $Term(T,p)_{op,k,k'} = 0$ if $|k-k'| \gtrsim 1$, we know that the same is true for $\mathbb{E}|H^{\tau_1\cdots\tau_K}_{Tkk'}|^2 = 0$.

Therefore, we complete the proof of this proposition. □

Lemma 2.4.37. *Let $Q = L^d T_{max}^{-1}$ be the same as in Proposition 3.3.22. Assume that α satisfies (2.1.2) and $n_{\text{in}} \in C_0^\infty(\mathbb{R}^d)$ is compactly supported. Then for any $\theta > 0$, we have*

$$\sup_k |Term(T,p)_{op,k,k'}| \le L^{O(l(T)\theta)} 2^{-\frac{1}{2}\sum_{j=1}^K \tau_j} Q^{l(T)} T_{max}^{2l(T)}. \tag{2.4.139}$$

and $Term(T,p)_{op,k,k'} = 0$ if $|k-k'| \gtrsim 1$.

Proof. By (2.4.137), we get

$$\begin{aligned} &Term(T,p)_{op,k,k'} \\ &= \sum_{k_1,\, k_2,\, \cdots,\, k_{l(T)}} \sum_{k_1',\, k_2',\, \cdots,\, k_{l(T)}'} H^{\tau_1\cdots\tau_K}_{Tk_1\cdots k_{l(T)}kk'} \overline{H^{\tau_1\cdots\tau_K}_{Tk_1'\cdots k_{l(T)}'kk'}} \delta_p(k_1,\cdots,k_{l(T)},k_1',\cdots,k_{l(T)}') \sqrt{n_{\text{in}}(k_1)} \cdots . \end{aligned} \tag{2.4.140}$$

Since n_{in} are compactly supported and there are bounded many of them in $Term(T,p)_{op,k,k'}$, by $k_1 + k_2 + \cdots + k_{l(T)} = k - k'$, we know that $Term(T,p)_{op,k,k'} = 0$ if $|k-k'| \gtrsim 1$.

By (2.4.120), we get

$$|H^{\tau_1\cdots\tau_K}_{Tk_1\cdots k_l kk'}| \lesssim \sum_{\{d_\mathfrak{n}\}_{\mathfrak{n}\in T_{\text{in}}} \in \{0,1\}^{l(T)}} \prod_{\mathfrak{n}\in T_{\text{in}}} \frac{2^{-\frac{\tau_\mathfrak{n}}{2}}}{|q_\mathfrak{n}| + T_{\max}^{-1}} \ \delta_{\cap_{\mathfrak{n}\in T_{\text{in}}}\{S_\mathfrak{n}=0\}} \prod_{\mathfrak{e}\in T_{\text{in}}} p_\mathfrak{e}. \tag{2.4.141}$$

Define $[c_{\mathfrak{n},\widetilde{\mathfrak{n}}}]$, $\mathscr{M}(T)$, $c(\Omega)$ in the same way as the proof of Lemma 2.4.27. We can apply the same derivation of (2.4.79) to obtain

$$\begin{aligned} |Term(T,p)_{op,k,k'}| \lesssim & \sum_{\substack{k_1,\cdots,k_{l(T)},k_1',\cdots,k_{l(T)}' \\ |k_j|,|k_j'|\lesssim 1,\forall j}} \sum_{c\in\mathscr{M}(T)} \prod_{\mathfrak{n}\in T_{\text{in}}} \frac{2^{-\frac{\tau_\mathfrak{n}}{2}}}{|c(\Omega)_\mathfrak{n}| + T_{\max}^{-1}} \ \delta_{\cap_{\mathfrak{n}\in T_{\text{in}}}\{S_\mathfrak{n}=0\}} \prod_{\mathfrak{e}\in T_{\text{in}}} p_\mathfrak{e} \\ & \sum_{c'\in\mathscr{M}(T)} \prod_{\mathfrak{n}'\in T_{\text{in}}} \frac{2^{-\frac{\tau_\mathfrak{n}}{2}}}{|c'(\Omega)_{\mathfrak{n}'}| + T_{\max}^{-1}} \prod_{\mathfrak{e}'\in T_{\text{in}}} p_{\mathfrak{e}'} \ \delta_{\cap_{\mathfrak{n}'\in T_{\text{in}}}\{S_{\mathfrak{n}'}=0\}} \delta_p(k_1,\cdots,k_{l(T)},k_1',\cdots,k_{l(T)}') \end{aligned} \tag{2.4.142}$$

We obviously have the following inequality

$$
\begin{aligned}
&|Term(T,p)_{op,k,k'}| \lesssim \sum_{k'} |Term(T,p)_{op,k,k'}| \\
\lesssim & \sum_{\substack{k_1,\cdots,k_{l(T)},k'_1,\cdots,k'_{l(T)},k' \\ |k_j|,|k'_j|\lesssim 1,\forall j,\ |k'-k|\lesssim 1}} \sum_{c\in\mathscr{M}(T)} \prod_{\mathfrak{n}\in T_{\text{in}}} \frac{2^{-\frac{\tau_{\mathfrak{n}}}{2}}}{|c(\Omega)_{\mathfrak{n}}|+T_{\max}^{-1}}\ \delta_{\cap_{\mathfrak{n}\in T_{\text{in}}}\{S_{\mathfrak{n}}=0\}} \prod_{\mathfrak{e}\in T_{\text{in}}} p_{\mathfrak{e}} \\
& \sum_{c'\in\mathscr{M}(T)} \prod_{\mathfrak{n}'\in T_{\text{in}}} \frac{2^{-\frac{\tau_{\mathfrak{n}}}{2}}}{|c'(\Omega)_{\mathfrak{n}'}|+T_{\max}^{-1}} \prod_{\mathfrak{e}'\in T_{\text{in}}} p_{\mathfrak{e}'}\ \delta_{\cap_{\mathfrak{n}'\in T_{\text{in}}}\{S_{\mathfrak{n}'}=0\}} \delta_p(k_1,\cdots,k_{l(T)},k'_1,\cdots,k'_{l(T)})
\end{aligned}
\tag{2.4.143}
$$

Switch the order of summations and products in (2.4.79), then we get

$$
\begin{aligned}
|Term(T,p)_{op,k,k'}| \lesssim & \sum_{\substack{k_1,\cdots,k_{l(T)},k'_1,\cdots,k'_{l(T)},k' \\ |k_j|,|k'_j|\lesssim 1,\forall j,\ |k'-k|\lesssim 1}} \sum_{c,c'\in\mathscr{M}(T)} \prod_{\mathfrak{n},\mathfrak{n}'\in T_{\text{in}}} \frac{2^{-\frac{\tau_{\mathfrak{n}}}{2}}}{|c(\Omega)_{\mathfrak{n}}|+T_{\max}^{-1}} \frac{2^{-\frac{\tau_{\mathfrak{n}}}{2}}}{|c'(\Omega)_{\mathfrak{n}'}|+T_{\max}^{-1}} \\
& \prod_{\mathfrak{e},\mathfrak{e}'\in T_{\text{in}}} (p_{\mathfrak{e}}p_{\mathfrak{e}'})\ \delta_{\cap_{\mathfrak{n},\mathfrak{n}'\in T_{\text{in}}}\{S_{\mathfrak{n}}=0,S_{\mathfrak{n}'}=0\}} \delta_p(k_1,\cdots,k_{l(T)},k'_1,\cdots,k'_{l(T)}).
\end{aligned}
\tag{2.4.144}
$$

Consider a tree T with a $\square$ nodes and a pairing $p\in\mathcal{P}(\{k_1,\cdots,k_{l(T)},k'_1,\cdots,k'_{l(T)}\})$. p can be viewed as a pairing of all star nodes of two copies of T. A couple $\mathcal{C}$ can be constructed by merging all paired star nodes according p and merging the two $\square$ nodes. As in (2.4.81), we can show that

$$
\sum_{c,c'\in\mathscr{M}(T)} = \sum_{c\in\mathscr{M}(\mathcal{C})}, \qquad \prod_{\mathfrak{n},\mathfrak{n}'\in T_{\text{in}}} = \prod_{\mathfrak{n}\in\mathcal{C}}, \qquad \prod_{\mathfrak{e},\mathfrak{e}'\in T_{\text{in}}} = \prod_{\mathfrak{e}\in\mathcal{C}_{\text{norm}}}, \qquad \cap_{\mathfrak{n},\mathfrak{n}'\in T_{\text{in}}} = \cap_{\mathfrak{n}\in\mathcal{C}}. \tag{2.4.145}
$$

The analog of (2.4.82) is

$$
|Term(T,p)_{op,k,k'}| \lesssim \sum_{\substack{k_1,\cdots,k_{l(T)},k'_1,\cdots,k'_{l(T)},k' \\ |k_j|,|k'_j|\lesssim 1,\forall j,\ |k'-k|\lesssim 1}} \sum_{c\in\mathscr{M}(\mathcal{C})} \prod_{\mathfrak{n}\in\mathcal{C}} \frac{2^{-\frac{\tau_{\mathfrak{n}}}{2}}}{|c(\Omega)_{\mathfrak{n}}|+T_{\max}^{-1}} \prod_{\mathfrak{e}\in\mathcal{C}_{\text{norm}}} p_{\mathfrak{e}}\ \delta_{\cap_{\mathfrak{n}\in\mathcal{C}}\{S_{\mathfrak{n}}=0\}} \tag{2.4.146}
$$

The rest part of the proof is exactly the same as the proof of Lemma 2.4.27 after (2.4.82). For completeness, we include a sketch.

As (2.4.83), we have

$$
\sum_{\substack{k_1,\cdots,k_{l(T)+1},k'_1,\cdots,k'_{l(T)+1} \\ \cap_{\mathfrak{n}\in\mathcal{C}}\{S_{\mathfrak{n}}=0\}}} = \sum_{\kappa_{\mathfrak{e}}\in\mathcal{D}(\alpha,1)} \sum_{\sigma_{\mathfrak{n}}\in\mathbb{Z}_{T_{\max}}} \sum_{Eq(\mathcal{C},\{\sigma_{\mathfrak{n}}\}_{\mathfrak{n}},\{\kappa_{\mathfrak{e}}\}_{\mathfrak{e}},k)}, \tag{2.4.147}
$$

which implies that

$$|Term(T,p)_{op,k,k'}| \lesssim \sum_{\kappa_{\mathfrak{e}}\in\mathcal{D}(\alpha,1)} \sum_{\sigma_{\mathfrak{n}}\in\mathbb{Z}_{T_{\max}}} \sum_{Eq(\mathcal{C},\{\sigma_{\mathfrak{n}}\}_{\mathfrak{n}},\{\kappa_{\mathfrak{e}}\}_{\mathfrak{e}},k)} \sum_{c\in\mathscr{M}(\mathcal{C})} \prod_{\mathfrak{n}\in\mathcal{C}} \frac{2^{-\frac{\tau_{\mathfrak{n}}}{2}}}{|c(\Omega)_{\mathfrak{n}}|+T_{\max}^{-1}} \prod_{\mathfrak{e}\in\mathcal{C}_{\text{norm}}} p_{\mathfrak{e}} \tag{2.4.148}$$

As (2.4.85), we get

$$\begin{aligned}
&Term(T,p)_k \\
\lesssim &\sum_{\kappa_{\mathfrak{e}}\in\mathcal{D}(\alpha,1)} \sum_{\sigma_{\mathfrak{n}}\in\mathbb{Z}_{T_{\max}}} \sum_{Eq(\mathcal{C},\{\sigma_{\mathfrak{n}}\}_{\mathfrak{n}},\{\kappa_{\mathfrak{e}}\}_{\mathfrak{e}},k)} \sum_{c\in\mathscr{M}(\mathcal{C})} \prod_{\mathfrak{n}\in\mathcal{C}} \frac{2^{-\frac{\tau_{\mathfrak{n}}}{2}}}{|c(\Omega)_{\mathfrak{n}}|+T_{\max}^{-1}} \prod_{\mathfrak{e}\in\mathcal{C}_{\text{norm}}} p_{\mathfrak{e}} \\
\lesssim &\sum_{c\in\mathscr{M}(\mathcal{C})} \sum_{\kappa_{\mathfrak{e}}\in\mathcal{D}(\alpha,1)} \sum_{\substack{\sigma_{\mathfrak{n}}\in\mathbb{Z}_{T_{\max}}\\ |\sigma_{\mathfrak{n}}|\lesssim 1}} \prod_{\mathfrak{n}\in\mathcal{C}} \frac{2^{-\frac{\tau_{\mathfrak{n}}}{2}}}{|c(\{\sigma_{\mathfrak{n}}\})_{\mathfrak{n}}|+T_{\max}^{-1}} \sum_{Eq(\mathcal{C},\{\sigma_{\mathfrak{n}}\}_{\mathfrak{n}},\{\kappa_{\mathfrak{e}}\}_{\mathfrak{e}},k)} 1 \prod_{\mathfrak{e}\in\mathcal{C}_{\text{norm}}} \frac{\kappa_{\mathfrak{e}}}{\kappa_{\mathfrak{e}}+1} \\
\lesssim &\sum_{c\in\mathscr{M}(\mathcal{C})} \sum_{\kappa_{\mathfrak{e}}\in\mathcal{D}(\alpha,1)} \sum_{\substack{\sigma_{\mathfrak{n}}\in\mathbb{Z}_{T_{\max}}\\ |\sigma_{\mathfrak{n}}|\lesssim 1}} \prod_{\mathfrak{n}\in\mathcal{C}} \frac{2^{-\frac{\tau_{\mathfrak{n}}}{2}}}{|c(\{\sigma_{\mathfrak{n}}\})_{\mathfrak{n}}|+T_{\max}^{-1}} \#Eq(\mathcal{C},\{\sigma_{\mathfrak{n}}\}_{\mathfrak{n}},\{\kappa_{\mathfrak{e}}\}_{\mathfrak{e}},k) \prod_{\mathfrak{e}\in\mathcal{C}_{\text{norm}}} \frac{\kappa_{\mathfrak{e}}}{\kappa_{\mathfrak{e}}+1} \\
\lesssim &\sum_{c\in\mathscr{M}(\mathcal{C})} \sum_{\kappa_{\mathfrak{e}}\in\mathcal{D}(\alpha,1)} \sum_{\substack{\sigma_{\mathfrak{n}}\in\mathbb{Z}_{T_{\max}}\\ |\sigma_{\mathfrak{n}}|\lesssim 1}} \prod_{\mathfrak{n}\in\mathcal{C}} \frac{2^{-\frac{\tau_{\mathfrak{n}}}{2}}}{|c(\{\sigma_{\mathfrak{n}}\})_{\mathfrak{n}}|+T_{\max}^{-1}} L^{O(n\theta)} Q^{\frac{n}{2}} \prod_{\mathfrak{e}\in\mathcal{C}_{\text{norm}}} \frac{\kappa_{\mathfrak{e}}}{\kappa_{\mathfrak{e}}+1} \prod_{\mathfrak{e}\in\mathcal{C}_{\text{norm}}} \kappa_{\mathfrak{e}}^{-1}
\end{aligned} \tag{2.4.149}$$

Here in the last inequality we applied (2.4.44) in Proposition 3.3.22.

After simplification, (2.4.149) gives us

$$\begin{aligned}
|Term(T,p)_{op,k,k'}| \lesssim &L^{O(n\theta)} Q^{\frac{n}{2}} \sum_{c\in\mathscr{M}(\mathcal{C})} \sum_{\kappa_{\mathfrak{e}}\in\mathcal{D}(\alpha,1)} \sum_{\substack{\sigma_{\mathfrak{n}}\in\mathbb{Z}_{T_{\max}}\\ |\sigma_{\mathfrak{n}}|\lesssim 1}} \prod_{\mathfrak{n}\in\mathcal{C}} \frac{2^{-\frac{\tau_{\mathfrak{n}}}{2}}}{|c(\{\sigma_{\mathfrak{n}}\})_{\mathfrak{n}}|+T_{\max}^{-1}} \\
\lesssim &L^{O(n\theta)} Q^{\frac{n}{2}} \left(\sum_{\kappa_{\mathfrak{e}}\in\mathcal{D}(\alpha,1)} 1\right) \sum_{c\in\mathscr{M}(\mathcal{C})} \sum_{\substack{\sigma_{\mathfrak{n}}\in\mathbb{Z}_{T_{\max}}\\ |\sigma_{\mathfrak{n}}|\lesssim 1}} \prod_{\mathfrak{n}\in\mathcal{C}} \frac{2^{-\frac{\tau_{\mathfrak{n}}}{2}}}{|c(\{\sigma_{\mathfrak{n}}\})_{\mathfrak{n}}|+T_{\max}^{-1}} \\
\lesssim &L^{O(n\theta)} Q^{\frac{n}{2}} \sum_{c\in\mathscr{M}(\mathcal{C})} \sum_{\substack{\sigma_{\mathfrak{n}}\in\mathbb{Z}_{T_{\max}}\\ |\sigma_{\mathfrak{n}}|\lesssim 1}} \prod_{\mathfrak{n}\in\mathcal{C}} \frac{2^{-\frac{\tau_{\mathfrak{n}}}{2}}}{|c(\{\sigma_{\mathfrak{n}}\})_{\mathfrak{n}}|+T_{\max}^{-1}}
\end{aligned} \tag{2.4.150}$$

Here in the first step we use the fact that $\prod_{\mathfrak{e}\in\mathcal{C}_{\text{norm}}} \frac{\kappa_{\mathfrak{e}}}{\kappa_{\mathfrak{e}}+1} \prod_{\mathfrak{e}\in\mathcal{C}_{\text{norm}}} \kappa_{\mathfrak{e}}^{-1} = \prod_{\mathfrak{e}\in\mathcal{C}_{\text{norm}}} \frac{1}{\kappa_{\mathfrak{e}}+1} \le 1$. The reason for other steps can be find in the derivation of (2.4.86).

We claim that

$$\sup_{c} \sum_{\substack{\sigma_{\mathfrak{n}}\in\mathbb{Z}_{T_{\max}}\\ |\sigma_{\mathfrak{n}}|\lesssim 1}} \prod_{\mathfrak{n}\in\mathcal{C}} \frac{2^{-\frac{\tau_{\mathfrak{n}}}{2}}}{|c(\{\sigma_{\mathfrak{n}}\})_{\mathfrak{n}}|+T_{\max}^{-1}} \lesssim L^{O(l(T)\theta)} 2^{-\frac{1}{2}\sum_{j=1}^{K}\tau_j} T_{\max}^{2l(T)} \tag{2.4.151}$$

Since there are only bounded many matrices in $\mathscr{M}(\mathcal{C})$. Given above claim, we know that

$$|Term(T,p)_{op,k,k'}| \lesssim L^{O(l(T)\theta)} 2^{-\frac{1}{2}\sum_{j=1}^{K}\tau_j} Q^{\frac{n}{2}} T_{\max}^{2l(T)}, \tag{2.4.152}$$

which proves the lemma since $n = 2l(T)$.

Now prove the claim. In a tree T, there are $l(T)$ branching nodes, so there are $l(T)$ nodes in T_{in}. Since all nodes of $\mathcal{C}$ comes the two copies of T_{in}, so there are $2l(T)$ nodes in $\mathcal{C}$. Label these nodes by $h = 1, \cdots, 2l(T)$ and denote $\sigma_{\mathfrak{n}}$ by σ_h if $\mathfrak{n}$ is labelled by h. Since $\sigma_h \in \mathbb{Z}_{T_{\max}}$, there exists $m_h \in \mathbb{Z}$ such that $\sigma_h = T_{\max}^{-1} m_h$. (2.4.151) is thus equivalent to

$$T_{\max}^{2l(T)} \sum_{\substack{m_h \in \mathbb{Z} \\ |m_h| \lesssim T_{\max}}} \prod_{h=1}^{2l(T)} \frac{2^{-\frac{\tau_{\mathfrak{n}}}{2}}}{|c(\{m_h\})_h| + 1} \lesssim L^{O(l(T)\theta)} 2^{-\frac{1}{2}\sum_{j=1}^{K}\tau_j} T_{\max}^{2l(T)} \tag{2.4.153}$$

To we prove (2.4.153). We just need to show that

$$\sum_{\substack{m_h \in \mathbb{Z} \\ |m_h| \lesssim T_{\max}}} \prod_{h=1}^{2l(T)} \frac{1}{|c(\{m_h\})_h| + 1} \lesssim L^{O(l(T)\theta)} \tag{2.4.154}$$

This can be proved by Euler-Maclaurin formula (4.2.1) as (2.4.91).

Now we complete the proof of the claim and thus the proof of the lemma. □

Proof of the operator norm bound

In this subsection, we finish the proof of Proposition 2.4.29.

Proof of Proposition 2.4.29. By Lemma 2.4.33, we have

$$\left(\prod_{j=1}^{K} \mathcal{P}_{T_j}^{\tau_j}(w)\right)_k (t) = \sum_{k'} \int_0^t H_{Tkk'}^{\tau_1\cdots\tau_K}(t,s) w_{k'}(s) ds \tag{2.4.155}$$

and the kernel $H_{Tkk'}^{\tau_1\cdots\tau_K}$ is a polynomial of Gaussian variables given by

$$H_{Tkk'}^{\tau_1\cdots\tau_K}(t,s) = \left(\frac{i\lambda}{L^d}\right)^l \sum_{k_1, k_2, \cdots, k_l} H_{Tk_1\cdots k_l kk'}^{\tau_1\cdots\tau_K} \xi_{k_1}\cdots\xi_{k_l}. \tag{2.4.156}$$

By Proposition 2.4.36, we have

$$\sup_k \mathbb{E}|H_{Tkk'}^{\tau_1\cdots\tau_K}|^2 \lesssim L^{O(l(T)\theta)} 2^{-\frac{1}{2}\sum_{j=1}^{K}\tau_j} \rho^{2l(T)}. \tag{2.4.157}$$

Then the large deviation estimate Lemma 2.4.28 gives

$$|H_{Tkk'}^{\tau_1\cdots\tau_K}(t,s)| \lesssim L^{\frac{n}{2}\theta}\sqrt{\mathbb{E}|H_{Tkk'}^{\tau_1\cdots\tau_K}|^2} \lesssim L^{O(l(T)\theta)}2^{-\frac{1}{4}\sum_{j=1}^{K}\tau_j}\rho^{l(T)}, \qquad L\text{-certainly.} \tag{2.4.158}$$

Summing over all $\tau_1, \cdots, \tau_K$ gives

$$\sum_{\tau_1,\cdots,\tau_K} |H_{Tkk'}^{\tau_1\cdots\tau_K}(t,s)| \lesssim L^{O(l(T)\theta)}\rho^{l(T)}, \qquad L\text{-certainly.} \tag{2.4.159}$$

Applying the epsilon net and union bound method as in (2.4.97), we obtain

$$\sup_{t,s}\sup_{|k|,|k'|\lesssim L^{2M}}\sum_{\tau_1,\cdots,\tau_K} |H_{Tkk'}^{\tau_1\cdots\tau_K}(t,s)| \lesssim L^{O(l(T)\theta)}\rho^{l(T)}, \qquad L\text{-certainly.} \tag{2.4.160}$$

For $w \in X^p_{L^{2M}}$, we have $|w_k(t)| \leq \sup_t ||w(t)||_{X^p_{L^{2M}}}\langle k\rangle^{-p}$. Since $w_{k'} = 0$ if $|k'| \gtrsim L^{2M}$ and $H_{Tkk'}^{\tau_1\cdots\tau_K} = 0$ if $|k-k'| \gtrsim 1$, we know that

$$\begin{aligned}\left|\left(\sum_{\tau_1,\cdots,\tau_K}\prod_{j=1}^{K}\mathcal{P}_{T_j}^{\tau_j}(w)\right)_k(t)\right| \leq& \sum_{|k'|\lesssim L^{2M}}\int_0^t\sum_{\tau_1,\cdots,\tau_K}|H_{Tkk'}^{\tau_1\cdots\tau_K}(t,s)||w_{k'}(s)|ds\\ \lesssim& L^{O(l(T)\theta)}\rho^{l(T)}t\sup_t||w(t)||_{X^p_{L^{2M}}}\sum_{\substack{|k'|\lesssim L^{2M}\\|k'-k|\lesssim 1}}\langle k'\rangle^{-p}\\ \lesssim& L^{O(1+l(T)\theta)}\rho^{l(T)}\sup_t||w(t)||_{X^p_{L^{2M}}}\langle k\rangle^{-p}.\end{aligned} \tag{2.4.161}$$

Since $l(T) = \sum_{j=1}^K l(T_j)$, L-certainly we have

$$\left|\left|\sum_{\tau_1,\cdots,\tau_K}\prod_{j=1}^{K}\mathcal{P}_{T_j}^{\tau_j}\right|\right|_{L_t^\infty X^p} \leq L^{O\left(1+\theta\sum_{j=1}^K l(T_j)\right)}\rho^{\sum_{j=1}^K l(T_j)}||w||_{L_t^\infty X^p_{L^{2M}}}. \tag{2.4.162}$$

Therefore, we complete the proof of Proposition 2.4.29. $\square$

2.4.7 Asymptotics of the main terms

In this section, we prove (2.1.5) in Theorem 2.1.1 which characterize the asymptotic behavior of $n^{(1)}(k)$.

Proposition 2.4.38. *Using the same notation as Theorem 2.1.1 (2), then we have*

$$n^{(1)}(k) = \begin{cases} \frac{t}{T_{\text{kin}}}\mathcal{K}(n_{\text{in}})(k) + O_{\ell^\infty_k}\left(L^{-\theta}\frac{T_{max}}{T_{\text{kin}}}\right) + \widetilde{O}_{\ell^\infty_k}\left(\epsilon_1 Err_D(k_x)\frac{T_{max}}{T_{\text{kin}}}\right) & \textit{for any } |k| \le \epsilon_1 l_d^{-1}, \\ 0, & \textit{for any } |k| \ge 2C_2 l_d^{-1} \end{cases} \tag{2.4.163}$$

Here

$$Err_D(k_x) = \begin{cases} D^{d+1}, & \textit{if } |k_x| \le D, \\ D^{d-1}(|k_x|^2 + D|k_x|), & \textit{if } |k_x| \ge D. \end{cases} \tag{2.4.164}$$

Proof. The second case of (2.1.5) is obvious. We divide the proof of the first case into several steps.

Step 1. (Calculation of $\psi_{app,k}$ and $n^{(1)}(k)$) The first three terms in the tree expansion (2.2.9) can be calculated explicitly.

$$\psi_{app,k} = \psi^{(0)}_{app,k} + \psi^{(1)}_{app,k} + \psi^{(2)}_{app,k} + \cdots \tag{2.4.165}$$

where $\psi^{(0)}_{app,k}$,$\psi^{(1)}_{app,k}$, $\psi^{(2)}_{app,k}$ are given by

$$\psi^{(0)}_{app,k} = \xi_k \tag{2.4.166}$$

$$\psi^{(1)}_{app,k} = \frac{i\lambda}{L^d}\sum_{k_1+k_2=k} k_x \xi_{k_1}\xi_{k_2}\int_0^t e^{is\Omega(k_1,k_2,k)-\nu|k|^2(t-s)}ds \tag{2.4.167}$$

$$\begin{aligned} \psi^{(2)}_{app,k} =& -2\left(\frac{\lambda}{L^d}\right)^2 \sum_{k_1+k_2+k_3=k} k_x(k_{2x}+k_{3x})\xi_{k_1}\xi_{k_2}\xi_{k_3}\times \\ & \int_{0\le r<s\le t} e^{is\Omega(k_1,k_2+k_3,k)-\nu|k|^2(t-s)}e^{ir\Omega(k_2,k_3,k_2+k_3)-\nu|k_2+k_3|^2(s-r)}dsdr \end{aligned} \tag{2.4.168}$$

By (2.2.35), we know that

$$n^{(1)}(k) = \mathbb{E}\left|\psi^{(1)}_{app,k}\right|^2 + 2\text{Re}\,\mathbb{E}\left(\psi^{(2)}_{app,k}\overline{\xi_k}\right). \tag{2.4.169}$$

Step 2. (Decomposition of $\psi_{app,k}$) $e^{-\nu|k|^2(t-s)}$ is supposed to be close to 1, so we have the decomposition $e^{-\nu|k|^2(t-s)} = 1 + (e^{-\nu|k|^2(t-s)} - 1) = 1 - \left(\int_0^1 e^{-a\nu|k|^2(t-s)}da\right)\nu|k|^2(t-s)$. We can also decompose $\psi^{(1)}_{app,k} = \psi^{(11)}_{app,k} + \psi^{(12)}_{app,k}$ and $\psi^{(2)}_{app,k} = \psi^{(21)}_{app,k} + \psi^{(22)}_{app,k}$ into $\psi^{(1)}_{app,k}$ accordingly.

$$\psi^{(11)}_{app,k} = \frac{i\lambda}{L^d}\sum_{k_1+k_2=k} k_x \xi_{k_1}\xi_{k_2}\int_0^t e^{is\Omega(k_1,k_2,k)}ds \tag{2.4.170}$$

$$\psi^{(12)}_{app,k} = -\nu|k|^2 \frac{i\lambda}{L^d} \int_0^1 \left(\sum_{k_1+k_2=k} k_x \xi_{k_1}\xi_{k_2} \int_0^t e^{is\Omega(k_1,k_2,k)-a\nu|k|^2(t-s)}(t-s)ds \right) da \tag{2.4.171}$$

$$\psi^{(21)}_{app,k} = -2\left(\frac{\lambda}{L^d}\right)^2 \sum_{k_1+k_2+k_3=k} k_x(k_{2x}+k_{3x})\xi_{k_1}\xi_{k_2}\xi_{k_3} \int_{0\le r<s\le t} e^{is\Omega(k_1,k_2+k_3,k)} e^{ir\Omega(k_2,k_3,k_2+k_3)} dsdr \tag{2.4.172}$$

$$\begin{aligned}\psi^{(22)}_{app,k} =& 2\left(\frac{\lambda}{L^d}\right)^2 \int_0^1 \Bigg(\sum_{k_1+k_2+k_3=k} k_x(k_{2x}+k_{3x}) \int_{0\le r<s\le t} (\nu|k|^2(t-s)+\nu|k_2+k_3|^2(s-r)) \\ & e^{is\Omega(k_1,k_2+k_3,k)-a\nu|k|^2(t-s)} e^{ir\Omega(k_2,k_3,k_2+k_3)-a\nu|k_2+k_3|^2(s-r)} dsdr \xi_{k_1}\xi_{k_2}\xi_{k_3} \Bigg) da\end{aligned} \tag{2.4.173}$$

$n^{(1)}(k)$ is also decomposed into $n^{(1)}(k) = n^{(11)}(k) + n^{(12)}(k)$

$$n^{(11)}(k) = \mathbb{E}\left|\psi^{(11)}_{app,k}\right|^2 + 2\mathrm{Re}\ \mathbb{E}\left(\psi^{(21)}_{app,k}\overline{\xi_k}\right). \tag{2.4.174}$$

$$n^{(12)}(k) = \mathbb{E}\left|\psi^{(12)}_{app,k}\right|^2 + 2\mathrm{Re}\ \mathbb{E}\left(\psi^{(11)}_{app,k}\overline{\psi^{(12)}_{app,k}}\right) + 2\mathrm{Re}\ \mathbb{E}\left(\psi^{(22)}_{app,k}\overline{\xi_k}\right). \tag{2.4.175}$$

Step 3. (Estimate of $n^{(12)}(k)$ for $|k| \le \epsilon_1 l_d^{-1}$) In this step, all constants in $\lesssim$ depend only on the dimension d. Define

$$H^{(11)}_{k_1k_2k} = k_x \int_0^t e^{is\Omega(k_1,k_2,k)} ds \tag{2.4.176}$$

$$H^{(12)}_{k_1k_2k} = -\nu|k|^2 k_x \int_0^t e^{is\Omega(k_1,k_2,k)-a\nu|k|^2(t-s)}(t-s)ds \tag{2.4.177}$$

$$H^{(21)}_{k_1k_2k_3k} = 2k_x(k_{2x}+k_{3x}) \int_{0\le r<s\le t} e^{is\Omega(k_1,k_2+k_3,k)} e^{ir\Omega(k_2,k_3,k_2+k_3)} dsdr \tag{2.4.178}$$

$$\begin{aligned}H^{(22)}_{k_1k_2k_3k} =& -2k_x(k_{2x}+k_{3x}) \int_{0\le r<s\le t} (\nu|k|^2(t-s)+\nu|k_2+k_3|^2(s-r)) \\ & e^{is\Omega(k_1,k_2+k_3,k)-a\nu|k|^2(t-s)} e^{ir\Omega(k_2,k_3,k_2+k_3)-a\nu|k_2+k_3|^2(s-r)} dsdr\end{aligned} \tag{2.4.179}$$

Then we have

$$\psi^{(11)}_{app,k} = \frac{i\lambda}{L^d} \sum_{k_1+k_2=k} H^{(11)}_{k_1k_2k}\xi_{k_1}\xi_{k_2} \tag{2.4.180}$$

$$\psi^{(12)}_{app,k} = \int_0^1 \frac{i\lambda}{L^d} \sum_{k_1+k_2=k} H^{(12)}_{k_1k_2k}\xi_{k_1}\xi_{k_2} da \tag{2.4.181}$$

$$\psi^{(21)}_{app,k} = \left(\frac{i\lambda}{L^d}\right)^2 \sum_{k_1+k_2+k_3=k} H^{(11)}_{k_1k_2k_3k}\xi_{k_1}\xi_{k_2}\xi_{k_3} \tag{2.4.182}$$

$$\psi^{(22)}_{app,k} = \int_0^1 \left(\frac{i\lambda}{L^d}\right)^2 \sum_{k_1+k_2+k_3=k} H^{(12)}_{k_1k_2k_3k}\xi_{k_1}\xi_{k_2}\xi_{k_3} da \tag{2.4.183}$$

To derive an upper bound for $n^{(12)}(k)$, it suffices to consider $\mathbb{E}\left|\psi^{(12)}_{app,k}\right|^2$, Re $\mathbb{E}\left(\psi^{(11)}_{app,k}\overline{\psi^{(12)}_{app,k}}\right)$ and Re $\mathbb{E}\left(\psi^{(22)}_{app,k}\overline{\xi_k}\right)$ separately.

Step 3.1. (Upper bounds of $\mathbb{E}\left|\psi^{(12)}_{app,k}\right|^2$ and Re $\mathbb{E}\left(\psi^{(11)}_{app,k}\overline{\psi^{(12)}_{app,k}}\right)$) We first derive upper bounds for $H^{(1j)}$. For $H^{(11)}$, we have

$$|H^{(11)}_{k_1k_2k}| \lesssim \left|\int_0^t \frac{k_x}{i(\Omega+T_{\max}^{-1}\mathrm{sgn}(\Omega))}e^{-isT_{\max}^{-1}\mathrm{sgn}(\Omega)}\frac{d}{ds}e^{is\Omega+is\,\mathrm{sgn}(\Omega)/T_{\max}}ds\right| \tag{2.4.184}$$

Integration by parts we get

$$\begin{aligned}|H^{(11)}_{k_1k_2k}| \lesssim &\left|\int_0^t \frac{T_{\max}^{-1}k_x}{\Omega+T_{\max}^{-1}\mathrm{sgn}(\Omega)}e^{is\Omega}ds\right| + \left|\left[\frac{k_x}{i(\Omega+T_{\max}^{-1}\mathrm{sgn}(\Omega)}e^{is\Omega}\right]_0^t\right| \\ \lesssim &\frac{|k_x|}{|\Omega|+T_{\max}^{-1}}\end{aligned} \tag{2.4.185}$$

By a similar integration by parts method, we get

$$|H^{(12)}_{k_1k_2k}| \lesssim \frac{|k|^2|k_x|\nu T_{\max}}{|\Omega|+T_{\max}^{-1}}. \tag{2.4.186}$$

By (2.4.181), we know that

$$\begin{aligned}\mathbb{E}\left|\psi^{(12)}_{app,k}\right|^2 \leq &\int_0^1 \frac{\lambda^2}{L^{2d}}\mathbb{E}\left|\sum_{k_1+k_2=k} H^{(12)}_{k_1k_2k}\xi_{k_1}\xi_{k_2}\right|^2 da \\ \leq &\frac{2\lambda^2}{L^{2d}}\sup_{a\in[0,1]}\sum_{k_1+k_2=k}\left|H^{(12)}_{k_1k_2k}\right|^2 n(k_1)n(k_2) \\ \leq &\frac{2\lambda^2}{L^{2d}}(|k|^2|k_x|\nu T_{\max})^2\sum_{k_1}\frac{n(k_1)n(k-k_1)}{(|\Omega(k_1,k-k_1,k)|+T_{\max}^{-1})^2} \\ \leq &\frac{2\epsilon_1^2\lambda^2}{L^{2d}}\max(|k_x|^2,D^2)T_{\max}L^dD^{d-1} = 2\epsilon_1^2\max(|k_x|^2,D^2)T_{\max}T_{\mathrm{kin}}^{-1}D^{d-1}\end{aligned} \tag{2.4.187}$$

Here in the second inequality we apply the Wick theorem to calculate the expectation. In the third inequality we apply (2.4.186). In the last line we apply (4.1.3) in Theorem 4.3.2 by taking $t=T_{\max}$, $g(x)=\frac{1}{(1+|x|)^2}$ and $F(k_1)=n(k_1)n(k-k_1)$. In the last line we also use the facts that $|k|\leq \epsilon_1 l_d^{-1}=\epsilon_1(\nu T_{\max})^{-\frac{1}{2}}$, $T_{\mathrm{kin}}=\frac{1}{8\pi\alpha^2}=\frac{L^d}{8\pi\lambda^2}$ and $|k_x|\leq |k_{x1}|+|k_{x2}|\lesssim D$.

By (2.4.185) and (2.4.186), we know that

$$\begin{aligned}\left|\mathrm{Re}\ \mathbb{E}\left(\psi^{(11)}_{app,k}\overline{\psi^{(12)}_{app,k}}\right)\right| \leq &\int_0^1 \frac{2\lambda^2}{L^{2d}}\mathrm{Re}\,\mathbb{E}\left(\sum_{k_1+k_2=k} H^{(11)}_{k_1k_2k}\xi_{k_1}\xi_{k_2}\sum_{k_1+k_2=k}\overline{H^{(12)}_{k_1k_2k}\xi_{k_1}\xi_{k_2}}\right)da\\ \leq&\frac{2\lambda^2}{L^{2d}}\sup_{a\in[0,1]}\sum_{k_1+k_2=k}\left|H^{(11)}_{k_1k_2k}\right|\left|H^{(12)}_{k_1k_2k}\right|n(k_1)n(k_2)\\ \leq&\frac{2\lambda^2}{L^{2d}}|k|^2|k_x|^2\nu T_{\max}\sum_{k_1}\frac{n(k_1)n(k-k_1)}{(|\Omega(k_1,k-k_1,k)|+T_{\max}^{-1})^2}D^{d-1}\\ \leq&\frac{2\epsilon_1\lambda^2}{L^{2d}}\max(|k_x|^2,D^2)T_{\max}L^d=\epsilon_1\max(|k_x|^2,D^2)T_{\max}T_{\mathrm{kin}}^{-1}D^{d-1}\end{aligned} \tag{2.4.188}$$

Here in the second inequality we apply the Wick theorem to calculate the expectation. In the third inequality we apply (2.4.185) and (2.4.186). In the last line we apply (4.1.3) in Theorem 4.3.2 by taking $t=T_{\max}$, $g(x)=\frac{1}{(1+|x|)^2}$ and $F(k_1)=n(k_1)n(k-k_1)$. In the last line we also use the facts that $|k|\leq\epsilon_1 l_d^{-1}=\epsilon_1(\nu T_{\max})^{-\frac{1}{2}}$, $T_{\mathrm{kin}}=\frac{1}{8\pi\alpha^2}=\frac{L^d}{8\pi\lambda^2}$ and $|k_x|\leq|k_{x1}|+|k_{x2}|\lesssim D$.

(2.4.187) and (2.4.188) give desire upper bounds of $\mathbb{E}\left|\psi^{(12)}_{app,k}\right|^2$ and $\mathrm{Re}\ \mathbb{E}\left(\psi^{(11)}_{app,k}\overline{\psi^{(12)}_{app,k}}\right)$.

Step 3.2. (Upper bound of $\mathrm{Re}\ \mathbb{E}\left(\psi^{(22)}_{app,k}\overline{\xi_k}\right)$) By (2.4.183), we have

$$\mathrm{Re}\ \mathbb{E}\left(\psi^{(22)}_{app,k}\overline{\xi_k}\right)=\int_0^1\left(\frac{i\lambda}{L^d}\right)^2\mathrm{Re}\sum_{k_1+k_2+k_3=k}H^{(22)}_{k_1k_2k_3k}\mathbb{E}\left(\xi_{k_1}\xi_{k_2}\xi_{k_3}\overline{\xi_k}\right)da \tag{2.4.189}$$

By Wick theorem, $\mathbb{E}\left(\xi_{k_1}\xi_{k_2}\xi_{k_3}\overline{\xi_k}\right)=\delta_{k_1=-k_2}\delta_{k_3=k}+\delta_{k_1=-k_3}\delta_{k_2=k}+\delta_{k_1=k}\delta_{k_2=-k_3}$. Therefore we get

$$\begin{aligned}\mathrm{Re}\ \mathbb{E}\left(\psi^{(22)}_{app,k}\overline{\xi_k}\right)=&\int_0^1\left(\frac{i\lambda}{L^d}\right)^2\mathrm{Re}\sum_{k_1+k_2+k_3=k}H^{(22)}_{k_1k_2k_3k}(\delta_{k_1=-k_2}\delta_{k_3=k}+\delta_{k_1=-k_3}\delta_{k_2=k}+0)da\\ =&\int_0^1 2\left(\frac{i\lambda}{L^d}\right)^2\sum_{k_1+k_2+k_3=k}\mathrm{Re}\left(H^{(22)}_{k_1,-k_1,k,k}\right)da\end{aligned} \tag{2.4.190}$$

Here in the first equality the term corresponding to $\delta_{k_1=k}\delta_{k_2=-k_3}$ vanishes because $H^{(22)}_{k,k_2,-k_2,k}=0$ and the two terms corresponding to $\delta_{k_1=-k_2}\delta_{k_3=k}$, $\delta_{k_1=-k_3}\delta_{k_2=k}$ are equal.

By (2.4.179), we get

$$\begin{aligned}H^{(22)}_{k_1,-k_1,k,k}=-2k_x(k_x-k_{1x})\int_{0\leq r<s\leq t}&(\nu|k|^2(t-s)+\nu|k-k_1|^2(s-r))\\ &e^{i(s-r)\Omega(k_1,k-k_1,k)-a\nu|k|^2(t-s)-a\nu|k-k_1|^2(s-r)}dsdr.\end{aligned} \tag{2.4.191}$$

We find upper bound of Re $\left(H^{(22)}_{k_1,-k_1,k,k}\right)$ using integration by parts.

$$\begin{aligned}
&\mathrm{Re}\left(H^{(22)}_{k_1,-k_1,k,k}\right) = 2k_x(k_x-k_{1x})\int_{0\le r<s\le t}(\nu|k|^2(t-s)+\nu|k-k_1|^2(s-r))\\
&\frac{e^{irT_{\max}^{-1}sgn\,\Omega}}{i(\Omega+T_{\max}^{-1}sgn\,\Omega)}\frac{d}{dr}e^{i(s-r)\Omega-irT_{\max}^{-1}sgn\,\Omega}e^{-a\nu|k|^2(t-s)-a\nu|k-k_1|^2(s-r)}dsdr\\
=&\mathrm{Re}\,\frac{2k_x(k_x-k_{1x})}{i(\Omega+T_{\max}^{-1}sgn\,\Omega)}\int_{0\le s\le t}\nu|k|^2(t-s)e^{-a\nu|k|^2(t-s)}ds\\
-&\mathrm{Re}\,\frac{2k_x(k_x-k_{1x})}{i(\Omega+T_{\max}^{-1}sgn\,\Omega)}\int_{0\le s\le t}(\nu|k|^2(t-s)+\nu|k-k_1|^2s)e^{-a\nu|k|^2(t-s)-a\nu|k-k_1|^2s}e^{is\Omega}ds\\
-&\mathrm{Re}\,\frac{2k_x(k_x-k_{1x})}{i(\Omega+T_{\max}^{-1}sgn\,\Omega)}\int_{0\le r<s\le t}\big[\nu|k-k_1|^2(\nu|k|^2(t-s)+\nu|k-k_1|^2(s-r)-1)\\
&-iT_{\max}^{-1}sgn\,\Omega\big]e^{i(s-r)\Omega-a\nu|k|^2(t-s)-a\nu|k-k_1|^2(s-r)}dsdr.
\end{aligned} \tag{2.4.192}$$

The first term on the right hands side equals to 0 after taking the real part. Using the same integration by parts argument as in (2.4.185), the second term can be bounded by

$$\frac{|k|^2|k_x|(|k_{1x}|+|k_x|)\nu T_{\max}}{(|\Omega(k_1,k-k_1,k)|+T_{\max}^{-1})^2}. \tag{2.4.193}$$

The last term can be bounded by integration by parts in the following integral

$$\begin{aligned}
&\int_{0\le r<s\le t}\big[\nu|k-k_1|^2(\nu|k|^2(t-s)+\nu|k-k_1|^2(s-r)-1)\\
&\qquad -iT_{\max}^{-1}sgn\,\Omega\big]\frac{e^{irT_{\max}^{-1}sgn\,\Omega}}{i(\Omega+T_{\max}^{-1}sgn\,\Omega)}\frac{d}{dr}e^{i(s-r)\Omega}e^{-a\nu|k|^2(t-s)-a\nu|k-k_1|^2(s-r)}dsdr
\end{aligned} \tag{2.4.194}$$

Integration by parts and bound the three resulting integral by taking the absolute value of the integrand, then we get

$$|(2.4.194)|\le\frac{|k|^2}{|\Omega|+T_{\max}^{-1}} \tag{2.4.195}$$

Therefore, the last term in (2.4.192) can also be bounded by (2.4.193).

Then we get

$$\mathrm{Re}\left(H^{(22)}_{k_1,-k_1,k,k}\right)\lesssim\nu T_{\max}|k|^2|k_x|\frac{|k_{1x}|+|k_x|}{(|\Omega(k_1,k-k_1,k)|+T_{\max}^{-1})^2}. \tag{2.4.196}$$

Substitute into (2.4.190), then we have

$$
\begin{aligned}
\operatorname{Re}\mathbb{E}\left(\psi^{(22)}_{app,k}\overline{\xi_k}\right) &\lesssim \frac{\lambda^2}{L^{2d}}|k|^2|k_x|\nu T_{\max} n(k)\sum_{k_1}\frac{|k_{1x}|+|k_x|}{(|\Omega(k_1,k-k_1,k)|+T_{\max}^{-1})^2}n(k_1)\\
&\lesssim \frac{\epsilon_1\lambda^2}{L^{2d}}|k_x|\sum_{k_1}\frac{D}{(|\Omega(k_1,k-k_1,k)|+T_{\max}^{-1})^2}n(k_1)\\
&+\frac{\epsilon_1\lambda^2}{L^{2d}}|k_x|^2\sum_{k_1}\frac{1}{(|\Omega(k_1,k-k_1,k)|+T_{\max}^{-1})^2}n(k_1)\\
&\lesssim \epsilon_1\frac{T_{\max}}{T_{\text{kin}}}\max(D^{d+1},|k_x|^2D^{d-1}+|k_x|D^d)
\end{aligned}
\tag{2.4.197}
$$

In the second inequality, we also use the facts that $|k|\le \epsilon_1 l_d^{-1}=\epsilon_1(\nu T_{\max})^{-\frac{1}{2}}$ and $T_{\text{kin}}=\frac{1}{8\pi\alpha^2}=\frac{L^d}{8\pi\lambda^2}$. In the last inequality we apply (4.1.3) in Theorem 4.3.2 by taking $t=T_{\max}$, $g(x)=\frac{1}{(1+|x|)^2}$, $F(k_1)=n(k_1)n(k-k_1)$ and $|k_x|\le|k_{x1}|+|k_{x2}|\lesssim D$.

Combining (2.4.187), (2.4.188) and (2.4.197), we get the following upper bound

$$
n^{(12)}(k)\lesssim \epsilon_1\frac{T_{\max}}{T_{\text{kin}}}\underbrace{\max(D^{d+1},|k_x|^2D^{d-1}+|k_x|D^d)}_{\text{Err}_D(k_x)}. \tag{2.4.198}
$$

Step 4. (Asymptotics of $n^{(11)}(k)$) By (2.4.174) and Wick theorem, we get

$$
\begin{aligned}
n^{(11)}(k) =&\frac{2\lambda^2}{L^{2d}}|k_x|^2\sum_{k_1+k_2=k}n(k_1)n(k_2)\left|\int_0^t e^{is\Omega(k_1,k_2,k)}ds\right|^2\\
&-\frac{8\lambda^2}{L^{2d}}\sum_{k_1}k_x(k_x-k_{1x})n(k_1)n(k)\operatorname{Re}\left(\int_{0\le r<s\le t}e^{i(s-r)\Omega(k_1,k-k_1,k)}dsdr\right)\\
=&\frac{2\lambda^2}{L^{2d}}|k_x|^2\sum_{k_1+k_2=k}n(k_1)n(k_2)\frac{4\sin^2\left(\frac{t}{2}\Omega(k_1,k_2,k)\right)}{\Omega^2(k_1,k_2,k)}\\
&-\frac{8\lambda^2}{L^{2d}}\sum_{k_1}k_x(k_x-k_{1x})n(k_1)n(k)\frac{2\sin^2\left(\frac{t}{2}\Omega(k_1,k-k_1,k)\right)}{\Omega^2(k_1,k-k_1,k)}\\
=&8\pi\alpha^2t|k_x|^2\int_{\substack{(k_1,k_2)\in\mathbb{R}^{2d}\\k_1+k_2=k}}n(k_1)n(k_2)\delta(|k_1|^2k_{1x}+|k_2|^2k_{2x}-|k|^2k_x)\,dk_1dk_2\\
&-16\pi\alpha^2t\,n(k)\int_{\mathbb{R}^d}k_x(k_x-k_{1x})n(k_1)\delta(|k_1|^2k_{1x}+|k_2|^2k_{2x}-|k|^2k_x)\,dk_1+O\left(L^{-\theta}\frac{T_{\max}}{T_{\text{kin}}}\right)\\
=&\frac{t}{T_{\text{kin}}}\mathcal{K}(n)(k)+O\left(L^{-\theta}\frac{T_{\max}}{T_{\text{kin}}}\right)
\end{aligned}
\tag{2.4.199}
$$

Here in the third equality we apply (4.1.1) in Theorem 4.3.2 by taking $t\to\frac{t}{2}$, $g(x)=\frac{\sin^2(x)}{x^2}$ and $F(k_1)=n(k_1)n(k-k_1)$ or $F(k_1)=n(k_1)$. In the third equality we also use the facts that

$T_{\text{kin}} = \frac{1}{8\pi\alpha^2} = \frac{L^d}{8\pi\lambda^2}$.

Combining (2.4.198) and (2.4.199), we complete the prove of the first case in (2.1.5). $\square$

Chapter 3

Analysis of four wave models

3.1 Introduction

3.1.1 Basic set-ups

In this chapter, we study the wave turbulence theory for the following Klein-Gordon type equation

$$\begin{cases} i\partial_t\psi - \Lambda(\nabla)\psi = \lambda^2\Lambda(\nabla)^{-\frac{1}{2}}\left(|\Lambda(\nabla)^{-\frac{1}{2}}\psi|^2\Lambda(\nabla)^{-\frac{1}{2}}\psi\right), \\ \psi(0,x) = \psi_{\mathrm{in}}(x), \quad x \in \mathbb{T}^d_{L_1\cdots L_d}. \end{cases} \tag{NKLG}$$

as an example of general four wave systems. Here $\Lambda(\xi) := \sqrt{1+|\xi|^2}$.

Here ψ is a complex value function. We consider the periodic boundary condition, which implies that the spatial domain is a torus $\mathbb{T}^d_L = [0,L]^d$.

We know that the Fourier coefficients of ψ lie on the lattice $\mathbb{Z}^d_L = \{k = \frac{K}{L} : K \in \mathbb{Z}^d\}$. Let n_{in} be a known function, we assume that

$$\psi_{\mathrm{in}}(x) = \frac{1}{L^d}\sum_{k\in\mathbb{Z}^d_L}\sqrt{n_{\mathrm{in}}(k)}\eta_k(\omega)\, e^{2\pi i k x} \tag{3.1.1}$$

where $\eta_k(\omega)$ are mean-zero and identically distributed complex Gaussian random variables satisfying $\mathbb{E}|\eta_k|^2 = 1$. To ensure ψ_{in} to be a real value function, we assume that $n_{\mathrm{in}}(k) = n_{\mathrm{in}}(-k)$ and $\eta_k = \overline{\eta_{-k}}$. Finally, we assume that η_k is independent of $\{\eta_{k'}\}_{k'\neq k,-k}$.

The energy spectrum $n(t,k)$ mentioned in previous section is defined to be $\mathbb{E}|\widehat{\psi}(t,k)|^2$, where $\psi(t,k)$ are Fourier coefficients of the solution. Although the initial data is assumed to be the

Gaussian random field, it is possible to develop a theory for other types of random initial data.

In the wave turbulence, the energy distribution $n(k)$ is supposed to evolve according to the following wave kinetic equation

$$\begin{aligned}\partial_t n(t,\xi) =&\mathcal{K}\left(n(t,\cdot)\right),\\ \mathcal{K}(\phi)(\xi) :=& \int_{\substack{(\xi_1,\xi_2,\xi_3)\in\mathbb{R}^{3d}\\ \xi_1-\xi_2+\xi_3=\xi}} \phi\phi_1\phi_2\phi_3\left(\frac{1}{\phi_1}-\frac{1}{\phi_2}+\frac{1}{\phi_3}-\frac{1}{\phi}\right)\\ &(\Lambda(k_1)\Lambda(k_2)\Lambda(k_3)\Lambda(k))^{-\frac{1}{2}}\delta_{\mathbb{R}}(|\xi_1|_\beta^2-|\xi_2|_\beta^2+|\xi_3|_\beta^2-|\xi|_\beta^2)\,d\xi_1 d\xi_2 d\xi_3,\end{aligned} \tag{WKE}$$

Now we explain the idea of deriving above wave kinetic equation. The derivation also uses Feynman diagram expansion. Note that in this thesis, we only consider the upper bound for most difficult terms in the diagram expansion. Therefore, the proof in this thesis is **incomplete**. We believe that all other terms can be handled by the existing technique developed by Deng-Hani [**?**] and we leave the full proof to be a future project.

3.1.2 Outline of this chapter

As in the previous chapter, in order to apply the Deng-Hani's argument to derive the (WKE), we need to analyze the perturbative expansion series. In this section, we explain the special structure of NLS and the difficulties of Deng-Hani's argument to general dispersive equation without this special structure.

The special structure of NLS

The NLS is given by the following equation

$$i\partial_t\psi - \Delta\psi = \lambda^2|\psi|^2\psi \tag{3.1.2}$$

Here we consider the focusing NLS and the sign of the nonlinear is not relevant to the argument below.

Its nonlinearity is given by $|u|^2u$. The Fourier coefficients of this nonlinearity are given by

$$\frac{\lambda^2}{L^{2d}}\sum_{\substack{(k_1,k_2,k_3)\in(\mathbb{Z}_L^d)^3\\ k-k_1+k_2-k_3=0}} \psi_{k_1}\overline{\psi}_{k_2}\psi_{k_3} \tag{3.1.3}$$

In order to prove (WKE), we need to show that the nonlinearity is small compared to the linear term. To show the smallness, we have to exploit the the square root cancellation given by randomness of u_k. But in $\sum_{k_1=k_2,k_3=k}$ or $\sum_{k_1=k,k_2=k_3}$, which are equal to $(\sum_{k_1}|u_{k_1}|^2)u_k$, we do not have the desired square root cancellation.

In NLS, $\sum_{k_1}|\psi_{k_1}|^2 = \fint_{\mathbb{T}_L^d}|\psi|^2$ is just the L^2 mass which is a conservative quantity. We just need to rewrite the NLS into the following form

$$i\partial_t\psi - \Delta\psi = 2\left(\lambda^2\fint_{\mathbb{T}_L^d}|\psi|^2\right)\psi + \lambda^2\left(|\psi|^2 - 2\fint_{\mathbb{T}_L^d}|\psi|^2\right)\psi \tag{3.1.4}$$

Due to the conservativeness and multiplier-freeness, the first term $2\lambda^2\fint_{\mathbb{T}_L^d}|\psi|^2\psi$ can be removed completely by a simple change phase argument $\psi \to e^{-2i\lambda^2\fint_{\mathbb{T}_L^d}|\psi|^2 t}\cdot\psi$.

After doing the change phase, $\lambda^2\left(|\psi|^2 - 2\fint_{\mathbb{T}_L^d}|\psi|^2\right)\psi$ has the desired square root cancellation property.

However, the Klein-Gordon equation (or many other PDEs from physics) does not have such a good structure. The Fourier coefficients of the nonlinearity $\lambda^2\Lambda(\nabla)^{-\frac{1}{2}}|\Lambda(\nabla)^{-\frac{1}{2}}\psi|^2\Lambda(\nabla)^{-\frac{1}{2}}\psi$ are given by

$$\frac{\lambda^2}{L^{2d}}\sum_{\substack{(k_1,k_2,k_3)\in(\mathbb{Z}_L^d)^3\\ k-k_1+k_2-k_3=0}}(\Lambda_{k_1}\Lambda_{k_2}\Lambda_{k_3})^{-\frac{1}{2}}\psi_{k_1}\overline{\psi}_{k_2}\psi_{k_3} \tag{3.1.5}$$

The sum $\sum_{k_1=k_2,k_3=k}$ or $\sum_{k_1=k,k_2=k_3}$ now equals to $(\sum_{k_1}\Lambda_{k_1}^{-1}|u_{k_1}|^2)\Lambda_k^{-1}u_k$. The corresponding L^2 term in this case equals to $\sum_{k_1}\Lambda_{k_1}^{-1}|u_{k_1}|^2$, which is not a conservative quantity. Even worse, this terms equals to $\left(\fint_{\mathbb{T}_L^d}|\Lambda(\nabla)^{-\frac{1}{2}}\psi|^2\right)\Lambda(\nabla)^{-1}\psi$, which contains derivative.

Therefore, the change phase argument does not work for (NKLG), since in the analogous change of variable $\psi_k(t) \to e^{\frac{2\lambda^2}{L^d}i\Lambda(k)^{-1}\int_0^t M(s)ds}\psi_k(t)$ the quantity $M(t) = \sum_{k_1}\Lambda_{k_1}^{-1}|\psi_{k_1}|^2$ is not a conservative quantity. More importantly, after change of variable the nonlinearity becomes

$$\frac{\lambda^2}{L^{2d}}\sum_{\substack{k-k_1+k_2-k_3=0\\ k_1\neq k_2,\ k_3\neq k}}\psi_{k_1}\overline{\psi}_{k_2}\psi_{k_3}e^{-\frac{2i\lambda^2}{L^d}\int_0^t M(s)ds\left(\Lambda(k_1)^{-1}-\Lambda(k_2)^{-1}+\Lambda(k_3)^{-1}-\Lambda(k)^{-1}\right)} \tag{3.1.6}$$

Due to randomness of M, the additional phase factor $e^{-\frac{2i\lambda^2}{L^d}\int_0^t M(s)ds(\cdots)}$ is a random variable which cause a serious problem.

The renormalized approximation series

In order to solve above problem we introduce a new change of variable $\psi_k(t) \to e^{\frac{2\lambda^2}{L^d}i\Lambda(k)^{-1}\int_0^t m(s)ds}\psi_k(t)$, where $m(t) = \mathbb{E}M(t)$. After doing this, the phase factor $e^{-\frac{2i\lambda^2}{L^d}\int_0^t m(s)ds(\cdots)}$ is not a random variable.

Our change of variable solves the problem coming from the randomness of M. The problem from time dependence of M will be solved later.

After the renormalization it can be shown that $\sum_{k_1=k_2,k_3=k}$ and $\sum_{k_1=k,k_2=k_3}$ is small, so in this section we ignore the contribution from them. For simplicity we also ignore the effect of renormalization. Then the equation of Fourier coefficients becomes

$$\begin{aligned} i\dot{\psi}_k = &\left(\Lambda(k) + \frac{2\lambda^2}{L^d} m(t)\Lambda(k)^{-2}\right)\psi_k + \frac{\lambda^2}{L^{2d}} \sum_{S_3=0}^{\times} \Lambda^{-1}_{k_1k_2k_3k}\, \psi_{k_1}\overline{\psi}_{k_2}\psi_{k_3} \\ &+ \frac{2\lambda^2}{L^{2d}} \left(\sum_{k_1\in\mathbb{Z}^d_L} \Lambda^{-2}_{k_1}\Big(|\psi_{k_1}|^2 - \mathbb{E}|\psi_{k_1}|^2\Big) \right) \Lambda^{-2}_k \psi_k \end{aligned} \tag{3.1.7}$$

Define new dynamical variable $e^{i\Lambda(k)t+\frac{2\lambda^2}{L^d}i\Lambda(k)^{-2}\int_0^t m(s)ds}\psi_k(t)$ and integrate (3.1.7) in time. Then (NKLG) with initial data (3.1.1) becomes

$$\begin{aligned} \phi_k = \xi_k &\underbrace{-\frac{i\lambda^2}{L^{2d}} \sum_{S_3=0}^{\times} \int_0^t \Lambda^{-1}_{k_1k_2k_3k}\, \phi_{k_1}\overline{\phi}_{k_2}\phi_{k_3} e^{-i(\Omega_3 t+\widetilde{\Omega}_3)} ds}_{\mathcal{T}_1(\phi,\phi,\phi)_k} \\ &\underbrace{-\frac{2i\lambda^2}{L^{2d}} \int_0^t \left(\sum_{k_1\in\mathbb{Z}^d_L} \Lambda^{-2}_{k_1} : |\phi_{k_1}|^2 : \right) \Lambda^{-1}_k \phi_k ds\,.}_{\mathcal{T}_2(\phi,\phi,\phi)_k} \end{aligned} \tag{3.1.8}$$

Here we introduce $S_{3,k} = k_1 - k_2 + k_3 - k$, $\Omega_{3,k} = \Lambda(k_1) - \Lambda(k_2) + \Lambda(k_3) - \Lambda(k)$, $\widetilde{\Omega}'_3(k_1,k_2,k_3,k) = \frac{2\lambda^2}{L^d}\int_0^t M(s)ds\left(\Lambda(k_1)^{-2} - \Lambda(k_2)^{-2} + \Lambda(k_3)^{-2} - \Lambda(k)^{-2}\right)$. ξ_k are the Fourier coefficients of the initial data of ψ defined by $\xi_k = \sqrt{n_{\text{in}}(k)}\,\eta_k(\omega)$.

The right hand side by $\mathcal{F}(\psi)_k = \xi_k + \mathcal{T}_1(\psi,\psi,\psi)_k + \mathcal{T}_2(\psi,\psi,\psi)_k$ As in three wave models in previous chapter, we can construct an approximate series by iteration. $\psi = \mathcal{F}(\psi) = \mathcal{F}(\mathcal{F}(\psi)) = \mathcal{F}(\mathcal{F}(\mathcal{F}(\psi))) = \cdots$

Define the approximate solution by $\psi_{app} = \mathcal{F}^N(\xi)$. By recursively expanding $\mathcal{F}^N$, we get

$$\begin{aligned} \psi_{app} =& \mathcal{F}^N(\xi) = \xi + \mathcal{T}_1(\mathcal{F}^{N-1}(\xi), \mathcal{F}^{N-1}(\xi), \mathcal{F}^{N-1}(\xi)) + \mathcal{T}_2(\mathcal{F}^{N-1}(\xi), \mathcal{F}^{N-1}(\xi), \mathcal{F}^{N-1}(\xi)) \\ =& \xi + \mathcal{T}_1(\xi,\xi,\xi) + \mathcal{T}_2(\xi,\xi,\xi) + \mathcal{T}_1(\mathcal{T}_1(\xi,\xi,\xi),\xi,\xi) + \cdots \end{aligned}$$

Since $\mathcal{T}_2$ contains the renormalization symbol $:\cdot:$, ψ_{app} is not a polynomial of ξ. Instead, it is a renormalized polynomial as defined by Definition 3.3.1 (3).

We need a good upper bound for each terms of ψ_{app}. To get this, we analyze Feynman diagrams.

The perturbative analysis

The analysis in previous section suggests that ψ_{app} should be a good approximation of ψ. In other words, the error of this approximation $w = \psi - \psi_{app}$ is very small. The strategy of proving this is the same as in previous chapter.

We use the follow equation of w which can be derived from (3.1.8):

$$w = Err(\xi) + Lw + B(w,w) + C(w,w,w) \tag{3.1.9}$$

Here $Err(\xi)$ is a polynomial of ξ whose degree $\leq N+1$ monomials vanish. Lw, $B(w,w)$, $C(w,w,w)$ are linear, quadratic, cubic in w respectively.

We prove the smallness of w using boostrap method.

Define $||w||_{X^p} = \sup_k \langle k \rangle^p |w_k|$. Starting from the assumption that $\sup_t ||w||_{X^p} \leq CL^{-M}$ ($C, M \gg 1$), in order to close the boostrap, we need to prove that $\sup_t ||w||_{X^p} \leq (1 + C/2)L^{-M} < CL^{-M}$. To prove $||w||_{X^p} \leq (1 + C/2)L^{-2}$, we use (3.1.9), which gives

$$||w||_{X^p} \leq ||Err(\xi)||_{X^p} + ||Lw||_{X^p} + ||B(w,w)||_{X^p} + ||C(w,w,w)||_{X^p} \tag{3.1.10}$$

In the rest part of the proof, we show that

$$||Err(\xi)||_{X^p} \leq L^{-M}, \quad ||B(w,w)||_{X^p} \leq C^2 L^{O_d(1)-2M}, \quad ||C(w,w,w)||_{X^p} \leq C^3 L^{O_d(1)-3M}. \tag{3.1.11}$$

Combining with a special treatment of Lw, above estimates imply that $||w||_{X^p} \leq (1 + C/2)L^{-M}$, which closes the boostrap.

The proof of inequalities of $B(w,w)$ and $C(w,w,w)$ is eas and will be given in later part of this section. The proof of inequalities of $Err(\xi)$ and Lw requires a complete analysis of terms in diagram expansion. As mentioned before, in this thesis, **we only consider the upper bound for those terms that are related to the issue of renormalization**, which should be the most difficult terms in the diagram expansion. We leave it a future project to adapt the the existing technique developed by Deng-Hani to get a full proof of the wave kinetic equation.

Lattice points counting and $||Err(\xi)||_{X^p}$

In this section we explain the idea of proving upper bound of $||Err(\xi)||_{X^p}$.

$(Err(\xi))_k$ is a sum of terms of the form

$$\begin{aligned} &\mathcal{J}^0_k(\xi) = \xi_k, \quad \mathcal{J}^1_k(\xi) = \frac{\lambda^2}{L^{2d}} \sum_{k_1-k_2+k_3-k=0} H^1_{k_1k_2k_3}[\xi_{k_1}\bar{\xi}_{k_2}\xi_{k_3}], \quad \cdots \\ &\mathcal{J}^l_k(\xi) = \left(\frac{\lambda^2}{L^{2d}}\right)^l \sum_{k_1-k_2+\cdots+k_{2l+1}-k=0} H^l_{k_1\cdots k_{2l+1}}[\xi_{k_1}\bar{\xi}_{k_2}\cdots\xi_{k_{2l+1}}], \quad \cdots \end{aligned} \tag{3.1.12}$$

Here the $[\cdot]$ indicates that this is a renormalized polynomial whose precise definition is given in Definition 3.3.1. According to section 3.2.2, each terms correspond to a Feynman diagram and their coefficient can be calculated from these diagrams.

As in the three wave case, from the Feynman diagram representation in section 3.2.2, we know that H^l is large near a surface given by $2l$ equations $S = \{S_3(T) = 0, \Omega_3(T) = 0, \cdots, \Omega_{2l+1}(T) = 0, \Omega_{2l+1}(T) = 0\}$. Then in order to estimate $\mathcal{J}^l_k(\xi)$ it suffices to upper bound the number of lattice points near this surface. Unlike the three wave case, we cannot obtain strong enough number theory estimate without renormalization symbol $[\cdot]$.

By the large deviation principle, to obtain upper bounds of Gaussian polynomials $\mathcal{J}^l_{T,k}(\xi)$, it suffices to calculate their variance. To calculate the variance of (3.1.12), we just need to calculate $\mathbb{E}[\xi_{k_1}\bar{\xi}_{k_2}\cdots\bar{\xi}_{k_{2l}}]$. This calculation is done in section 3.3.2, where we introduce a novel renormalized Wick theorem (Theorem 3.1.1 or Theorem 3.3.5).

The usual Wick theorem says that $\mathbb{E}\xi_{k_1}\bar{\xi}_{k_2}\cdots\bar{\xi}_{k_{2l}} = \sum_{p\in\mathcal{P}}\delta_p$ where $\mathcal{P}$ is a set of pairing whose definition will be given in Definition 3.3.3. Our renormalized Wick theorem says that with the Renormalization symbol $\mathbb{E}[\xi_{k_1}\bar{\xi}_{k_2}\cdots\bar{\xi}_{k_{2l}}] = \sum_{p\in\mathcal{P}_F}\delta_p$ where $\mathcal{P}_F$ is a subset of all pairing which exclude many bad pairings.

Combining the heuristic argument in section **??** and a cancellation identity (**??**), we can get upper bounds for all diagram terms.

In conclusion, combining lattice points counting and renormalization argument we can show that, for any M, we can take N large enough so that $||Err(\xi)||_{X^p} \le L^{-M}$.

Upper bounds for $||B(w,w)||_{X^p}$ and $||C(w,w,w)||_{X^p}$

$||B(w,w)||_{X^p}$ is a sum of terms of the form

$$\frac{\lambda^2}{L^{2d}}\int_0^t \sum_{k_1-k_2+k_3-k=0} B_{k_1k_2k_3}(s)\mathcal{J}^l_{k_1}(\xi)\bar{w}_{k_2}w_{k_3} \tag{3.1.13}$$

$||C(w,w,w)||_{X^p}$ is a cubic polynomial of w of the form

$$\frac{\lambda^2}{L^{2d}}\int_0^t \sum_{k_1-k_2+k_3-k=0} C_{k_1k_2k_3}(s)w_{k_1}\bar{w}_{k_2}w_{k_3} \tag{3.1.14}$$

By assumptions and proofs in this paper, we know that $t \leq \alpha^{-3/2} \leq L^{O_d(1)}$, $|B_{k_1k_2k_3}(s)|, |C_{k_1k_2k_3}(s)| \lesssim 1$ and $|\mathcal{J}^l_{k_1}(\xi)| \lesssim \langle k\rangle^{-p}$. By boostrap assumption, $\sup_k \langle k\rangle^p |w_k| \leq CL^{-M}$. Therefore we have following estimate of $||B(w,w)||_{X^p}$ and $||C(w,w,w)||_{X^p}$

$$\begin{aligned}
||B(w,w)||_{X^p} =& \sup_k \langle k\rangle^p \left|\frac{\lambda^2}{L^{2d}}\int_0^t \sum_{k_1-k_2+k_3-k=0} B_{k_1k_2k_3}(s)\mathcal{J}^l_{k_1}(\xi)\bar{w}_{k_2}w_{k_3}\right| \\
\lesssim& \sup_k \langle k\rangle^p \frac{\lambda^2}{L^{2d}} L^{O_d(1)} \sum_{k_1-k_2+k_3-k=0} \langle k_1\rangle^{-p} L^{-M}\langle k_2\rangle^{-p} L^{-M}\langle k_3\rangle^{-p} \\
\leq& C^2 L^{O_d(1)-2M},
\end{aligned} \tag{3.1.15}$$

$$\begin{aligned}
||C(w,w,w)||_{X^p} =& \sup_k \langle k\rangle^p \left|\frac{\lambda^2}{L^{2d}}\int_0^t \sum_{k_1-k_2+k_3-k=0} C_{k_1k_2k_3}(s)w_{k_1}\bar{w}_{k_2}w_{k_3}\right| \\
\lesssim& \sup_k \langle k\rangle^p \frac{\lambda^2}{L^{2d}} L^{O_d(1)} \sum_{k_1-k_2+k_3-k=0} L^{-M}\langle k_1\rangle^{-p} L^{-M}\langle k_2\rangle^{-p} L^{-M}\langle k_3\rangle^{-p} \\
\leq& C^3 L^{O_d(1)-3M}
\end{aligned} \tag{3.1.16}$$

Therefore, we get the desire upper bounds for $||B(w,w)||_{X^p}$ and $||C(w,w,w)||_{X^p}$.

A random matrix bound and Lw and proof of the main theorem

This part of the proof is the same as the three wave case, if we can control all diagram terms.

3.1.3 Informal statement of the main theorems

Now we introduce the main theorems of this chapter.

Theorem 3.1.1. *this is the informal version, the formal version can be founed in ...*

Theorem 3.1.2. *this is the informal version, the formal version can be founed in ...*

Theorem 3.1.3. *this is the informal version, the formal version can be founed in ...*

Theorem 3.1.4. *this is the informal version, the formal version can be founed in ...*

3.1.4 Notations

Universal constants: In this paper, universal constants are constants that just depend on dimension d, diameter D of the support of n_{in}.

$O(\cdot)$, $\ll$, $\lesssim$, $\sim$: Throughout this paper, we frequently use the notation, $O(\cdot)$, $\ll$, $\lesssim$. $A = O(B)$ or $A \lesssim B$ means that there exists C such that $A \lesssim CB$. $A \ll B$ means that there exists a small constant c such that $A \lesssim cB$. $A \sim B$ means that there exist two constant c, C such that $cB \lesssim A \lesssim CB$. Here the meaning of constant depends on the context. If they appear in conditions involving k, Λ, Ω, etc., like $|k| \lesssim 1$, $\iota_{\mathfrak{e}_1} k_{\mathfrak{e}_1} + \iota_{\mathfrak{e}_2} k_{\mathfrak{e}_2} + \iota_{\mathfrak{e}} k_{\mathfrak{e}} = 0$, then they are universal constants. If these constants appear in an estimate which gives upper bound of some quantity, like $||L^K||_{X^p \to X^p} \ll 1$ or $\sup_t \sup_k |(\mathcal{J}_T)_k| \lesssim L^{O(l(T)\theta)} \rho^{l(T)}$, then in addition to the quantities that universal constants depend, they can also depend on the quantities θ, ε, K, M, N, ϵ_1.

Order of constants: Here is the order of all constants which can appear in the exponential or superscript of L. These constants are θ, ε, K, M, N, ϵ_1.

All the constants are small compared to L in the sense they are less than L^θ for arbitrarily small $\theta > 0$.

ε can be an arbitrarily small constant less than 0.5, the reader is encouraged to assume it to be 0.01. The order of other constants can be decided by the relations $\theta \ll \varepsilon$, $K = O(\theta^{-1})$, $M \gg K$, $N \geq M/\theta$, here the constants in $\ll$, $O(\cdot)$ are universal.

$\mathbb{Z}_L^d$: $\mathbb{Z}_L^d = \{k = \frac{K}{L} : K \in \mathbb{Z}^d\}$

k_x, $k_\perp$: Given any vector k, let k_x be its first component and $k_\perp$ be the vector formed by the rest components.

$\Lambda(k)$, $\Lambda(\nabla)$: $\Lambda(k) \coloneqq k_1(k_1^2 + \cdots k_d^2)$ and $\Lambda(\nabla) = i|\nabla|^2 \partial_{x_1}$

Fourier series: The spatial Fourier series of a function $u : \mathbb{T}_L^d \to \mathbb{C}$ is defined on $\mathbb{Z}_L^d := L^{-1}\mathbb{Z}^d$ by

$$u_k = \int_{\mathbb{T}_L^d} u(x) e^{-2\pi i k \cdot x}, \quad \text{so that} \quad u(x) = \frac{1}{L^d} \sum_{k \in \mathbb{Z}_L^d} u_k \, e^{2\pi i k \cdot x}. \tag{3.1.17}$$

Given any function F, let F_k or $(F)_k$ be its Fourier coefficients.

Order of L: In this paper, L is assumed to be a constant which is much larger than all the universal constants and θ, ε, K, M, N, ϵ_1.

L-certainty: If some statement S involving ω is true with probability $\geq 1 - O_\theta(e^{-L^\theta})$, then we say this statement S is L-certain.

3.2 The Perturbation Expansion

In this section, we calculate the renormalized approximate series and introduce Feynman diagrams to represent terms in this series. We will give an incomplete proof of these propositions in the rest part of the paper.

3.2.1 The renormalization argument

In this section, we derive the equation for Fourier coefficients and specify the renormalization argument mentioned in the introduction.

Let ψ_k be the Fourier coefficient of ψ. Then in term of ψ_k equation (NKLG) becomes

$$\begin{cases} i\dot{\psi}_k = \Lambda(k)\psi_k + \frac{\lambda^2}{L^{2d}} \sum\limits_{\substack{(k_1,k_2,k_3)\in(\mathbb{Z}_L^d)^3 \\ k-k_1+k_2-k_3=0}} \Lambda^{-1}_{k_1k_2k_3k}\, \psi_{k_1}\overline{\psi}_{k_2}\psi_{k_3} \\ \\ \psi_k(0) = \xi_k = \sqrt{n_{\mathrm{in}}(k)}\,\eta_k(\omega), \end{cases} \tag{3.2.1}$$

where $\Lambda_{k_1k_2k_3k} = \Lambda_{k_1}\Lambda_{k_2}\Lambda_{k_3}\Lambda_k$.

As explained in section 3.1.2, the contribution from $\sum_{k_1=k_2,k_3=k}$ or $\sum_{k_1=k,k_2=k_3}$ is very large. This suggests us to split the sum in (3.2.1) into

$$\begin{aligned} \sum_{\substack{(k_1,k_2,k_3)\in(\mathbb{Z}_L^d)^3 \\ k-k_1+k_2-k_3=0}} &= \sum_{\substack{k_1,k_3\neq k \\ k-k_1+k_2-k_3=0}} + \sum_{k_1=k_2,k_3=k} + \sum_{k_1=k,k_2=k_3} - \sum_{k_1=k_2=k_3=k} \\ &=2\sum_{k_1=k_2,k_3=k} - \sum_{k_1=k_2=k_3=k} + \sum_{\substack{k_1,k_3\neq k \\ k-k_1+k_2-k_3=0}} \end{aligned} \tag{3.2.2}$$

Here in the second equality we use symmetry to conclude that $\sum_{k_1=k_2,k_3=k} = \sum_{k_1=k,k_2=k_3}$.

Apply this splitting to (3.2.1),

$$
\begin{aligned}
i\dot{\psi}_k =& \Lambda(k)\psi_k + \frac{2\lambda^2}{L^{2d}}\left(\sum_{k_1\in\mathbb{Z}_L^d}\Lambda_{k_1}^{-2}|\psi_{k_1}|^2\right)\Lambda_k^{-2}\psi_k - \frac{\lambda^2}{L^{2d}}\Lambda_k^{-4}|\psi_k|^2\psi_k + \frac{\lambda^2}{L^{2d}}\sum_{\substack{k_1,k_3\neq k\\ S_3=0}}\Lambda_{k_1k_2k_3k}^{-1}\,\psi_{k_1}\overline{\psi}_{k_2}\psi_{k_3}\\
=& \left(\Lambda(k) + \frac{2\lambda^2}{L^d}M(t)\Lambda(k)^{-2}\right)\psi_k + \frac{\lambda^2}{L^{2d}}\sum_{S_3=0}^{\times}\Lambda_{k_1k_2k_3k}^{-1}\,\psi_{k_1}\overline{\psi}_{k_2}\psi_{k_3}
\end{aligned}
\tag{3.2.3}
$$

Here we have introduced the notation $\Lambda_{k_1k_2k_3k} = \Lambda_{k_1}\Lambda_{k_2}\Lambda_{k_3}\Lambda_k$, $M(t) = \frac{1}{L^d}\sum\limits_{k_1\in\mathbb{Z}_L^d}\Lambda_{k_1}^{-2}|\psi_{k_1}|^2$ and $\sum^{\times} = \sum\limits_{\substack{k_1,k_3\neq k\\ S_3=0}} - \sum_{k_1=k_2=k_3=k}$. In what follows, **a '$\times$' on the summation symbol** indicates that there are constrains of the form $k_1, k_3 \neq k$ and $k_1 = k_2 = k_3 = k$. The exact form of inequality constrains or equal summation indices depends on the the context.

<u>Change phase argument in previous paper:</u> In [7], the L^2 phase problem was trivial and was solved by a simple change phase argument. Let us explain why this argument does not work for general four wave systems.

In [7], the authors view $\left(\Lambda(k) + \frac{2\lambda^2}{L^d}M(t)\Lambda(k)^{-2}\right)\psi_k$ as the linear part of (3.2.3) and remove it by rewriting the equation in term of the linear profile $\phi_k(t) := e^{i\Lambda(k)t + \frac{2\lambda^2}{L^d}i\Lambda(k)^{-2}\int_0^t M(s)ds}\psi_k(t)$.

$$
\begin{cases}
i\dot{\phi}_k = \frac{\lambda^2}{L^{2d}}\sum\limits_{S_3=0}^{\times}\Lambda_{k_1k_2k_3k}^{-1}\,\phi_{k_1}\overline{\phi}_{k_2}\phi_{k_3}e^{-i(\Omega_3 t+\widetilde{\Omega}_3')(k_1,k_2,k_3,k)}\\
\\
\phi_k(0) = \xi_k = \sqrt{n_{\mathrm{in}}(k)}\,\eta_k(\omega)
\end{cases}
\tag{3.2.4}
$$

where

$$
\begin{aligned}
&S_3(k_1,k_2,k_3,k) = k_1 - k_2 + k_3 - k,\\
&\Omega_3(k_1,k_2,k_3,k) = \Lambda(k_1) - \Lambda(k_2) + \Lambda(k_3) - \Lambda(k),\\
&\widetilde{\Omega}_3'(k_1,k_2,k_3,k) = \frac{2\lambda^2}{L^d}\int_0^t M(s)ds\left(\Lambda(k_1)^{-2} - \Lambda(k_2)^{-2} + \Lambda(k_3)^{-2} - \Lambda(k)^{-2}\right).
\end{aligned}
\tag{3.2.5}
$$

Unlike [7], in the Klein-Gordon equation, M is not a conservative quantity, and the linear term of M contains a multiplier $\Lambda(k)^{-2}$. Due to these facts, we have an additional resonant phase $\widetilde{\Omega}_3'$. As explained in section 3.1.2, since this resonant phase is a random variable (it contains M which is a random variable), we can't apply the Wick theorem to calculate the correlation functions in the perturbative expansion.

<u>Our renormalization method:</u> To avoid the above difficulties, we define $m(t) = \mathbb{E}M(t)$. Notice

that a good approximation of a random variable is its expectation, so we can decompose M into $m + (M - m)$. Therefore, we can replace M with m in the definition of the linear profile. Then we remove the randomness of $\widetilde{\Omega}'_3$ at the cost of introducing an error $M - m$ in the equation.

We now explain the renormalization method. Decomposing M in (3.2.3) gives

$$i\dot{\psi}_k = \left(\Lambda(k) + \frac{2\lambda^2}{L^d} m(t)\Lambda(k)^{-2}\right)\psi_k + \frac{\lambda^2}{L^{2d}} \sum_{S_3=0}^{\times} \Lambda^{-1}_{k_1k_2k_3k}\, \psi_{k_1}\overline{\psi}_{k_2}\psi_{k_3} + \frac{2\lambda^2}{L^d}(M(t) - m(t))\Lambda(k)^{-2}\psi_k \tag{3.2.6}$$

By the definition of M and m, the above equation is equivalent to

$$\begin{aligned} i\dot{\psi}_k = & \left(\Lambda(k) + \frac{2\lambda^2}{L^d} m(t)\Lambda(k)^{-2}\right)\psi_k + \frac{\lambda^2}{L^{2d}} \sum_{S_3=0}^{\times} \Lambda^{-1}_{k_1k_2k_3k}\, \psi_{k_1}\overline{\psi}_{k_2}\psi_{k_3} \\ & + \frac{2\lambda^2}{L^{2d}} \left(\sum_{k_1\in\mathbb{Z}^d_L} \Lambda^{-2}_{k_1}\left(|\psi_{k_1}|^2 - \mathbb{E}|\psi_{k_1}|^2\right)\right) \Lambda^{-2}_k \psi_k \end{aligned} \tag{3.2.7}$$

View $\left(\Lambda(k) + \frac{2\lambda^2}{L^d} M(t)\Lambda(k)^{-2}\right)\psi_k$ as the linear part and define the new linear profile by

$$\phi_k(t) := e^{i\Lambda(k)t + \frac{2\lambda^2}{L^d} i\Lambda(k)^{-2}\int_0^t m(s)ds}\psi_k(t) \tag{3.2.8}$$

Rewriting (3.2.7) in terms of ϕ_k gives

$$\begin{aligned} i\dot{\phi}_k = & \frac{\lambda^2}{L^{2d}} \sum_{S_3=0}^{\times} \Lambda^{-1}_{k_1k_2k_3k}\, \phi_{k_1}\overline{\phi}_{k_2}\phi_{k_3} e^{-i(\Omega_3 t + \widetilde{\Omega}_3)(k_1,k_2,k_3,k)} \\ & + \frac{2\lambda^2}{L^{2d}} \left(\sum_{k_1\in\mathbb{Z}^d_L} \Lambda^{-2}_{k_1}\left(|\phi_{k_1}|^2 - \mathbb{E}|\phi_{k_1}|^2\right)\right) \Lambda^{-2}_k \phi_k \end{aligned} \tag{3.2.9}$$

where

$$\widetilde{\Omega}_3(k_1,k_2,k_3,k) = \frac{2\lambda^2}{L^d}\int_0^t m(s)ds\left(\Lambda(k_1)^{-2} - \Lambda(k_2)^{-2} + \Lambda(k_3)^{-2} - \Lambda(k)^{-2}\right) \tag{3.2.10}$$

We will work with the renormalized equation (3.2.9) in the rest part of this paper.

3.2.2 The approximation series and Feynman diagrams

In this section, we construct the approximate solution.

For the ease of notation, we define $:X: := X - \mathbb{E}X$.

Integrating (3.2.9) gives

$$\begin{aligned}\phi_k = \xi_k & \underbrace{- \frac{i\lambda^2}{L^{2d}} \sum_{S_3=0}^{\times} \int_0^t \Lambda^{-1}_{k_1 k_2 k_3 k}\, \phi_{k_1} \overline{\phi}_{k_2} \phi_{k_3} e^{-i(\Omega_3 t + \widetilde{\Omega}_3)} ds}_{\mathcal{T}_1(\phi,\phi,\phi)_k} \\ & \underbrace{- \frac{2i\lambda^2}{L^{2d}} \int_0^t \left(\sum_{k_1 \in \mathbb{Z}^d_L} \Lambda^{-2}_{k_1} : |\phi_{k_1}|^2 : \right) \Lambda^{-1}_k \phi_k ds}_{\mathcal{T}_2(\phi,\phi,\phi)_k}.\end{aligned} \tag{3.2.11}$$

Denote the second and the third term of the right hand side by $\mathcal{T}_1(\phi,\phi,\phi)_k$, $\mathcal{T}_2(\phi,\phi,\phi)_k$ respectively. Then the right hand side equals to $\mathcal{F}(\phi)_k = \xi_k + \mathcal{T}_1(\phi,\phi,\phi)_k + \mathcal{T}_2(\phi,\phi,\phi)_k$. With these notations, (3.2.11) becomes $\phi = \mathcal{F}(\phi)_k$.

We construct the approximation series by iteration: $\phi = \mathcal{F}(\phi) = \mathcal{F}(\mathcal{F}(\phi)) = \mathcal{F}(\mathcal{F}(\mathcal{F}(\phi))) = \cdots$. To estimate this approximation series, we need a compact graphical notation to represent the huge amount of terms generated from iteration. This is done by introducing the concept of Feynman diagrams.

Some basic definitions from graph theory

In this section, we introduce the concept of ternary trees used in this paper. Some concepts below maybe hard to understand, but we will provide a dictionary between these concepts and coefficients of series expansion in Table 3.2, from which the motivation of these concepts should be obvious.

Definition 3.2.1. In this paper, we use following concepts from the graph theory

1. **Ternary trees:** A ternary tree T is a tree in which each node has 3 or 0 children. An example of ternary trees used in this paper is shown in Figure 3.1.

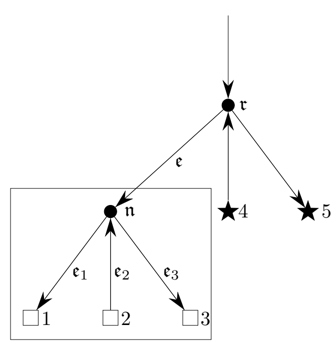


Figure 3.1: Subtrees and node decoration.

2. **Root and children:** The root of a trees and children of nodes are defined in the usual way. The three children from left to right of a node are called the primary, secondary and tertiary child respectively. For example, in Figure 3.1, the node $\mathfrak{r}$ is the root, and nodes 1, 2, 3 are the the primary, secondary and tertiary child of $\mathfrak{n}$ respectively.

3. **Branching nodes:** A branching node in a ternary tree is a node which has 3 children. The number of all branching nodes in a tree T is denoted by $l(T)$. In Figure 3.1, $\mathfrak{r}$ and $\mathfrak{n}$ are branching nodes and $l(T) = 2$.

4. **Leaves and leaf edges:** A leaf of a tree T is a node which has no children. An edge connected to a leaf is called a leaf edge. In Figure 3.1, all $\star$ nodes and $\square$ nodes are leaves and the edges connected to them are leaf edges.

5. **Subtrees:** If any child of any node in a subset T' of a tree T is also contained in T' then T' also forms a tree, we call T' a subtree of T. If the root node of T' is $\mathfrak{n} \in T$, we say T' is the subtree rooted at $\mathfrak{n}$ or subtree of $\mathfrak{n}$ and denote it by $T_{\mathfrak{n}}$. In Figure 3.1, the tree inside the box is the subtree rooted at node $\mathfrak{n}$.

6. **Node decoration:** In Figure 3.1, each node is associated with a symbol in $\{\bullet, \circ, \star, \square\}$. If a node $\mathfrak{n}$ has pattern $\bullet$ (similarly $\circ$, $\star$, $\square$), we say $\mathfrak{n}$ is decorated by $\bullet$ ($\circ$, $\star$, $\square$) or $\mathfrak{n}$ has decoration $\bullet$ ($\circ$, $\star$, $\square$). In what follows,we adopt the convention that leaves always have decoration $\star$ or $\square$ and nodes other than leaves always have decoration $\bullet$ or $\circ$.

7. **Expanding and final leaves:** Leaves denoted by $\square$ are called expanding leaves. Other leaves denoted by $\star$ are called final leaves. The notion of expanding leaves is useful in the construction of trees, in which the presence of $\square$ means that the construction is not finishing and $\square$ denotes leaves that may be replaced by a branching node later.

   The concept of expanding leaves and $\square$ is only used in section 3.2.2, so the readers can safely forget it after that section.

8. **Leg:** The edge that is only connected to the root node is called the leg of the tree. In Figure 3.1, the edge on top of the tree is the leg.

9. **Nodes and children of an edge:** Let $\mathfrak{n}_u$ and $\mathfrak{n}_l$ be two endpoints of an edge $\mathfrak{e}$ and assume that $\mathfrak{n}_l$ is a children of $\mathfrak{n}_u$. We define $\mathfrak{n}_u$ (resp. $\mathfrak{n}_l$) to the upper node (resp. lower node) of $\mathfrak{e}$. Let $\mathfrak{n}_1$, $\mathfrak{n}_2$, $\mathfrak{n}_3$ be the three nodes of $\mathfrak{n}_l$ and let $\mathfrak{e}_1$(resp. $\mathfrak{e}_2$, $\mathfrak{e}_3$) be the edge between two nodes

$\mathfrak{n}_l$ and $\mathfrak{n}_1$(resp. $\mathfrak{n}_2$, $\mathfrak{n}_3$). $\mathfrak{e}_1$, $\mathfrak{e}_2$, $\mathfrak{e}_3$ are defined to be the three children edges of $\mathfrak{e}$. We also define the primary, secondary and tertiary children in the same way as in Definition 3.2.1 (2).

For example, in Figure 3.1, if we take $\mathfrak{e}$ to be the edge between $\mathfrak{r}$ and $\mathfrak{n}$, then $\mathfrak{n}_u = \mathfrak{r}$ (resp. $\mathfrak{n}_l = \mathfrak{n}$) are the upper node (resp. lower node of $\mathfrak{e}$). $\mathfrak{e}_1$, $\mathfrak{e}_2$, $\mathfrak{e}_3$ are the three children edges of $\mathfrak{e}$, which are primary, secondary and tertiary respectively.

10. **Direction of edges and conjugated tree:** In Figure 3.1, each edge is assoicated with a direction by the following rule.

    The leg is chosen to point downwards (or upwards) to the root. The primary and tertiary children edges has the opposite direction as their parent, while the secondary child edge has the same direction.

    The tree in Figure 3.1 is an example of this rule. By default, the leg in most trees in this paper points downwards, except when we pair two trees in Definition 3.3.6. Two isomorphic trees with opposite direction assoication are called conjugated.

11. **Labelling of leaves:** Each leaves are labelled by 1, 2, $\cdots$, $2l(T)+1$ from left to right. An edge from a leaf $\mathfrak{l}$ is also labelled by j if $\mathfrak{l}$ is labelled by j. The tree in Figure 3.1 is an example of this labelling rule.

12. **Sign of edges and leaves:** Each edge $\mathfrak{e}$ in a tree can be assigned with a sign $\iota_{\mathfrak{e}} \in \{\pm\}$ by following rule. If the edge points downwards (resp. upwards), then the edge is associate with a $+$ sign (a $-$ sign). The sign of a leaf equals to the sign of the edge connected to it. For example, in Figure 3.1, leaves labelled by 1, 3, 5 are assigned with a $+$ sign, while leaves labelled by 2, 4 are assigned with a $-$ sign.

Connection between iteration and trees

In this section, we explain non-rigorously the connection between perturbation expansion and trees. Rigorous argument can be find in the next section.

For the ease of notation, we first assume that $\mathcal{F}(\phi)_k = \xi_k + \mathcal{T}(\phi,\phi,\phi)_k$. In this case, we do not need the decoration $\circ$. We will explain how to deal with two nonlinearity $\mathcal{T}_1$ and $\mathcal{T}_2$ later on. This iteration process can be described as the following,

$$
\begin{aligned}
\phi =& \mathcal{F}(\phi) = \xi + \mathcal{T}(\phi,\phi,\phi) \\
=& \xi + \mathcal{T}\Big(\xi + \mathcal{T}(\phi,\phi,\phi),\cdots,\cdots\Big) = \xi + \mathcal{T}(\xi,\xi,\xi) + \mathcal{T}\Big(\mathcal{T}(\phi,\phi,\phi),\xi,\xi\Big)\cdots \\
=& \xi + \mathcal{T}(\xi,\xi,\xi) + \mathcal{T}(\mathcal{T}(\xi,\xi,\xi),\xi,\xi) + \mathcal{T}(\xi,\mathcal{T}(\xi,\xi,\xi),\xi) + \mathcal{T}(\xi,\xi,\mathcal{T}(\xi,\xi,\xi)) + \cdots
\end{aligned}
$$

In above iteration, we recursively choose one ϕ, replace it by $\xi + \mathcal{T}(\phi,\phi,\phi)$ and use the linearity of $\mathcal{T}$ to expand into two terms.

$$
\begin{aligned}
&\mathcal{T}\Big(\cdots,\mathcal{T}(\cdots,\mathcal{T}(\xi,\underline{\phi},\cdots),\cdots),\cdots\Big) \to \mathcal{T}\Big(\cdots,\mathcal{T}(\cdots,\mathcal{T}(\xi,\underline{\xi + \mathcal{T}(\phi,\phi,\phi)},\cdots),\cdots),\cdots\Big) \\
=&\underbrace{\mathcal{T}\Big(\cdots,\mathcal{T}(\cdots,\mathcal{T}(\xi,\underline{\xi},\cdots),\cdots),\cdots\Big)}_{I} + \underbrace{\mathcal{T}\Big(\cdots,\mathcal{T}(\cdots,\mathcal{T}(\xi,\underline{\mathcal{T}(\phi,\phi,\phi)},\cdots),\cdots),\cdots\Big)}_{II}
\end{aligned}
\tag{3.2.12}
$$

Here I and II are obtained by replacing ϕ by ξ and $\mathcal{T}(\phi,\phi,\phi)$ respectively.

In summary, all terms in the expansion can be generated by following steps

- **Step** 0**.** Add a term ϕ in the summation $\mathcal{J}$.
- **Step** i **(**$i \geq 1$**).** Assume that **Step** $i-1$ has been finished which produce a sum of terms $\mathcal{J}$, then choose a term in $\mathcal{J}$ which has least number of ξ and ϕ, remove this term from $\mathcal{J}$ and add the two terms in $\mathcal{J}$ constructed in (3.2.12).

This process is very similar to the construction of ternary trees, in which we recursively replace a chosen branching node by a leaf or branch.

- **Step** 0**.** Start from a branching root node $\square$.
- **Step** i **(**$i \geq 1$**).** Assume that we have finish the **Step** $i-1$ which produce a collection of trees $\mathscr{T}$, then choose a tree in $\mathscr{T}$ which has least number of expanding leaves $\square$ and final leaves $\star$, remove this tree from $\mathscr{T}$ and add two new trees in $\mathscr{T}$. In these two new trees, we replace an expanding leaf $\square$ by a final leaf $\star$ or a branched node $\bullet$ with three branching children leaves $\square$. This construction is illustrated by Figure 3.2.

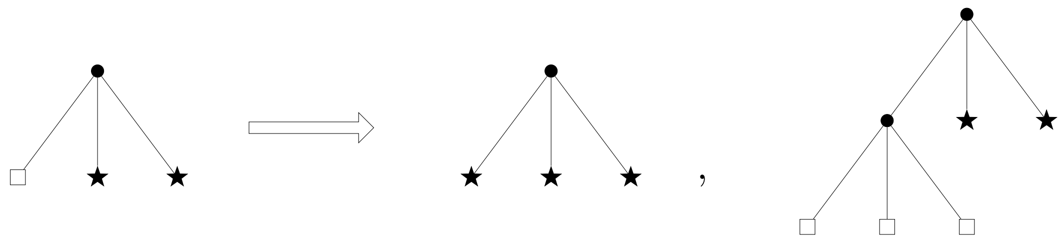


Figure 3.2: One step in the construction of binary trees

By comparing the above two process, we can make the connection between terms and trees more explicit. Each node $\bullet$ other than leaf in the tree T corresponds to a $\mathcal{T}(\cdots,\cdots,\cdots)$ in a term $\mathcal{J}_T$. Each final leaf $\star$ and expanding leaf $\square$ corresponds to ξ and ϕ respectively. The **Step** i of replacing ϕ by ξ or $\mathcal{T}(\phi,\phi,\phi)$ corresponds to replacing $\square$ by $\star$ or a branching node with three children $\square$.

We have following recursive formula for calculating a term $\mathcal{J}_T$ from a ternary tree T.

If T has only one node then $\mathcal{J}_T = \xi$. Otherwise let $\bullet_1$, $\bullet_2$, $\bullet_3$ be three children of the root node $\bullet$, let $T_{\bullet_1}$, $T_{\bullet_2}$, $T_{\bullet_3}$ be the subtrees of T rooted at above nodes. If $\mathcal{J}_{T_{\bullet_1}}$, $\mathcal{J}_{T_{\bullet_2}}$, $\mathcal{J}_{T_{\bullet_3}}$ have been recursively calculated, then $\mathcal{J}_T$ can be calculated by

$$\mathcal{J}_T = \mathcal{T}(\mathcal{J}_{T_{\bullet_1}}, \mathcal{J}_{T_{\bullet_2}}, \mathcal{J}_{T_{\bullet_3}}). \tag{3.2.13}$$

The formal power series obtained by iterate $\phi = \mathcal{F}(\phi)$ can be calculated from trees by $\sum_{T\in\mathscr{T}} \mathcal{J}_T$.

Let $l(T)$ be the number of branches in T, then it can be shown that $\mathcal{J}_T$ is a degree $2l(T)+1$ polynomial of ξ. We define the approximation series to be a finite degree truncation of the formal power series which equals to $\sum_{l(T)\le N} \mathcal{J}_T$.

Now we explain how to generalize above ideas to the case in which two nonlinearity $\mathcal{T}_1$ and $\mathcal{T}_2$ appears. Notice that in above argument, each node decorated by $\bullet$ corresponds to a $\mathcal{T}(\cdots,\cdots,\cdots)$ in a term $\mathcal{J}_T$. In present case, we shall introduce two decoration $\bullet$, $\circ$ correspond to $\mathcal{T}_1$ and $\mathcal{T}_2$ respectively. To adapt these changes, we should generalize the recursive formula (3.2.13) to (3.2.14) introduced in the next section.

Feynman diagrams and construction of the approximation solution

In this section we present the rigorous argument equivalent to that in above section.

In the construction of trees, finally all $\square$ nodes will be replaced by $\bullet$, $\circ$, $\star$, so in what follows, we only consider trees whose nodes are decorated by $\bullet$, $\circ$, $\star$.

Definition 3.2.2. Given a ternary tree T whose nodes are decorated by $\bullet$, $\circ$, $\star$, we inductively define the quantity $\mathcal{J}_T$ by:

$$\mathcal{J}_T = \begin{cases} \xi, & \text{if } T \text{ has only one node } \star. \\ \mathcal{T}_1(\mathcal{J}_{T_{\mathfrak{n}_1}}, \mathcal{J}_{T_{\mathfrak{n}_2}}, \mathcal{J}_{T_{\mathfrak{n}_3}}), & \text{if the root node } \mathfrak{r} \text{ is decorated by } \bullet. \\ \mathcal{T}_2(\mathcal{J}_{T_{\mathfrak{n}_1}}, \mathcal{J}_{T_{\mathfrak{n}_2}}, \mathcal{J}_{T_{\mathfrak{n}_3}}), & \text{if the root node } \mathfrak{r} \text{ is decorated by } \circ. \end{cases} \tag{3.2.14}$$

Here $\mathfrak{n}_1$, $\mathfrak{n}_2$, $\mathfrak{n}_3$ are three children of the root node $\mathfrak{r}$ and $T_{\mathfrak{n}_1}$, $T_{\mathfrak{n}_2}$, $T_{\mathfrak{n}_3}$ are the subtrees of T rooted at these nodes.

Definition 3.2.3. Given a large number N, define the approximate solution ϕ_{app} by

$$\phi_{app} = \sum_{l(T)\le N} \mathcal{J}_T \tag{3.2.15}$$

Section 3.2.2 explains why the approximation series should equal to (3.2.15), a sum of many tree terms, but if we know this fact, we can directly prove it, and forget all the motivations. The lemma below prove that ϕ_{app} defined by above expression is an approximate solution.

Lemma 3.2.4. *Define*

$$Err = \mathcal{F}(\phi_{app}) - \phi_{app}, \tag{3.2.16}$$

then we have

$$Err = \sum_{T\in\mathcal{T}^*_{>N}} \mathcal{J}_T, \tag{3.2.17}$$

where $\mathcal{T}^*_{>N}$ is defined by

$$\begin{aligned} \mathcal{T}^*_{>N} = \{T : l(T) > N,\ & l(T_{\mathfrak{n}_1}) \le N, l(T_{\mathfrak{n}_2}) \le N, l(T_{\mathfrak{n}_3}) \le N, \\ & T_{\mathfrak{n}_1},\ T_{\mathfrak{n}_2},\ T_{\mathfrak{n}_3} \text{ are the subtrees defined in Definition 3.2.2}\} \end{aligned} \tag{3.2.18}$$

Remark 3.2.5. Notice that all terms in $\sum_{T\in\mathcal{T}^*_{>N}}$ are polynomials of degree $> N$. Therefore, the approximation error of ϕ_{app} is of very high order, which proves that ϕ_{app} is an appropriate approximation solution.

Proof. By (3.2.15), we get

$$\begin{aligned}
Err =&\mathcal{F}(\phi_{app}) - \phi_{app}\\
=&\xi + \mathcal{T}_1(\phi_{app}, \phi_{app}, \phi_{app}) + \mathcal{T}_2(\phi_{app}, \phi_{app}, \phi_{app}) - \phi_{app}\\
=&\xi + \sum_{l(T_1), l(T_2), l(T_3) \le N} \Big(\mathcal{T}_1(\mathcal{J}_{T_1}, \mathcal{J}_{T_2}, \mathcal{J}_{T_3}) + \mathcal{T}_2(\mathcal{J}_{T_1}, \mathcal{J}_{T_2}, \mathcal{J}_{T_3})\Big) - \sum_{l(T)\le N} \mathcal{J}_T
\end{aligned} \tag{3.2.19}$$

Let T be a tree constructed by connecting the root nodes $\mathfrak{n}_1$, $\mathfrak{n}_2$, $\mathfrak{n}_3$ of T_1, T_2, T_3 to a new node $\mathfrak{r}$. We define $\mathfrak{r}$ to be the root node of T.

Then by (3.2.14), we have

$$\mathcal{J}_T = \begin{cases} \mathcal{T}_1(\mathcal{J}_{T_1}, \mathcal{J}_{T_2}, \mathcal{J}_{T_3}), & \textit{if } \mathfrak{r} \textit{ is decorated by } \bullet. \\ \mathcal{T}_2(\mathcal{J}_{T_1}, \mathcal{J}_{T_2}, \mathcal{J}_{T_3}), & \textit{if } \mathfrak{r} \textit{ is decorated by } \circ. \end{cases} \tag{3.2.20}$$

and

$$\sum_{l(T_1), l(T_2), l(T_3) \le N} \Big(\mathcal{T}_1(\mathcal{J}_{T_1}, \mathcal{J}_{T_2}, \mathcal{J}_{T_3}) + \mathcal{T}_2(\mathcal{J}_{T_1}, \mathcal{J}_{T_2}, \mathcal{J}_{T_3})\Big) = \sum_{\substack{l(T)\ge 1 \\ l(T_1), l(T_2), l(T_3) \le N}} \mathcal{J}_T \tag{3.2.21}$$

By (3.2.19), we get

$$\begin{aligned}
Err =&\xi + \sum_{\substack{l(T)\ge 1 \\ l(T_1), l(T_2), l(T_3) \le N}} \mathcal{J}_T - \sum_{l(T)\le N} \mathcal{J}_T\\
=& \sum_{\substack{T_1, T_2, T_3 \text{ are subtrees of } \mathfrak{r} \\ l(T_1), l(T_2), l(T_3) \le N}} \mathcal{J}_T - \sum_{\substack{l(T)\le N \\ l(T_1), l(T_2), l(T_3) \le N}} \mathcal{J}_T.\\
=& \sum_{T \in \mathcal{T}^*_{>N}} \mathcal{J}_T
\end{aligned} \tag{3.2.22}$$

Here in the second equality, we use the fact that $\sum_{l(T)\le N} = \sum_{\substack{l(T)\le N \\ l(T_1), l(T_2), l(T_3) \le N}}$.

Therefore, we complete the proof of this lemma. □

Estimates of tree terms

By (3.2.17), in order to control the approximation error Err, it suffices to get upper bounds of tree terms $\mathcal{J}_T$. We state the upper bound in the proposition below and delay its proof to section 2.4.5.

Let us introduce a definition before state the proposition.

Definition 3.2.6. Given a property A, we say A happens L-certainly if the probability that A

happens satisfies $P(A) \geq 1 - Ke^{-L^\theta}$ for some $K, \theta > 0$.

Proposition 3.2.7. *We have L-certainly that*

$$\sup_t \sup_k |(\mathcal{J}_T)_k| \lesssim L^{O(l(T)\theta)} \rho^{l(T)}. \tag{3.2.23}$$

Here $(\mathcal{J}_T)_k$ is the Fourier coefficients of $\mathcal{J}_T$ and

$$\rho = \min(\alpha^{\frac{3}{2}} t, \alpha L). \tag{3.2.24}$$

3.3 Lattice points counting and convergence results

In this section, we give an incomplete proof of Proposition 3.2.7 which gives upper bounds for some tree terms $\mathcal{J}_{T,k}$. The proof of is divided into several steps.

In section 3.3.1, we calculate the coefficients of $\mathcal{J}_{T,k}$ as polynomials of Gaussian random variables.

Large deviation theory suggests that an upper bound of a Gaussian polynomial can be derived from an upper bound of its expectation and variance.

In section 3.3.2, we introduce the concept of couples which is a graphical method of calculating the expectation of Gaussian polynomials. In this section, we also introduce and prove the formal version of our renormalized Wick theorem.

In section 3.3.3, we use couple to establish an lattice points counting result. This result says gives upper bound of numbers of the lattice points near the resonance surface and this bound is strong enough to derive the wave kinetic equation up to epsilon loss, **if the number of good closed couples is more than the number of bad closed couples**.

In section **??**, we introduce a cancellation identity and explain **heuristically** why Wick theorem is sufficient to exclude all bad couples.

we obtain upper bounds for the coefficients of these Gaussian polynomials.

3.3.1 Refined expression of coefficients

In this section, we calculate the coefficients of $\mathcal{J}_{T,k}$ using the definition (3.2.14) of them.

From (3.2.11) and (3.2.14), it is easy to show that $\mathcal{J}_{T,k}$ are polynomials of ξ if we remove the renormalization symbol $: \cdot :$. Given this renormalization symbol, we know that $\mathcal{J}_{T,k}$ should be a renormalized polynomial of ξ that will be defined in the following definition. An example of renormalized polynomial is $(:(:\xi_{k_1}\bar{\xi}_{k_2}:)\xi_{k_3}\bar{\xi}_{k_4}:)\bar{\xi}_{k_5}\xi_{k_6}$.

Definition 3.3.1. 1. **The renormalization symbol.** For any random variable X, define the renormalization symbol by $:X: = X - \mathbb{E}X$.

2. **Some definitions about forests.** Given a forest F, a node $\mathfrak{r}$ is a root if it does not have a parent. Notice that a forest may possess multiple roots. Given a node $\mathfrak{n} \in T$, its subtree $T_{\mathfrak{n}}$ is defined in the same way as the subtrees of trees. Assume that all the **non-leaf** children of $\mathfrak{n}$ are $\mathfrak{n}_1, \cdots, \mathfrak{n}_k$, then the subforest $F_{\mathfrak{n}}$ of $\mathfrak{n}$ is defined as the union of all subtrees $T_{\mathfrak{n}_1}, \cdots, T_{\mathfrak{n}_k}$. The set of leaves of $F_{\mathfrak{n}}$ (resp. $T_{\mathfrak{n}}$) is denoted by $L(F_{\mathfrak{n}})$ (resp. $L(T_{\mathfrak{n}})$). Each leaf of the forest is labelled by a number, called the label of this leaf. The example of roots, subtrees and subforests can be find in Figure 3.3.

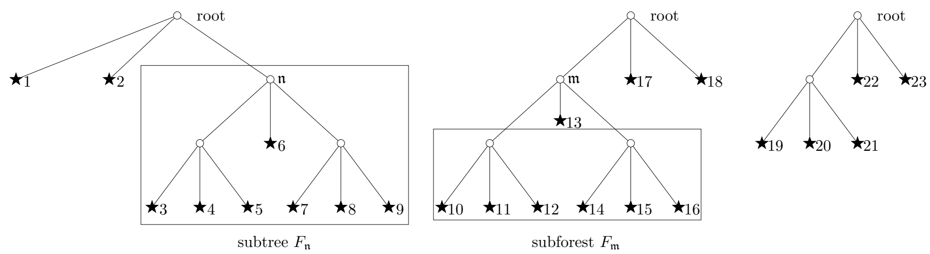


Figure 3.3: Example of roots, subtrees and subforests

3. **Renormalization by a forest.** Given a forest F and a monomial $\xi_{k_1} \cdots \xi_{k_{2m}}$, we define $[\xi_{k_1} \cdots \xi_{k_{2m}}]_F$, the renormalization of $\xi_{k_1} \cdots \xi_{k_{2m}}$ by F, inductively. If $F = \emptyset$, we define $[\xi_{k_1} \cdots \xi_{k_{2m}}]_F = \xi_{k_1} \cdots \xi_{k_{2m}}$. If the root nodes of F are $\mathfrak{r}_1, \cdots, \mathfrak{r}_r$ and their subforests are $F_{\mathfrak{r}_1}, \cdots, F_{\mathfrak{r}_r}$, assume that the renormalizations by these subforests have been defined, then we define $[\xi_{k_1} \cdots \xi_{k_{2m}}]_F = \prod_{i \notin L(F)} \xi_{k_i} \prod_{j=1}^{r} : \left(\left[\prod_{i_j \in L(T_{\mathfrak{r}_j})} \xi_{k_{i_j}} \right]_{F_{\mathfrak{r}_j}} \right) :$, where the number $i_j \in L(T_{\mathfrak{r}_j}$ if and only if the leaf labelled by i_j belongs to $L(T_{\mathfrak{r}_j})$.

   We say $[\xi_{k_1} \cdots \xi_{k_{2m}}]_F$ to be a renormalized monomial and a sum of renormalized monomial to be a renormalized polynomial.

4. **Renormalization forests.** Given a ternary tree T whose nodes are decorated by $\bullet$, $\circ$, $\star$, then the associated renormalization forest $R(T)$ of T is defined to be the unique forest determined by there conditions. First, it is formed by all $\circ$, $\star$ nodes in T. Second, a node $\mathfrak{n}$ in $R(T)$ is a parent of another node $\mathfrak{n}'$ if and only if $\mathfrak{n}$ is the closest one in the ancestors of $\mathfrak{n}'$ that satisfies the property that the unique path from $\mathfrak{n}$ to $\mathfrak{n}'$ contains at least one primary or secondary child edge of a $\circ$ node. (Actually, in this case, this child edge is unique and is at the beginning of this path.) Third, remove all nodes whose path to the root does not contain any primary or secondary child edge.

One example of this construction can be found in Figure 3.4.

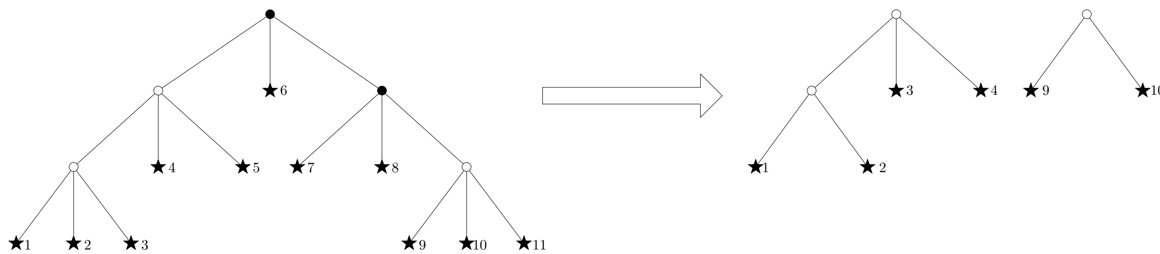


Figure 3.4: Construction of renormalization forests

Now we show that $\mathcal{J}_{T,k}$ is a renormalized polynomial and calculate the coefficients of $\mathcal{J}_{T,k}$.

Lemma 3.3.2. *Given a tree T of depth $l = l(T)$, denote by T_{in} the tree formed by all non-leaf nodes $\mathfrak{n}$, then associate each node $\mathfrak{n} \in T_{in}$ and edge $\mathfrak{l} \in T$ with variables $t_{\mathfrak{n}}$ and $k_{\mathfrak{l}}$ respectively. Given a labelling of all leaves by $1,\ 2,\ \cdots,\ 2l+1$, we identify $k_{\mathfrak{e}}$ with k_j if $\mathfrak{e}$ connects a leaf labelled by j. Given a non-leaf node $\mathfrak{n}$, let $\mathfrak{e}_1,\ \mathfrak{e}_2,\ \mathfrak{e}_3,\ \mathfrak{e}$ be the four edges from or pointing to $\mathfrak{n}$ ($\mathfrak{e}$ is the parent of $\mathfrak{e}_1,\ \mathfrak{e}_2,\ \mathfrak{e}_3$), $\mathfrak{n}_1,\ \mathfrak{n}_2,\ \mathfrak{n}_3$ be children of $\mathfrak{n}$ and $\hat{\mathfrak{n}}$ be the parent of $\mathfrak{n}$.*

Let $\mathcal{J}_T$ be terms defined in Definition 3.2.2, then their Fourier coefficients $\mathcal{J}_{T,k}$ are degree $2l+1$ renormalized polynomials of ξ given by the following formula

$$\mathcal{J}_{T,k} = \left(\frac{-i\lambda^2}{L^{2d}}\right)^l \sum_{k_1,\, k_2,\, \cdots,\, k_{2l+1}} H^T_{k_1\cdots k_{2l+1}} [\xi_{k_1}\bar{\xi}_{k_2}\cdots\xi_{k_{2l+1}}]_{R(T)} \tag{3.3.1}$$

where $H^T_{k_1\cdots k_{2l+1}}$ is given by

$$H^T_{k_1\cdots k_{2l+1}} = \int_{\cup_{\mathfrak{n}\in T_{in}} A_{\mathfrak{n}}} e^{-i\sum_{\mathfrak{n}\in T_{in}}(t_{\mathfrak{n}}\Omega_{\mathfrak{n}}+\widetilde{\Omega}_{\mathfrak{n}})} \prod_{\mathfrak{n}\in T_{in}} dt_{\mathfrak{n}}\ \delta_{\cap_{\mathfrak{n}\in T_{in}} S_{\mathfrak{n}}}\ \prod \Lambda^{-1}(k_{\mathfrak{e}}), \tag{3.3.2}$$

and $\iota_{\mathfrak{e}}$ is defined by Definition 3.2.1 (12), $A_{\mathfrak{n}}$, $S_{\mathfrak{n}}$, $\Omega_{\mathfrak{n}}$, $\widetilde{\Omega}_{\mathfrak{n}}$ are defined by

$$A_{\mathfrak{n}} = \begin{cases} \{t_{\mathfrak{n}_1},\, t_{\mathfrak{n}_2},\, t_{\mathfrak{n}_3} \le t_{\mathfrak{n}}\} & \text{if } \mathfrak{n} \neq \text{ the root } \mathfrak{r} \\ \{t_{\mathfrak{r}} \le t\} & \text{if } \mathfrak{n} = \mathfrak{r} \end{cases} \tag{3.3.3}$$

$$S_{\mathfrak{n}} = \begin{cases} \{k_{\mathfrak{e}_1} - k_{\mathfrak{e}_2} + k_{\mathfrak{e}_3} - k_{\mathfrak{e}} = 0,\ k_{\mathfrak{e}_1} \neq k_{\mathfrak{e}_2} \neq k_{\mathfrak{e}_3} & \text{if } \mathfrak{n} \text{ is decorated by } \bullet \\ \qquad \text{or } k_{\mathfrak{e}_1} = k_{\mathfrak{e}_2} = k_{\mathfrak{e}_3} = k\}, & \\ \{k_{\mathfrak{e}_1} = k_{\mathfrak{e}_2},\ k_{\mathfrak{e}_3} = k_{\mathfrak{e}}\} & \text{if } \mathfrak{n} \text{ is decorated by } \circ \end{cases} \tag{3.3.4}$$

$$\Omega_{\mathfrak{n}} = \begin{cases} \iota_{\mathfrak{e}_1}\Lambda_{k_{\mathfrak{e}_1}} + \iota_{\mathfrak{e}_2}\Lambda_{k_{\mathfrak{e}_2}} + \iota_{\mathfrak{e}_3}\Lambda_{k_{\mathfrak{e}_3}} + \iota_{\mathfrak{e}}\Lambda_{k_{\mathfrak{e}}} & \textit{if } \mathfrak{n} \textit{ is decorated by } \bullet \\ 0 & \textit{if } \mathfrak{n} \textit{ is decorated by } \circ \end{cases} \tag{3.3.5}$$

$$\widetilde{\Omega}_{\mathfrak{n}} = \begin{cases} 2\alpha \int_0^t m(s)ds \left(\iota_{\mathfrak{e}_1}\Lambda^{-1}_{k_{\mathfrak{e}_1}} + \iota_{\mathfrak{e}_2}\Lambda^{-1}_{k_{\mathfrak{e}_2}} + \iota_{\mathfrak{e}_3}\Lambda^{-1}_{k_{\mathfrak{e}_3}} + \iota_{\mathfrak{e}}\Lambda^{-1}_{k_{\mathfrak{e}}}\right) & \textit{if } \mathfrak{n} \textit{ is decorated by } \bullet \\ 0 & \textit{if } \mathfrak{n} \textit{ is decorated by } \circ \end{cases} \tag{3.3.6}$$

For root node $\mathfrak{r}$, we impose the constrain that $k_{\mathfrak{r}} = k$ and $t_{\widehat{\mathfrak{r}}} = t$ (notice that $\mathfrak{r}$ does not have a parent so $\widehat{\mathfrak{r}}$ is not well defined).

Proof. We can check that $\mathcal{J}_T$ defined by (3.3.1) and (3.3.2) satisfies the recursive formula (3.2.14) by a direct substitution, so they are the unique solution of that recursive formula, and this proves Lemma 3.3.2. $\square$

The following table provide a dictionary between concepts in Definition 3.2.1 in previous section and expressions in tree terms $\mathcal{J}_{T,k}$ in Lemma 3.3.2.

Concepts	Corresponding expressions
Ternary tree T	Tree term $\mathcal{J}_{T,k}$
Edge $\mathfrak{e}$	Summation index $k_{\mathfrak{e}}$
Leaf edges $\mathfrak{l}$ labelled by j	Summation index k_j
$\star$ node $\mathfrak{n}$	Two expressions $\widetilde{\Omega}_{\mathfrak{n}}$ and $k_{\mathfrak{e}_1} - k_{\mathfrak{e}_2} + k_{\mathfrak{e}_3} - k_{\mathfrak{e}} = 0$
$\circ$ node $\mathfrak{n}$	Expression $k_{\mathfrak{e}_1} = k_{\mathfrak{e}_2},\ k_{\mathfrak{e}_3} = k_{\mathfrak{e}}$
Leaf $\mathfrak{n}_{\mathfrak{l}}$ labelled by j with a sign	ξ_{k_j} or $\bar{\xi}_{k_j}$ depending on the sign
Direction of edges	$\iota_{\mathfrak{e}}$
Leg	The fixed index k
Conjugated tree $\overline{T}$	Tree term $\overline{\mathcal{J}_{T,k}}$

Table 3.2: A dictionary between concepts of trees and expressions in tree terms.

Lemma 3.3.2 suggests that the coefficients is small when $|\Omega_{\mathfrak{n}}| \gg \omega$, ω is supposed to be $\frac{1}{T_{\mathrm{kin}}}$. Therefore, in order to bound $\mathcal{J}_{T,k}$, we should count the lattice points on $|\Omega_{\mathfrak{n}}| \lesssim \omega$

$$\{k_{\mathfrak{e}} \in \mathbb{Z}^d,\ |k_{\mathfrak{e}}| \lesssim 1,\ \forall \mathfrak{e} : |\Omega_{\mathfrak{n}}| \lesssim \omega,\ \forall \mathfrak{n}.\ \{k_{\mathfrak{e}}\}_{\mathfrak{e}} \in \cap_{\mathfrak{n}\in T_{\mathrm{in}}} S_{\mathfrak{n}}.\ k_{\mathfrak{l}} = k\} \tag{3.3.7}$$

(3.3.7) can be read from the tree diagrams T. As in Figure 3.5, each edge corresponds to a variable $k_{\mathfrak{e}}$. The leg $\mathfrak{l}$ corresponds to equation $k_{\mathfrak{l}} = k$. Each node $\mathfrak{n}$ is connected with four edges $\mathfrak{e}_1$, $\mathfrak{e}_2$, $\mathfrak{e}_3$, $\mathfrak{e}$ whose corresponding variables $k_{\mathfrak{e}_1}$, $k_{\mathfrak{e}_2}$, $k_{\mathfrak{e}_3}$, $k_{\mathfrak{e}}$ satisfy the momentum conservation equation

$$k_{\mathfrak{e}_1} - k_{\mathfrak{e}_2} + k_{\mathfrak{e}_3} - k_{\mathfrak{e}} = 0,\ k_{\mathfrak{e}_1} \neq k_{\mathfrak{e}_2} \neq k_{\mathfrak{e}_3},\ \text{or } k_{\mathfrak{e}_1} = k_{\mathfrak{e}_2} = k_{\mathfrak{e}_3} = k_{\mathfrak{e}} \tag{3.3.8}$$

and the energy conservation equation (if the node is decorated by $\bullet$ or $\circ$)

$$\begin{gathered} \Lambda_{k_{\mathfrak{e}_1}} - \Lambda_{k_{\mathfrak{e}_2}} + \Lambda_{k_{\mathfrak{e}_3}} - \Lambda_{k_{\mathfrak{e}}} = 0. \\ \text{or } k_{\mathfrak{e}_1} = k_{\mathfrak{e}_2},\ k_{\mathfrak{e}_3} = k_{\mathfrak{e}} \text{ (depending on decoration of } \mathfrak{n}) \end{gathered} \tag{3.3.9}$$

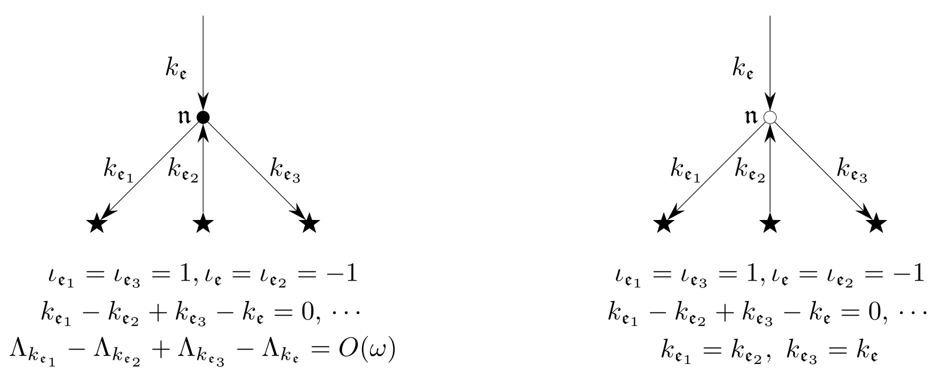


Figure 3.5: Equations of a node $\mathfrak{n}$

3.3.2 Couples and the renormalized Wick theorem

In this section, we introduce the renormalized Wick theorem and calculate $\mathbb{E}|\mathcal{J}_{T,k}|^2$ using this theorem. We also introduce another type of diagrams, the couple diagrams, to represent the result.

By the upper bound in the last section, the coefficients $H^T_{k_1\cdots k_{2l+1}}$ concentrate near the surface $\Omega_{\mathfrak{n}} = 0$, $\forall \mathfrak{n}$. But to get an upper bound of $\mathcal{J}_{T,k}$, we need upper bound of their variance $\mathbb{E}|\mathcal{J}_{T,k}|^2$. The coefficients of $\mathbb{E}|\mathcal{J}_{T,k}|^2$ also concentrate near a surface whose expression is similar to (3.3.7).

Let's derive the expression of the coefficients of $\mathbb{E}|\mathcal{J}_{T,k}|^2$ and its concentration surface.

By Lemma 3.3.2, we know that $\mathcal{J}_{T,k}$ is a polynomial of ξ which are proportional to i.i.d Gaussians. Therefore,

$$\begin{aligned}\mathbb{E}|\mathcal{J}_{T,k}|^2 =&\mathbb{E}(\mathcal{J}_{T,k}\overline{\mathcal{J}_{T,k}}) = \left(\frac{\lambda^2}{L^{2d}}\right)^{2l(T)} \sum_{k_1,\, k_2,\, \cdots,\, k_{2l(T)+1}} \sum_{k_1',\, k_2',\, \cdots,\, k_{2l(T)+1}'} \\ & H^T_{k_1\cdots k_{2l(T)+1}} \overline{H^T_{k_1'\cdots k_{2l(T)+1}'}} \mathbb{E}\Big([\xi_{k_1}\bar{\xi}_{k_2}\cdots\xi_{k_{2l(T)+1}}]_{R(T)}[\bar{\xi}_{k_1'}\xi_{k_2'}\cdots\bar{\xi}_{k_{2l(T)+1}'}]_{R(T)}\Big)\end{aligned} \tag{3.3.10}$$

We just need to calculate

$$\mathbb{E}\Big([\xi_{k_1}\bar{\xi}_{k_2}\cdots\xi_{k_{2l(T)+1}}]_{R(T)}[\bar{\xi}_{k_1'}\xi_{k_2'}\cdots\bar{\xi}_{k_{2l(T)+1}'}]_{R(T)}\Big). \tag{3.3.11}$$

Notice that $\xi_k = \sqrt{n_{\text{in}}(k)}\,\eta_k(\omega)$ and η_k are i.i.d Gaussians. We can apply the Wick theorem to calculate above expectations.

To introduce the renormalized Wick theorem, we need the following definition.

Definition 3.3.3. 1. **Sign map.** Given a set A, a map $\tau : A \to \{-1, 1\}$ is said to be a sign map of A. $\tau(a) \in \{-1, 1\}$ is said to be the sign of a.

2. **Balanced set.** A set A with sign map τ is balanced if A has equal amount of elements of positive ($\tau(a) = 1$) and negative sign ($\tau(a) = -1$).

3. **Pairing.** Suppose that we have a balanced set $A = \{a_1, \cdots, a_m, b_1, \cdots, b_m\}$. Let a_1, $\cdots$, a_m (resp. b_1, $\cdots$, b_m) be elements that have positive sign (resp. negative sign). A balanced pairing or simply balanced pairing is a partition of $A = \{a_{i_1}, b_{j_1}\} \cup \cdots \cup \{a_{i_m}, b_{j_m}\}$ into m subsets which have exactly two elements of different sign. Given a pairing p, elements a_{i_k}, b_{j_k} in the same subset of p are called paired with each other, which is denoted by $a_{i_k} \sim_p b_{j_k}$.

4. **The pairing sets $\mathcal{P}(A)$ and $\mathcal{P}(B, A)$.** Denote by $\mathcal{P}(A)$ the set of all pairings of A. If B is a subset of A, let $\mathcal{P}(B, A) = \{p \in \mathcal{P}(A) : \forall a \in B,\ b \sim_p a \Rightarrow b \in B\}$, i.e. set of pairings that pair elements in B with elements in B.

5. **The renormalized pairing sets $\mathcal{P}_F(A)$.** Given a forest F and a set A that contains the leaf set $L(F)$ as its subset, we define the renormalized pairing set $\mathcal{P}_F(A)$ by $\mathcal{P}_F(A) = \{p \in \mathcal{P} : \forall \mathfrak{n} \in F, p \notin \mathcal{P}(L(T_\mathfrak{n}), A)\}$. $\mathcal{P}_F(A)$ is the set of pairings that do not pair all leaves of any subtree with leaves of the same subtree.

6. **Concatenation of two pairings.** Given two pairing $p_1 \in \mathcal{P}(A_1)$ and $p_2 \in \mathcal{P}(A_2)$, (see

Definition 3.3.3 (4) for the definition of $\mathcal{P}(A,B)$) we can define their concatenation $p_1 \vee p_2 \in \mathcal{P}(A_1) \cup \mathcal{P}(A_2)$ as the following.

Assume that $A_1 = \{a_1^{(1)}, \cdots, a_m^{(1)}, b_1^{(1)}, \cdots, b_m^{(1)}\}$, $A_2 = \{a_1^{(2)}, \cdots, a_m^{(2)}, b_1^{(2)}, \cdots, b_m^{(2)}\}$, p_1 is given by the partition $\{a_{i_1}^{(1)}, b_{j_1}^{(1)}\} \cup \cdots \cup \{a_{i_m}^{(1)}, b_{j_m}^{(1)}\}$ and p_2 is given by the partition $\{a_{i_1}^{(2)}, b_{j_1}^{(2)}\} \cup \cdots \cup \{a_{i_m}^{(2)}, b_{j_m}^{(2)}\}$, then $p_1 \vee p_2$ is given by the partition $\{a_{i_1}^{(1)}, b_{j_1}^{(1)}\} \cup \cdots \cup \{a_{i_m}^{(1)}, b_{j_m}^{(1)}\} \cup \{a_{i_1}^{(2)}, b_{j_1}^{(2)}\} \cup \cdots \cup \{a_{i_m}^{(2)}, b_{j_m}^{(2)}\}$.

Similarly given pairings p_1, $\cdots$, p_n, we can define the concatenation $p_1 \vee p_2 \vee \cdots \vee p_n$

The original Wick theorem is important in the proof of the renormalized Wick theorem.

Theorem 3.3.4 (Wick theorem)**.** *Let $\{\eta_k\}_{k\in\mathbb{Z}^d}$ be i.i.d complex Gaussian random variable. Assume that τ is a sign map of a balanced set $A = \{k_1, k_2, \cdots, k_{2m}\}$. Define $\tau_j = \tau(k_j)$. Let $\eta_{k_j}^{\tau_j} = \eta_{k_j}$ if $\tau_j = 1$ and $\eta_{k_j}^{\tau_j} = \bar{\eta}_{k_j}$ if $\tau_j = -1$. Let $\mathcal{P}$ be the set of all pairings of $\{k_1, k_2, \cdots, k_{2m}\}$, then*

$$\mathbb{E}(\eta_{k_1}^{\tau_1} \cdots \eta_{k_{2m}}^{\tau_{2m}}) = \sum_{p\in\mathcal{P}} \delta_p(k_1, \cdots, k_{2m}), \tag{3.3.12}$$

where

$$\delta_p = \begin{cases} 1 & \text{if } k_i = k_j \text{ for all } k_i \sim_p k_j, \\ 0 & \text{otherwise.} \end{cases} \tag{3.3.13}$$

Proof. This Wick theorem is a direct corollary of the complex Isserlis' theorem proved in Lemma A.2 of [6]. □

Theorem 3.3.5 (Renormalized Wick theorem)**.** *In addition to assumptions in Theorem 3.3.4, define $\mathcal{P}_F = \mathcal{P}_F(A)$ (remember that $A = \{k_1, k_2, \cdots, k_{2m}\}$ and $\mathcal{P}_F(A)$ is defined in Definition 3.3.3 (5)), then*

$$\mathbb{E}([\eta_{k_1}^{\tau_1} \cdots \eta_{k_{2m}}^{\tau_{2m}}]_F) = \sum_{p\in\mathcal{P}_F} \delta_p(k_1, \cdots, k_{2m}) \tag{3.3.14}$$

Proof. Before starting the formal proof, we give a proof for an example to demonstrate the idea.

Proof for a special case. Consider the following forest F in Figure 3.6.

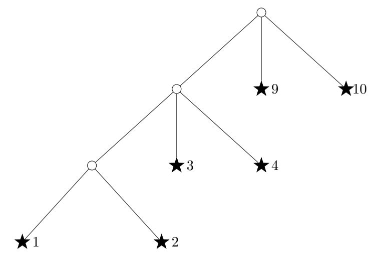


Figure 3.6: An example of forests

Let us calculate $\mathbb{E}([\xi_{k_1}\bar{\xi}_{k_2}\xi_{k_3}\bar{\xi}_{k_4}\bar{\xi}_{k_5}\xi_{k_6})]_F) = \mathbb{E}((:(:\xi_{k_1}\bar{\xi}_{k_2}:)\xi_{k_3}\bar{\xi}_{k_4}:)\bar{\xi}_{k_5}\xi_{k_6})$.

Expanding the renormalization symbol gives

$$\begin{aligned}
&\mathbb{E}((:(:\xi_{k_1}\bar{\xi}_{k_2}:)\xi_{k_3}\bar{\xi}_{k_4}:)\bar{\xi}_{k_5}\xi_{k_6}) \\
=&\mathbb{E}(\xi_{k_1}\bar{\xi}_{k_2}\xi_{k_3}\bar{\xi}_{k_4}\bar{\xi}_{k_5}\xi_{k_6}) - \mathbb{E}(\xi_{k_1}\bar{\xi}_{k_2})\mathbb{E}(\xi_{k_3}\bar{\xi}_{k_4}\bar{\xi}_{k_5}\xi_{k_6}) - \\
&\mathbb{E}(\xi_{k_1}\bar{\xi}_{k_2}\xi_{k_3}\bar{\xi}_{k_4})\mathbb{E}(\bar{\xi}_{k_5}\xi_{k_6}) + \mathbb{E}(\xi_{k_1}\bar{\xi}_{k_2})\mathbb{E}(\xi_{k_3}\bar{\xi}_{k_4})\mathbb{E}(\bar{\xi}_{k_5}\xi_{k_6}) \\
=&\sum_{\text{all pairings}} \delta_p - \sum_{A=\{p \text{ pairs } (12) \text{ and } (3456)\}} \delta_p - \sum_{B=\{p \text{ pairs } (1234) \text{ and } (56)\}} \delta_p + \sum_{A\cap B} \delta_p \\
=&\sum_{A^c\cap B^c} \delta_p = \sum_{p\in\mathcal{P}_F} \delta_p
\end{aligned} \tag{3.3.15}$$

Here in the second equality, we have applied the Theorem 3.3.4 and the inclusion-exclusion principle. The last equality follows from the definition of $\mathcal{P}_F$.

Proof for the general case. The proof for the general case also uses an analogous inclusion-exclusion argument. Since the renormalization is defined by recursion in Definition 3.3.1 (3), we will prove the theorem by induction.

Step 1. (The induction assumption) For the ease of notation let $\eta_{k_j} = \eta_{k_j}^{\tau_j}$. Let $A = \{k_1, k_2, \cdots, k_{2m}\}$. Then we need to show that $\mathbb{E}([\eta_{k_1}\cdots\eta_{k_{2m}}]_F) = \sum_{p\in\mathcal{P}_F}\delta_p$.

We prove the theorem by induction on $Depth(F)$.

If $Depth(F) = 0$, i.e. $F = \emptyset$, then $\mathbb{E}([\eta_{k_1}\cdots\eta_{k_{2m}}]_F) = \mathbb{E}(\eta_{k_1}\cdots\eta_{k_{2m}})$ and the theorem follows from the standard Wick theorem (Theorem 3.3.4).

Assume that the theorem is true for $Depth(F) < l$, we show that it is true for $Depth(F) = l$ in next steps.

Step 2. (Expanding the product) Let $\{\mathfrak{r}_1, \cdots, \mathfrak{r}_r\}$ be roots of F and $F_{\mathfrak{r}_1}, \cdots, F_{\mathfrak{r}_r}$ be their subforests, then by Definition 3.3.1 (3)

$$\mathbb{E}([\eta_{k_1}\cdots\eta_{k_{2m}}]_F)=\mathbb{E}\left(\prod_{k_i\notin L(F)}\eta_{k_i}\prod_{j=1}^{r}\left(\Big[\prod_{k_{i_j}\in L(T_{\mathfrak{r}_j})}\eta_{k_{i_j}}\Big]_{F_{\mathfrak{r}_j}}-\mathbb{E}\Big[\prod_{k_{i_j}\in L(T_{\mathfrak{r}_j})}\eta_{k_{i_j}}\Big]_{F_{\mathfrak{r}_j}}\right)\right) \tag{3.3.16}$$

Expanding the product $\prod_{j=1}^{r}$ gives

$$\begin{aligned}
&\mathbb{E}([\eta_{k_1}\cdots\eta_{k_{2m}}]_F)\\
=&\mathbb{E}\left(\sum_{S\subseteq\{1,\cdots,r\}}(-1)^{|S|}\prod_{k_i\notin L(F)}\eta_{k_i}\prod_{j'\notin S}\Big[\prod_{k_{i_{j'}}\in L(T_{\mathfrak{r}_{j'}})}\eta_{k_{i_{j'}}}\Big]_{F_{\mathfrak{r}_{j'}}}\prod_{j\in S}\mathbb{E}\Big[\prod_{k_{i_j}\in L(T_{\mathfrak{r}_j})}\eta_{k_{i_j}}\Big]_{F_{\mathfrak{r}_j}}\right)\\
=&\sum_{S\subseteq\{1,\cdots,r\}}(-1)^{|S|}\underbrace{\mathbb{E}\left(\prod_{k_i\notin L(F)}\eta_{k_i}\prod_{j'\notin S}\Big[\prod_{k_{i_{j'}}\in L(T_{\mathfrak{r}_{j'}})}\eta_{k_{i_{j'}}}\Big]_{F_{\mathfrak{r}_{j'}}}\right)}_{E}\prod_{j\in S}\underbrace{\mathbb{E}\Big[\prod_{k_{i_j}\in L(T_{\mathfrak{r}_j})}\eta_{k_{i_j}}\Big]_{F_{\mathfrak{r}_j}}}_{E_{F_{\mathfrak{r}_j}}}
\end{aligned} \tag{3.3.17}$$

Here in the second line, we have used the identity

$$\prod_{j=1}^{r}(a_j-b_j)=\sum_{S\subseteq\{1,\cdots,r\}}(-1)^{|S|}\prod_{j\in S}a_j\prod_{j'\notin S}b_j \tag{3.3.18}$$

Step 3. (Applying the induction assumption) Notice that E in the last line of (3.3.17) is the expectation of the polynomial $\prod_{k_i\notin L(F) \text{ or } k_i\in\cup_{j'\notin S}L(T_{\mathfrak{r}_j})}\eta_{k_i}$ renormalized by the forest $\cup_{j'\notin S}F_{\mathfrak{r}_j}$. $E_{F_{\mathfrak{r}_j}}$ is the expectation of the polynomial $\prod_{k_{i_j}\in L(T_{\mathfrak{r}_j})}\eta_{k_{i_j}}$ renormalized by the forest $F_{\mathfrak{r}_j}$.

In other words,

$$E=\mathbb{E}\left(\Big[\prod_{k_i\notin L(F) \text{ or } k_i\in\cup_{j'\notin S}L(T_{\mathfrak{r}_j})}\eta_{k_i}\Big]_{\cup_{j'\notin S}F_{\mathfrak{r}_j}}\right),\qquad E_{F_{\mathfrak{r}_j}}=\mathbb{E}\left(\Big[\prod_{k_{i_j}\in L(T_{\mathfrak{r}_j})}\eta_{k_{i_j}}\Big]_{F_{\mathfrak{r}_j}}\right) \tag{3.3.19}$$

Since $Depth(F_{\mathfrak{r}_j})<l$ and $Depth(\cup_{j'\notin S}F_{\mathfrak{r}_j})<l$ for any j, we can thus apply the induction assumption to calculate E and $E_{F_{\mathfrak{r}_j}}$.

$$\begin{aligned}
E=&\sum_{p\in\mathcal{P}_{\cup_{j'\notin S}F_{\mathfrak{r}_{j'}}}(\{k_i\notin L(F) \text{ or } k_i\in\cup_{j'\notin S}L(T_{\mathfrak{r}_j})\})}\delta_p\\
=&\sum_{p\in\mathcal{P}_{\cup_{j'\notin S}F_{\mathfrak{r}_{j'}}}((\cup_{j\in S}L(T_{\mathfrak{r}_j}))^c)}\delta_p,
\end{aligned} \tag{3.3.20}$$

and

$$E_{F_{\mathfrak{r}_j}} = \prod_{j\in S} \sum_{q\in\mathcal{P}_{F_{\mathfrak{r}_j}}(T_{\mathfrak{r}_j})} \delta_{q_j}, \tag{3.3.21}$$

where in (3.3.20), we have used the fact that $\{k_i \notin L(F) \text{ or } k_i \in \cup_{j'\notin S} L(T_{\mathfrak{r}_j})\} = (\cup_{j\in S} L(T_{\mathfrak{r}_j}))^c$, and in (3.3.20) and (3.3.21), we haved used the notation $\mathcal{P}_F(B)$ in Definition 3.3.3 (5). In (3.3.20), $\mathcal{P}_{\cup_{j'\notin S} F_{\mathfrak{r}_{j'}}}((\cup_{j\in S} L(T_{\mathfrak{r}_j}))^c)$ is of the form $\mathcal{P}_F(B)$, in which $F = \cup_{j'\notin S} F_{\mathfrak{r}_{j'}}$ and $B = (\cup_{j\in S} L(T_{\mathfrak{r}_j}))^c$.

Substituting (3.3.20) and (3.3.21) into (3.3.17), we get

$$\mathbb{E}([\eta_{k_1}\cdots\eta_{k_{2m}}]_F) = \sum_{S\subseteq\{1,\cdots,r\}} (-1)^{|S|} \left(\sum_{p\in\mathcal{P}_{\cup_{j'\notin S} F_{\mathfrak{r}_{j'}}}((\cup_{j\in S} L(T_{\mathfrak{r}_j}))^c)} \delta_p \right) \left(\prod_{j\in S} \sum_{q\in\mathcal{P}_{F_{\mathfrak{r}_j}}(T_{\mathfrak{r}_j})} \delta_{q_j} \right) \tag{3.3.22}$$

Step 4. (Concatenating short pairings) Expanding the product, we get

$$\begin{aligned}\mathbb{E}([\eta_{k_1}\cdots\eta_{k_{2m}}]_F) &= \sum_{S\subseteq\{1,\cdots,r\}} (-1)^{|S|} \left(\sum_{p\in\mathcal{P}_{\cup_{j'\notin S} F_{\mathfrak{r}_{j'}}}((\cup_{j\in S} L(T_{\mathfrak{r}_j}))^c)} \delta_p \right) \left(\prod_{j\in S} \sum_{q\in\mathcal{P}_{F_{\mathfrak{r}_j}}(T_{\mathfrak{r}_j})} \delta_{q_j} \right) \\ &= \sum_{S\subseteq\{1,\cdots,r\}} (-1)^{|S|} \sum_{\substack{p\in\mathcal{P}_{\cup_{j'\notin S} F_{\mathfrak{r}_{j'}}}((\cup_{j\in S} L(T_{\mathfrak{r}_j}))^c) \\ q_j\in\mathcal{P}_{F_{\mathfrak{r}_j}}(T_{\mathfrak{r}_j})\ \forall j}} \left(\delta_p \prod_{j\in S} \delta_{q_j} \right)\end{aligned} \tag{3.3.23}$$

For any tree or forest T, define $\mathcal{P}[T] = \mathcal{P}(L(T), A)$ (see Definition 3.3.3 (4) for the definition of $\mathcal{P}(A, B)$), where $A = \{k_1, k_2, \cdots, k_{2m}\}$. For trees or forests $\{T_j\}_{j\in S}$, define $\mathcal{P}[T_j, j \in S] = \cap_{j\in S}\mathcal{P}[L(F_j)]$.

Claim. Given $p \in \mathcal{P}_{\cup_{j'\notin S} F_{\mathfrak{r}_{j'}}}((\cup_{j\in S} L(T_{\mathfrak{r}_j}))^c)$ and $q_j \in \mathcal{P}_{F_{\mathfrak{r}_j}}(T_{\mathfrak{r}_j})$ $\forall j$, then their concatenation satisfies $p \vee (\vee_{j\in S} q_j) \in \mathcal{P}[T_j, j \in S] \cap \mathcal{P}_{\cup_{j=1}^r F_{\mathfrak{r}_j}}$. Conversely, all pairings in $p \vee (\vee_{j\in S} q_j) \in \mathcal{P}[T_j, j \in S] \cap \mathcal{P}_{\cup_{j=1}^r F_{\mathfrak{r}_j}}$ can be generated in this way.

Proof of the claim. $p \in \mathcal{P}_{\cup_{j'\notin S} F_{\mathfrak{r}_{j'}}}((\cup_{j\in S} L(T_{\mathfrak{r}_j}))^c)$ means that p does not pair all leaves of any subtree of $\cup_{j'\notin S} F_{\mathfrak{r}_{j'}}$ with leaves of the same subtree. $q_j \in \mathcal{P}_{F_{\mathfrak{r}_j}}(T_{\mathfrak{r}_j})$ $\forall j$ means that q_j does not pair all leaves of any subtree of $F_{\mathfrak{r}_j}$ with leaves of the same subtree. Therefore, $p \vee (\vee_{j\in S} q_j)$ does not pair all leaves of any subtree of $\cup_{j=1}^r F_{\mathfrak{r}_j}$ with leaves of the same subtree, which implies that $\mathcal{P}_{\cup_{j=1}^r F_{\mathfrak{r}_j}}$.

It it not difficult to show that $p \vee (\vee_{j\in S} q_j) \in \mathcal{P}[T_j, j \in S]$, so we have $p \vee (\vee_{j\in S} q_j) \in \mathcal{P}[T_j, j \in S] \cap \mathcal{P}_{\cup_{j=1}^r F_{\mathfrak{r}_j}}$.

Given a pairing $p' \in \mathcal{P}[T_j, j \in S] \cap \mathcal{P}_{\cup_{j=1}^r F_{\mathfrak{r}_j}}$, we can restrict p' to subsets and define $p = p'|_{(\cup_{j\in S} L(T_{\mathfrak{r}_j}))^c}$ and $q_j = p|_{\cup_{j\in S} L(T_{\mathfrak{r}_j})}$. It is not hard to show that $p' = p \vee (\vee_{j\in S} q_j)$, $q_j \in \mathcal{P}_{F_{\mathfrak{r}_j}}(T_{\mathfrak{r}_j})$ $\forall j$

and $p \in \mathcal{P}_{\cup_{j'\notin S} F_{\mathfrak{r}_{j'}}}((\cup_{j\in S} L(T_{\mathfrak{r}_j}))^c)$.

Therefore, we have finished the proof of the claim. $\square$

From above claim and (3.3.23), we know that

$$\mathbb{E}([\eta_{k_1}\cdots\eta_{k_{2m}}]_F) = \sum_{S\subseteq\{1,\cdots,r\}} (-1)^{|S|} \sum_{p\vee(\vee_{j\in S} q_j)\in\mathcal{P}[T_j, j\in S]\cap\mathcal{P}_{\cup_{j=1}^r F_{\mathfrak{r}_j}}} \delta_{p\vee(\vee_{j\in S} q_j)} \tag{3.3.24}$$

Step 5. (The inclusion-exclusion principle) Now we apply the following inclusion-exclusion principle.

$$\mathbb{1}_{A_1^c\cap\cdots\cap A_m^c} = \sum_{S\subseteq\{1,\cdots,r\}} (-1)^{|S|} \mathbb{1}_{\cap_{j\in S} A_j} \tag{3.3.25}$$

Since $\mathcal{P}[T_j, j\in S] = \cap_{j\in S}\mathcal{P}[T_j]$, we get

$$\begin{aligned}
\mathbb{E}([\eta_{k_1}\cdots\eta_{k_{2m}}]_F) &= \sum_{S\subseteq\{1,\cdots,r\}} (-1)^{|S|} \sum_p \mathbb{1}_{\cap_{j\in S}\mathcal{P}[T_j]\cap\mathcal{P}_{\cup_{j=1}^r F_{\mathfrak{r}_j}}} \delta_p \\
&= \sum_p \left(\sum_{S\subseteq\{1,\cdots,r\}} (-1)^{|S|} \mathbb{1}_{\cap_{j\in S}\mathcal{P}[T_j]\cap\mathcal{P}_{\cup_{j=1}^r F_{\mathfrak{r}_j}}} \right) \delta_p \\
&= \sum_p \left(\sum_{S\subseteq\{1,\cdots,r\}} (-1)^{|S|} \mathbb{1}_{\cap_{j\in S}\mathcal{P}[T_j]} \right) \mathbb{1}_{\mathcal{P}_{\cup_{j=1}^r F_{\mathfrak{r}_j}}} \delta_p \\
&= \sum_p \mathbb{1}_{\cap_{j\in S}(\mathcal{P}[T_j])^c\cap\mathcal{P}_{\cup_{j=1}^r F_{\mathfrak{r}_j}}} \delta_p = \sum_p \mathbb{1}_{\mathcal{P}_F} \delta_p \\
&= \sum_{p\in\mathcal{P}_F} \delta_p
\end{aligned} \tag{3.3.26}$$

Here in the fourth line we have used (3.3.25) and the following identity

$$\mathcal{P}_F = \cap_{j\in S}(\mathcal{P}[T_j])^c \cap \mathcal{P}_{\cup_{j=1}^r F_{\mathfrak{r}_j}}. \tag{3.3.27}$$

Now we prove the above identity. By definition (Definition 3.3.3 (5)),

$$\begin{aligned}
\mathcal{P}_{\cup_{j=1}^r F_{\mathfrak{r}_j}} &= \{p\in\mathcal{P} : \forall \mathfrak{n}\in\cup_{j=1}^r F_{\mathfrak{r}_j}, p\notin\mathcal{P}(L(T_{\mathfrak{n}}), A)\} \\
&= \{p\in\mathcal{P} : \forall \mathfrak{n}\in F\backslash\{\mathfrak{r}_1,\cdots,\mathfrak{r}_r\}, p\notin\mathcal{P}(L(T_{\mathfrak{n}}), A)\},
\end{aligned} \tag{3.3.28}$$

and

$$\mathcal{P}_F = \{p\in\mathcal{P} : \forall \mathfrak{n}\in F, p\notin\mathcal{P}(L(T_{\mathfrak{n}}), A)\}. \tag{3.3.29}$$

By definition of $\mathcal{P}[T_j]$ (it is in the paragraph above the claim),

$$(\mathcal{P}[T_j])^c = (\mathcal{P}(L(T_{\mathfrak{r}_j}), A))^c = \{p \in \mathcal{P} : \mathfrak{r}_j, p \notin \mathcal{P}(L(T_{\mathfrak{r}_j}), A)\}. \tag{3.3.30}$$

Therefore, we have

$$\begin{aligned} &\cap_{j\in S} (\mathcal{P}[T_j])^c \cap \mathcal{P}_{\cup_{j=1}^r F_{\mathfrak{r}_j}} \\ =&\cap_{j\in S} (\{p \in \mathcal{P} : \mathfrak{r}_j, p \notin \mathcal{P}(L(T_{\mathfrak{r}_j}), A)\}) \cap \{p \in \mathcal{P} : \forall \mathfrak{n} \in F\backslash\{\mathfrak{r}_1, \cdots, \mathfrak{r}_r\}, p \notin \mathcal{P}(L(T_{\mathfrak{n}}), A)\} \\ =&\{p \in \mathcal{P} : \forall \mathfrak{n} \in F, p \notin \mathcal{P}(L(T_{\mathfrak{n}}), A)\} = \mathcal{P}_F. \end{aligned} \tag{3.3.31}$$

Now we finished the proof of (3.3.27) and thus the proof of the renormalized Wick theorem. $\square$

Denote by $R(T) \cup R(T)$ the union of two copies of $R(T)$, then we have

$$\begin{aligned} &\mathbb{E}\Big([\xi_{k_1}\bar{\xi}_{k_2}\cdots\xi_{k_{2l(T)+1}}]_{R(T)}[\bar{\xi}_{k'_1}\xi_{k'_2}\cdots\bar{\xi}_{k'_{2l(T)+1}}]_{R(T)}\Big) \\ =&\mathbb{E}\Big([\xi_{k_1}\bar{\xi}_{k_2}\cdots\xi_{k_{2l(T)+1}}\bar{\xi}_{k'_1}\xi_{k'_2}\cdots\bar{\xi}_{k'_{2l(T)+1}}]_{R(T)\cup R(T)}\Big) \end{aligned} \tag{3.3.32}$$

Applying Wick theorem to (3.3.10), we get

$$\begin{aligned} \mathbb{E}|\mathcal{J}_{T,k}|^2 = \left(\frac{\lambda^2}{L^{2d}}\right)^{2l(T)} &\sum_{p\in\mathcal{P}_F(\{k_1,\cdots,k_{2l(T)+1},k'_1,\cdots,k'_{2l(T)+1}\})} \\ &\underbrace{\sum_{\substack{k_1,\,k_2,\,\cdots,\,k_{2l(T)+1}\\ k'_1,\,k'_2,\,\cdots,\,k'_{2l(T)+1}}} H^T_{k_1\cdots k_{2l(T)+1}} H^T_{k'_1\cdots k'_{2l(T)+1}} \delta_p(k_1,\cdots,k_{2l(T)+1},k'_1,\cdots,k'_{2l(T)+1}) \sqrt{n_{\text{in}}(k_1)}\cdots}_{Term(T,p)}. \end{aligned} \tag{3.3.33}$$

We see that the correlation of two tree terms is a sum of smaller expressions $Term(T,p)$. By (3.3.7), the coefficients $H^T_{k_1\cdots k_{2l(T)+1}} H^T_{k'_1\cdots k'_{2l(T)+1}}$ of $Term(T,p)$ concentrate near the subset

$$\{k_{\mathfrak{e}}, k'_{\mathfrak{e}} \in \mathbb{Z}^d,\ |k_{\mathfrak{e}}|, |k'_{\mathfrak{e}}| \lesssim 1,\ \forall \mathfrak{e} : |\Omega_{\mathfrak{n}}|, |\Omega'_{\mathfrak{n}}| \lesssim \omega,\ \forall \mathfrak{n}.\ \{k_{\mathfrak{e}}\}_{\mathfrak{e}}, \{k'_{\mathfrak{e}}\}_{\mathfrak{e}} \in \cap_{\mathfrak{n}\in T_{\text{in}}} S_{\mathfrak{n}}.\ k_{\mathfrak{l}} = k'_{\mathfrak{l}} = k\}. \tag{3.3.34}$$

The pairing p in Wick theorem introduces new equations $k_i = k'_j$ (defined in (3.3.13)) and the coefficients $H^T_{k_1\cdots k_{2l(T)+1}} H^T_{k'_1\cdots k'_{2l(T)+1}} \delta_p$ concentrate near the subset

$$\begin{aligned}\{k_{\mathfrak{e}}, k'_{\mathfrak{e}} \in \mathbb{Z}^d,\ |k_{\mathfrak{e}}|, |k'_{\mathfrak{e}}| \lesssim 1,\ \forall \mathfrak{e} : |\Omega_{\mathfrak{n}}|, |\Omega'_{\mathfrak{n}}| \lesssim \omega,\ \forall \mathfrak{n}.\ \{k_{\mathfrak{e}}\}_{\mathfrak{e}}, \{k'_{\mathfrak{e}}\}_{\mathfrak{e}} \in \cap_{\mathfrak{n}\in T_{\mathrm{in}}} S_{\mathfrak{n}}.\ k_{\mathfrak{l}} = k'_{\mathfrak{l}} = k.\\ k_i = k'_j \textit{ (and } k_i = k_j,\ k'_i = k'_j \textit{) for all } k_i \sim_p k'_j \textit{ (and } k_i \sim_p k_j,\ k'_i \sim_p k'_j\textit{)}\}.\end{aligned} \tag{3.3.35}$$

As in the case of (3.3.7), there is a graphical representation of (3.3.35). To explain this, we need the concept of couples.

Definition 3.3.6 (Construction of couples)**.** Given two trees T and T', we flip the orientation of all edges in T' (as in the two left trees in Figure 3.7). We also label their leaves by $1, 2, \cdots, 2l(T) + 1$ and $1, 2, \cdots, 2l(T') + 1$ so that the corresponding variables of these leaves are $k_1, k_2, \cdots, k_{2l(T)+1}$ and $k_1, k_2, \cdots, k_{2l(T')+1}$. Assume that we have a pairing p of the set $\{k_1, k_2, \cdots, k_{2l(T)+1}, k_1, k_2, \cdots,$ $k_{2l(T')+1}\}$, then this pairing induces a pairing between leaves (if $k_i \sim_p k_j$ then define *the i-th leaf* $\sim_p$ *the j-th leaf*). Given this pairing of leaves, we define the following procedure which glues two trees T and T' into a couple $\mathcal{C}(T, T, p)$. Some example of pairing can be find in Figure 3.7.

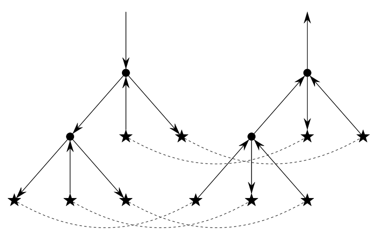


Figure 3.7: Example of pairings between trees.

1. **Merging edges connected to leaves.** Given two edges with opposite orientation connected to two paired leaves, these two edges can be <u>merged</u> into one edge as in Figure 3.8. Since the pairing in this paper is a balanced pairing, only edges with opposite orientation can be paired.

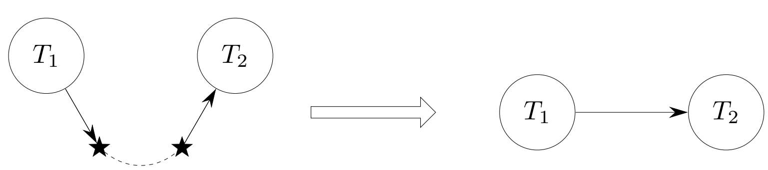


Figure 3.8: Pairing and merging of two edges

We know that two edges connected to leaves correspond to two indices k_i, k_j. Merging two such edges is a graphical interpretation that $k_i = k_j = k$.

2. **Splitting of $\circ$ nodes:** Given a $\circ$ node $\mathfrak{n}$, we split it into two edges as in Figure 3.9.

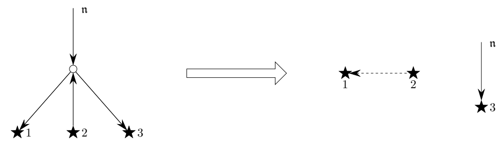


Figure 3.9: Splitting of a $\circ$ node

One of this two edges is usual and the other one is dotted (called dotted edge). If the parent of $\circ$ is $\mathfrak{n}$ and the children of $\circ$ are $\mathfrak{n}_1$, $\mathfrak{n}_2$, $\mathfrak{n}_3$ from left to right, then we require the usual edge connects $\mathfrak{n}$ and $\mathfrak{n}_3$ and the dotted edge connects $\mathfrak{n}_1$ and $\mathfrak{n}_2$. Figure 3.10 is an example of splitting $\circ$ in a tree

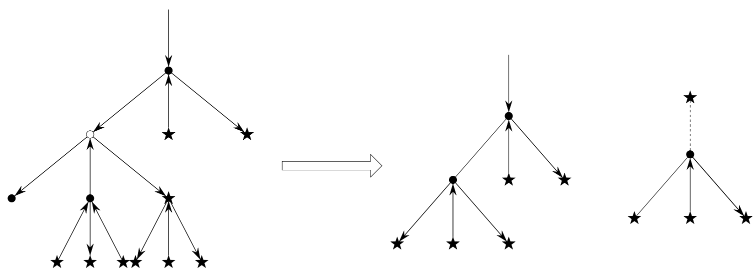


Figure 3.10: An example of splitting a $\circ$ node

We know that the $\circ$ node corresponds to equation $S_{\mathfrak{n}} = \{k_{\mathfrak{e}_1} = k_{\mathfrak{e}_2},\ k_{\mathfrak{e}_3} = k_{\mathfrak{e}}\}$. The corresponding variable $k_{\mathfrak{e}_d}$ of the dotted edge $\mathfrak{e}_d$ is defined to be the common value of $k_{\mathfrak{e}_1}$ and $k_{\mathfrak{e}_2}$. The corresponding variable $k_{\mathfrak{e}_n}$ of the usual edge $\mathfrak{e}_n$ is defined to be the common value of $k_{\mathfrak{e}_3}$ and $k_{\mathfrak{e}}$.

3. **Pairing of trees and couples.** Given a pairing p of the set of leaves in T, T' we merge all edges paired by p as in Figure 3.11 and the resulting combinatorial structure is called a couple.

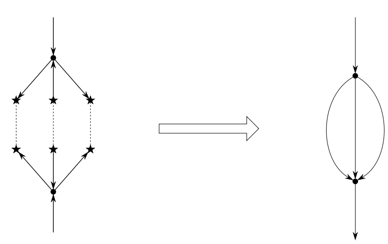


Figure 3.11: The construction of a couple

We know that each edge connected to leaf corresponds to a variable k_i. A pairing p of $\{k_1, k_2, \cdots, k_{2m}\}$ in (3.3.35) induces a pairing of edges connected to leaves. Merging paired edges corresponds to $k_i = k'_j$ for all $k_i \sim_p k'_j$ in (3.3.35).

Proposition 3.3.7. (3.3.35) *can be read from a couple diagram $\mathcal{C}(T, T, p)$. Each edge corresponds to a variable $k_{\mathfrak{e}}$. The leg $\mathfrak{l}$ corresponds to equation $k_{\mathfrak{l}} = k$. Each node corresponds to a momentum conservation equation*

$$k_{\mathfrak{e}_1} - k_{\mathfrak{e}_2} + k_{\mathfrak{e}_3} - k_{\mathfrak{e}} = 0,\ k_{\mathfrak{e}_1} \neq k_{\mathfrak{e}_2} \neq k_{\mathfrak{e}_3},\ \text{or } k_{\mathfrak{e}_1} = k_{\mathfrak{e}_2} = k_{\mathfrak{e}_3} = k, \tag{3.3.36}$$

and a energy conservation equation

$$\Lambda_{k_{\mathfrak{e}_1}} - \Lambda_{k_{\mathfrak{e}_2}} + \Lambda_{k_{\mathfrak{e}_3}} - \Lambda_{k_{\mathfrak{e}}} = O(\omega). \tag{3.3.37}$$

Remark 3.3.8. In a couple diagram, we only have nodes decorated by $\bullet$. Nodes decorated by $\circ$ and $\star$ have been removed in (2), (3) of Definition 3.3.6.

Remark 3.3.9. Through the process of (2), (3) in Definition 3.3.6, a couple diagram can automatically encode the equation $k_i = k'_j$ for all $k_i \sim_p k'_j$ and $k_{\mathfrak{e}_1} = k_{\mathfrak{e}_2},\ k_{\mathfrak{e}_3} = k_{\mathfrak{e}}$. Therefore, they do not appear in Proposition 3.3.7.

Proof. This directly follows from the definition of couples. □

The calculations of this section are summarized in the following proposition.

Proposition 3.3.10. *(1) Define $Term(T, p)$ in the same way as in* (3.3.33),

$$\begin{aligned} Term(T, p) = \sum_{k_1, k_2, \cdots, k_{2l(T)+1}} \sum_{k'_1, k'_2, \cdots, k'_{2l(T)+1}} \\ H^T_{k_1 \cdots k_{2l(T)+1}} H^T_{k'_1 \cdots k'_{2l(T)+1}} \delta_p(k_1, \cdots, k_{2l(T)+1}, k'_1, \cdots, k'_{2l(T)+1}) \sqrt{n_{in}(k_1)} \cdots \sqrt{n_{in}(k'_1)} \cdots \end{aligned} \tag{3.3.38}$$

then $\mathbb{E}|\mathcal{J}_{T,k}|^2$ is a sum of $Term(T, p)$ for all $p \in \mathcal{P}_F$, (in (3.3.33) *the sum is over set of all possible pairing $\mathcal{P}$)*

$$\mathbb{E}|\mathcal{J}_{T,k}|^2 = \left(\frac{\lambda^2}{L^{2d}}\right)^{2l(T)} \sum_{p \in \mathcal{P}_F(\{k_1, \cdots, k_{2l(T)+1}, k'_1, \cdots, k'_{2l(T)+1}\})} Term(T, p). \tag{3.3.39}$$

(2) Since the definition (3.3.38) *of $Term(T, p)$ do not change, it still concentrates near the subset* (3.3.35) *which has a simple graphical representation given by Proposition 3.3.7.*

Proof. The proof of (1), (2) is easy and thus skipped. □

3.3.3 Counting lattice points

In this section, we apply the connection between couple and concentration subset (3.3.35) to count the number of solutions of a generalized version of (3.3.35),

$$\{k_{\mathfrak{e}}, k'_{\mathfrak{e}} \in \mathbb{Z}^d,\ |k_{\mathfrak{e}}|, |k'_{\mathfrak{e}}| \lesssim 1,\ \forall \mathfrak{e} : |\Omega_{\mathfrak{n}} - \sigma_{\mathfrak{n}}|, |\Omega'_{\mathfrak{n}} - \sigma'_{\mathfrak{n}}| \lesssim \omega,\ \forall \mathfrak{n}.\ \{k_{\mathfrak{e}}\}_{\mathfrak{e}}, \{k'_{\mathfrak{e}}\}_{\mathfrak{e}} \in \cap_{\mathfrak{n}\in T'} S_{\mathfrak{n}}.\ k_{\mathfrak{l}} = k'_{\mathfrak{l}} = k.$$
$$k_i = k'_j \textit{ (and } k_i = k_j,\ k'_i = k'_j\textit{) for all } k_i \sim_p k'_j \textit{ (and } k_i \sim_p k_j,\ k'_i \sim_p k'_j\textit{)}\}. \tag{3.3.40}$$

(3.3.40) is obtained by replacing $\Omega_{\mathfrak{n}}$, $\Omega'_{\mathfrak{n}}$ by $\Omega_{\mathfrak{n}} - \sigma_{\mathfrak{n}}$, $\Omega'_{\mathfrak{n}} - \sigma'_{\mathfrak{n}}$ in (3.3.35). $\sigma_{\mathfrak{n}}$ and $\sigma'_{\mathfrak{n}}$ are some given constants. The counterpart of Proposition 3.3.7 in this case is

Proposition 3.3.11. (3.3.40) *can be read from a couple diagram $\mathcal{C} = \mathcal{C}(T, T, p)$. Each edge corresponds to a variable $k_{\mathfrak{e}}$. The leg $\mathfrak{l}$ corresponds to equation $k_{\mathfrak{l}} = k$. Each node corresponds to a momentum conservation equation*

$$k_{\mathfrak{e}_1} - k_{\mathfrak{e}_2} + k_{\mathfrak{e}_3} - k_{\mathfrak{e}} = 0,\ k_{\mathfrak{e}_1} \neq k_{\mathfrak{e}_2} \neq k_{\mathfrak{e}_3},\ \textit{or } k_{\mathfrak{e}_1} = k_{\mathfrak{e}_2} = k_{\mathfrak{e}_3} = k, \tag{3.3.41}$$

and a energy conservation equation

$$\Lambda_{k_{\mathfrak{e}_1}} - \Lambda_{k_{\mathfrak{e}_2}} + \Lambda_{k_{\mathfrak{e}_3}} - \Lambda_{k_{\mathfrak{e}}} = \sigma_{\mathfrak{n}} + O(\omega). \tag{3.3.42}$$

Denote the momentum and energy conservation equations by $MC_{\mathfrak{n}}$ and $EC_{\mathfrak{n}}$ respectively, then (3.3.40) *can be rewritten as*

$$(3.3.40) = \{k_{\mathfrak{e}} \in \mathbb{Z}^d,\ |k_{\mathfrak{e}}| \lesssim 1,\ \forall \mathfrak{e} \in \mathcal{C} : MC_{\mathfrak{n}},\ EC_{\mathfrak{n}},\ \forall \mathfrak{n} \in \mathcal{C}.\ k_{\mathfrak{l}} = k'_{\mathfrak{l}} = k.\} \tag{3.3.43}$$

Proof. This directly follows from Proposition 3.3.7. □

Basic ideas for counting lattice points

In this section, we explain the basic idea used in this paper to counting lattices points. To do this, we need following definitions related to couples.

Definition 3.3.12.
1. **Connected couples.** A couple $\mathcal{C}$ is a connected couple if it is connected as a graph. The connected components is also defined in the same way as the graph theory.
2. **Equations of a couple $Eq(\mathcal{C})$:** Given a couple $\mathcal{C}$ and constants k, $\sigma_{\mathfrak{n}}$, let $Eq(\mathcal{C}, \{\sigma_{\mathfrak{n}}\}_{\mathfrak{n}}, k)$ (or

simply $Eq(\mathcal{C})$) be the system of equation (3.3.43) constructed in Proposition 3.3.11. For any system of equations Eq, let $\#(Eq)$ be its number of solutions.

The main goal of this section is to prove an upper bound of $\#Eq(\mathcal{C})$. The main idea of proving this is to decompose a large couple $\mathcal{C}$ into smaller pieces and then prove this for smaller piece using induction hypothesis. To explain the idea, let us first focus on an example. Let $\mathcal{C}$ be the left couple in the following picture. (The corresponding variables of each edge are labelled near these edges.)

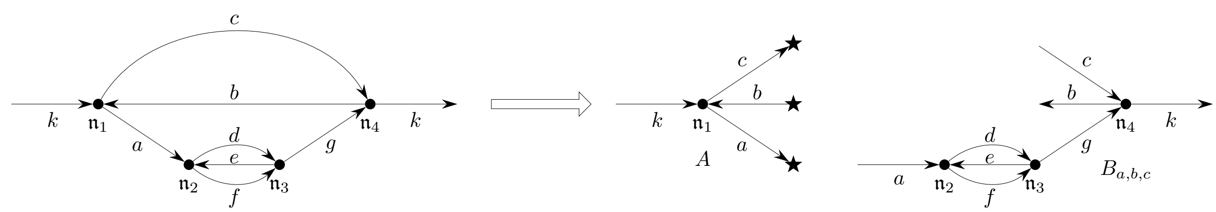


Figure 3.12: An example of decomposing a couple

By (3.3.43), we know that the couple $\mathcal{C}$ corresponds to the following equations.

$$\begin{aligned}
&\{(a,b,c,d,e,f,g): \ (|a| \text{ to } |g|) \lesssim 1,\\
&\ a-b+c=k,\ \Lambda(a)-\Lambda(b)+\Lambda(c)-\Lambda(k)=\sigma_1+O(\omega)\\
&\ d-e+f=a,\ \Lambda(d)-\Lambda(e)+\Lambda(f)-\Lambda(a)=\sigma_2+O(\omega)\\
&\ d-e+f=g,\ \Lambda(d)-\Lambda(e)+\Lambda(f)-\Lambda(g)=\sigma_3+O(\omega)\\
&\ c-b+g=k,\ \Lambda(c)-\Lambda(b)+\Lambda(g)-\Lambda(k)=\sigma_4+O(\omega)\}
\end{aligned} \tag{3.3.44}$$

We know that (3.3.44) can be rewritten into the form $\bigcup_{a,e\in A} B_{a,e}$, where

$$A=\{a,b,c: \ |a|,|b|,|c|\lesssim 1,\ a-b+c=k,\ \Lambda(a)-\Lambda(b)+\Lambda(c)-\Lambda(k)=\sigma_1+O(\omega)\} \tag{3.3.45}$$

$$\begin{aligned}
B_{a,b,c}=\{d,e,f,g: \ &|d|,|e|,|f|,|g|\lesssim 1,\\
&d-e+f=a,\ \Lambda(d)-\Lambda(e)+\Lambda(f)-\Lambda(a)=\sigma_2+O(\omega)\\
&d-e+f=g,\ \Lambda(d)-\Lambda(e)+\Lambda(f)-\Lambda(g)=\sigma_3+O(\omega)\\
&c-b+g=k,\ \Lambda(c)-\Lambda(b)+\Lambda(g)-\Lambda(k)=\sigma_4+O(\omega)\}
\end{aligned} \tag{3.3.46}$$

Since an upper bound of $\#Eq(\mathcal{C})$ can be derived from upper bounds of $\#A$, $\#B_{a,b,c}$, we just need to consider A, $B_{a,b,c}$ which are systems of equations of smaller size. We can reduce the size of systems of equations in this way and prove upper bounds by induction.

One problem of applying induction argument is that A, $B_{a,b,c}$ cannot be represented by couple defined by Definition 3.3.6 that can contain at most two legs (an edge just connected to one node). In Definition 3.3.6, a leg is used to represent a variable which is fixed, as in the condition $k_{\mathfrak{l}} = k'_{\mathfrak{l}} = k$ in (3.3.43). The definition of $\#B_{a,b,c}$ contains four fixed variables a, b, c, k which cannot be represented by just two legs. Therefore, we has to define a new type of couple that allows multiple legs.

Except for the lack of legs, we also have the problem of representing free variables. We know that the couple representation of A should contain one node and four edges if we insist on the rule that a node corresponds to an equation and the variables in the equation correspond to edges connected to this node. All these edges are legs, but three of four edges correspond to variable a, b, c which are not fixed. Therefore, we have to define a type of legs that can correspond to unfixed variables.

To solve the above problems, we introduce the following definition.

Definition 3.3.13. 1. **Couples with multiple legs:** A graph in which all nodes have degree 1 or 4 is called a couples with multiple legs. The graph A and $B_{a,b,c}$ in Figure 3.12 are examples of this definition.

2. **Legs:** In a couple with multiple leg, an edge connected to a degree one node is called a leg. Remember that we have encounter this concept in the second paragraph of section 3.3.1 and in what follows, we call the leg defined there the root leg of a tree.

3. **Free legs and fixed legs:** In a couple with multiple leg, we use two types of node decoration for degree 1 nodes as in Figure 3.13. One is $\star$ and the other one is invisible.

Figure 3.13: Node decoration of degree one nodes

An edge connected to a $\star$ or invisible nodes is called a free leg or fixed leg respectively. They correspond to free variables or fixed variables $k_{\mathfrak{e}}$ respectively.

4. **Equations of a couple $Eq(\mathcal{C}, \{c_{\mathfrak{l}}\}_{\mathfrak{l}})$:** We define the corresponding equations for a couples with multiple legs.

$$Eq(\mathcal{C}, \{c_{\mathfrak{l}}\}_{\mathfrak{l}}) = \{k_{\mathfrak{e}} \in \mathbb{Z}^d_L,\ |k_{\mathfrak{e}}| \lesssim 1\ \forall \mathfrak{e} \in \mathcal{C} :\ MC_{\mathfrak{n}},\ EC_{\mathfrak{n}},\ \forall \mathfrak{n} \in \mathcal{C}.\ k_{\mathfrak{l}} = c_{\mathfrak{l}},\ \forall \mathfrak{l}.\} \tag{3.3.47}$$

In this representation, the corresponding variable of a fixed leg $\mathfrak{l}$ is fixed to be the constant $c_{\mathfrak{l}}$

and the corresponding variable of a free leg $\mathfrak{l}$ is not fixed.

With the above definition, it's easy to show that the couple A and $B_{a,b,c}$ in Figure 3.12 correspond to the system of equations (3.3.45) and (3.3.46) respectively.

The cutting operation and its properties

In this section, we give the formal definition of cutting and explain how $\#Eq(\mathcal{C})$ changes after cutting.

Definition 3.3.14. 1. **Cutting an edge:** Given an edge $\mathfrak{e}$, we can cut it into two edges (a fixed and a free leg) as in Figure 3.14.

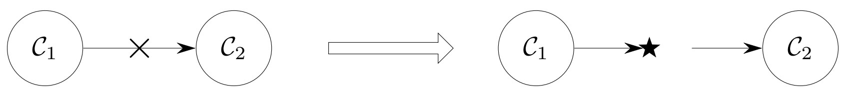


Figure 3.14: An example of cutting an edge

2. **Cut:** A cut c of a couple $\mathcal{C}$ is a set of edges such that $\mathcal{C}$ is disconnected after cutting all edges in c, together with a map $\mathrm{rc} : c \to \{\text{left}, \text{right}\}$. For each $\mathfrak{e} \in c$, if $\mathrm{rc}(\mathfrak{e}) = \text{left}$ (resp. right), then as in Figure 3.14 the left node (resp. right node) produced by cutting $\mathfrak{e}$ is a $\star$ node (resp. invisible node). The map rc describes which one should be the free or fixed leg in the two legs produced by cutting an edge. An admissible cut is defined to be the cut such that the new legs after cutting are all free in one component and are fixed in another component.

3. $c(\mathfrak{e})$**,** $c(\mathfrak{n})$ **and** $c(\mathfrak{l})$**:** Given an edge $\mathfrak{e}$ that is not a leg, define $c(\mathfrak{e})$ to be the cut that contains only one edge $\mathfrak{e}$. Given a node $\mathfrak{n} \in \mathcal{C}$, let $\{\mathfrak{e}_i\}$ be edges that are connected to $\mathfrak{n}$, then define $c(\mathfrak{n})$ to be the cut that consists of edges $\{\mathfrak{e}_i\}$. Given an leg $\mathfrak{l}$, let $\mathfrak{n}$ be the unique node connected to it, then define $c(\mathfrak{l})$ to be the cut $c(\mathfrak{n})$. An example of cutting $c(\mathfrak{e})$ is give by Figure 3.14. The following picture gives an example of cutting $c(\mathfrak{n})$ or $c(\mathfrak{l})$ (in this picture $\mathfrak{n} = \mathfrak{n}_1$ and $\mathfrak{l}$ is the leg labelled by k.)

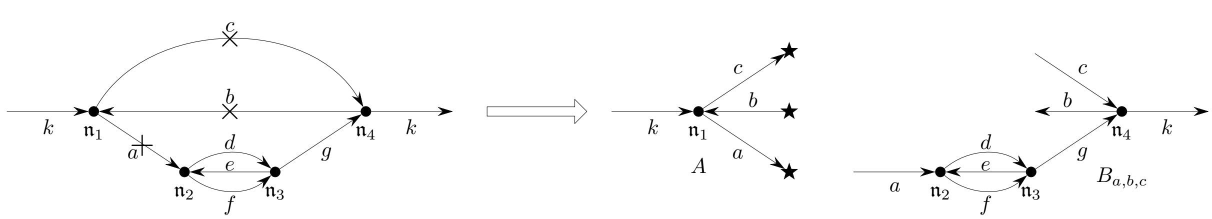


Figure 3.15: An example of cuts, $c(\mathfrak{n})$ and $c(\mathfrak{l})$

Remark 3.3.15. Explicitly writing down the full definition of rc is often complicated, so in what follows, when defining rc, we will only describe which one should be the free or fixed leg in the two legs produced by cutting an edge.

Let us explain how does $Eq(\mathcal{C})$ and $\#Eq(\mathcal{C})$ changes after cutting. The result is summarized in the following lemma.

Lemma 3.3.16. *Let c be an admissible cut of $\mathcal{C}$ that consists of edges $\{\mathfrak{e}_i\}$ and $\mathcal{C}_1$, $\mathcal{C}_2$ be two components after cutting. Let $\mathfrak{e}_i^{(1)} \in \mathcal{C}_1$, $\mathfrak{e}_i^{(2)} \in \mathcal{C}_2$ be two edges obtained by cutting $\mathfrak{e}_i$. The rc map is defined by assigning $\{\mathfrak{e}_i^{(1)}\}$ to be free legs and $\{\mathfrak{e}_i^{(2)}\}$ to be fixed legs. Then we have*

$$Eq(\mathcal{C}, \{c_{\mathfrak{l}}\}_{\mathfrak{l}}) = \left\{(k_{\mathfrak{e}_1}, k_{\mathfrak{e}_2}) : \ k_{\mathfrak{e}_1} \in Eq(\mathcal{C}_1, \{c_{\mathfrak{l}_1}\}), \ k_{\mathfrak{e}_2} \in Eq\left(\mathcal{C}_2, \{c_{\mathfrak{l}_2}\}, \left\{k_{\mathfrak{e}_i^{(1)}}\right\}_i\right)\right\}. \tag{3.3.48}$$

and

$$\sup_{\{c_{\mathfrak{l}}\}_{\mathfrak{l}}} \#Eq(\mathcal{C}, \{c_{\mathfrak{l}}\}_{\mathfrak{l}}) \leq \sup_{\{c_{\mathfrak{l}_1}\}_{\mathfrak{l}_1 \in leg(\mathcal{C}_1)}} \#Eq(\mathcal{C}_1, \{c_{\mathfrak{l}_1}\}) \sup_{\{c_{\mathfrak{l}_2}\}_{\mathfrak{l}_2 \in leg(\mathcal{C}_2)}} \#Eq(\mathcal{C}_2, \{c_{\mathfrak{l}_2}\}). \tag{3.3.49}$$

Here $leg(\mathcal{C})$ is the set of fixed legs in $\mathcal{C}$ (not the set of all legs!).

Proof. By definition (3.3.47), we have

$$\begin{aligned} Eq(\mathcal{C}, \{c_{\mathfrak{l}}\}_{\mathfrak{l}}) =& \{k_{\mathfrak{e}} \in \mathbb{Z}^d, \ |k_{\mathfrak{e}}| \lesssim 1, \ \forall \mathfrak{e} \in \mathcal{C} : MC_{\mathfrak{n}}, \ EC_{\mathfrak{n}}, \ \forall \mathfrak{n} \in \mathcal{C}. \ k_{\mathfrak{l}} = c_{\mathfrak{l}}, \ \forall \mathfrak{l} \in \mathrm{leg}(\mathcal{C}).\} \\ =& \{(k_{\mathfrak{e}_1}, k_{\mathfrak{e}_2}) : \ |k_{\mathfrak{e}_1}| \lesssim 1, \ MC_{\mathfrak{n}_1}, \ EC_{\mathfrak{n}_1}. \ \forall \mathfrak{e}_1, \mathfrak{n}_1 \in \mathcal{C}_1. \ k_{\mathfrak{l}_1} = c_{\mathfrak{l}_1}, \ \forall \mathfrak{l}_1 \in \mathrm{leg}(\mathcal{C}) \cap \mathrm{leg}(\mathcal{C}_1) \\ |k_{\mathfrak{e}_2}| \lesssim 1, \ MC_{\mathfrak{n}_2}, \ EC_{\mathfrak{n}_2}. \ \forall \mathfrak{e}_2, \mathfrak{n}_2 \in \mathcal{C}_2. \ k_{\mathfrak{l}_2} = c_{\mathfrak{l}_2}, \ &\forall \mathfrak{l}_2 \in \mathrm{leg}(\mathcal{C}) \cap \mathrm{leg}(\mathcal{C}_2), \ k_{\mathfrak{e}_i^{(2)}} = k_{\mathfrak{e}_i^{(1)}}, \ \forall \mathfrak{e}_i \in c\} \\ =& \left\{(k_{\mathfrak{e}_1}, k_{\mathfrak{e}_2}) : \ k_{\mathfrak{e}_1} \in Eq(\mathcal{C}_1, \{c_{\mathfrak{l}_1}\}), \ k_{\mathfrak{e}_2} \in Eq\left(\mathcal{C}_2, \{c_{\mathfrak{l}_2}\}, \left\{k_{\mathfrak{e}_i^{(1)}}\right\}_i\right)\right\} \end{aligned} \tag{3.3.50}$$

Here in $Eq\left(\mathcal{C}_2, \{c_{\mathfrak{l}_2}\}, \left\{k_{\mathfrak{e}_i^{(1)}}\right\}_i\right)$, $k_{\mathfrak{e}_i^{(1)}}$ are view as a constant value and $k_{\mathfrak{e}_i^{(2)}}$ are fixed to be this constant value.

$$\begin{aligned} Eq(\mathcal{C}, \{c_{\mathfrak{l}}\}_{\mathfrak{l}}) =& \{k_{\mathfrak{e}} \in \mathbb{Z}^d, \ |k_{\mathfrak{e}}| \lesssim 1, \ \forall \mathfrak{e} \in \mathcal{C} : MC_{\mathfrak{n}}, \ EC_{\mathfrak{n}}, \ \forall \mathfrak{n} \in \mathcal{C}. \ k_{\mathfrak{l}} = c_{\mathfrak{l}}, \ \forall \mathfrak{l} \in \mathrm{leg}(\mathcal{C}).\} \\ =& \{(k_{\mathfrak{e}_1}, k_{\mathfrak{e}_2}) : \ |k_{\mathfrak{e}_1}| \lesssim 1, \ MC_{\mathfrak{n}_1}, \ EC_{\mathfrak{n}_1}. \ \forall \mathfrak{e}_1, \mathfrak{n}_1 \in \mathcal{C}_1. \ k_{\mathfrak{l}_1} = c_{\mathfrak{l}_1}, \ \forall \mathfrak{l}_1 \in \mathrm{leg}(\mathcal{C}) \cap \mathrm{leg}(\mathcal{C}_1) \\ |k_{\mathfrak{e}_2}| \lesssim 1, \ MC_{\mathfrak{n}_2}, \ EC_{\mathfrak{n}_2}. \ \forall \mathfrak{e}_2, \mathfrak{n}_2 \in \mathcal{C}_2. \ k_{\mathfrak{l}_2} = c_{\mathfrak{l}_2}, \ &\forall \mathfrak{l}_2 \in \mathrm{leg}(\mathcal{C}) \cap \mathrm{leg}(\mathcal{C}_2), \ k_{\mathfrak{e}_i^{(2)}} = k_{\mathfrak{e}_i^{(1)}}, \ \forall \mathfrak{e}_i \in c\} \\ =& \left\{(k_{\mathfrak{e}_1}, k_{\mathfrak{e}_2}) : \ k_{\mathfrak{e}_1} \in Eq(\mathcal{C}_1, \{c_{\mathfrak{l}_1}\}), \ k_{\mathfrak{e}_2} \in Eq\left(\mathcal{C}_2, \{c_{\mathfrak{l}_2}\}, \left\{k_{\mathfrak{e}_i^{(1)}}\right\}_i\right)\right\} \end{aligned} \tag{3.3.51}$$

Here in $Eq\left(\mathcal{C}_2, \{c_{\mathfrak{l}_2}\}, \left\{k_{\mathfrak{e}_i^{(1)}}\right\}_i\right)$, $k_{\mathfrak{e}_i^{(1)}}$ are view as a constant value and $k_{\mathfrak{e}_i^{(2)}}$ are fixed to be this constant value.

Therefore, we have the following identity of $Eq(\mathcal{C}, \{c_{\mathfrak{l}}\}_{\mathfrak{l}})$

$$Eq(\mathcal{C}, \{c_{\mathfrak{l}}\}_{\mathfrak{l}}) = \left\{(k_{\mathfrak{e}_1}, k_{\mathfrak{e}_2}) : \ k_{\mathfrak{e}_1} \in Eq(\mathcal{C}_1, \{c_{\mathfrak{l}_1}\}), \ k_{\mathfrak{e}_2} \in Eq\left(\mathcal{C}_2, \{c_{\mathfrak{l}_2}\}, \left\{k_{\mathfrak{e}_i^{(1)}}\right\}_i\right)\right\}. \tag{3.3.52}$$

which proves (3.3.48).

We can also find the relation between $\#Eq(\mathcal{C}_1)$, $\#Eq(\mathcal{C}_2)$ and $\#Eq(\mathcal{C})$. Applying (3.3.48),

$$\begin{aligned}
\#Eq(\mathcal{C}, \{c_{\mathfrak{l}}\}_{\mathfrak{l}}) &= \sum_{(k_{\mathfrak{e}_1}, k_{\mathfrak{e}_2}) \in \#Eq(\mathcal{C}, \{c_{\mathfrak{l}}\}_{\mathfrak{l}})} 1 \\
&= \sum_{\left\{(k_{\mathfrak{e}_1}, k_{\mathfrak{e}_2}) :\ k_{\mathfrak{e}_1} \in Eq(\mathcal{C}_1, \{c_{\mathfrak{l}_1}\}),\ k_{\mathfrak{e}_2} \in Eq\left(\mathcal{C}_2, \{c_{\mathfrak{l}_2}\}, \left\{k_{\mathfrak{e}_i^{(1)}}\right\}_i\right)\right\}} 1 \\
&= \sum_{k_{\mathfrak{e}_1} \in Eq(\mathcal{C}_1, \{c_{\mathfrak{l}_1}\})} \ \sum_{k_{\mathfrak{e}_2} \in Eq\left(\mathcal{C}_2, \{c_{\mathfrak{l}_2}\}, \left\{k_{\mathfrak{e}_i^{(1)}}\right\}_i\right)} 1 \\
&= \sum_{k_{\mathfrak{e}_1} \in Eq(\mathcal{C}_1, \{c_{\mathfrak{l}_1}\})} \#Eq\left(\mathcal{C}_2, \{c_{\mathfrak{l}_2}\}, \left\{k_{\mathfrak{e}_i^{(1)}}\right\}_i\right)
\end{aligned} \tag{3.3.53}$$

Take sup in the above equation

$$\begin{aligned}
&\sup_{\{c_{\mathfrak{l}}\}_{\mathfrak{l}}} \#Eq(\mathcal{C}, \{c_{\mathfrak{l}}\}_{\mathfrak{l}}) = \sup_{\{c_{\mathfrak{l}}\}_{\mathfrak{l}}} \sum_{k_{\mathfrak{e}_1} \in Eq(\mathcal{C}_1, \{c_{\mathfrak{l}_1}\})} \#Eq\left(\mathcal{C}_2, \{c_{\mathfrak{l}_2}\}, \left\{k_{\mathfrak{e}_i^{(1)}}\right\}_i\right) \\
\le & \sup_{\{c_{\mathfrak{l}_1}\}_{\mathfrak{l}_1 \in \mathrm{leg}(\mathcal{C}) \cap \mathrm{leg}(\mathcal{C}_1)}} \sum_{k_{\mathfrak{e}_1} \in Eq(\mathcal{C}_1, \{c_{\mathfrak{l}_1}\})} \sup_{\{c_{\mathfrak{l}_2}\}_{\mathfrak{l}_2 \in \mathrm{leg}(\mathcal{C}) \cap \mathrm{leg}(\mathcal{C}_2)}} \#Eq\left(\mathcal{C}_2, \{c_{\mathfrak{l}_2}\}, \left\{k_{\mathfrak{e}_i^{(1)}}\right\}_i\right) \\
\le & \sup_{\{c_{\mathfrak{l}_1}\}_{\mathfrak{l}_1 \in \mathrm{leg}(\mathcal{C}_1)}} \sum_{k_{\mathfrak{e}_1} \in Eq(\mathcal{C}_1, \{c_{\mathfrak{l}_1}\})} \sup_{\{c_{\mathfrak{l}_2}\}_{\mathfrak{l}_2 \in \mathrm{leg}(\mathcal{C}_2)}} \#Eq(\mathcal{C}_2, \{c_{\mathfrak{l}_2}\}) \\
= & \sup_{\{c_{\mathfrak{l}_1}\}_{\mathfrak{l}_1 \in \mathrm{leg}(\mathcal{C}_1)}} \#Eq(\mathcal{C}_1, \{c_{\mathfrak{l}_1}\}) \sup_{\{c_{\mathfrak{l}_2}\}_{\mathfrak{l}_2 \in \mathrm{leg}(\mathcal{C}_2)}} \#Eq(\mathcal{C}_2, \{c_{\mathfrak{l}_2}\})
\end{aligned} \tag{3.3.54}$$

This proves (3.3.49). $\square$

Combinatorial properties of couples

In this section, we prove several combinatorial properties of couples. These properties are important in the counting argument.

The following lemma gives a relation between numbers of non-leg edges, legs and nodes.

Lemma 3.3.17. *For any couple $\mathcal{C}$, let $n(\mathcal{C})$ be the total number of nodes in $\mathcal{C}$ and $n_e(\mathcal{C})$ (resp.*

$n_{fx}(\mathcal{C})$, $n_{fr}(\mathcal{C})$) be the total number of non-leg edges (resp. fixed legs, free legs). Then we have

(1) $n(\mathcal{C})$, $n_e(\mathcal{C})$, $n_{fx}(\mathcal{C})$ and $n_{fr}(\mathcal{C})$ satisfies the following relation

$$2n_e(\mathcal{C}) + n_{fx}(\mathcal{C}) + n_{fr}(\mathcal{C}) = 4n(\mathcal{C}) \tag{3.3.55}$$

(2) For any couple $\mathcal{C}$, define $\chi(\mathcal{C}) = n_e(\mathcal{C}) + n_{fr}(\mathcal{C}) - n(\mathcal{C})$. Let c, $\mathcal{C}_1$, $\mathcal{C}_2$ be the same as in Lemma 3.3.16 and also assume that $\{\mathfrak{e}_i^{(1)}\}$ are free legs and $\{\mathfrak{e}_i^{(2)}\}$ are fixed legs. Then

$$\chi(\mathcal{C}) = \chi(\mathcal{C}_1) + \chi(\mathcal{C}_2). \tag{3.3.56}$$

Proof. We first prove (1). Consider the set $\mathcal{S} = \{(\mathfrak{n}, \mathfrak{e}) \in \mathcal{C} : \mathfrak{n} \textit{ is an end point of } \mathfrak{e}\}$, then

$$\#\mathcal{S} = \sum_{\substack{(\mathfrak{n},\mathfrak{e})\in\mathcal{C} \\ \mathfrak{n} \textit{ is an end point of } \mathfrak{e}}} 1 \tag{3.3.57}$$

First sum over $\mathfrak{e}$ and then over $\mathfrak{n}$, we get

$$\#\mathcal{S} = \sum_{\mathfrak{n}} \sum_{\substack{\mathfrak{e}\in\mathcal{C} \\ \mathfrak{n} \textit{ is an end point of } \mathfrak{e}}} 1 = \sum_{\mathfrak{n}} 4 = 4n(\mathcal{C}). \tag{3.3.58}$$

In the second equality, $\sum_{\mathfrak{e}\in\mathcal{C}:\mathfrak{n} \textit{ is an end point of } \mathfrak{e}} 1 = 4$ because for each node $\mathfrak{n}$ there are 4 edges connected to it.

Switch the order of summation, we get

$$\begin{aligned}\#\mathcal{S} =& \sum_{\mathfrak{e} \textit{ is a non-leg edges}} \sum_{\substack{\mathfrak{n}\in\mathcal{C} \\ \mathfrak{n} \textit{ is an end point of } \mathfrak{e}}} 1 + \sum_{\mathfrak{e} \textit{ is a leg}} \sum_{\substack{\mathfrak{n}\in\mathcal{C} \\ \mathfrak{n} \textit{ is an end point of } \mathfrak{e}}} 1 \\ =& \sum_{\mathfrak{e} \textit{ is a non-leg edges}} 2 + \sum_{\mathfrak{e} \textit{ is a leg}} 1 \\ =& 2n_e(\mathcal{C}) + n_{fx}(\mathcal{C}) + n_{fr}(\mathcal{C})\end{aligned} \tag{3.3.59}$$

In the second equality, $\sum_{\substack{\mathfrak{n}\in\mathcal{C} \\ \mathfrak{n} \textit{ is an end point of } \mathfrak{e}}} 1$ equals to 1 or 2 because for each non-leg edge (resp. leg) there are 2 (resp. 1) nodes connected to it.

Because the value of $\#\mathcal{S}$ does not depend on the order of summation, we conclude that $2n_e(\mathcal{C}) + n_{fx}(\mathcal{C}) + n_{fr}(\mathcal{C}) = 4n(\mathcal{C})$, which proves (1).

We now prove (2). Since cutting does change the number of nodes, we have $n(\mathcal{C}) = n(\mathcal{C}_1) + n(\mathcal{C}_2)$. Let c be the cut that consists of edges $\{\mathfrak{e}_i\}$ and $n(c)$ be the number of edges in c. When cutting $\mathcal{C}$ into

$\mathcal{C}_1$ and $\mathcal{C}_2$, $n(c)$ non-leg edges are cut into pairs of free and fixed legs, so $n_e(\mathcal{C}) = n(\mathcal{C}_1)+n(\mathcal{C}_2)+n(c)$. Because we have $n(c)$ additional free legs after cutting, so $n_{fr}(\mathcal{C}) = n_{fr}(\mathcal{C}_1)+n_{fr}(\mathcal{C}_2)-n(c)$. Therefore, we get

$$\begin{aligned}\chi(\mathcal{C}) =& n_e(\mathcal{C}) + n_{fr}(\mathcal{C}) - n(\mathcal{C})\\ =& n(\mathcal{C}_1) + n(\mathcal{C}_2) + n(c) + n_{fr}(\mathcal{C}_1) + n_{fr}(\mathcal{C}_2) - n(c) - (n(\mathcal{C}_1) + n(\mathcal{C}_2))\\ =& (n(\mathcal{C}_1) + n_{fr}(\mathcal{C}_1) - n(\mathcal{C}_1)) + (n(\mathcal{C}_2) + n_{fr}(\mathcal{C}_2) - n(\mathcal{C}_2))\\ =& \chi(\mathcal{C}_1) + \chi(\mathcal{C}_2).\end{aligned} \tag{3.3.60}$$

We complete the proof of (2). $\square$

The following lemma gives a relation between variables of legs.

Lemma 3.3.18. *Given a connected couple $\mathcal{C}$ with multiple legs, then we have the following conclusions.*

(1) Let $\{k_{\mathfrak{l}_i}\}_{i=1,\cdots,n_{leg}}$ be the variables corresponding to legs in couple $\mathcal{C}$. Let $\iota_{\mathfrak{e}}$ be the same as Definition 3.2.1 (12), then $Eq(\mathcal{C})$ implies the following momentum conservation equation

$$\sum_{i=1}^{n_{leg}} \iota_{\mathfrak{l}_i} k_{\mathfrak{l}_i} = 0, \tag{3.3.61}$$

(2) Assume that there is exactly one free leg $\mathfrak{l}_{i_0}$ in $\mathcal{C}$ and all other variables $\{k_{\mathfrak{l}_i}\}_{i\neq i_0}$ corresponding to fix legs are fixed to be constants $\{c_{\mathfrak{l}_i}\}_{i\neq i_0}$. For any $i_1 = 1,\cdots,n_{leg}$, we can construct a new couple $\widehat{\mathcal{C}}$ by replacing the i_0 leg by a fix leg and i_1 leg by a free leg. If $i \neq i_0, i_1$, fix $k_{\mathfrak{l}_i}$ to be the constant $c_{\mathfrak{l}_i}$, if $i = i_0$, fix $k_{\mathfrak{l}_{i_0}}$ to be the constant $-\iota_{\mathfrak{l}_{i_0}} \sum_{i\neq i_0} \iota_{\mathfrak{l}_i} k_{\mathfrak{l}_i}$. Under the above assumptions, we have

$$Eq(\mathcal{C}, \{c_{\mathfrak{l}_i}\}_{i\neq i_0}) = Eq\left(\widehat{\mathcal{C}}, \{c_{\mathfrak{l}_i}\}_{i\neq i_0,i_1} \cup \{-\iota_{\mathfrak{l}_{i_0}} \sum_{i\neq i_0} \iota_{\mathfrak{l}_i} k_{\mathfrak{l}_i}\}\right). \tag{3.3.62}$$

(3) Assume that there is no free leg in $\mathcal{C}$ and all $\{k_{\mathfrak{l}_i}\}_{i\neq i_0}$ are fixed to be constants $\{c_{\mathfrak{l}_i}\}_{i\neq i_0}$. For any $i_1 = 1,\cdots,n_{leg}$, we can construct a new couple $\widehat{\mathcal{C}}$ by replacing the i_0 leg by a free leg. Then we have

$$Eq(\mathcal{C}, \{c_{\mathfrak{l}_i}\}_i) = Eq(\widehat{\mathcal{C}}, \{c_{\mathfrak{l}_i}\}_{i\neq i_0}). \tag{3.3.63}$$

(4) If the couple $\mathcal{C}$ contains any leg, then it contains at least two legs.

Proof. We first prove (1). Given a node $\mathfrak{n}$ and an edge $\mathfrak{e}$ connected to it, we define $\iota_{\mathfrak{e}}(\mathfrak{n})$ by the following rule

$$\iota_{\mathfrak{e}}(\mathfrak{n}) = \begin{cases} +1 & \textit{if } \mathfrak{e} \textit{ pointing towards } \mathfrak{n} \\ \\ -1 & \textit{if } \mathfrak{e} \textit{ pointing outwards from } \mathfrak{n} \end{cases} \tag{3.3.64}$$

For a leg $\mathfrak{l}$, since it is connected to just one node, we may omit the $(\mathfrak{n})$ and just write $\iota_{\mathfrak{l}}$ as in the statement of the lemma.

For each node $\mathfrak{n}$, let $\mathfrak{e}_1(\mathfrak{n})$, $\mathfrak{e}_2(\mathfrak{n})$, $\mathfrak{e}(\mathfrak{n})$ be the three edges connected to it. For each edge $\mathfrak{e}$, let $\mathfrak{n}_1(\mathfrak{e})$, $\mathfrak{n}_2(\mathfrak{e})$ be the two nodes connected to it. Then we know that $\iota_{\mathfrak{e}}(\mathfrak{n}_1(\mathfrak{e})) + \iota_{\mathfrak{e}}(\mathfrak{n}_2(\mathfrak{e}))$, since $\mathfrak{n}_1(\mathfrak{e})$ and $\mathfrak{n}_2(\mathfrak{e})$ have the opposite direction.

Since $k_{\mathfrak{e}}$ satisfy $Eq(\mathcal{C})$, by (3.3.41), we get

$$\iota_{\mathfrak{e}_1(\mathfrak{n})}(\mathfrak{n})k_{\mathfrak{e}_1(\mathfrak{n})} + \iota_{\mathfrak{e}_2(\mathfrak{n})}(\mathfrak{n})k_{\mathfrak{e}_2(\mathfrak{n})} + \iota_{\mathfrak{e}_3(\mathfrak{n})}(\mathfrak{n})k_{\mathfrak{e}_3(\mathfrak{n})} + \iota_{\mathfrak{e}(\mathfrak{n})}(\mathfrak{n})k_{\mathfrak{e}(\mathfrak{n})} = 0. \tag{3.3.65}$$

Summing over $\mathfrak{n}$ gives

$$\begin{aligned} 0 =& \sum_{\mathfrak{n}\in\mathcal{C}} \iota_{\mathfrak{e}_1(\mathfrak{n})}(\mathfrak{n})k_{\mathfrak{e}_1(\mathfrak{n})} + \iota_{\mathfrak{e}_2(\mathfrak{n})}(\mathfrak{n})k_{\mathfrak{e}_2(\mathfrak{n})} + \iota_{\mathfrak{e}_3(\mathfrak{n})}(\mathfrak{n})k_{\mathfrak{e}_3(\mathfrak{n})} + \iota_{\mathfrak{e}(\mathfrak{n})}(\mathfrak{n})k_{\mathfrak{e}(\mathfrak{n})} \\ =& \sum_{\mathfrak{e} \text{ is not a leg}} (\iota_{\mathfrak{e}}(\mathfrak{n}_1(\mathfrak{e})) + \iota_{\mathfrak{e}}(\mathfrak{n}_2(\mathfrak{e})))k_{\mathfrak{e}} + \sum_{\mathfrak{l} \text{ is a leg}} \iota_{\mathfrak{l}} k_{\mathfrak{l}} \\ =& \sum_{i=1}^{n_{\text{leg}}} \iota_{\mathfrak{l}_i} k_{\mathfrak{l}_i} \end{aligned} \tag{3.3.66}$$

This proves (3.3.61) and thus proves (1).

Now we prove (2). Since in $Eq\left(\widehat{\mathcal{C}}, \{c_{\mathfrak{l}_i}\}_{i\neq i_0,i_1} \cup \{-\iota_{\mathfrak{l}_{i_0}} \sum_{i\neq i_0} \iota_{\mathfrak{l}_i} k_{\mathfrak{l}_i}\}\right)$, $\{k_{\mathfrak{l}_i}\}_{i\neq i_0,i_1}$ are fixed to be constants $\{c_{\mathfrak{l}_i}\}_{i\neq i_0,i_1}$ and $k_{\mathfrak{l}_{i_0}}$ is fixed to be the constant $-\iota_{\mathfrak{l}_{i_0}} \sum_{i\neq i_0} \iota_{\mathfrak{l}_i} k_{\mathfrak{l}_i}$, by (3.3.61), we know that

$$\iota_{\mathfrak{l}_{i_0}} \left(-\iota_{\mathfrak{l}_{i_0}} \sum_{i\neq i_0} \iota_{\mathfrak{l}_i} k_{\mathfrak{l}_i} \right) + \iota_{\mathfrak{l}_{i_1}} k_{\mathfrak{l}_{i_1}} + \sum_{i\neq i_0,i_1} \iota_{\mathfrak{l}_i} c_{\mathfrak{l}_i} = 0. \tag{3.3.67}$$

This implies that $k_{\mathfrak{l}_{i_1}} = c_{\mathfrak{l}_{i_1}}$ in $Eq(\widehat{\mathcal{C}})$. Therefore, equations in $Eq(\widehat{\mathcal{C}})$ automatically imply $k_{\mathfrak{l}_{i_1}} = c_{\mathfrak{l}_{i_1}}$. Notice that whether or not containing $k_{\mathfrak{l}_{i_1}} = c_{\mathfrak{l}_{i_1}}$ is the only difference between $Eq(\mathcal{C})$ and $Eq(\widehat{\mathcal{C}})$. We conclude that $Eq(\mathcal{C}) = Eq(\widehat{\mathcal{C}})$. We thus complete the proof of (2).

The proof of (3) is similar to (2). Whether or not containing $k_{\mathfrak{l}_{i_0}} = c_{\mathfrak{l}_{i_0}}$ is the only difference between $Eq(\mathcal{C})$ and $Eq(\widehat{\mathcal{C}})$. But if $\{k_{\mathfrak{l}_i}\}_{i\neq i_0}$ are fixed to be constants $\{c_{\mathfrak{l}_i}\}_{i\neq i_0}$, by momentum conservation we know that

$$k_{\mathfrak{l}_{i_0}} = -\iota_{\mathfrak{l}_{i_0}} \sum_{i\neq i_0} \iota_{\mathfrak{l}_i} c_{\mathfrak{l}_i}. \tag{3.3.68}$$

Therefore, $k_{\mathfrak{l}_{i_0}}$ is fixed to be the constant $-\iota_{\mathfrak{l}_{i_0}} \sum_{i \neq i_0} \iota_{\mathfrak{l}_i} c_{\mathfrak{l}_i}$ in $Eq(\widehat{\mathcal{C}})$ and we conclude that $Eq(\mathcal{C}) = Eq(\widehat{\mathcal{C}})$. We thus complete the proof of (3).

By Lemma 3.3.17, we know that the total number of legs $n_{leg} = n_{fx}(\mathcal{C}) + n_{fr}(\mathcal{C})$ is an even number. This proves (4). $\square$

After splitting all $\circ$ nodes in Definition 3.3.6, a couple may have several connected components. The following properties of connected components are important in the counting argument.

Definition 3.3.19. 1. **Open and closed couples.** A connected couple is said to be open (resp. closed) if it (resp. does not) contains legs. A connected components of a couple is a open component (resp. closed component) if it is open (resp. closed).

2. **Good couples and bad couples.** A closed couple is a good couple (resp. bad couple) if it (resp. does not) contains at least two dotted edges (see Definition 3.3.6 (2)).

   Figure 3.16 provides some examples of couples that are good or bad.

Figure 3.16: Example of good couple (left) and bad couple (right)

3. **Good nodes and bad nodes in renormalization forest.** Given a pairing p and a node $\mathfrak{n}$ in a renormalization forest F, let $T_{\mathfrak{n}}$ be the subtree F rooted at $\mathfrak{n}$ and $\mathrm{Child}_{\mathfrak{n}}$ be the set of all children of $\mathfrak{n}$, then $\mathfrak{n}$ is said to be bad if the pairing $p \notin \mathcal{P}(L(T_{\mathfrak{n}}) \backslash \cup_{n' \in \mathrm{Child}_{\mathfrak{n}}} L(T_{\mathfrak{n}'}), A)$. Otherwise, $\mathfrak{n}$ is said to be good. Here $A = \{k_1, k_2, \cdots, k_{2l+1}\}$, $\mathcal{P}(B, A)$ is defined in Definition 3.3.3 (4)

   In other word, a node is bad if p does pair all leaves in $L(T_{\mathfrak{n}}) \backslash \cup_{n' \in \mathrm{Child}_{\mathfrak{n}}} L(T_{\mathfrak{n}'})$ with leaves in the this set. For example, consider the following renormalization forest (this forest is the same as the forest in Figure 3.4).

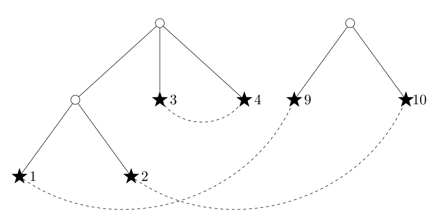


Figure 3.17: Good nodes and bad nodes

Assume that the pairing is $p = \{1,9\} \cup \{2,10\} \cup \{3,4\}$, then the top left node $\mathfrak{n}_0$ is bad because $L(T_{\mathfrak{n}_0}) \setminus \cup_{n' \in \mathrm{Child}_{\mathfrak{n}_0}} L(T_{\mathfrak{n}'}) = \{3,4\}$ are paired. The other two nodes are good nodes.

4. **Chain of bad circle nodes.** Given a renormalization forest F, a chain of bad nodes is a sequence of bad nodes $\mathfrak{n}_1, \mathfrak{n}_2, \cdots, \mathfrak{n}_l$ such that this sequence have the property that for each $1 \le i \le l-1$, $\mathfrak{n}_i$ only have one child $\mathfrak{n}_{i+1}$ and this sequence is one of the maximal sequence satisfies this property. We say a chain of bad node is long or short, if $l > 1$ or $l = 1$ respectively.

   For example, consider the following renormalization forest, the pairing is $p = \{1,7\} \cup \{2,8\} \cup \{3,4\} \cup \{5,6\}$, then n_1, n_2 form a long chain of bad nodes, and n_2 or n_3 form two short chain of bad nodes

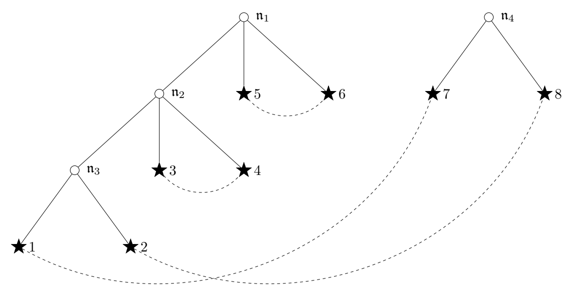


Figure 3.18: Chain of bad nodes

The counting result in the next section says that we lose a factor for each bad nodes but gain a factor for each good nodes. We have the following lemma that gives uppper bounds of number of chain of bad nodes in terms of good nodes.

Lemma 3.3.20. *Given two trees T, T' and a pairing $p \in \mathcal{P}_F$ of leaves in the renormalized pairing set $\mathcal{P}_F$, let F be the forest which is the union of two forests $R(T)$ and $R(T')$. Then the couple $\mathcal{C} = \mathcal{C}(T,T',p)$ constructed from T, T', p have the following properties.*

Let $n_G(F)$ be the number of good and bad nodes of F respectively. Let $n_{BC}(F)$ be the number of chain of bad nodes of F. Then we have the following inequality.

$$n_{BC}(F) + 1 \le 2n_G(F) \tag{3.3.69}$$

Remark 3.3.21. It is crucial that the couple is constructed from a pairing p from $\mathcal{P}_F$. For a general pairing, the lemma is not true.

Proof. We first consider the following claim.

Claim 1. If a circle node $\mathfrak{n}$ in F has no circle child, then $\mathfrak{n}$ is good.

Proof. Under this assumption, $\cup_{n' \in \mathrm{Child}_{\mathfrak{n}}} L(T_{\mathfrak{n}'}) = \emptyset$ and $L(T_{\mathfrak{n}}) = L(T_{\mathfrak{n}}) \setminus \cup_{n' \in \mathrm{Child}_{\mathfrak{n}}} L(T_{\mathfrak{n}'})$. By the definition of $\mathcal{P}_F$, Definition 3.3.3 (5), $p \notin \mathcal{P}(L(T_{\mathfrak{n}}), A)$. Therefore, $\mathfrak{n}$ is a good nodes. □

Claim 2. The last nodes of a chain of bad circle nodes must either have a good circle child or have at least two bad children.

Proof. If the claim is wrong, the last node have at most one circle child and this child is a bad circle nodes. If the last node does not have circle children, by claim 1, the last node must be good, which gives a contradiction. If the last node has exacatly one circle children which is bad, then this contradicts with the maximality of the chain with respect to its defining property. □

If a chain of bad nodes has no bad ancestors, we call it top chain of bad nodes. Denote all top chain of bad nodes by $\mathfrak{c}_1, \cdots, \mathfrak{c}_k$. Denote by $T_{\mathfrak{c}_1} \cdots T_{\mathfrak{c}_k}$ the subtree of the first node of this chains. Then we know that

$$n_{BC}(F) = n_{BC}(T_{\mathfrak{c}_1}) + \cdots + n_{BC}(T_{\mathfrak{c}_k}) \tag{3.3.70}$$

$$n_G(T_{\mathfrak{c}_1}) + \cdots + n_G(T_{\mathfrak{c}_k}) \leq n_G(F) \tag{3.3.71}$$

By these two inequalities, we know that it suffices to prove (3.3.69) for trees T whose root is a bad circle nodes.

We prove (3.3.69) for a tree T with a bad root by induction on the total number n of bad chains.

If $n = 1$, by claim 2, it have a good circle children, so $n_G(T) \geq 1$. We get $n_G(T) + 1 \geq 2 = 2n_{BC}(T)$.

Assume that the Lemma is true for $< n$. We prove the case n.

For the top chain of bad nodes, by claim 2, it either have a good child or two bad children.

The one good child case. In the first case, remove the top chain $\mathfrak{c}$ from T to get a forest, then take the subtrees $T_{\mathfrak{c}'_1}, \cdots, T_{\mathfrak{c}'_{k'}}$ of top chains of the new forest. We have

$$n_{BC}(T) = n_{BC}(T_{\mathfrak{c}'_1}) + \cdots + n_{BC}(T_{\mathfrak{c}'_{k'}}) + 1 \tag{3.3.72}$$

$$n_G(T_{\mathfrak{c}'_1}) + \cdots + n_G(T_{\mathfrak{c}'_{k'}}) + 1 \leq n_G(T). \tag{3.3.73}$$

Here we have the $+1$ in (3.3.72) for n_G because in the first case $\mathfrak{c}$ has a good children.

By induction assumption, we have

$$n_{BC}(T_{\mathfrak{c}'_i}) + 1 \leq 2n_G(T_{\mathfrak{c}'_i}). \tag{3.3.74}$$

Summing over i and applying (3.3.72) and (3.3.73), we get

$$n_{BC}(T) + k' - 1 \leq 2(n_G(T) - 1). \tag{3.3.75}$$

Since $k' \geq 0$, this implies (3.3.69) in the first case.

The two bad child case. Now we consider the second case. Remove the top chain $\mathfrak{c}$ from T to get a forest, then take the subtrees $T_{\mathfrak{c}'_1}, \cdots, T_{\mathfrak{c}'_{k'}}$ of top chains of the new forest. We have

$$n_{BC}(T) = n_{BC}(T_{\mathfrak{c}'_1}) + \cdots + n_{BC}(T_{\mathfrak{c}'_{k'}}) + 1 \tag{3.3.76}$$

$$n_G(T_{\mathfrak{c}'_1}) + \cdots + n_G(T_{\mathfrak{c}'_{k'}}) \leq n_G(T). \tag{3.3.77}$$

By induction assumption, we have

$$n_{BC}(T_{\mathfrak{c}'_i}) + 1 \leq 2n_G(T_{\mathfrak{c}'_i}). \tag{3.3.78}$$

Summing over i and applying (3.3.72) and (3.3.73), we get

$$n_{BC}(T) + k' - 1 \leq 2n_G(T). \tag{3.3.79}$$

Since $k' \geq 2$, this implies (3.3.69) in the second case, which complete the proof of the lemma. $\square$

Counting results and their proof

The following proposition gives an upper bound of number of solutions of (3.3.40) (or (3.3.43)).

Proposition 3.3.22. *Let $\mathcal{C} = \mathcal{C}(T, T', p)$ be an open admissible couple, n be the total number of nodes in $\mathcal{C}$, n_c be the number of closed components of $\mathcal{C}$ and $Q = L^d\sqrt{\omega}$. We fix $k \in \mathbb{R}$ for the fixed legs $\mathfrak{l}$, $\mathfrak{l}'$ and $\sigma_{\mathfrak{n}} \in \mathbb{R}$ for each $\mathfrak{n} \in \mathcal{C}$. Assume that $\omega \geq L^{-1}$ (or $\omega \geq L^{-2}$ in case that the ratio of periods is generic in the sense of Theorem 4.3.2). Then we have following conclusions*

(1) The number of solutions M of (3.3.40) *(or* (3.3.43)*) is bounded by*

$$M \leq L^{O(\theta)} Q^n L^{dn_c}. \tag{3.3.80}$$

(2) We know that n equals to the total number of $\bullet$ nodes in T and T'. Let n_G and n_B be the number of good and bad nodes in T and T' respectively. Then we have

$$\begin{cases} 2n_G \geq n_{BC} + 1 \\ n_G \geq 2(n_c - n_B) \end{cases} \tag{3.3.81}$$

(3) Let l and l' be the total number of nodes in T and T' respectively. Assume that there is no long chain of bad circle nodes and $\omega \geq L^{-\frac{d}{2}}$. Then we also have

$$M/Q^{l+l'} \leq L^{O(\theta)}. \tag{3.3.82}$$

This inequality is important because the order of magnitude of $Term(T, T', p)$ is supposed to be $M/Q^{l+l'}$ when $\omega = T_{kin}^{-1}$. As long as we can remove the condition that there is no long chain of bad circle nodes, up to epsilon loss the wave kinetic equation can be derived.

Proof. We first prove (2). The fact that n equals to the total number of $\bullet$ nodes in T and T' follows from the definition. The first inequality of (3.3.81) follows from Lemma 3.3.20.

Now we prove the second inequality of (3.3.81). Remember from the definition of dotted edges, Definition 3.3.6 (2), each dotted edge correspond to a circle node. By definition of bad circle nodes, component arising from them can only contains one dotted edge, so $n_c - n_B$ are the number of couples contain at least two dotted edges. All these couples correspond to at least to good circle nodes and thus the number of them is less than $n_G/2$. Therefore, we proved the desired inequality.

Then we prove (3) from (1) and (2). Let n_{circ} be the total number of circle nodes in T and T'. Since $l + l'$ and n are the total number of nodes and $\bullet$ nodes in T and T' respectively, we know that $l + l' = n + n_{\text{circ}}$. By definition of n_G and n_B, we know that $n = n_G + n_B$.

By assumption that there is no long chain of bad circle nodes, we know that $n_B = n_{BC}$. Therefore, the first inequality of (3.3.81) becomes

$$n_G \geq n_B. \tag{3.3.83}$$

Multiply this inequality by $\frac{1}{3}$, the second inequality of (3.3.81) $\frac{2}{3}$ and then sum them. We get the following inequality.

$$n_G \geq \frac{4}{3}n_c - n_B \qquad \Leftrightarrow \qquad n_c \leq \frac{3}{4}(n_G + n_B) \tag{3.3.84}$$

By (1), we get

$$\begin{aligned} M/Q^{l+l'} \leq & L^{O(\theta)} Q^n L^{dn_c}/Q^{n+n_G+n_B} \\ = & L^{O(\theta)} L^{dn} (\sqrt{\omega})^n L^{dn_c}/(L^{d(n+n_G+n_B)}(\sqrt{\omega})^{n+n_G+n_B}) \\ = & L^{O(\theta)} (\sqrt{\omega})^{-n_G-n_B} L^{d(n_c-n_G-n_B)} \\ = & L^{O(\theta)} (L^{-\frac{d}{4}}\sqrt{\omega}^{-1})^{n_G+n_B} \leq L^{O(\theta)}. \end{aligned} \tag{3.3.85}$$

Here in the third inequality, we applied (3.3.84) and $\omega \geq L^{-\frac{d}{2}}$ in the hypothesis of (3).

Therefore, we have finished the proof of (3)

The proof of (1) is lengthy and therefore divided into several steps. To present its proof, let us first introduce several useful notations.

Given a couple $\mathcal{C}$ and k, $\sigma_{\mathfrak{n}}$, let $Eq(\mathcal{C})$ be the system of equation (3.3.43) constructed in Proposition 3.3.11. For any system of equations Eq, let $\#(Eq)$ be its number of solutions.

Now we prove (1).

Step 1. In this step, we reduce Proposition 3.3.22 to the case of connected couples.

By Lemma (3.3.48), we know that

$$\#(Eq(\mathcal{C})) = \#(Eq(\widetilde{\mathcal{C}})) \prod_{j=1}^{n_c} \#(Eq(\mathcal{C}_j)). \tag{3.3.86}$$

Here $\widetilde{\mathcal{C}}$ is the open component of $\mathcal{C}$ (it is not difficult to show that $\mathcal{C}$ only has one open component), $\mathcal{C}_j$, $1 \leq j \leq n_c$ are the closed components of $\mathcal{C}$ and we have $\mathcal{C} = \widetilde{\mathcal{C}} \cup (\cup_{j=1}^{n_c} \mathcal{C}_j)$.

Therefore, we just need to prove the following proposition which claim that Proposition 3.3.22 is true for connected couples. Note that in the statement of the following proposition, we have an extra factor L^d for each closed component. In total, we get a factor L^{dn_c}. This how we get the L^{dn_c} in $M \leq L^{O(\theta)} Q^n L^{dn_c}$ of Proposition 3.3.22.

Proposition 3.3.23. *Let $\mathcal{C}$ be a connected couple and n be the total number of nodes in $\mathcal{C}$. We fix $\sigma_{\mathfrak{n}} \in \mathbb{R}$ for each $\mathfrak{n} \in \mathcal{C}$ and $k \in \mathbb{R}$ if $\mathcal{C}$ is open. Assume that $\omega \geq L^{-1}$ (or $\omega \geq L^{-2}$ in case that the* *ratio of periods is generic in the sense of Theorem 4.3.2). Then (recall that $Q = L^d\sqrt{\omega}$)*

$$\#Eq(\mathcal{C}) \leq \begin{cases} L^{O(\theta)}Q^n, & \text{if } C \text{ is open,} \\ L^{d+O(\theta)}Q^n, & \text{if } C \text{ is closed.} \end{cases} \tag{3.3.87}$$

Step 2-4 are devoted to the proof of Proposition 3.3.23 using a cutting edge argument.

Step 2. Although all components can have at most two legs at the beginning, after cutting, the resulting couples can have more than two legs. In this step, we introduce a stronger version of Proposition 3.3.23 that covers legs of more than two legs.

Lemma 3.3.24. *Let $\mathcal{C}$ be a connected couple with multiple legs, n be the total number of nodes in $\mathcal{C}$ and n_e (resp. n_{fx}, n_{fr}) be the total number of non-leg edges (resp. fixed legs, free legs). We fix $\sigma_{\mathfrak{n}} \in \mathbb{R}$ for each $\mathfrak{n} \in \mathcal{C}$ and $c_{\mathfrak{l}} \in \mathbb{R}$ for each fixed leg $\mathfrak{l}$. Assume that $\omega \geq L^{-1}$ (or $\omega \geq L^{-2}$ in case that the ratio of periods is generic in the sense of Theorem 4.3.2). We assume that $n_{fx}, n_{fr} \neq 0$. Then*

$$\sup_{\{c_{\mathfrak{l}}\}_{\mathfrak{l}}} \#Eq(\mathcal{C}, \{c_{\mathfrak{l}}\}_{\mathfrak{l}}) \leq L^{O(\theta)}Q^{\chi(\mathcal{C})} = L^{O(\theta)}Q^{n_e+n_{fr}-n}. \tag{3.3.88}$$

Proposition 3.3.24 will be proved in Step 3, 4. In the rest part of this step we show that Proposition 3.3.23 is a corollary of Proposition 3.3.24.

Since the couples in Proposition 3.3.23 and Proposition 3.3.22 have no free leg, these two propositions are not obvious corollaries of Proposition 3.3.24.

Proof of Proposition 3.3.23. The closed couple case of Proposition 3.3.23 can be derived from the open couple case by following argument.

Let $\mathcal{C}$ be a closed couple, choose an edge $\mathfrak{e}_* \in \mathcal{C}$ which connects $\mathfrak{n}_1$, $\mathfrak{n}_2$. Cut this edge into two legs $\mathfrak{l}_1$ and $\mathfrak{l}_2$, then the couple $\widetilde{\mathcal{C}}$ obtained in this way is an open couple. Therefore, we have

$$
\begin{aligned}
\#Eq(\mathcal{C}) =& \#\{k_{\mathfrak{e}} \in \mathbb{Z}^d,\ |k_{\mathfrak{e}}| \lesssim L^+,\ \forall \mathfrak{e} \in \mathcal{C} : MC_{\mathfrak{n}},\ EC_{\mathfrak{n}},\ \forall \mathfrak{n} \in \mathcal{C}\} \\
=& \#\{k_{\mathfrak{e}} \in \mathbb{Z}^d,\ |k_{\mathfrak{e}}| \lesssim L^+,\ \forall \mathfrak{e} \in \mathcal{C} : MC_{\mathfrak{n}},\ EC_{\mathfrak{n}},\ \forall \mathfrak{n} \in \mathcal{C}. MC_{n_1}(k_{\mathfrak{e}_*}, \cdots),\ MC_{n_2}(k_{\mathfrak{e}_*}, \cdots)\} \\
=& \# \sum_{k_{\mathfrak{e}_*}} \{k_{\mathfrak{e}} \in \mathbb{Z}^d,\ |k_{\mathfrak{e}}| \lesssim L^+,\ \forall \mathfrak{e} \in \mathcal{C} : MC_{\mathfrak{n}},\ EC_{\mathfrak{n}},\ \forall \mathfrak{n} \in \mathcal{C}. MC_{n_1}(k_{\mathfrak{l}_1}, \cdots), \\
& \qquad MC_{n_2}(k_{\mathfrak{l}_2}, \cdots),\ k_{\mathfrak{l}_1} = k_{\mathfrak{l}_2} = k_{\mathfrak{e}_*}\} \\
=& \sum_{|k_{\mathfrak{e}_*}| \lesssim 1} \#Eq(\widetilde{\mathcal{C}}, \{k_{\mathfrak{e}_*}, k_{\mathfrak{e}_*}\})
\end{aligned}
\tag{3.3.89}
$$

If we can prove the open couple case, we have $\#Eq(\widetilde{\mathcal{C}}, \{k_{\mathfrak{e}_*}, k_{\mathfrak{e}_*}\}) \leq L^{O(\theta)} Q^n$. Then by (3.3.89), we can prove the closed couple case:

$$
\#Eq(\mathcal{C}) = \sum_{|k_{\mathfrak{e}_*}| \lesssim 1} \#Eq(\widetilde{\mathcal{C}}, \{k_{\mathfrak{e}_*}, k_{\mathfrak{e}_*}\}) \lesssim L^{d+O(\theta)} Q^n. \tag{3.3.90}
$$

The open couple case can be proved by Lemma 3.3.18.

An open couple has two fixed legs $\mathfrak{l}$, $\mathfrak{l}'$ with opposite sign, so by momentum conservation equation (3.3.61), we have $k_{\mathfrak{l}} = k_{\mathfrak{l}'}$, and $k_{\mathfrak{l}} = k_{\mathfrak{l}}' = k$ is a consequence of $k_{\mathfrak{l}} = k$. Therefore, we have

$$
\begin{aligned}
Eq(\mathcal{C}) =& \{k_{\mathfrak{e}} \in \mathbb{Z}^d,\ |k_{\mathfrak{e}}| \lesssim L^+,\ \forall \mathfrak{e} \in \mathcal{C} : MC_{\mathfrak{n}},\ EC_{\mathfrak{n}},\ \forall \mathfrak{n} \in \mathcal{C}.\ k_{\mathfrak{l}} = k_{\mathfrak{l}}' = k.\} \\
=& \{k_{\mathfrak{e}} \in \mathbb{Z}^d,\ |k_{\mathfrak{e}}| \lesssim L^+,\ \forall \mathfrak{e} \in \mathcal{C} : MC_{\mathfrak{n}},\ EC_{\mathfrak{n}},\ \forall \mathfrak{n} \in \mathcal{C}.\ k_{\mathfrak{l}} = k.\} \\
=& Eq(\widehat{\mathcal{C}}, k)
\end{aligned}
\tag{3.3.91}
$$

where $\widehat{\mathcal{C}}$ is the couple obtained by replacing a fixed leg by a free leg.

In $\widehat{\mathcal{C}}$, $n_{fx} = n_{fr} = 1 \neq 0$, so by (3.3.55), we have $n_e + 1 = 2n$. Applying (3.3.88), we get

$$
Eq(\mathcal{C}) = Eq(\widehat{\mathcal{C}}, k) \lesssim L^{O(\theta)} Q^{n_e + n_{fr} - n} = L^{O(\theta)} Q^n. \tag{3.3.92}
$$

This proves Proposition 3.3.23. □

Step 3. In this step, we prove Proposition 3.3.24 for basic counting units.

To prove Proposition 3.3.24, we keep cutting nodes connected to legs from the couple $\mathcal{C}$. Suppose that $\mathcal{C}$ has n nodes and we cut a node $\mathfrak{n}$ from $\mathcal{C}$ using cut $c(\mathfrak{n})$. Let $\mathcal{C}_{\mathfrak{n}}$ be the smaller components contains fixed leg $\mathfrak{l}$ with 1 nodes $\mathfrak{n}$ and $\mathcal{C}' = \mathcal{C} \backslash \mathcal{C}_{\mathfrak{n}}$ be the larger components with $n - 1$ nodes.

As in Figure 3.19, there are three possibilities of $\mathcal{C}_{\mathfrak{n}}$. We label them by $\mathcal{C}_I$, $\mathcal{C}_{II}$, $\mathcal{C}_{III}$.

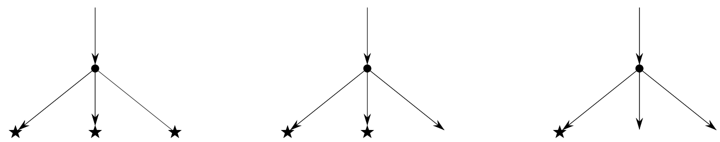


Figure 3.19: Three possibilities of $\mathcal{C}_{\mathfrak{l}}$.

Lemma 3.3.25. *$\mathcal{C}_I$, $\mathcal{C}_{II}$, $\mathcal{C}_{III}$ satisfy the bound (3.3.88) in Proposition 3.3.24. In other words, fix c_1 (resp. c_1, c_3 and c_1, c_2, c_3) for the legs of $\mathcal{C}_I$ (resp. $\mathcal{C}_{II}$ and $\mathcal{C}_{III}$), then we have*

$$\#Eq(\mathcal{C}_I) \leq L^{\theta} Q^2, \qquad \#Eq(\mathcal{C}_{II}) \leq L^{\theta} Q, \qquad \#Eq(\mathcal{C}_{III}) \leq Q^0 = 1. \tag{3.3.93}$$

Proof. Given c_1, c_2, c_3 the equation of $\mathcal{C}_{III}$ is

$$\begin{cases} k_{\mathfrak{e}_1} - k_{\mathfrak{e}_2} + k_{\mathfrak{e}_3} - k_{\mathfrak{e}} = 0, \ k_{\mathfrak{e}_2} = c_2, \ k_{\mathfrak{e}_3} = c_3, \ k_{\mathfrak{e}} = c_1, \\ \Lambda_{k_{\mathfrak{e}_1}} - \Lambda_{k_{\mathfrak{e}_2}} + \Lambda_{k_{\mathfrak{e}_3}} - \Lambda_{k_{\mathfrak{e}}} = \sigma_{\mathfrak{n}} + O(\omega) \end{cases} \tag{3.3.94}$$

It's obvious that there is at most one solution to this system of equations.

Given c_1, the equation of $\mathcal{C}_I$ is

$$\begin{cases} k_{\mathfrak{e}_1} - k_{\mathfrak{e}_2} + k_{\mathfrak{e}_3} - k_{\mathfrak{e}} = 0, \ k_{\mathfrak{e}} = c_1, \\ \Lambda_{k_{\mathfrak{e}_1}} - \Lambda_{k_{\mathfrak{e}_2}} + \Lambda_{k_{\mathfrak{e}_3}} - \Lambda_{k_{\mathfrak{e}}} = \sigma_{\mathfrak{n}} + O(\omega) \end{cases} \tag{3.3.95}$$

By Theorem 4.3.1 or 4.3.2 (4.3.1) or (4.3.4), corresponding to general or generic case, the number of solutions of the above system of equations can be bounded by $L^{\theta} Q^2$.

Given c_1, c_2, the equation of $\mathcal{C}_{II}$ is

$$\begin{cases} k_{\mathfrak{e}_1} - k_{\mathfrak{e}_2} + k_{\mathfrak{e}_3} - k_{\mathfrak{e}} = 0, \ k_{\mathfrak{e}} = c_1, \ k_{\mathfrak{e}_3} = c_3, \\ \Lambda_{k_{\mathfrak{e}_1}} - \Lambda_{k_{\mathfrak{e}_2}} + \Lambda_{k_{\mathfrak{e}_3}} - \Lambda_{k_{\mathfrak{e}}} = \sigma_{\mathfrak{n}} + O(\omega) \end{cases} \tag{3.3.96}$$

By Theorem 4.3.1 or 4.3.2 (4.3.3) or (4.3.5), corresponding to general or generic case, the number of solutions of the above system of equations can be bounded by $L^{\theta} Q$

Therefore, we complete the proof of this lemma. □

Step 4. In this step, we apply the edge cutting argument to prove Proposition 3.3.24 by induction.

If $\mathcal{C}$ has only one node ($n = 1$), then $\mathcal{C}$ equals to $\mathcal{C}_I$, $\mathcal{C}_{II}$ or $\mathcal{C}_{III}$ and Proposition 3.3.24 in this case follows from Lemma 3.3.25.

Suppose that Proposition 3.3.24 holds true for couples with number of nodes $\leq n-1$. We prove it for couples with number of nodes n.

Since $n_{fx}, n_{fr} \neq 0$, there exists both fixed and free legs. We will cut a node $\mathfrak{n}_* \in \mathcal{C}$ which is connected to a fixed leg. Denoted by $\mathcal{N}_{leg}$ the set of all such nodes and by $\mathcal{C}_{\mathfrak{n}_*}$ the components containing $\mathfrak{n}_*$ after cutting. There are several different situations in this cutting.

Case 1. Assume that for all $\mathfrak{n} \in \mathcal{N}_{leg}$, after cutting $\mathfrak{n}$, $\mathcal{C}\backslash\mathcal{C}_{\mathfrak{n}}$ is still connected.

Case 1.1. Assume that not all legs are connected to one node. One example of this case is shown in Figure 3.20.

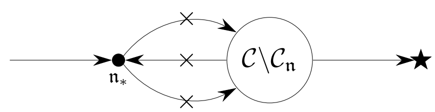


Figure 3.20: Example of case 1.1.

In this case, we have the following claim.

Claim. There exists a node $\mathfrak{n}_* \in \mathcal{N}_{leg}$ such that not all free legs are connected to this node.

Proof. If there are only one node in $\mathfrak{n} \in \mathcal{N}_{leg}$, then all fixed legs are connect to this node. By this, we know that not all free legs are connected to this node, because by assumption of case 1.1, not all legs are connected to one node. We choose $\mathfrak{n}_*$ to be this node.

If there are two node $\mathfrak{n}_1, \mathfrak{n}_2 \in \mathcal{N}_{leg}$, and if all free legs are connected to one of them, then choose $\mathfrak{n}_*$ to be the other one. Otherwise, choose $\mathfrak{n}_*$ to either one of $\mathfrak{n}_1, \mathfrak{n}_2$ works. $\square$

We cut the node $\mathfrak{n}_*$ in this claim from $\mathcal{C}$. Let $\mathcal{C}_{\mathfrak{n}}$, $\mathcal{C}_1 = \mathcal{C}\backslash\mathcal{C}_{\mathfrak{n}}$ be two components after cutting. As in Lemma 3.3.16, let $\{\mathfrak{e}_i^{(1)}\} \subseteq \mathcal{C}_{\mathfrak{n}}$ be free legs and $\{\mathfrak{e}_i^{(2)}\} \subseteq \mathcal{C}_1$ be fixed legs, then applying Lemma 3.3.16 gives

$$\begin{aligned}\sup_{\{c_{\mathfrak{l}}\}_{\mathfrak{l}}} \#Eq(\mathcal{C}, \{c_{\mathfrak{l}}\}_{\mathfrak{l}}) \leq & \sup_{\{c_{\mathfrak{l}_1}\}_{\mathfrak{l}_1 \in \mathrm{leg}(\mathcal{C}_{\mathfrak{n}})}} \#Eq(\mathcal{C}_{\mathfrak{n}}, \{c_{\mathfrak{l}_1}\}) \sup_{\{c_{\mathfrak{l}_2}\}_{\mathfrak{l}_2 \in \mathrm{leg}(\mathcal{C}_1)}} \#Eq(\mathcal{C}_1, \{c_{\mathfrak{l}_2}\}) \\ \lesssim & L^{\theta} Q^{\chi(\mathcal{C}_{\mathfrak{n}})} L^{O(\theta)} Q^{\chi(\mathcal{C}_1)} = L^{O(\theta)} Q^{\chi(\mathcal{C}_{\mathfrak{n}})+\chi(\mathcal{C}_1)} \\ = & L^{O(\theta)} Q^{\chi(\mathcal{C})}.\end{aligned} \tag{3.3.97}$$

Here the second inequality follows from induction assumption and Lemma 3.3.25. Because it can be verified that in both $\mathcal{C}_{\mathfrak{n}}$ and $\mathcal{C}_1$, n_{fx} and n_{fr} are not equal to 0, the induction assumption and

Lemma 3.3.25 are applicable.

Now we provide a detailed analysis of why in both $\mathcal{C}_{\mathfrak{n}}$ and $\mathcal{C}_1$, n_{fx} and n_{fr} are not equal to 0.

$\mathcal{C}_{\mathfrak{n}}$ contains a fixed leg because $\mathfrak{n}_* \in \mathcal{N}_{leg}$ and by definition of $\mathcal{N}_{leg}$, $\mathfrak{n}_*$ is connected to a fixed leg. $\mathcal{C}_{\mathfrak{n}}$ contains a free leg because it contains free legs $\{\mathfrak{e}_i^{(1)}\} \subseteq \mathcal{C}_{\mathfrak{n}}$.

By the claim, since not all free legs are connected to $\mathfrak{n}_*$, $\mathcal{C}_1$ contains a free leg. $\mathcal{C}_1$ contains a fixed leg because it contains fixed legs $\{\mathfrak{e}_i^{(2)}\} \subseteq \mathcal{C}_1$.

Case 1.2. Assume that all legs are connected to one node $\mathfrak{n}_*$. One example of this case is shown in Figure 3.21

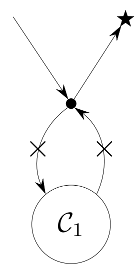


Figure 3.21: Example of case 1.2.

We cut $\mathfrak{n}_*$ from $\mathcal{C}$. Let $\mathcal{C}_{\mathfrak{n}}$, $\mathcal{C}_1 = \mathcal{C}\backslash\mathcal{C}_{\mathfrak{n}}$ be two components after cutting.

By (3.3.55), we know that $n_{fx} + n_{fr}$ is an even number, so it must equal to 2. Then we know that $n_{fx} = n_{fr} = 1$ and $\mathcal{C}$ has one fixed leg $\mathfrak{l}$ and one free leg $\mathfrak{le}$. The other two edges $\mathfrak{e}', \mathfrak{e}''$ from $\mathfrak{n}_*$ should be connected to $\mathcal{C}_1$.

Unlike case 1.1 we cannot simply apply induction assumption to bound $\#Eq(\mathcal{C}_1, \{c_{\mathfrak{l}_2}\})$. Because in this case $\mathcal{C}_1$ does not have free legs ($n_{fx} = 2$, $n_{fr} = 0$). See Figure 3.21.

As in Lemma 3.3.16, $\mathfrak{e}', \mathfrak{e}''$ are cut into $\mathfrak{e}'^{(1)}, \mathfrak{e}''^{(1)}$ and $\mathfrak{e}'^{(2)}, \mathfrak{e}''^{(2)}$. $\mathfrak{e}'^{(1)}, \mathfrak{e}''^{(1)} \in \mathcal{C}_{\mathfrak{n}}$ are free legs and $\mathfrak{e}'^{(2)}, \mathfrak{e}''^{(2)} \in \mathcal{C}_1$ are fixed legs, then applying Lemma 3.3.16 (3.3.48) gives

$$Eq(\mathcal{C}, c_{\mathfrak{l}}) = \{(k_{\mathfrak{e}_1}, k_{\mathfrak{e}_2}) : \ k_{\mathfrak{e}_1} \in Eq(\mathcal{C}_{\mathfrak{n}}, c_{\mathfrak{l}}), \ k_{\mathfrak{e}_2} \in Eq\left(\mathcal{C}_1, \{k_{\mathfrak{e}'^{(1)}}, k_{\mathfrak{e}''^{(1)}}\}_i\right)\}. \tag{3.3.98}$$

By (3.3.61), we know that $k_{\mathfrak{e}'^{(1)}} = k_{\mathfrak{e}''^{(1)}}$. Therefore, we have

$$Eq(\mathcal{C}, c_{\mathfrak{l}}) = \{(k_{\mathfrak{e}_1}, k_{\mathfrak{e}_2}) : \ k_{\mathfrak{e}_1} \in Eq(\mathcal{C}_{\mathfrak{n}}, c_{\mathfrak{l}}) \cap \{k_{\mathfrak{e}'^{(1)}} = k_{\mathfrak{e}''^{(1)}}\}, \ k_{\mathfrak{e}_2} \in Eq\left(\mathcal{C}_1, \{k_{\mathfrak{e}'^{(1)}}, k_{\mathfrak{e}''^{(1)}}\}_i\right)\}. \tag{3.3.99}$$

By definition, we know that

$$\begin{aligned}
Eq(\mathcal{C}_{\mathfrak{n}}, c_{\mathfrak{l}}) =&\{(k_{\mathfrak{l}}, k_{\mathfrak{l}\mathfrak{f}}, k_{\mathfrak{e}'}, k_{\mathfrak{e}''}) : |k_{\mathfrak{l}}|, |k_{\mathfrak{l}\mathfrak{f}}|, |k_{\mathfrak{e}'}|, |k_{\mathfrak{e}''}| \lesssim L^+,\ MC_{\mathfrak{n}_*},\ EC_{\mathfrak{n}_*},\ k_{\mathfrak{l}} = c_{\mathfrak{l}}\} \\
=&\{(k_{\mathfrak{l}}, k_{\mathfrak{l}\mathfrak{f}}, k_{\mathfrak{e}'}, k_{\mathfrak{e}''}) : |k_{\mathfrak{l}\mathfrak{f}}|, |k_{\mathfrak{e}'}|, |k_{\mathfrak{e}''}| \lesssim L^+, \\
&\quad k_{\mathfrak{e}'} - k_{\mathfrak{e}''} + k_{\mathfrak{l}\mathfrak{f}} - k_{\mathfrak{l}} = 0,\ k_{\mathfrak{l}\mathfrak{f}} \neq k_{\mathfrak{e}'} \neq k_{\mathfrak{e}''},\ \text{or } k_{\mathfrak{e}'} = k_{\mathfrak{e}''} = k_{\mathfrak{l}\mathfrak{f}} = k_{\mathfrak{l}} \\
&\quad \Lambda_{k_{\mathfrak{e}'}} - \Lambda_{k_{\mathfrak{e}''}} + \Lambda_{k_{\mathfrak{l}\mathfrak{f}}} - \Lambda_{k_{\mathfrak{l}}} = \sigma_{\mathfrak{n}} + O(\omega),\ k_{\mathfrak{l}} = c_{\mathfrak{l}}\}
\end{aligned} \tag{3.3.100}$$

Therefore, we have

$$Eq(\mathcal{C}_{\mathfrak{n}}, c_{\mathfrak{l}}) \cap \{k_{\mathfrak{e}'(1)} = k_{\mathfrak{e}''(1)}\} = \{(k_{\mathfrak{l}}, k_{\mathfrak{l}\mathfrak{f}}, k_{\mathfrak{e}'}, k_{\mathfrak{e}''}) : k_{\mathfrak{e}'} = k_{\mathfrak{e}''} = k_{\mathfrak{l}\mathfrak{f}} = k_{\mathfrak{l}} = c_{\mathfrak{l}}\}. \tag{3.3.101}$$

By (3.3.99), we have

$$\#Eq(\mathcal{C}, c_{\mathfrak{l}}) = \#Eq(\mathcal{C}_1, \{c_{\mathfrak{l}}, c_{\mathfrak{l}}\}). \tag{3.3.102}$$

Let $\widehat{\mathcal{C}_1}$ be the couple obtained by replacing a fixed leg by a free leg in $\mathcal{C}_1$. Using a similar argument to (3.3.91) and (3.3.92) and then applying the induction assumption, we get

$$\begin{aligned}
\#Eq(\mathcal{C}, c_{\mathfrak{l}}) =&\#Eq(\mathcal{C}_1, \{c_{\mathfrak{l}}, c_{\mathfrak{l}}\}) = \#Eq(\widehat{\mathcal{C}_1}, \{c_{\mathfrak{l}}\}) \\
\lesssim& L^{O(\theta)} Q^{\chi(\widehat{\mathcal{C}_1})} = L^{O(\theta)} Q^{\chi(\mathcal{C})-1} \leq L^{O(\theta)} Q^{\chi(\mathcal{C})}
\end{aligned} \tag{3.3.103}$$

Here the last step is because $\chi(\mathcal{C}) = \chi(\mathcal{C}_{\mathfrak{n}}) + \chi(\mathcal{C}_1) = 2 + \chi(\mathcal{C}_1) = 1 + \chi(\widehat{\mathcal{C}_1})$.

Case 2. Assume that for for some $\mathfrak{n}_* \in \mathcal{N}_{leg}$, after cutting $\mathfrak{n}_*$, $\mathcal{C} \backslash \mathcal{C}_{\mathfrak{n}}$ has exactly three components. One example of this case is shown in Figure 3.22

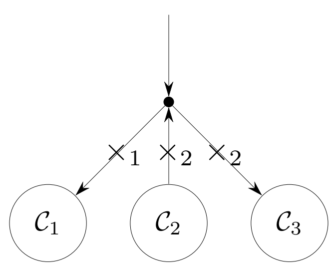


Figure 3.22: Example of case 2. $\times_1$ and $\times_2$ indicate the edges cut in the first and second time respectively.

We cut $\mathfrak{n}_*$ from $\mathcal{C}$. Let $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ be the three components of $\mathcal{C} \backslash \mathcal{C}_{\mathfrak{n}}$ after cutting.

Since $\mathfrak{n}_* \in \mathcal{N}_{leg}$, there should be a fixed leg $\mathfrak{l}$ that is connected to $\mathfrak{n}_*$. The other three edges $\mathfrak{e}'$, $\mathfrak{e}''$, $\mathfrak{e}'''$ from $\mathfrak{n}_*$ should be connected to $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ respectively.

Since there exists at least one free leg in $\mathcal{C}$ ($n_{fr} \neq 0$), at least one of $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ contain a free leg.

Without loss of generality let us assume that $\mathcal{C}_1$ contains a free leg.

Let us first cut $\mathfrak{e}'$ into $\mathfrak{e}'^{(1)} \in \mathcal{C}\backslash\mathcal{C}_1 = \mathcal{C}_{\mathfrak{n}_*} \cup \mathcal{C}_2 \cup \mathcal{C}_3$ and $\mathfrak{e}'^{(2)} \in \mathcal{C}_1$. Let $\mathfrak{e}'^{(1)}$ be a free leg and $\mathfrak{e}'^{(2)}$ be a fixed leg. Therefore, $\mathcal{C}_1$ contains both fixed and free legs. Applying Lemma 3.3.16 gives

$$\begin{aligned}\sup_{\{c_{\mathfrak{l}}\}_{\mathfrak{l}}} \#Eq(\mathcal{C},\{c_{\mathfrak{l}}\}_{\mathfrak{l}}) \leq & \sup_{\{c_{\mathfrak{l}_1}\}_{\mathfrak{l}_1\in \mathrm{leg}(\mathcal{C}\backslash\mathcal{C}_1)}} \#Eq(\mathcal{C}\backslash\mathcal{C}_1,\{c_{\mathfrak{l}_1}\}) \sup_{\{c_{\mathfrak{l}_2}\}_{\mathfrak{l}_2\in \mathrm{leg}(\mathcal{C}_1)}} \#Eq(\mathcal{C}_1,\{c_{\mathfrak{l}_2}\}) \\ \lesssim & L^{O(\theta)}Q^{\chi(\mathcal{C}_1)} \sup_{\{c_{\mathfrak{l}_1}\}_{\mathfrak{l}_1\in \mathrm{leg}(\mathcal{C}\backslash\mathcal{C}_1)}} \#Eq(\mathcal{C}\backslash\mathcal{C}_1,\{c_{\mathfrak{l}_1}\}).\end{aligned} \tag{3.3.104}$$

Here in the second inequality, we can apply the induction assumption to $\mathcal{C}_1$ because it contains both fixed and free legs.

Then we cut $\mathfrak{e}''$ into $\mathfrak{e}''^{(1)} \in \mathcal{C}_{\mathfrak{n}_*} \cup \mathcal{C}_3$ and $\mathfrak{e}''^{(2)} \in \mathcal{C}_2$. Let $\mathfrak{e}''^{(1)}$ be a free leg (or fix leg) and $\mathfrak{e}''^{(2)}$ be a fixed leg (or a free leg) if $\mathcal{C}_2$ contains a free leg (or a fixed leg). Therefore, $\mathcal{C}_2$ contains both fixed and free legs. Applying Lemma 3.3.16 and (3.3.104) gives

$$\begin{aligned}&\sup_{\{c_{\mathfrak{l}}\}_{\mathfrak{l}}} \#Eq(\mathcal{C},\{c_{\mathfrak{l}}\}_{\mathfrak{l}}) \lesssim L^{O(\theta)}Q^{\chi(\mathcal{C}_1)} \sup_{\{c_{\mathfrak{l}_1}\}_{\mathfrak{l}_1\in \mathrm{leg}(\mathcal{C}\backslash\mathcal{C}_1)}} \#Eq(\mathcal{C}\backslash\mathcal{C}_1,\{c_{\mathfrak{l}_1}\}) \\ \lesssim & L^{O(\theta)}Q^{\chi(\mathcal{C}_1)} \sup_{\{c_{\mathfrak{l}_1}\}_{\mathfrak{l}_1\in \mathrm{leg}(\mathcal{C}_{\mathfrak{n}_*}\cup\mathcal{C}_3)}} \#Eq(\mathcal{C}_{\mathfrak{n}_*}\cup\mathcal{C}_3,\{c_{\mathfrak{l}_1}\}) \sup_{\{c_{\mathfrak{l}_2}\}_{\mathfrak{l}_2\in \mathrm{leg}(\mathcal{C}_2)}} \#Eq(\mathcal{C}_2,\{c_{\mathfrak{l}_2}\}) \\ \lesssim & L^{\theta}Q^{\chi(\mathcal{C}_1)+\chi(\mathcal{C}_2)} \sup_{\{c_{\mathfrak{l}_1}\}_{\mathfrak{l}_1\in \mathrm{leg}(\mathcal{C}_{\mathfrak{n}_*}\cup\mathcal{C}_3)}} \#Eq(\mathcal{C}_{\mathfrak{n}_*}\cup\mathcal{C}_3,\{c_{\mathfrak{l}_1}\})\end{aligned} \tag{3.3.105}$$

Here in the third inequality, we can apply the induction assumption to $\mathcal{C}_2$ because it contains both fixed and free legs.

The argument for $\mathcal{C}_3$ is similar to that of $\mathcal{C}_2$. We cut $\mathfrak{e}'''$ into $\mathfrak{e}'''^{(1)} \in \mathcal{C}_{\mathfrak{n}_*}$ and $\mathfrak{e}'''^{(2)} \in \mathcal{C}_3$. Let $\mathfrak{e}'''^{(1)}$ be a free leg (or a fixed leg) and $\mathfrak{e}'''^{(2)}$ be a fixed leg (or a free leg) if $\mathcal{C}_3$ contains a free leg (or a fixedleg). Therefore, $\mathcal{C}_3$ contains both fixed and free legs. Applying Lemma 3.3.16 and (3.3.105) gives

$$\begin{aligned}\sup_{\{c_{\mathfrak{l}}\}_{\mathfrak{l}}} \#Eq(\mathcal{C},\{c_{\mathfrak{l}}\}_{\mathfrak{l}}) \lesssim & L^{O(\theta)}Q^{\chi(\mathcal{C}_1)+\chi(\mathcal{C}_2)} \sup_{\{c_{\mathfrak{l}_1}\}_{\mathfrak{l}_1\in \mathrm{leg}(\mathcal{C}_{\mathfrak{n}_*}\cup\mathcal{C}_3)}} \#Eq(\mathcal{C}_{\mathfrak{n}_*}\cup\mathcal{C}_3,\{c_{\mathfrak{l}_1}\}) \\ \lesssim & L^{O(\theta)}Q^{\chi(\mathcal{C}_1)+\chi(\mathcal{C}_2)} \sup_{\{c_{\mathfrak{l}_1}\}_{\mathfrak{l}_1\in \mathrm{leg}(\mathcal{C}_{\mathfrak{n}_*}}} \#Eq(\mathcal{C}_{\mathfrak{n}_*},\{c_{\mathfrak{l}_1}\}) \sup_{\{c_{\mathfrak{l}_2}\}_{\mathfrak{l}_2\in \mathrm{leg}(\mathcal{C}_3)}} \#Eq(\mathcal{C}_3,\{c_{\mathfrak{l}_2}\}) \\ \lesssim & L^{O(\theta)}Q^{\chi(\mathcal{C}_1)+\chi(\mathcal{C}_2)}L^{O(\theta)}Q^{\chi(\mathcal{C}_{\mathfrak{n}_*})}L^{\theta}Q^{\chi(\mathcal{C}_3)}\end{aligned} \tag{3.3.106}$$

Here in the third inequality, we can apply Lemma 3.3.25 and the induction assumption to $\mathcal{C}_{\mathfrak{n}_*}$ and $\mathcal{C}_3$ because they contains both fixed and free legs. $\mathcal{C}_{\mathfrak{n}_*}$ contains fixed legs (or free legs) because

$\mathfrak{n}_* \in \mathcal{N}_{leg}$ (or $\mathfrak{e}'^{(1)}$ is a free legs from $\mathfrak{n}_*$).

Case 3. Assume that for all $\mathfrak{n} \in \mathcal{N}_{leg}$, after cutting $\mathfrak{n}$, $\mathcal{C}\backslash\mathcal{C}_{\mathfrak{n}}$ has exactly two components.

Choose arbitrarily $\mathfrak{n}_* \in \mathcal{N}_{leg}$. We cut $\mathfrak{n}_*$ from $\mathcal{C}$. Let $\mathcal{C}_1$, $\mathcal{C}_2$ be the three components of $\mathcal{C}\backslash\mathcal{C}_{\mathfrak{n}}$ after cutting. Since $\mathfrak{n}_* \in \mathcal{N}_{leg}$, there should be a fixed leg $\mathfrak{l}$ that is connected to $\mathfrak{n}_*$. There are several possiblities of the distribution of other three edges $\mathfrak{e}'$, $\mathfrak{e}''$, $\mathfrak{e}'''$ from $\mathfrak{n}_*$.

Case 3.1. One of $\mathfrak{e}'$, $\mathfrak{e}''$, $\mathfrak{e}'''$ connects $\mathfrak{n}_*$ and $\mathcal{C}_1$ (w.l.o.g. assume that it's $\mathfrak{e}'$), another one connects $\mathfrak{n}_*$ and $\mathcal{C}_2$ (w.l.o.g. assume that it's $\mathfrak{e}''$) and the last one is a leg (w.l.o.g. assume that it's $\mathfrak{e}'''$). One example of this case is shown in Figure 3.23

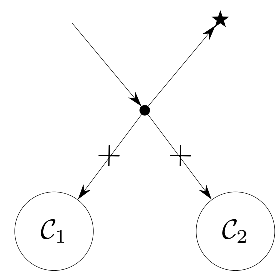


Figure 3.23: Example of case 3.1.

Case 3.1.1. Assume that $\mathfrak{e}'''$ is an fixed leg. Then one of $\mathcal{C}_1$ and $\mathcal{C}_2$ contains a free leg (w.l.o.g. assume that it's $\mathcal{C}_1$).

We apply the same argument as case 2 in this case. Cut $\mathfrak{e}'$ into $\mathfrak{e}'^{(1)} \in \mathcal{C}_{\mathfrak{n}_*} \cup \mathcal{C}_2$, and $\mathfrak{e}'^{(2)} \in \mathcal{C}_1$ and then cut $\mathfrak{e}''$ into $\mathfrak{e}''^{(1)} \in \mathcal{C}_{\mathfrak{n}_*}$ and $\mathfrak{e}''^{(2)} \in \mathcal{C}_2$. Let $\mathfrak{e}'^{(1)}$ be a free leg and $\mathfrak{e}'^{(2)}$ be a fixed leg. Let $\mathfrak{e}''^{(1)}$ be a free leg (or a fixed leg) and $\mathfrak{e}''^{(2)}$ be a fixed leg (or a free leg) if $\mathcal{C}_2$ contains a free leg (or a fixed leg). Cutting in this way, all couples contain both fixed and free legs.

As in case 2, we have

$$\begin{aligned}
\sup_{\{c_{\mathfrak{l}}\}_{\mathfrak{l}}} \#Eq(\mathcal{C}, \{c_{\mathfrak{l}}\}_{\mathfrak{l}}) \leq & \sup_{\{c_{\mathfrak{l}_1}\}_{\mathfrak{l}_1 \in \mathrm{leg}(\mathcal{C}_{\mathfrak{n}_*} \cup \mathcal{C}_2)}} \#Eq(\mathcal{C}_{\mathfrak{n}_*} \cup \mathcal{C}_2, \{c_{\mathfrak{l}_1}\}) \sup_{\{c_{\mathfrak{l}_2}\}_{\mathfrak{l}_2 \in \mathrm{leg}(\mathcal{C}_1)}} \#Eq(\mathcal{C}_1, \{c_{\mathfrak{l}_2}\}) \\
\lesssim & L^{O(\theta)} Q^{\chi(\mathcal{C}_1)} \sup_{\{c_{\mathfrak{l}_1}\}_{\mathfrak{l}_1 \in \mathrm{leg}(\mathcal{C}_{\mathfrak{n}_*} \cup \mathcal{C}_2)}} \#Eq(\mathcal{C}_{\mathfrak{n}_*} \cup \mathcal{C}_2, \{c_{\mathfrak{l}_1}\}) \\
\lesssim & L^{O(\theta)} Q^{\chi(\mathcal{C}_1)} \sup_{\{c_{\mathfrak{l}_1}\}_{\mathfrak{l}_1 \in \mathrm{leg}(\mathcal{C}_{\mathfrak{n}_*})}} \#Eq(\mathcal{C}_{\mathfrak{n}_*}, \{c_{\mathfrak{l}_1}\}) \sup_{\{c_{\mathfrak{l}_2}\}_{\mathfrak{l}_2 \in \mathrm{leg}(\mathcal{C}_2)}} \#Eq(\mathcal{C}_2, \{c_{\mathfrak{l}_2}\}) \\
\lesssim & L^{O(\theta)} Q^{\chi(\mathcal{C}_1) + \chi(\mathcal{C}_2) + \chi(\mathcal{C}_{\mathfrak{n}_*})} = L^{O(\theta)} Q^{\chi(\mathcal{C})}.
\end{aligned} \tag{3.3.107}$$

Case 3.1.2. Assume that $\mathfrak{e}'''$ is an free leg.

We apply the same argument as case 2 and case 3.1.1 in this case. Cut $\mathfrak{e}'$ into $\mathfrak{e}'^{(1)} \in \mathcal{C}_{\mathfrak{n}_*} \cup \mathcal{C}_2$

and $\mathfrak{e}'^{(2)} \in \mathcal{C}_1$, and then cut $\mathfrak{e}''$ into $\mathfrak{e}''^{(1)} \in \mathcal{C}_{\mathfrak{n}_*}$ and $\mathfrak{e}''^{(2)} \in \mathcal{C}_2$. Let $\mathfrak{e}'^{(1)}$ be a free leg (or a fixed leg) and $\mathfrak{e}'^{(2)}$ be a fixed leg (or a free leg) if $\mathcal{C}_1$ contains a free leg (or a fixed leg). Let $\mathfrak{e}''^{(1)}$ be a free leg (or a fixed leg) and $\mathfrak{e}''^{(2)}$ be a fixed leg (or a free leg) if $\mathcal{C}_2$ contains a free leg (or a fixed leg). Cutting in this way, all couples contain both fixed and free legs.

As in case 3.1.1, we have

$$\begin{aligned}\sup_{\{c_{\mathfrak{l}}\}_{\mathfrak{l}}} \#Eq(\mathcal{C}, \{c_{\mathfrak{l}}\}_{\mathfrak{l}}) \leq & \sup_{\{c_{\mathfrak{l}_1}\}_{\mathfrak{l}_1 \in \mathrm{leg}(\mathcal{C}_{\mathfrak{n}_*} \cup \mathcal{C}_2)}} \#Eq(\mathcal{C}_{\mathfrak{n}_*} \cup \mathcal{C}_2, \{c_{\mathfrak{l}_1}\}) \sup_{\{c_{\mathfrak{l}_2}\}_{\mathfrak{l}_2 \in \mathrm{leg}(\mathcal{C}_1)}} \#Eq(\mathcal{C}_1, \{c_{\mathfrak{l}_2}\}) \\ \lesssim & L^{O(\theta)} Q^{\chi(\mathcal{C}_1)} \sup_{\{c_{\mathfrak{l}_1}\}_{\mathfrak{l}_1 \in \mathrm{leg}(\mathcal{C}_{\mathfrak{n}_*} \cup \mathcal{C}_2)}} \#Eq(\mathcal{C}_{\mathfrak{n}_*} \cup \mathcal{C}_2, \{c_{\mathfrak{l}_1}\}) \\ \lesssim & L^{O(\theta)} Q^{\chi(\mathcal{C}_1)} \sup_{\{c_{\mathfrak{l}_1}\}_{\mathfrak{l}_1 \in \mathrm{leg}(\mathcal{C}_{\mathfrak{n}_*})}} \#Eq(\mathcal{C}_{\mathfrak{n}_*}, \{c_{\mathfrak{l}_1}\}) \sup_{\{c_{\mathfrak{l}_2}\}_{\mathfrak{l}_2 \in \mathrm{leg}(\mathcal{C}_2)}} \#Eq(\mathcal{C}_2, \{c_{\mathfrak{l}_2}\}) \\ \lesssim & L^{O(\theta)} Q^{\chi(\mathcal{C}_1) + \chi(\mathcal{C}_2) + \chi(\mathcal{C}_{\mathfrak{n}_*})} = L^{O(\theta)} Q^{\chi(\mathcal{C})}.\end{aligned} \tag{3.3.108}$$

Case 3.2. Two of $\mathfrak{e}'$, $\mathfrak{e}''$, $\mathfrak{e}'''$ (w.l.o.g. assume that they are $\mathfrak{e}'$, $\mathfrak{e}''$) connects $\mathfrak{n}_*$ and one of the two components (w.l.o.g. assume that it's $\mathcal{C}_1$). The last one (w.l.o.g. assume that it's $\mathfrak{e}'''$) connects $\mathfrak{n}_*$ and the last components (w.l.o.g. assume that it's $\mathcal{C}_2$). In this case $\mathcal{C}_2$ must contain a leg by (3.3.55).

One example of this case is shown in Figure 3.24

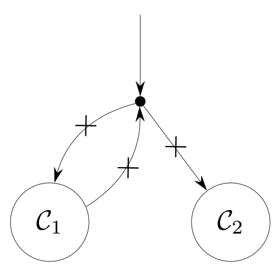


Figure 3.24: Example of case 3.2.

Case 3.2.1. Assume that $\mathcal{C}_1$ contains a leg.

We apply the same argument as case 2 or case 3.1 in this case. Cut $\mathfrak{e}'$, $\mathfrak{e}''$ into $\mathfrak{e}'^{(1)}, \mathfrak{e}''^{(1)} \in \mathcal{C}_{\mathfrak{n}_*} \cup \mathcal{C}_2$ and $\mathfrak{e}'^{(2)}, \mathfrak{e}''^{(2)} \in \mathcal{C}_1$, and then cut $\mathfrak{e}'''$ into $\mathfrak{e}'''^{(1)} \in \mathcal{C}_{\mathfrak{n}_*}$ and $\mathfrak{e}'''^{(2)} \in \mathcal{C}_2$. Let $\mathfrak{e}'^{(1)}$, $\mathfrak{e}''^{(1)}$ be free legs (or fixed legs) and $\mathfrak{e}'^{(2)}$, $\mathfrak{e}''^{(2)}$ be fixed legs (or a free legs) if $\mathcal{C}_1$ contains a free leg (or a fixed leg). Let $\mathfrak{e}'''^{(1)}$ be a free leg (or a fixed leg) and $\mathfrak{e}'''^{(2)}$ be a fixed leg (or a free leg) if $\mathcal{C}_2$ contains a free leg (or a fixed leg). Cutting in this way, all couples contain both fixed and free legs.

As in case 3.1, we have

$$\begin{aligned}
\sup_{\{c_{\mathfrak{l}}\}_{\mathfrak{l}}} \#Eq(\mathcal{C}, \{c_{\mathfrak{l}}\}_{\mathfrak{l}}) \leq & \sup_{\{c_{\mathfrak{l}_1}\}_{\mathfrak{l}_1 \in \mathrm{leg}(\mathcal{C}_{\mathfrak{n}_*} \cup \mathcal{C}_2)}} \#Eq(\mathcal{C}_{\mathfrak{n}_*} \cup \mathcal{C}_2, \{c_{\mathfrak{l}_1}\}) \sup_{\{c_{\mathfrak{l}_2}\}_{\mathfrak{l}_2 \in \mathrm{leg}(\mathcal{C}_1)}} \#Eq(\mathcal{C}_1, \{c_{\mathfrak{l}_2}\}) \\
\lesssim & L^{O(\theta)} Q^{\chi(\mathcal{C}_1)} \sup_{\{c_{\mathfrak{l}_1}\}_{\mathfrak{l}_1 \in \mathrm{leg}(\mathcal{C}_{\mathfrak{n}_*} \cup \mathcal{C}_2)}} \#Eq(\mathcal{C}_{\mathfrak{n}_*} \cup \mathcal{C}_2, \{c_{\mathfrak{l}_1}\}) \\
\lesssim & L^{O(\theta)} Q^{\chi(\mathcal{C}_1)} \sup_{\{c_{\mathfrak{l}_1}\}_{\mathfrak{l}_1 \in \mathrm{leg}(\mathcal{C}_{\mathfrak{n}_*})}} \#Eq(\mathcal{C}_{\mathfrak{n}_*}, \{c_{\mathfrak{l}_1}\}) \sup_{\{c_{\mathfrak{l}_2}\}_{\mathfrak{l}_2 \in \mathrm{leg}(\mathcal{C}_2)}} \#Eq(\mathcal{C}_2, \{c_{\mathfrak{l}_2}\}) \\
\lesssim & L^{O(\theta)} Q^{\chi(\mathcal{C}_1) + \chi(\mathcal{C}_2) + \chi(\mathcal{C}_{\mathfrak{n}_*})} = L^{O(\theta)} Q^{\chi(\mathcal{C})}.
\end{aligned} \tag{3.3.109}$$

Case 3.2.2. Assume that $\mathcal{C}_1$ contains no legs. Then $\mathcal{C}_2$ must contain free legs, since $\mathcal{C}$ contains free legs.

We apply the same argument as case 2 and case 3.1. Cut $\mathfrak{e}'''$ into $\mathfrak{e}'''^{(1)} \in \mathcal{C}_{\mathfrak{n}_*} \cup \mathcal{C}_1$ and $\mathfrak{e}'''^{(2)} \in \mathcal{C}_2$. Let $\mathfrak{e}'''^{(1)}$ be a free leg and $\mathfrak{e}'''^{(2)}$ be a fixed leg. Cutting in this way, all couples contain both fixed and free legs.

As in case 3.1, we have

$$\begin{aligned}
\sup_{\{c_{\mathfrak{l}}\}_{\mathfrak{l}}} \#Eq(\mathcal{C}, \{c_{\mathfrak{l}}\}_{\mathfrak{l}}) \leq & \sup_{\{c_{\mathfrak{l}_1}\}_{\mathfrak{l}_1 \in \mathrm{leg}(\mathcal{C}_{\mathfrak{n}_*} \cup \mathcal{C}_1)}} \#Eq(\mathcal{C}_{\mathfrak{n}_*} \cup \mathcal{C}_1, \{c_{\mathfrak{l}_1}\}) \sup_{\{c_{\mathfrak{l}_2}\}_{\mathfrak{l}_2 \in \mathrm{leg}(\mathcal{C}_2)}} \#Eq(\mathcal{C}_2, \{c_{\mathfrak{l}_2}\}) \\
\lesssim & L^{O(\theta)} Q^{\chi(\mathcal{C}_2)} \sup_{\{c_{\mathfrak{l}_1}\}_{\mathfrak{l}_1 \in \mathrm{leg}(\mathcal{C}_{\mathfrak{n}_*} \cup \mathcal{C}_1)}} \#Eq(\mathcal{C}_{\mathfrak{n}_*} \cup \mathcal{C}_1, \{c_{\mathfrak{l}_1}\}).
\end{aligned} \tag{3.3.110}$$

$\mathcal{C}_{\mathfrak{n}_*} \cup \mathcal{C}_1$ has only one free leg $\mathfrak{e}'''^{(1)}$ and a fixed leg $\mathfrak{l}$ and both of these two edges are connected to $\mathfrak{n}_*$. Therefore, $\mathcal{C}_{\mathfrak{n}_*} \cup \mathcal{C}_1$ satisfies exactly the condition of case 1.2. Applying the conclusion there gives

$$\sup_{\{c_{\mathfrak{l}_1}\}_{\mathfrak{l}_1 \in \mathrm{leg}(\mathcal{C}_{\mathfrak{n}_*} \cup \mathcal{C}_1)}} \#Eq(\mathcal{C}_{\mathfrak{n}_*} \cup \mathcal{C}_1, \{c_{\mathfrak{l}_1}\}) \lesssim L^{O(\theta)} Q^{\chi(\mathcal{C}_{\mathfrak{n}_*} \cup \mathcal{C}_1)}. \tag{3.3.111}$$

Combine (3.3.110) and (3.3.111). We get

$$\begin{aligned}
\sup_{\{c_{\mathfrak{l}}\}_{\mathfrak{l}}} \#Eq(\mathcal{C}, \{c_{\mathfrak{l}}\}_{\mathfrak{l}}) \lesssim & L^{O(\theta)} Q^{\chi(\mathcal{C}_2)} \sup_{\{c_{\mathfrak{l}_1}\}_{\mathfrak{l}_1 \in \mathrm{leg}(\mathcal{C}_{\mathfrak{n}_*} \cup \mathcal{C}_1)}} \#Eq(\mathcal{C}_{\mathfrak{n}_*} \cup \mathcal{C}_1, \{c_{\mathfrak{l}_1}\}) \\
\lesssim & L^{O(\theta)} Q^{\chi(\mathcal{C}_2) + \chi(\mathcal{C}_{\mathfrak{n}_*} \cup \mathcal{C}_1)} = L^{O(\theta)} Q^{\chi(\mathcal{C})}.
\end{aligned} \tag{3.3.112}$$

Therefore, we complete the proof of Proposition 3.3.24 and thus the proof of Proposition 3.3.23 and 3.3.22.

□

3.3.4 An upper bound of coefficients in expansion series

In this section, we derive an upper bound for coefficients $H^T_{k_1\cdots k_{2l+1}}$.

Notice that in (3.3.1), $H^T_{k_1\cdots k_{l+1}}$ are integral of some oscillatory functions. An upper bound can be derived by the standard integration by parts arguments.

Associate each $\mathfrak{n} \in T_{\text{in}}$ with two variables $a_\mathfrak{n}$, $b_\mathfrak{n}$. Then we define

$$F_T(t, \{a_\mathfrak{n}\}_{\mathfrak{n}\in T_{\text{in}}}, \{b_\mathfrak{n}\}_{\mathfrak{n}\in T_{\text{in}}}) = \int_{\cup_{\mathfrak{n}\in T_{\text{in}}} A_\mathfrak{n}} e^{\sum_{\mathfrak{n}\in T_{\text{in}}} it_\mathfrak{n} a_\mathfrak{n} - \nu(t_{\hat{\mathfrak{n}}} - t_\mathfrak{n}) b_\mathfrak{n}} \prod_{\mathfrak{n}\in T_{\text{in}}} dt_\mathfrak{n} \tag{3.3.113}$$

Lemma 3.3.26. *If $c'(t_\mathfrak{n}) \lesssim 1$, then we have the following upper bound for $F_T(t, \{a_\mathfrak{n}\}_\mathfrak{n}, \{b_\mathfrak{n}\}_\mathfrak{n})$,*

$$\sup_{\{b_\mathfrak{n}\}_\mathfrak{n} \lesssim 1} |F_T(t, \{a_\mathfrak{n}\}_\mathfrak{n}, \{b_\mathfrak{n}\}_\mathfrak{n})| \lesssim \sum_{\{d_\mathfrak{n}\}_{\mathfrak{n}\in T_{in}} \in \{0,1\}^{l(T)}} \prod_{\mathfrak{n}\in T_{in}} \frac{t\alpha}{|q_\mathfrak{n}| + \alpha}. \tag{3.3.114}$$

Fix a sequence $\{d_\mathfrak{n}\}_{\mathfrak{n}\in T_{in}}$ whose elements $d_\mathfrak{n}$ takes boolean values $\{0, 1\}$. We define the sequence $\{q_\mathfrak{n}\}_{\mathfrak{n}\in T_{in}}$ by the following recursive formula

$$q_\mathfrak{n} = \begin{cases} a_\mathfrak{r}, & \text{if } \mathfrak{n} = \text{the root } \mathfrak{r}. \\ a_\mathfrak{n} + d_\mathfrak{n} q_{\mathfrak{n}'}, & \text{if } \mathfrak{n} \neq \mathfrak{r} \text{ and } \mathfrak{n}' \text{ is the parent of } \mathfrak{n}. \end{cases} \tag{3.3.115}$$

Proof. The lemma is proved by induction.

For a tree T contains only one node $\mathfrak{r}$, $F_T = 1$ and (3.3.114) is obviously true.

Assume that (3.3.114) is true for trees with $\le n-1$ nodes. We prove the n nodes case.

For general T, let T_1, T_2, T_3 be the three subtrees and $\mathfrak{n}_1$, $\mathfrak{n}_2$, $\mathfrak{n}_3$ be the three children of the root $\mathfrak{r}$, then by the definition of F_T(3.3.113), we get

$$\begin{aligned} F_T(t) = &\int_{\cup_{\mathfrak{n}\in T_{\text{in}}} A_\mathfrak{n}} e^{-i\sum_{\mathfrak{n}\in T_{\text{in}}}(t_\mathfrak{n} a_\mathfrak{n} + \alpha c(t_\mathfrak{n}) b_\mathfrak{n})} \prod_{\mathfrak{n}\in T_{\text{in}}} dt_\mathfrak{n} \\ = &\int_{\cup_{\mathfrak{n}\in T_{\text{in}}} A_\mathfrak{n}} e^{-i(t_\mathfrak{r} a_\mathfrak{r} + \alpha c(t_\mathfrak{r}) b_\mathfrak{r})} e^{-i\sum_{\mathfrak{n}\in T_{1,\text{in}}\cup T_{2,\text{in}}\cup T_{3,\text{in}}}(t_\mathfrak{n} a_\mathfrak{n} + \alpha c(t_\mathfrak{n}) b_\mathfrak{n})} \left(dt_\mathfrak{r} \prod_{j=1}^{3} \prod_{\mathfrak{n}\in T_{j,\text{in}}} dt_\mathfrak{n}\right) \\ = &\int_{\cup_{\mathfrak{n}\in T_{\text{in}}} A_\mathfrak{n}} e^{-i(t_\mathfrak{r} a_\mathfrak{r} + \alpha\,\text{sgn}(a_\mathfrak{r}))} e^{-i\alpha(c(t_\mathfrak{n}) b_\mathfrak{n} - \text{sgn}(a_\mathfrak{r}))} e^{-i\sum_{\mathfrak{n}\in \mathfrak{n}\in T_{1,\text{in}}\cup T_{2,\text{in}}\cup T_{3,\text{in}}}(t_\mathfrak{n} a_\mathfrak{n} + \alpha c(t_\mathfrak{n}) b_\mathfrak{n})} \left(dt_\mathfrak{r} \prod_{j=1}^{3} \prod_{\mathfrak{n}\in T_{j,\text{in}}} dt_\mathfrak{n}\right) \end{aligned} \tag{3.3.116}$$

We do integration by parts in above integrals using Stokes formula. Notice that for $t_\mathfrak{r}$, there are four inequality constrains, $t_\mathfrak{r} \le t$ and $t_\mathfrak{r} \ge t_{\mathfrak{n}_1}, t_{\mathfrak{n}_2}, t_{\mathfrak{n}_3}$.

$$
\begin{aligned}
F_T(t) = &\frac{i}{a_{\mathfrak{r}} + \alpha \mathrm{sgn}(a_{\mathfrak{r}})} \int_{\cup_{\mathfrak{n}\in T_{\text{in}}} A_{\mathfrak{n}}} \frac{d}{dt_{\mathfrak{r}}} e^{-i(t_{\mathfrak{r}} a_{\mathfrak{r}} + \alpha\,\mathrm{sgn}(a_{\mathfrak{r}}))} \\
&\qquad e^{-i\alpha(c(t_{\mathfrak{n}})b_{\mathfrak{n}} - \mathrm{sgn}(a_{\mathfrak{r}}))} e^{-i\sum_{\mathfrak{n}\in T_{1,\text{in}}\cup T_{2,\text{in}}\cup T_{3,\text{in}}}(t_{\mathfrak{n}}a_{\mathfrak{n}} + \alpha c(t_{\mathfrak{n}})b_{\mathfrak{n}})} \left(dt_{\mathfrak{r}} \prod_{j=1}^{3} \prod_{\mathfrak{n}\in T_{j,\text{in}}} dt_{\mathfrak{n}} \right) \\
= &\frac{i}{a_{\mathfrak{r}} + \alpha \mathrm{sgn}(a_{\mathfrak{r}})} \left(\int_{\cup_{\mathfrak{n}\in T_{\text{in}}} A_{\mathfrak{n}},\ t_{\mathfrak{r}}=t} - \int_{\cup_{\mathfrak{n}\in T_{\text{in}}} A_{\mathfrak{n}},\ t_{\mathfrak{r}}=t_{\mathfrak{n}_1}} - \int_{\cup_{\mathfrak{n}\in T_{\text{in}}} A_{\mathfrak{n}},\ t_{\mathfrak{r}}=t_{\mathfrak{n}_2}} - \int_{\cup_{\mathfrak{n}\in T_{\text{in}}} A_{\mathfrak{n}},\ t_{\mathfrak{r}}=t_{\mathfrak{n}_3}} \right) \\
&e^{-i(t_{\mathfrak{r}} a_{\mathfrak{r}} + \alpha\,\mathrm{sgn}(a_{\mathfrak{r}}))} e^{-i\alpha(c(t_{\mathfrak{n}})b_{\mathfrak{n}} - \mathrm{sgn}(a_{\mathfrak{r}}))} e^{-i\sum_{\mathfrak{n}\in T_{1,\text{in}}\cup T_{2,\text{in}}\cup T_{3,\text{in}}}(t_{\mathfrak{n}}a_{\mathfrak{n}} + \alpha c(t_{\mathfrak{n}})b_{\mathfrak{n}})} \left(dt_{\mathfrak{r}} \prod_{j=1}^{3} \prod_{\mathfrak{n}\in T_{j,\text{in}}} dt_{\mathfrak{n}} \right)
\end{aligned}
\tag{3.3.117}
$$

$$
\begin{aligned}
&- \frac{i}{a_{\mathfrak{r}} + \alpha \mathrm{sgn}(a_{\mathfrak{r}})} \int_{\cup_{\mathfrak{n}\in T_{\text{in}}} A_{\mathfrak{n}}} e^{-i(t_{\mathfrak{r}} a_{\mathfrak{r}} + \alpha\,\mathrm{sgn}(a_{\mathfrak{r}}))} \\
&\qquad \frac{d}{dt_{\mathfrak{r}}} e^{-i\alpha(c(t_{\mathfrak{n}})b_{\mathfrak{n}} - \mathrm{sgn}(a_{\mathfrak{r}}))} e^{-i\sum_{\mathfrak{n}\in T_{1,\text{in}}\cup T_{2,\text{in}}\cup T_{3,\text{in}}}(t_{\mathfrak{n}}a_{\mathfrak{n}} + \alpha c(t_{\mathfrak{n}})b_{\mathfrak{n}})} \left(dt_{\mathfrak{r}} \prod_{j=1}^{3} \prod_{\mathfrak{n}\in T_{j,\text{in}}} dt_{\mathfrak{n}} \right) \\
= &\frac{i}{a_{\mathfrak{r}} + \alpha \mathrm{sgn}(a_{\mathfrak{r}})} (F_I - F_{T^{(1)}} - F_{T^{(2)}} - F_{T^{(3)}} - F_{II})
\end{aligned}
$$

Here $T^{(j)}$, $j = 1, 2, 3$ are trees that is obtained by deleting the root $\mathfrak{r}$, adding edges that connecting $\mathfrak{n}_j$ with other two nodes and defining $\mathfrak{n}_j$ to be the new root. For $T^{(j)}$, we can define the term $F_{T^{(j)}}$ by (3.3.113). It can be shown that $F_{T^{(j)}}$ defined in this way is the same as the $\int_{\cup_{\mathfrak{n}\in T_{\text{in}}} A_{\mathfrak{n}},\ t_{\mathfrak{r}}=t_{\mathfrak{n}_j}}$ term in the second equality of (3.3.117), so the last equality of (3.3.117) is true. F_I is the $\int_{\cup_{\mathfrak{n}\in T_{\text{in}}} A_{\mathfrak{n}},\ t_{\mathfrak{r}}=t}$ term and F_{II} is the last term containing $\frac{d}{dt_{\mathfrak{r}}}$.

We can apply the induction assumption to $F_{T^{(j)}}$ and show that $\frac{i}{a_{\mathfrak{r}}+\alpha\mathrm{sgn}(a_{\mathfrak{r}})} F_{T^{(j)}}$ can be bounded by the right hand side of (3.3.114).

A direct calculation gives that

$$F_I(t) = e^{-i(ta_{\mathfrak{r}} + \alpha c(t) b_{\mathfrak{r}})} F_{T_1}(t) F_{T_2}(t) F_{T_3}(t). \tag{3.3.118}$$

Then the induction assumption implies that $\frac{i}{a_{\mathfrak{r}}+\alpha\mathrm{sgn}(a_{\mathfrak{r}})} F_I$ can be bounded by the right hand side of (3.3.114).

Another direct calculation gives that

$$F_{II}(t) = \int_0^t e^{-i(t_{\mathfrak{r}} a_{\mathfrak{r}} + \alpha c(t_{\mathfrak{r}}) b_{\mathfrak{r}})} \frac{d}{dt_{\mathfrak{r}}} e^{-i\alpha(c(t_{\mathfrak{n}})b_{\mathfrak{n}} - \mathrm{sgn}(a_{\mathfrak{r}}))} F_{T_1}(t_{\mathfrak{r}}) F_{T_2}(t_{\mathfrak{r}}) F_{T_3}(t_{\mathfrak{r}}) dt_{\mathfrak{r}}. \tag{3.3.119}$$

Applying the induction assumption

$$\begin{aligned}\left|\frac{i}{a_{\mathfrak{r}}+\alpha\mathrm{sgn}(a_{\mathfrak{r}})}F_{II}(t)\right| \leq & \frac{\alpha t}{|a_{\mathfrak{r}}|+\alpha}\prod_{j=1}^{3}\left(\sum_{\{d_{\mathfrak{n}}\}_{\mathfrak{n}\in T_{j,\mathrm{in}}}\in\{0,1\}^{l(T)}}\prod_{\mathfrak{n}\in T_{j,\mathrm{in}}}\frac{t\alpha}{|q_{\mathfrak{n}}|+\alpha}\right) \\ \leq & \sum_{\{d_{\mathfrak{n}}\}_{\mathfrak{n}\in T_{\mathrm{in}}}\in\{0,1\}^{l(T)}}\prod_{\mathfrak{n}\in T_{\mathrm{in}}}\frac{t\alpha}{|q_{\mathfrak{n}}|+\alpha}.\end{aligned} \tag{3.3.120}$$

.

Combining the bounds of F_I, $F_{T^{(1)}}$, $F_{T^{(2)}}$, $F_{T^{(3)}}$, F_{II}, we conclude that F_T can be bounded by the right hand side of (3.3.114) and thus complete the proof of Lemma 3.3.26. □

A straight forward application of above lemma gives following upper bound of the coefficients $H^T_{k_1\cdots k_{2l+1}}$.

Lemma 3.3.27. *Let $m(t)=\mathbb{E}M(t)$, then by mass conservation $m(t_{\mathfrak{n}})\lesssim 1$ for all $t_{\mathfrak{n}}$. We have the following upper bound for $H^T_{k_1\cdots k_{2l+1}}$,*

$$H^T_{k_1\cdots k_{2l+1}} \lesssim \sum_{\{d_{\mathfrak{n}}\}_{\mathfrak{n}\in T_{in}}\in\{0,1\}^{l(T)}}\prod_{\mathfrak{n}\in T_{in}}\frac{t\alpha}{|q_{\mathfrak{n}}|+\alpha}\ \delta_{\cap_{\mathfrak{n}\in T_{in}}S_{\mathfrak{n}}}. \tag{3.3.121}$$

Fix a sequence $\{d_{\mathfrak{n}}\}_{\mathfrak{n}\in T_{in}}$ whose elements $d_{\mathfrak{n}}$ takes boolean values $\{0,1\}$. We define the sequence $\{q_{\mathfrak{n}}\}_{\mathfrak{n}\in T_{in}}$ by the following recursive formula

$$q_{\mathfrak{n}}=\begin{cases}\Omega_{\mathfrak{r}}, & \text{if } \mathfrak{n}= \text{ the root } \mathfrak{r}.\\ \Omega_{\mathfrak{n}}+d_{\mathfrak{n}}q_{\mathfrak{n}'}, & \text{if } \mathfrak{n}\neq\mathfrak{r} \text{ and } \mathfrak{n}' \text{ is the parent of } \mathfrak{n}.\end{cases} \tag{3.3.122}$$

Proof. This is a direct corollary of Lemma 3.3.114 if we take $a_{\mathfrak{n}}=\Omega_{\mathfrak{n}}$, $b_{\mathfrak{n}}=\widetilde{\Omega}_{\mathfrak{n}}/(2\alpha\int_0^t m(s)ds)$ and $c(t_{\mathfrak{n}})=2\int_0^t m(s)ds$. □

Chapter 4

Counting results in wave kinetic theory

4.1 Number theoretic results in three wave models

The mains results of this appendix is to prove Theorem 4.3.1 and Theorem 4.3.2

Theorem 4.1.1. *Let $\Lambda(k) = (\beta_x k_x^2 + \beta_2 k_2^2 + \cdots \beta_d k_d^2) k_x$, $d \geq 3$, then for all $\beta \in [1,2]^d$, the following number theory estimate is true*

$$\sup_{\substack{k,\sigma\in\mathbb{Z}_L^d\\ k\neq 0,\ T\leq L}} |k_x| T\# \left\{ \begin{matrix} k_1, k_2 \in \mathbb{Z}_L^d \\ |k_1| \lesssim 1 \end{matrix} : \begin{matrix} k_1 + k_2 = k \\ \Lambda(k_1) + \Lambda(k_2) = \Lambda(k) + \sigma + O(T^{-1}) \end{matrix} \right\} \leq L^{2d}. \tag{4.1.1}$$

Remark 4.1.2. The restriction $T \leq L$ is not optimal. The optimal result is expected to be $T \leq L^2$ for general β and $T \leq L^d$ for generic β. It is possible to apply circle method and probabilistic method in number theory method to prove the optimal result.

Remark 4.1.3. 4.3.1 and 4.3.2 is unlikely to be true when $d = 2$. Because in this case, the quadratic term of k_1 in (4.1.11) becomes $3k_x k_{1x}^2 + k_x |k_{1y}|^2 + 2k_y \cdot k_{1x} k_{1y}$ ($k_{1\perp}$ becomes $k_{1\perp} = k_y$), which is degenerate when $k_y^2 = 3k_x^2$.

Theorem 4.1.4. *Let $\Omega_k(k_1) \coloneqq \Lambda(k_1) + \Lambda(k - k_1) - \Lambda(k)$ and t be a large number, then given any smooth compactly supported $F(k)$ and smooth $g(s)$ satisfying $|g|(s) + |g'|(s) \lesssim 1/(1 + s^2)$ and*

$\int_{\mathbb{R}} g(s)ds = c$, *we have*

$$\sum_{k_1\in\mathbb{Z}_L^d} g(t\Omega_k(k_1))F(k_1) = cL^d t^{-1}\int F(k_1)\delta(\Omega_k(k_1))dk_1 + O(L^{d-1}). \tag{4.1.2}$$

in particular

$$\sum_{k_1\in\mathbb{Z}_L^d} g(t\Omega_k(k_1))F(k_1) \leq 2ct^{-1}L^d D^{d-1}. \tag{4.1.3}$$

Proof of Theorem 4.3.1. (4.1.1) is a corollary of the following lemma.

Lemma 4.1.5. *For any $d \geq 4$ and β*

$$\sup_{\substack{k,\sigma\in\mathbb{Z}_L^d\\ k\neq 0}} |k_x|\#\{k_1\in\mathbb{Z}_L^d,\ |k_1|\lesssim 1 : \Lambda(k_1)+\Lambda(k-k_1)-\Lambda(k) = \sigma + O(L^{-1})\} \lesssim L^{d-1}. \tag{4.1.4}$$

Proof. Define $\mathcal{D}_{k,\sigma} = \#\{k_1\in\mathbb{Z}_L^d,\ |k_1|\lesssim 1 : \Lambda(k_1)+\Lambda(k-k_1)-\Lambda(k) = \sigma+O(L^{-1})\}$, then we just need to show that

$$\sup_{\substack{k,\sigma\in\mathbb{Z}_L^d\\ k\neq 0}} \#\mathcal{D}_{k,\sigma} \lesssim L^{d-1}. \tag{4.1.5}$$

We prove (4.1.5) using volume bound. The proof is divided into three steps.

Step 1. In this step, we show that

$$\#\mathcal{D}_{k,\sigma} \leq L^d \mathrm{vol}(\mathcal{D}_{k,\sigma}^{\mathbb{R}}), \tag{4.1.6}$$

where $\mathcal{D}_{k,\sigma}^{\mathbb{R}} = \{k_1\in\mathbb{R}^d,\ |k_1|\lesssim 1 : \Lambda(k_1)+\Lambda(k-k_1)-\Lambda(k) = \sigma+O(L^{-1})\}$.

(4.1.6) can be proved from the following claim.

Claim. If $k_1\in\mathcal{D}_{k,\sigma}$, then $D_{1/(2L)}(k_1)\subseteq\mathcal{D}_{k,\sigma}^{\mathbb{R}}$. Here $D_r(k_1) = \{k_1'\in\mathbb{R}^d : \sup_{i=1,\cdots,d}|(k_1')_i - (k_1)_i| \leq r\}$ ($(k_1)_i$ are the components of k_1).

We prove the claim now. $x\in\mathcal{D}_{k,\sigma}$ is equivalent to $\Lambda(k_1)+\Lambda(k-k_1)-\Lambda(k) = \sigma+O(L^{-1})$. For any $k_1'\in D_{1/(2L)}(k_1)$, $|k_1'-k_1|\lesssim 1/L$, because Λ is a Lifschitz function, we have $|\Lambda(k_1)-\Lambda(k_1')|\lesssim L^{-1}$. Therefore,

$$\begin{aligned}
&|\Lambda_\beta(k_1')+\Lambda_\beta(k-k_1')-\Lambda(k)-\sigma|\\
\leq&|\Lambda_\beta(k_1)+\Lambda_\beta(k-k_1)-\Lambda(k)-\sigma| + |\Lambda_\beta(k_1)-\Lambda_\beta(k_1')| + |\Lambda_\beta(k-k_1)-\Lambda_\beta(k-k_1')|\\
\lesssim&L^{-1}.
\end{aligned} \tag{4.1.7}$$

Therefore, we have $\Lambda(k_1') + \Lambda(k - k_1') - \Lambda(k) = \sigma + O(L^{-1})$ and thus $k_1' \in \mathcal{D}^{\mathbb{R}}_{k,\sigma}$. This is true for any $k_1' \in D_{1/(2L)}(k_1)$, so $D_{1/(2L)}(k_1) \subseteq \mathcal{D}^{\mathbb{R}}_{k,\sigma}$.

Now we prove (4.1.6). Since for different $k_1, k_1' \in \mathcal{D}_{k,\sigma}$, $D_{1/(2L)}(k_1) \cap D_{1/(2L)}(k_1') = \emptyset$, we have

$$\sum_{k_1 \in \mathcal{D}_{k,\sigma}} \mathrm{vol}(D_{1/(2L)}(k_1)) = \mathrm{vol}\left(\bigcup_{k_1 \in \mathcal{D}_{k,\sigma}} D_{1/(2L)}(k_1) \right) \leq \mathrm{vol}(\mathcal{D}^{\mathbb{R}}_{k,\sigma}). \tag{4.1.8}$$

The left hand side equals to $L^{-d} \# \mathcal{D}_{k,\sigma}$, so we get

$$L^{-d} \# \mathcal{D}_{k,\sigma} \leq \mathrm{vol}(\mathcal{D}^{\mathbb{R}}_{k,\sigma}), \tag{4.1.9}$$

which implies (4.1.6).

Step 2. In this step, we show that

$$\mathrm{vol}(\mathcal{D}^{\mathbb{R}}_{k,\sigma}) \leq L^{-1} |k_x|^{-1}. \tag{4.1.10}$$

Combining (4.1.6) and (4.1.10), we get (4.1.5), which proves Lemma 4.3.3.

Given a vector k, we denote the first component of k by k_x and the vector formed by other components by $k_\perp$. Then $k = (k_x, k_\perp)$, $k_1 = (k_{1x}, k_{1\perp})$ and a simple calculation suggests that

$$\begin{aligned} &\Lambda(k_1) + \Lambda(k - k_1) - \Lambda(k) \\ =&3k_x k_{1x}^2 + k_x |k_{1\perp}|^2 + 2k_\perp \cdot k_{1\perp} k_{1x} - (3k_x^2 + |k_\perp|^2) k_{1x} - 2k_x k_\perp \cdot k_{1\perp} \end{aligned} \tag{4.1.11}$$

If we fix k_{1x} to be a constant c in $\Lambda(k_1) + \Lambda(k - k_1) - \Lambda(k)$ and denote the resulting function by $F_{k,\sigma,c}$, then

$$F_{k,\sigma,c}(k_{1\perp}) = k_x |k_{1\perp}|^2 + 2(c - k_x) k_\perp \cdot k_{1\perp} + 3k_x c^2 - (3k_x^2 + |k_\perp|^2) c. \tag{4.1.12}$$

Therefore, if we define $\mathcal{D}^{\mathbb{R}}_{k,\sigma}(k_{1x} = c)$ by

$$\mathcal{D}^{\mathbb{R}}_{k,\sigma}(k_{1x} = c) = \{k_{1\perp} \in \mathbb{R}^{d-1},\ |k_{1\perp}| \lesssim 1 : F_{k,\sigma,c}(k_{1\perp}) = \sigma + O(L^{-1})\} \tag{4.1.13}$$

Then

$$\mathcal{D}^{\mathbb{R}}_{k,\sigma} = \cup_{|c| \lesssim 1} \mathcal{D}^{\mathbb{R}}_{k,\sigma}(k_{1x} = c) \tag{4.1.14}$$

By Fubini theorem (or coarea formula), we get

$$\mathrm{vol}(\mathcal{D}^{\mathbb{R}}_{k,\sigma}) = \int_{|c|\lesssim 1} \mathrm{vol}(\mathcal{D}^{\mathbb{R}}_{k,\sigma}(k_{1x} = c))dc. \tag{4.1.15}$$

To prove (4.1.10), it suffices to find an upper of $\mathrm{vol}(\mathcal{D}^{\mathbb{R}}_{k,\sigma}(k_{1x} = c))$. Since $F_{k,\sigma,c}(k_{1\perp})$ is a quadratic function in $k_{1\perp}$ whose degree 2 term is $k_x|k_{1\perp}|^2$, a translation $k_{1\perp} \to k_{1\perp} - (c-k_x)k_{1\perp}/k_x$ transforms $F_{k,\sigma,c}(k_{1\perp}) = \sigma + O(L^{-1})$ to $k_x|k_{1\perp}|^2 = C_{k,\sigma,c} + O(L^{-1})$.

Therefore,

$$\begin{aligned}\mathrm{vol}(\mathcal{D}^{\mathbb{R}}_{k,\sigma}(k_{1x} = c)) \leq& \mathrm{vol}(\{k_{1\perp} \in \mathbb{R}^{d-1} : k_x|k_{1\perp}|^2 = C_{k,\sigma,c} + O(L^{-1})\}) \\ =& \mathrm{vol}(\{k_{1\perp} \in \mathbb{R}^{d-1} : |k_{1\perp}|^2 = C_{k,\sigma,c}/k_x + O(L^{-1}|k_x|^{-1})\}) \\ \lesssim& L^{-1}|k_x|^{-1}\end{aligned} \tag{4.1.16}$$

Here the last inequality follows from the elementary fact that $\mathrm{vol}(\{x \in \mathbb{R}^n : |x|^2 = R^2 + O(\eta)\}) \lesssim \eta$ when $n \geq 2$. Notice here the dimension of $k_{1\perp}$ is $d-1$ which is greater than 2, so this fact is applicable.

Combining (4.1.15) and (4.1.15), we proves (4.1.10). Therefore, we complete the proof of Lemma 4.3.3 □

We now return to the proof of Theorem 4.3.1. Define

$$\mathcal{D}_{k,\sigma}([-T^{-1}, T^{-1}]) = \{k_1 \in \mathbb{Z}^d_L,\ |k_1| \lesssim 1 : \Lambda(k_1) + \Lambda(k-k_1) - \Lambda(k) = \sigma + O(L^{-1})\} \tag{4.1.17}$$

Let us prove (4.1.1). For any $T \leq L$, let $N = [L/T]+1$, then $\cup_{j=-N}^{N}[jL^{-1}, (j+1)L^{-1}]$ is a cover of $[-T^{-1}, T^{-1}]$. Therefore, $\cup_{j=-N}^{N}\mathcal{D}_{k,\sigma}([jL^{-1}, (j+1)L^{-1}])$ is a cover of $\mathcal{D}_{k,\sigma}([-T^{-1}\ , T^{-1}])$

By Lemma 4.3.3 we get

$$\#\mathcal{D}_{k,\sigma}([-T^{-1}, T^{-1}]) \lesssim \sum_{j=-N}^{N} \#\mathcal{D}_{k,\sigma}([jL^{-1}, (j+1)L^{-1}]) \lesssim NL^{d-1}|k_x|^{-1} \lesssim L^dT^{-1}|k_x|^{-1}. \tag{4.1.18}$$

This proves (4.1.1). □

Proof of Theorem 4.3.2. The inequality in Theorem 4.3.2 follows from the equality because $\int F(k_1)\delta(\Omega_k(k_1))dk_1 \leq D^{d-1}$, $t \leq L^{-\theta}\alpha^{-2} \leq L^{1-\theta}$ and $D^{d-1} \lesssim L^{\theta}$ if L is large enough.

Let $k_1 = \frac{K_1}{L}$ and $k = \frac{K}{L}$. Apply the high dimensional Euler-Maclaurin formula, we know that

$$\begin{aligned}\sum_{k_1\in\mathbb{Z}_L^d} g(t\Omega_k(k_1))F(k_1) =& \int g(t\Omega_k(K_1/L))F\left(\frac{K_1}{L}\right)dK_1 \\ &+ \sum_{|J|_\infty=1}\int_{\mathbb{R}^d}\{K_1\}^J\partial_{K_1}^J\left(g(t\Omega_k(K_1/L))F\left(\frac{K_1}{L}\right)\right)dK_1\end{aligned} \tag{4.1.19}$$

We have the following estimates

$$\int g(t\Omega_k(K_1/L))F\left(\frac{K_1}{L}\right)dK_1 = L^d\int g(t\Omega_k(k_1))F(k_1)dk_1 \tag{4.1.20}$$

and

$$\begin{aligned}&\int_{\mathbb{R}^d}\{K_1\}^J\partial_{K_1}^J\left(g(t\Omega_k(K_1/L))F\left(\frac{K_1}{L}\right)\right)dK_1 \\ =&L^{d-|J|}\int_{\mathbb{R}^d}\{Lk_1\}^J\partial_{k_1}^J\left(g(t\Omega_k(k_1))F(k_1)\right)dk_1 \\ =&O\left(L^{d-|J|}\int_{\mathbb{R}^d}|\partial_{k_1}^J\left(g(t\Omega_k(k_1))F(k_1)|\right)dk_1\right) \\ =&O\left(\sum_{j=1}^{|J|}L^{d-j}t^j\int_{\mathbb{R}^d}g^{(j)}(t\Omega_k(k_1))F^{(j)}(k_1)dk_1\right)\end{aligned} \tag{4.1.21}$$

Here in the second equality we apply the fact that $\{Lk_1\}\le 1$. In the last line we define $g^{(j)} = \sum_{j'\le j}\left|\frac{d^{j'}}{d^{j'}s}g(s)\right|$ and $F^{(j)}(k_1) = \sum_{|J'|\le j}|\partial_{k_1}^{J'}F(k_1)|$ (Here J' is a multi-index) and we also use the fact that $|\partial_{k_1}^{j}(g(t\Omega_k(k_1)))|\le t^j g^{(j)}(t\Omega_k(k_1))$.

Now we just need to show that

$$\int g(t\Omega_k(k_1))F(k_1)dk_1 = ct^{-1}\int F(k_1)\delta(\Omega_k(k_1))dk_1 + O(t^{-2}) \tag{4.1.22}$$

and

$$\int_{\mathbb{R}^d} g^{(j)}(t\Omega_k(k_1))F^{(j)}(k_1)dk_1 \le t^{-1} \tag{4.1.23}$$

In fact, if we substitute (4.1.22) into (4.1.20), we get the main term in (4.1.2). If we substitute (4.1.23) into (4.1.21), we know that the error terms come from (4.1.21) can be bounded by $\sum_{j=1}^{|J|}L^{d-j}t^{j-1}\le |J|L^{d-1}$. Therefore, (4.1.22) and (4.1.23) implies the lemma.

By coarea formula we know that

$$\int g(t\Omega_k(k_1))F(k_1)dk_1 = \int g(t\omega)\underbrace{\left(\int F(k_1)\delta(\Omega_k(k_1)-\omega)dk_1\right)}_{h(\omega)}d\omega \tag{4.1.24}$$

Notice that $h(\omega)$ is differentiable, then we have

$$\begin{aligned}\int g(t\omega)h(\omega)d\omega =& \int g(t\omega)(h(\omega)-h(0))d\omega + h(0)\int g(t\omega)d\omega \\ =& ct^{-1}h(0)+O(t^{-2}).\end{aligned} \tag{4.1.25}$$

Therefore,

$$\begin{aligned}\int g(t\Omega_k(k_1))F(k_1)dk_1 =& \int g(t\omega)h(\omega)d\omega \\ =& ct^{-1}h(0)+O(t^{-2}) = ct^{-1}\left(\int F(k_1)\delta(\Omega_k(k_1))dk_1\right)+O(t^{-2})\end{aligned} \tag{4.1.26}$$

Using the same argument we can also prove (4.1.23), so we complete the proof of the Theorem 4.3.2. $\square$

4.2 High Dimensional Euler-Maclaurin Formula

Theorem 4.2.1. *Assume that $J=(j_1,\cdots,j_d)$ is a multi-index. Given a vector $K=(K^{(1)},\cdots,K^{(d)})$, define $K^J = \left(K^{(1)}\right)^{j_1}\cdots\left(K^{(d)}\right)^{j_d}$. Given a number a, $\{a\}=a-[a]$ is its fractional part, $\{K\}:=(\{K^{(1)}\},\cdots,\{K^{(d)}\})$. We also define $|J|_\infty = \sup_{1\le n\le d} j_n$, $|J|=\sum_{1\le n\le d} j_n$. Then we have*

$$\sum_{K\in\mathbb{Z}^d} f(K) = \int_{\mathbb{R}^d} f(K)dK + \sum_{|J|_\infty=1}\int_{\mathbb{R}^d}\{K\}^J\partial_K^J f(K)dK \tag{4.2.1}$$

Proof. This can be proved by induction.

When $d=1$, (4.2.1) becomes

$$\sum_{K\in\mathbb{Z}} f(K) = \int_{\mathbb{R}} f(K)dK + \int_{\mathbb{R}}\{K\}\partial_K f(K)dK \tag{4.2.2}$$

This is the standard Euler-Maclaurin formula which can be proved by integration by parts in the second integral of the right hand side.

If the formula is true in dimension d, we prove it for dimension $d+1$. Assume that $K=$

$(\widetilde{K}, K^{(d+1)})$ is a $d+1$ dimensional vector, then apply the induction assumption

$$\sum_{\widetilde{K}\in\mathbb{Z}^d} f(\widetilde{K}, K^{(d+1)}) = \int_{\mathbb{R}^d} f(\widetilde{K}, K^{(d+1)})d\widetilde{K} + \sum_{|\widetilde{J}|_\infty=1} \int_{\mathbb{R}^d} \{\widetilde{K}\}^{\widetilde{J}} \partial_{\widetilde{K}}^{\widetilde{J}} f(\widetilde{K}, K^{(d+1)})d\widetilde{K}. \tag{4.2.3}$$

Sum over $K^{(d+1)}$ and apply $d=1$ Euler-Maclaurin formula we get

$$\begin{aligned}
&\sum_{K^{(d+1)}\in\mathbb{Z}} \sum_{\widetilde{K}\in\mathbb{Z}^d} f(\widetilde{K}, K^{(d+1)}) \\
=&\int_{\mathbb{R}^d} \sum_{K^{(d+1)}\in\mathbb{Z}} f(\widetilde{K}, K^{(d+1)})d\widetilde{K} + \sum_{|\widetilde{J}|_\infty=1} \int_{\mathbb{R}^d} \{\widetilde{K}\}^{\widetilde{J}} \partial_{\widetilde{K}}^{\widetilde{J}} \sum_{K^{(d+1)}\in\mathbb{Z}} f(\widetilde{K}, K^{(d+1)})d\widetilde{K}. \\
=&\int_{\mathbb{R}^d}\int_{\mathbb{R}} f(\widetilde{K}, K^{(d+1)})dK^{(d+1)}d\widetilde{K} + \int_{\mathbb{R}^d}\int_{\mathbb{R}} \{K^{(d+1)}\}\partial_{K^{(d+1)}} f(\widetilde{K}, K^{(d+1)})dK^{(d+1)}d\widetilde{K} \\
&+ \sum_{|\widetilde{J}|_\infty=1} \int_{\mathbb{R}^d}\int_{\mathbb{R}} \{\widetilde{K}\}^{\widetilde{J}} \partial_{\widetilde{K}}^{\widetilde{J}} f(\widetilde{K}, K^{(d+1)})dK^{(d+1)}d\widetilde{K} \\
&+ \sum_{|\widetilde{J}|_\infty=1} \int_{\mathbb{R}^d}\int_{\mathbb{R}} \{K^{(d+1)}\}\partial_{K^{(d+1)}} \{\widetilde{K}\}^{\widetilde{J}} \partial_{\widetilde{K}}^{\widetilde{J}} f(\widetilde{K}, K^{(d+1)})dK^{(d+1)}d\widetilde{K}. \\
=&\int_{\mathbb{R}^{d+1}} f(K)dK + \sum_{|J|_\infty=1} \int_{\mathbb{R}^d} \{K\}^J \partial_K^J f(K)dK
\end{aligned} \tag{4.2.4}$$

In the last step, $J = (\widetilde{J}, j^{(d+1)})$. In the second equality, the second term corresponds to $\widetilde{J}=0$ and $j^{(d+1)}=1$, the third term corresponds to $\widetilde{J}\neq 0$ and $j^{(d+1)}=0$ and the fourth term corresponds to $\widetilde{J}\neq 0$ and $j^{(d+1)}=1$. $\square$

4.3 Number theoretic results in four wave models

The mains results of this appendix is to prove Theorem 4.3.1 and Theorem 4.3.2

Theorem 4.3.1. *Let $\Lambda(k) = \sqrt{1+|k|^2}$, then for all $\beta\in[1,2]^d$, the following number theory estimate is true*

$$\sup_{\substack{c,n\in\mathbb{Z}_L^d\\ T\leq L}} T\#\left\{\begin{matrix} k_1,k_2,k_3\in\mathbb{Z}_L^d \\ |k_1|,|k_2|,|k_3|\lesssim 1\end{matrix} : \begin{matrix} k_1-k_2+k_3=c \\ \Lambda(k_1)-\Lambda(k_2)+\Lambda(k_3)=\Lambda(c)+n+O(T^{-1})\end{matrix}\right\} \leq L^{2d}. \tag{4.3.1}$$

$$\sup_{\substack{c_1,c_2,n\in\mathbb{Z}_L^d\\ T\leq L}} T\#\left\{\begin{matrix} k_1,k_2\in\mathbb{Z}_L^d \\ |k_1|,|k_2|\lesssim 1\end{matrix} : \begin{matrix} k_1+k_2=c_1+c_2 \\ \Lambda(k_1)+\Lambda(k_2)=\Lambda(c_1)+\Lambda(c_2)+n+O(T^{-1})\end{matrix}\right\} \leq L^{2d}. \tag{4.3.2}$$

$$\sup_{\substack{c_1,c_2,n\in\mathbb{Z}_L^d\\ T\le L^2}} T\#\left\{\begin{matrix} k_1,k_2\in\mathbb{Z}_L^d \\ |k_1|,|k_2|\lesssim 1\end{matrix} : \begin{matrix} k_1-k_2=c_1-c_2,\ k_1\neq k_2 \\ \Lambda(k_1)-\Lambda(k_2)=\Lambda(c_1)-\Lambda(c_2)+n+O(T^{-1})\end{matrix}\right\} \le L^{2d+1}. \tag{4.3.3}$$

More generally, the proof of this theorem works for many other choices of $\Lambda(k)$.

Theorem 4.3.2. *Let $\Lambda(k)=\sqrt{1+|k|^2}$, then for sufficiently generic $\beta\in[1,2]^d$, the following number theory estimate is true for all $\theta\ll 1$*

$$\sup_{\substack{c,n\in\mathbb{Z}_L^d\\ T\le L^2}} T\#\left\{\begin{matrix} k_1,k_2,k_3\in\mathbb{Z}_L^d \\ |k_1|,|k_2|,|k_3|\le L^{\theta}\end{matrix} : \begin{matrix} k_1-k_2+k_3=c \\ \Lambda(k_1)-\Lambda(k_2)+\Lambda(k_3)=\Lambda(c)+n+O(T^{-1})\end{matrix}\right\} \le L^{2d+O(\theta)} \tag{4.3.4}$$

$$\sup_{\substack{c_1,c_2,n\in\mathbb{Z}_L^d\\ T\le L^2}} T\#\left\{\begin{matrix} k_1,k_2\in\mathbb{Z}_L^d \\ |k_1|,|k_2|\le L^{\theta}\end{matrix} : \begin{matrix} k_1\pm k_2=c_1\pm c_2,\ k_1\neq k_2 \\ \Lambda(k_1)\pm\Lambda(k_2)=\Lambda(c_1)\pm\Lambda(c_2)+n+O(T^{-1})\end{matrix}\right\} \le L^{2d+O(\theta)} \tag{4.3.5}$$

More generally, the proof of this theorem works for many other choices of $\Lambda(k)$.

Proof of Theorem 4.3.1. (4.3.1) and (4.3.2) are corollaries of the following lemma.

Lemma 4.3.3. *For any β*

$$\sup_{\substack{m,n\\ m\lesssim 1}} \#\left\{\begin{matrix} x,y\in\mathbb{Z}_L^d \\ |x|,|y|\lesssim 1\end{matrix} : \begin{matrix} x+y=m \\ \Lambda_\beta(x)+\Lambda_\beta(y)=n+O(L^{-1})\end{matrix}\right\} \lesssim_\beta L^{d-2}. \tag{4.3.6}$$

Proof. It is easy to show that

$$\begin{aligned}&\#\left\{\begin{matrix} x,y\in\mathbb{Z}_L^d \\ |x|,|y|\lesssim 1\end{matrix} : \begin{matrix} x+y=m \\ \Lambda_\beta(x)+\Lambda_\beta(y)=n+O(L^{-1})\end{matrix}\right\} \\ =&\#\left\{x\in\mathbb{Z}_L^d,\ |x|\lesssim 1:\Lambda_\beta(x)+\Lambda_\beta(m-x)=n+O(L^{-1})\right\}\end{aligned} \tag{4.3.7}$$

Define $\mathcal{D}_{m,n}=\left\{x\in\mathbb{Z}_L^d,\ |x|\lesssim 1:\Lambda_\beta(x)+\Lambda_\beta(m-x)=n+O(L^{-1})\right\}$, then we just need to show that

$$\sup_{\substack{m,n\\ m\lesssim 1}} \#\mathcal{D}_{m,n} \lesssim_\beta L^{d-1}. \tag{4.3.8}$$

We prove (4.3.8) using volume bound. The proof is divided into two steps.

Step 1. In this step, we show that

$$\#\mathcal{D}_{m,n} \le L^d \mathrm{vol}(\mathcal{D}^{\mathbb{R}}_{m,n}), \tag{4.3.9}$$

where $\mathcal{D}^{\mathbb{R}}_{m,n} = \big\{x \in \mathbb{R}^d,\ |x| \lesssim 1 : \Lambda_\beta(x) + \Lambda_\beta(m-x) = n + O(L^{-1})\big\}$.

(4.3.9) can be proved from the following claim.

Claim. If $x \in \mathcal{D}_{m,n}$, then $D_{1/(2L)}(x) \subseteq \mathcal{D}^{\mathbb{R}}_{m,n}$. Here $D_r(x) = \{x' \in \mathbb{R}^d : \sup_{i=1,\cdots,d} |x'_i - x_i| \le r\}$ (x_i are the components of x).

We prove the claim now. $x \in \mathcal{D}_{m,n}$ is equivalent to $\Lambda_\beta(x) + \Lambda_\beta(m-x) = n + O(L^{-1})$. For any $x' \in D_{1/(2L)}(x)$, $|x'-x| \lesssim 1/L$, because Λ is a Lifschitz function, we have $|\Lambda_\beta(x) - \Lambda_\beta(x')| \lesssim L^{-1}$. Therefore,

$$\begin{aligned} &|\Lambda_\beta(x') + \Lambda_\beta(m-x') - n| \\ \le &|\Lambda_\beta(x) + \Lambda_\beta(m-x) - n| + |\Lambda_\beta(x) - \Lambda_\beta(x')| + |\Lambda_\beta(m-x) - \Lambda_\beta(m-x')| \\ \lesssim &L^{-1}. \end{aligned} \tag{4.3.10}$$

Therefore, we have $\Lambda_\beta(x') + \Lambda_\beta(m-x') - n = O(L^{-1})$ and thus $x' \in \mathcal{D}^{\mathbb{R}}_{m,n}$. This is true for any $x' \in D_{1/(2L)}(x)$, so $D_{1/(2L)}(x) \subseteq \mathcal{D}^{\mathbb{R}}_{m,n}$.

Since for different $x_1, x_2 \in \mathcal{D}_{m,n}$, $D_{1/(2L)}(x_1) \cap D_{1/(2L)}(x_2) = \emptyset$, we have

$$\sum_{x\in\mathcal{D}_{m,n}} \mathrm{vol}(D_{1/(2L)}(x)) = \mathrm{vol}\left(\bigcup_{x\in\mathcal{D}_{m,n}} D_{1/(2L)}(x)\right) \le \mathrm{vol}(\mathcal{D}^{\mathbb{R}}_{m,n}). \tag{4.3.11}$$

The left hand side equals to $L^{-d}\#\mathcal{D}_{m,n}$, so we get

$$L^{-d}\#\mathcal{D}_{m,n} \le \mathrm{vol}(\mathcal{D}^{\mathbb{R}}_{m,n}), \tag{4.3.12}$$

which implies (4.3.9).

Step 2. In this step, we show that

$$\mathrm{vol}(\mathcal{D}^{\mathbb{R}}_{m,n}) \le L^{-1}. \tag{4.3.13}$$

Combining (4.3.9) and (4.3.13), we get (4.3.8), which proves Lemma 4.3.3.

By calculating the Hessian we can show that $F_{\beta,m,n} = \Lambda_\beta(x) + \Lambda_\beta(m-x) - n$ is a convex function. Let $x_{\beta,m}$ be its unique critical point. By Morse lemma, we know that there exists r_0 such that $F_{\beta,m,n} = |y|^2 - a$ after a change of variable.

In $\mathcal{D}^{\mathbb{R}}_{m,n} \backslash B_{r_0}(x_{\beta,m})$, $|\nabla F_{\beta,m,n}| \gtrsim 1$. Then by the coarea formula, we get

$$\begin{aligned}
\mathrm{vol}(\mathcal{D}^{\mathbb{R}}_{m,n} \backslash B_{r_0}(x_{\beta,m})) =& \mathrm{vol}(\{|F_{\beta,m,n}| \lesssim L^{-1}\} \backslash B_{r_0}(x_{\beta,m})) \\
=& \int_{-CL^{-1}}^{CL^{-1}} \int_{\{F_{\beta,m,n}=s\} \backslash B_{r_0}(x_{\beta,m})} \frac{1}{|\nabla F_{\beta,m,n}|} d\sigma ds \\
\lesssim& \int_{-CL^{-1}}^{CL^{-1}} \mathrm{Area}(\{F_{\beta,m,n} = s\} \backslash B_{r_0}(x_{\beta,m})) ds \\
\lesssim& L^{-1}.
\end{aligned} \tag{4.3.14}$$

In $\mathcal{D}^{\mathbb{R}}_{m,n} \cap B_{r_0}(x_{\beta,m})$, $F_{\beta,m,n} = |y|^2 - a$ after a change of variable $y = y(x)$ and $y(B_{r_0}(x_{\beta,m})) \subseteq B_{Cr_0}(0)$

$$\mathrm{vol}(\mathcal{D}^{\mathbb{R}}_{m,n} \cap B_{r_0}(x_{\beta,m})) \lesssim \mathrm{vol}(\{||y|^2 - a| \lesssim L^{-1}\} \cap B_{Cr_0}(0)) \lesssim L^{-1}. \tag{4.3.15}$$

Here the proof of the second inequality is elementary and is thus skipped.

Therefore, we complete the proof of Lemma 4.3.3 $\square$

We now return to the proof of Theorem 4.3.1. Define

$$\mathcal{D}^{(1)}_{m,n}([-T^{-1}, T^{-1}]) = \left\{ \begin{matrix} k_1, k_2, k_3 \in \mathbb{Z}^d_L \\ |k_1|, |k_2|, |k_3| \lesssim 1 \end{matrix} : \begin{matrix} k_1 - k_2 + k_3 = c \\ \Lambda(k_1) - \Lambda(k_2) + \Lambda(k_3) = \Lambda(c) + n + O(T^{-1}) \end{matrix} \right\} \tag{4.3.16}$$

$$\mathcal{D}^{(2)}_{m,n}([-T^{-1}, T^{-1}]) = \left\{ \begin{matrix} k_1, k_2 \in \mathbb{Z}^d_L \\ |k_1|, |k_2| \lesssim 1 \end{matrix} : \begin{matrix} k_1 + k_2 = c_1 + c_2 \\ \Lambda(k_1) + \Lambda(k_2) = \Lambda(c_1) + \Lambda(c_2) + n + O(T^{-1}) \end{matrix} \right\} \tag{4.3.17}$$

Let us first prove (4.3.2). For any $T \leq L$, let $N = [L/T] + 1$, then $\cup_{j=-N}^{N} [jL^{-1}, (j+1)L^{-1}]$ is a cover of $[-T^{-1}, T^{-1}]$. Therefore, $\cup_{j=-N}^{N} \mathcal{D}^{(2)}_{m,n}([jL^{-1}, (j+1)L^{-1}])$ is a cover of $\mathcal{D}^{(2)}_{m,n}([-T^{-1}, T^{-1}])$

By Lemma 4.3.3 we get

$$\#\mathcal{D}^{(2)}_{m,n}([-T^{-1}, T^{-1}]) \lesssim \sum_{j=-N}^{N} \#\mathcal{D}^{(2)}_{m,n}([jL^{-1}, (j+1)L^{-1}]) \lesssim NL^{d-1} \lesssim L^d T^{-1}. \tag{4.3.18}$$

This proves (4.3.2).

Now we prove (4.3.1).

$$\begin{aligned}
\#\mathcal{D}^{(1)}_{m,n}([-T^{-1},T^{-1}]) =& \# \left\{ \begin{matrix} k_1,k_2,k_3 \in \mathbb{Z}^d_L \\ |k_1|,|k_2|,|k_3| \lesssim 1 \end{matrix} : \begin{matrix} k_1 - k_2 + k_3 = c \\ \Lambda(k_1) - \Lambda(k_2) + \Lambda(k_3) = \Lambda(c) + n + O(T^{-1}) \end{matrix} \right\} \\
=& \sum_{|c_2|\lesssim 1} \# \left\{ \begin{matrix} k_1,k_3 \in \mathbb{Z}^d_L \\ |k_1|,|k_3| \lesssim 1 \end{matrix} : \begin{matrix} k_1 + k_3 = c + c_2 \\ \Lambda(k_1) + \Lambda(k_3) = \Lambda(c) + \Lambda(c_2) + n + O(T^{-1}) \end{matrix} \right\} \\
=& \sum_{|c_2|\lesssim 1} \mathcal{D}^{(2)}_{m,n}([-T^{-1},T^{-1}]) \lesssim \sum_{|c_2|\lesssim 1} L^d T^{-1} \\
=& L^{2d} T^{-1}.
\end{aligned} \tag{4.3.19}$$

This proves (4.3.1).

(4.3.3) is a simple corollary of the following lemma.

Lemma 4.3.4. *For any β*

$$\sup_{\substack{m,n \\ 0\neq m \lesssim 1}} \# \left\{ \begin{matrix} x,y \in \mathbb{Z}^d_L \\ |x|,|y| \lesssim 1 \end{matrix} : \begin{matrix} x - y = m \neq 0 \\ \Lambda_\beta(x) - \Lambda_\beta(y) = n + O(L^{-2}) \end{matrix} \right\} \lesssim_\beta L^{d-1}. \tag{4.3.20}$$

Proof. The proof of this lemma is similar to that of Lemma 4.3.3.

Since $m \in \mathbb{Z}^d_L$ and $m \neq 0$, we have $|m| \geq L^{-1}$. Therefore, we have

$$\begin{aligned}
&\left\{ x \in \mathbb{Z}^d_L,\ |x| \lesssim 1 : \Lambda_\beta(x) - \Lambda_\beta(x-m) - n = O(L^{-2}) \right\} \\
\subseteq &\left\{ x \in \mathbb{Z}^d_L,\ |x| \lesssim 1 : |m|^{-1}(\Lambda_\beta(x) - \Lambda_\beta(x-m) - n) = O(L^{-1}) \right\}
\end{aligned} \tag{4.3.21}$$

Define $\mathcal{D}_{m,n} = \left\{ x \in \mathbb{Z}^d_L,\ |x| \lesssim 1 : |m|^{-1}(\Lambda_\beta(x) - \Lambda_\beta(x-m) - n) = O(L^{-1}) \right\}$, then we just need to show that

$$\sup_{\substack{m,n \\ m\lesssim 1}} \#\mathcal{D}_{m,n} \lesssim_\beta L^{d-1}. \tag{4.3.22}$$

We prove (4.3.22) using volume bound. The proof is divided into two steps.

Step 1. In this step, we show that

$$\#\mathcal{D}_{m,n} \leq L^d \mathrm{vol}(\mathcal{D}^{\mathbb{R}}_{m,n}), \tag{4.3.23}$$

where $\{x \in \mathbb{R}^d,\ |x| \lesssim 1 : |m|^{-1}(\Lambda_\beta(x) - \Lambda_\beta(x-m) - n) = O(L^{-1})\}$.

Since the Lifschitz norm of the function $F_{\beta,m,n}(x) = |m|^{-1}(\Lambda_\beta(x) - \Lambda_\beta(x-m) - n)$ is bounded by a universal constant, the same argument of step 1 in the proof of Lemma 4.3.3 works and we do not repeat it.

Step 2. In this step, we show that

$$\mathrm{vol}(\mathcal{D}^{\mathbb{R}}_{m,n}) \leq L^{-1}. \tag{4.3.24}$$

Combining (4.3.23) and (4.3.24), we get (4.3.22), which proves Lemma 4.3.4.

By calculating the Hessian we can show that $\Lambda_\beta(x)$ is a convex function, so

$$|m|^{-1} |\nabla\Lambda_\beta(x) - \nabla\Lambda_\beta(x-m)| \gtrsim 1. \tag{4.3.25}$$

Above equation implies that $|\nabla F_{\beta,m,n}| \gtrsim 1$ By the coarea formula, we get

$$\begin{aligned}\mathrm{vol}(\mathcal{D}^{\mathbb{R}}_{m,n}) =& \mathrm{vol}(\{|F_{\beta,m,n}| \lesssim L^{-1}\}) \\ =& \int_{-CL^{-1}}^{CL^{-1}} \int_{\{F_{\beta,m,n}=s\}} \frac{1}{|\nabla F_{\beta,m,n}|} d\sigma ds \\ \lesssim& \int_{-CL^{-1}}^{CL^{-1}} \mathrm{Area}(\{F_{\beta,m,n} = s\}) ds \\ \lesssim& L^{-1}.\end{aligned} \tag{4.3.26}$$

Therefore, we complete the proof of Lemma 4.3.4

□

Finally we prove (4.3.3). Define

$$\mathcal{D}^{(3)}_{m,n}([-T^{-1}, T^{-1}]) = \left\{\begin{matrix} k_1, k_2 \in \mathbb{Z}^d_L \\ |k_1|, |k_2| \lesssim 1 \end{matrix} : \begin{matrix} k_1 - k_2 = c_1 - c_2,\ k_1 \neq k_2 \\ \Lambda(k_1) - \Lambda(k_2) = \Lambda(c_1) - \Lambda(c_2) + n + O(T^{-1}) \end{matrix}\right\} \tag{4.3.27}$$

Apply the same argument of getting (4.3.18),

$$\#\mathcal{D}^{(3)}_{m,n}([-T^{-1}, T^{-1}]) \lesssim \sum_{j=-N}^{N} \#\mathcal{D}^{(3)}_{m,n}([jL^{-1}, (j+1)L^{-1}]) \lesssim NL^{d-1} \lesssim L^d T^{-1}. \tag{4.3.28}$$

This proves (4.3.3) and we complete the proof of Theorem 4.3.1. □

Proof of Theorem 4.3.2. The $-$ case of (4.3.5) is equivalent to (4.3.3).

(4.3.4) and the + case of (4.3.5) are corollaries of the following lemma.

Lemma 4.3.5. *For any θ, for almost all β*

$$\sup_{\substack{m,n \\ m\lesssim 1}} \# \left\{ \begin{array}{ccc} x, y \in \mathbb{Z}_L^d & : & x+y=m \\ |x|, |y| \lesssim L^\theta & & \Lambda_\beta(x)+\Lambda_\beta(y)=n+O(L^{-2+\theta}) \end{array} \right\} \lesssim_{\theta,\beta} L^{d-2+\theta}. \tag{4.3.29}$$

Proof. Let

$$\#_{m,n,L,\beta} = \# \left\{ \begin{array}{ccc} x, y \in \mathbb{Z}_L^d & : & x+y=m \\ |x|, |y| \lesssim L^\theta & & \Lambda_\beta(x)+\Lambda_\beta(y)=n+O(L^{-2+\theta}) \end{array} \right\} \tag{4.3.30}$$

Then the lemma is equivalent to

$$\mathbb{P}_\beta \left(\sup_{\substack{m,n \\ m\lesssim 1}} \#_{m,n,L,\beta} \lesssim_{\theta,\beta} L^{d-2+\theta} \right) = 1. \tag{4.3.31}$$

which is also equivalent to

$$\mathbb{P}_\beta \left(\sup_L \sup_{\substack{m,n \\ m\lesssim 1}} L^{-(d-2+\theta)} \#_{m,n,L,\beta} < \infty \right) = 1. \tag{4.3.32}$$

The proof of (4.3.32) is divided into several steps.

Step 1. In this step, we show that in the proof of (4.3.32), we may replace $\sup_L$ and $\sup_{m,n}$ by supremum over discrete set $\mathcal{L}$ and $\mathcal{M}$, $\mathcal{N}$, i.e. $\sup_{L\in\mathcal{L}}$ and $\sup_{m\in\mathcal{M}, n\in\mathcal{N}}$.

Since $x, y \in \mathbb{Z}_L^d$, we may assume that $m \in \mathbb{Z}_L^d$ and we take $\mathcal{M} = \mathbb{Z}_L^d$.

Since if $|n-n'| \lesssim L^{-2}$, then $\Lambda_\beta(x)+\Lambda_\beta(y)=n+O(L^{-2+\theta}) \Rightarrow \Lambda_\beta(x)+\Lambda_\beta(y)=n'+O(L^{-2+\theta})$, we know that (4.3.32) is true for all $n \in \mathbb{Z}_{L^2}^d$ implies that it is true for all $n' \in \mathbb{R}^d$. We can thus take $\mathcal{N} = \mathbb{Z}_{L^2}^d$.

Let $K = [\log_2 L]$ and $l = L/2^K$, then $L = l \cdot 2^K$ and $l \in [1,2]$. Since if $|l-l'| \lesssim L^{-2}$, then $\Lambda_\beta(x)+\Lambda_\beta(y)=n+O(L^{-2+\theta}) \Rightarrow \Lambda_\beta(x)+\Lambda_\beta(y)=n'+O(L^{-2+\theta})$, we know that (4.3.32) is true for all $l \in [1,2]$ implies that it is true for all $l \in [1,2] \cap \mathbb{Z}_{L^2}$. We can thus take $\mathcal{L} = \{L = l\cdot 2^K :\ l \in [1,2] \cap \mathbb{Z}_{L^2},\ K \in \mathbb{N}_+\}$.

Therefore, to prove (4.3.32), it suffices to show that

$$\mathbb{P}_\beta \left(\sup_{L\in\mathcal{L}} \sup_{\substack{m\in\mathcal{M}, m\lesssim 1 \\ n\in\mathcal{N}}} L^{-(d-2+\theta)} \#_{m,n,L,\beta} < \infty \right) = 1. \tag{4.3.33}$$

In what follows, we show that

$$\mathbb{P}_\beta\left(\sup_{L\in\mathcal{L}} L^{-(d-2+\theta)}\#_{m,n,L,\beta} < \infty\right) = 1. \tag{4.3.34}$$

(4.3.34) implies that $\sup_{L\in\mathcal{L}} L^{-(d-2+\theta)}\#_{m,n,L,\beta} = \infty$ is a null set. Since a countable union of null sets is still a null set, (4.3.33) is a corollary of (4.3.34).

Step 2. In this step, we derive (4.3.34) from the following expectation bound.

$$\mathbb{E}(\#_{m,n,L,\beta}) \lesssim L^{d-2+\theta/2}. \tag{4.3.35}$$

(4.3.35) implies that

$$\mathbb{E}(\#_{m,n,l\cdot 2^K,\beta}) \lesssim 2^{(d-2)K+\theta K/2}. \tag{4.3.36}$$

Therefore, we have

$$\mathbb{E}\left(\sum_K 2^{-(d-2+\theta)K}\#_{m,n,l\cdot 2^K,\beta}\right) \lesssim \sum_K 2^{-\theta K/2} \le C_\theta < \infty. \tag{4.3.37}$$

This implies that almost surely

$$\sum_K 2^{-(d-2+\theta)K}\#_{m,n,l\cdot 2^K,\beta} < \infty. \tag{4.3.38}$$

Let $C_{\beta,\theta} = \sum_K 2^{-(d-2+\theta)K}\#_{m,n,l\cdot 2^K,\beta}$ which is finite almost surely. We get

$$\#_{m,n,l\cdot 2^K,\beta} \le C_{\beta,\theta}2^{(d-2+\theta)K}, \tag{4.3.39}$$

which is equivalent to (4.3.34).

Step 3. In this step, we prove the expectation bound (4.3.35).

Let's calculate the expectation.

$$
\begin{aligned}
\mathbb{E}(\#_{m,n,L,\beta}) =& \mathbb{E}\left(\sum_{\substack{x,y\in\mathbb{Z}_L^d, x+y=m \\ |x|,|y|\lesssim L^\theta}} \chi_{\Lambda_\beta(x)+\Lambda_\beta(y)=n+O(L^{-2})} \right) \\
=& \sum_{\substack{x,y\in\mathbb{Z}_L^d, x+y=m \\ |x|,|y|\lesssim L^\theta}} \mathbb{P}_\beta(|\Lambda_\beta(x)+\Lambda_\beta(y)-n| \lesssim L^{-2}) \\
=& \sum_{\substack{x,y\in\mathbb{Z}_L^d, x+y=m \\ |x|,|y|\lesssim L^\theta}} \mathbb{P}_\beta(|F_{x,y}(\beta)| \lesssim L^{-2})
\end{aligned}
\tag{4.3.40}
$$

Here we define $F_{x,y}(\beta) = \Lambda_\beta(x) + \Lambda_\beta(y) - n$.

We estimate $\mathbb{P}_\beta(|F_{x,y}(\beta)| \lesssim L^{-2})$ using coarea formula.

$$
\begin{aligned}
\mathbb{P}_\beta(|F_{x,y}(\beta)| \lesssim L^{-2}) =& \mathrm{Area}(|F_{x,y}(\beta)| \lesssim L^{-2}) \\
=& \int_{-CL^{-2}}^{CL^{-2}} \int_{\{F_{x,y}=s\}} \frac{1}{|\nabla F_{x,y}|} d\sigma ds \\
\lesssim& L^{-2} \sup |\nabla F_{x,y}|^{-1} \mathrm{Area}(F_{x,y}=s) \\
\lesssim& L^{-2} \sup |\nabla F_{x,y}|^{-1}
\end{aligned}
\tag{4.3.41}
$$

We thus need to get lower bound of $|\nabla F_{x,y}|$.

A simple calculation gives

$$
\nabla_\beta F_{x,y}(\beta) = \frac{1}{2\Lambda_\beta(x)} \begin{bmatrix} x_1^2 \\ \vdots \\ x_d^2 \end{bmatrix} + \frac{1}{2\Lambda_\beta(y)} \begin{bmatrix} y_1^2 \\ \vdots \\ y_d^2 \end{bmatrix}, \tag{4.3.42}
$$

which implies that

$$
\begin{aligned}
|\nabla_\beta F_{x,y}(\beta)|^2 =& \frac{1}{4\Lambda_\beta^2(x)}(x_1^4 + \cdots x_d^4) + \frac{1}{4\Lambda_\beta^2(y)}(y_1^4 + \cdots y_d^4) + \frac{1}{2\Lambda_\beta(x)\Lambda_\beta(y)}(x_1^2y_1^2 + \cdots x_d^2y_d^2) \\
\gtrsim& L^{-}(|x|^2+|y|^2)^2.
\end{aligned}
\tag{4.3.43}
$$

Therefore, we have

$$
|\nabla_\beta F_{x,y}(\beta)| \gtrsim L^{-}(|x|^2+|y|^2). \tag{4.3.44}
$$

By (4.3.41), we get for $(x, y) \neq (0,0)$

$$\mathbb{P}_\beta(|F_{x,y}(\beta)| \lesssim L^{-2}) \lesssim L^{-2+}(|x|^2 + |y|^2)^{-1}. \tag{4.3.45}$$

and for $x = y = 0$

$$\mathbb{P}_\beta(|F_{0,0}(\beta)| \leq 1. \tag{4.3.46}$$

By (4.3.40),

$$\begin{aligned}\mathbb{E}(\#_{m,n,L,\beta}) \lesssim& 1 + L^{-2+} \sum_{\substack{x,y\in\mathbb{Z}_L^d, x+y=m \\ |x|,|y|\lesssim L^\theta}} (|x|^2 + |y|^2)^{-1} \\ \lesssim& L^{d-2+\theta/2}.\end{aligned} \tag{4.3.47}$$

Therefore, we complete the proof of the expectation bound (4.3.35) and thus the proof of Lemma 4.3.5 □

The derivation of (4.3.4) and the + case of (4.3.5) from Lemma 4.3.5 can be done by applying a similar argument of (4.3.18) and (4.3.19). □

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