

# Boundary Value Problems for Partial Differential Equations

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## 1 Introduction and preliminaries

### 1.1 Definitions of partial differential equations

**Definition 1.1** (Notations of partial derivatives). *For  $f(x)$  with one variable  $x$ , we know  $f'(x) = \frac{df}{dx}$ . For  $u(x, y)$ , we introduce partial derivatives as*

$$\frac{\partial u}{\partial x} = \left. \frac{du}{dx} \right|_{y \text{ is fixed}} = \partial_x u = u_x. \quad (1.1)$$

*Similarly,*

$$\frac{\partial^2 u}{\partial x^2} = \partial_x^2 u = \partial_{xx} u = u_{xx} \quad (1.2)$$

*Example 1.1.* For  $u(x, y) = xy^2$ , we have

$$\begin{aligned} \frac{\partial u}{\partial x} &= \partial_x u = u_x = y^2, & \frac{\partial u}{\partial y} &= \partial_y u = u_y = 2xy, \\ \frac{\partial^2 u}{\partial x^2} &= \partial_x^2 u = \partial_{xx} u = u_{xx} = 0, & \frac{\partial^2 u}{\partial y^2} &= \partial_y^2 u = \partial_{yy} u = u_{yy} = 2x \end{aligned}$$

**Definition 1.2** (Definition of general PDEs). *Given a function  $u = u(x, y)$  of two variables, (similarly  $u = u(x_1, \dots, x_n)$  of  $n$  variables) and an expression  $F(u_{xx}, u_{xy}, u_{yy}, u_x, u_y, u, x, y)$  of partial derivatives of  $u$ , the following equation is a partial differential equation, abbreviated as PDE.*

$$F(u_{xx}, u_{xy}, u_{yy}, u_x, u_y, u, x, y) = 0 \quad (1.3)$$

*In the future, we may also use the notation  $F[u]$  to represent  $F(u_{xx}, u_{xy}, u_{yy}, u_x, u_y, u, x, y)$ . And (1.3) can be rewritten as*

$$F[u] = 0. \quad (1.4)$$

*Remark 1.3.*  $F[u]$  may also involve derivatives of order  $\geq 2$ , but we do not discuss it in this course.

*Example 1.2* (Examples of PDEs). Here are some examples of PDEs.

$$\begin{aligned}
u_{xx} - u_y &= 0 && \text{(the heat equation)} \\
u_{xx} - u_{yy} &= 0 && \text{(the wave equation)} \\
u_{xx} + u_{yy} &= 0 && \text{(Laplace's equation)} \\
u_x + u_y &= 0 && \text{(the transport equation)} \\
u_x + uu_y &= 0 && \text{(the Burgers equation)}
\end{aligned} \tag{1.5}$$

**Definition 1.4** (Order of PDEs). The order of a PDE is the order of the highest-order derivative in the equation. In (1.5), the first three PDEs are second order, and the last two are first order.

**Definition 1.5** (Linear PDEs). Given a PDE  $F[u] = 0$ , if it satisfies

$$F[u + v] = F[u] + F[v] \text{ and } F[cu] = cF[u], \tag{1.6}$$

then we say that  $F[u] = 0$  is a linear PDE. In (1.5), the first four PDEs are linear, while the last one is not.

We have the following proposition which characterizes all second order linear PDEs,

**Proposition 1.6.** The second-order linear PDEs can always be written as

$$a(x, y)u_{xx} + b(x, y)u_{xy} + c(x, y)u_{yy} + d(x, y)u_x + e(x, y)u_y + f(x, y)u = g(x, y), \tag{1.7}$$

We assume  $a^2 + b^2 + c^2 \neq 0$  for any  $x, y$  (at least one of  $a, b, c$  is nonzero).

**Definition 1.7.** We call  $a, b, c, d, e, f$  coefficients and  $g$  source term.

## 1.2 Classification of second-order PDEs

In this course, we will mainly consider second-order linear PDEs.

$$a(x, y)u_{xx} + b(x, y)u_{xy} + c(x, y)u_{yy} + d(x, y)u_x + e(x, y)u_y + f(x, y)u = g(x, y), \tag{1.8}$$

These equations are classified as follows by the coefficients  $a, b, c$ .

**Definition 1.8** (Classification of PDEs). *The second-order linear PDEs (1.8) are classified as elliptic, parabolic and hyperbolic by the following,*

$$\begin{cases} 4ac - b^2 > 0 & \text{elliptic} \\ 4ac - b^2 = 0 & \text{parabolic} \\ 4ac - b^2 < 0 & \text{hyperbolic} \end{cases} \quad (1.9)$$

where we note that  $a, b, c$  are functions of  $x, y$  and the inequalities in (1.9) is required to be true for any  $x, y$ .

### 1.3 Review of ODEs

**Definition 1.9** (Separable ODEs). *The following ODE is the separable ODE*

$$y' + p(x)y = 0. \quad (1.10)$$

**Theorem 1.10.** *Separable ODE can be solved in the following way*

$$y' + p(x)y = 0 \Rightarrow \frac{dy}{dx} + p(x)y = 0 \Rightarrow \frac{dy}{y} = -p(x)dx \Rightarrow \int \frac{dy}{y} = -\int p(x)dx$$

and the solution is

$$y(x) = Ce^{-\int p(x)dx} \quad (1.11)$$

*Example 1.3.* Let us solve the following ODEs

$$y' = -6xy$$

Apply the procedure in Theorem 1.10

$$y' = -6xy \Rightarrow \frac{dy}{y} = -6xdx \Rightarrow \int \frac{dy}{y} = -\int 6xdx \Rightarrow \ln|y| = -3x^2 + C'$$

Therefore, the solution is

$$y = Ce^{-3x^2}.$$

where  $C(= \pm e^{C'})$  is an arbitrary constant.

**Definition 1.11** (Linear ODEs). *The following ODE is the linear ODE*

$$y' + p(x)y = q(x). \quad (1.12)$$

**Theorem 1.12.** *Linear ODE can be solved by the following procedure*

1. Solve the corresponding separable equation  $y' - p(x)y = 0$  to obtain a solution  $\hat{y} = e^{\int p(x)dx}$ .

2. Multiply the linear ODE by  $\hat{y}$  and rewrite the ODE

$$y' + p(x)y = q(x) \quad \Rightarrow \quad \hat{y}(y' + p(x)y) = \hat{y}q(x) \quad \Rightarrow \quad (\hat{y}y)' = \hat{y}q(x)$$

3. Integrate the above equation

$$\begin{aligned} \hat{y}y &= \int \hat{y}q(x)dx + C \quad \Rightarrow \quad y = \frac{1}{\hat{y}} \left( \int \hat{y}q(x)dx + C \right) \\ &\Rightarrow \quad y = e^{-\int p(x)dx} \left( \int q(x)e^{\int p(x)dx}dx + C \right) \end{aligned}$$

*Remark 1.13.* In (2), we applied the following equation

$$(\hat{y}y)' = \hat{y}(y' + p(x)y)$$

which is a corollary of the Leibniz rule.

$$(\hat{y}y)' = \hat{y}'y + \hat{y}y' = \hat{y}y' + p(x)\hat{y}y = \hat{y}(y' + p(x)y)$$

*Example 1.4.*  $(x^2 + 1)y' + 3xy = 6x, y(0) = 3$  is solved as  $y(x) = 2 + (x^2 + 1)^{-3/2}$  by the procedure in Theorem 1.12.

1. Divide both sides by  $(x^2 + 1)$ .

$$(x^2 + 1)y' + 3xy = 6x \quad \Rightarrow \quad y' + \frac{3x}{x^2 + 1}y = \frac{6x}{x^2 + 1}$$

2. Solve the corresponding separable equation  $y' - \frac{3x}{x^2+1}y = 0$  to obtain a solution  $(x^2 + 1)^{\frac{3}{2}}$ .

3. Multiply the linear ODE by  $(x^2 + 1)^{\frac{3}{2}}$  and rewrite the ODE

$$y' + \frac{3x}{x^2 + 1}y = \frac{6x}{x^2 + 1} \quad \Rightarrow \quad ((x^2 + 1)^{\frac{3}{2}}y)' = 6x(x^2 + 1)^{\frac{1}{2}}$$

4. Integrate the above equation

$$\begin{aligned} ((x^2 + 1)^{\frac{3}{2}}y)' &= 6x(x^2 + 1)^{\frac{1}{2}} \quad \Rightarrow \quad y = (x^2 + 1)^{-\frac{3}{2}} \left( \int 6x(x^2 + 1)^{\frac{1}{2}} + C \right) \\ &\Rightarrow \quad y = 2 + C(x^2 + 1)^{-\frac{3}{2}} \end{aligned}$$

**Definition 1.14** (Second order ODEs). *The constant coefficient second order ODEs are the following equations*

$$ay'' + by' + cy = 0 \quad (a \neq 0). \quad (1.13)$$

**Theorem 1.15.** *Constant coefficient second order ODEs can be solved by the following procedure*

1. Solve the characteristic equation  $a\lambda^2 + b\lambda + c = 0$  to get two solutions  $\lambda_1$  and  $\lambda_2$ .

2. If  $\lambda_1 \neq \lambda_2$ , the general solution is

$$y(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} \quad (1.14)$$

3. If  $\lambda_1 = \lambda_2 = \lambda$ , the general solution is

$$y(x) = (C_1 + C_2 x) e^{\lambda x} \quad (1.15)$$

4. If  $\lambda_1, \lambda_2$  are complex roots  $\alpha \pm i\beta$ , apply the Euler's formula to rewrite (1.14)

$$y(x) = C_1 e^{(\alpha+i\beta)x} + C_2 e^{(\alpha-i\beta)x} = e^{\alpha x} (c_1 \cos \beta x + c_2 \sin \beta x) \quad (1.16)$$

where  $c_1 = C_1 + C_2$ ,  $c_2 = i(C_1 - C_2)$ .

*Example 1.5.*  $y'' - 3y' + 2y = 0$  is solved as  $y(x) = C_1 e^x + C_2 e^{2x}$  by the procedure in Theorem 1.15.

1. Solve the characteristic equation  $\lambda^2 - 3\lambda + 2 = 0$  to get two solutions  $\lambda_1 = 1$  and  $\lambda_2 = 2$ .

2. Since  $\lambda_1 \neq \lambda_2$ , by (1.14), the general solution is

$$y(x) = C_1 e^x + C_2 e^{2x} \quad (1.17)$$

*Example 1.6.*  $y'' + y = 0$  is solved as  $y(x) = C_1 \cos(x) + C_2 \sin(x)$  by the procedure in Theorem 1.15.

1. Solve the characteristic equation  $\lambda^2 + 1 = 0$  to get two solutions  $\lambda_1 = i$  and  $\lambda_2 = -i$ .

2. Since  $\lambda_1, \lambda_2$  are complex, by (1.16), the general solution is

$$y(x) = C_1 \cos x + C_2 \sin x \quad (1.18)$$

*Example 1.7.*  $y'' + 2y' + y = 0$  is solved as  $y(x) = (C_1 + C_2 x) e^{-x}$  by the procedure in Theorem 1.15.

1. Solve the characteristic equation  $\lambda^2 + 2\lambda + 1 = 0$  to get  $\lambda_1 = \lambda_2 = -1$ .

2. Since  $\lambda_1, \lambda_2$  are equal, by (1.15), the general solution is

$$y(x) = (C_1 + C_2 x) e^{-x} \quad (1.19)$$

## 1.4 Boundary value problem

TODO:

## 1.5 Separation of variables

Many linear PDEs can be reduced to linear ODEs with the method of separation of variables, described below.

We take the Laplace's equation

$$u_{xx} + u_{yy} = 0 \tag{1.20}$$

with boundary condition

$$u(0, y) = 0, \quad u(L, y) = 0, \quad u(x, 0) = 0, \quad u(x, L) = \varphi(x). \tag{1.21}$$

as an example.

We are looking for a separated solution. Substitute into (1.20), then we get

$$X''Y + XY'' = 0 \quad \Rightarrow \quad \frac{X''}{X} = -\frac{Y''}{Y}$$

The following lemma implies that  $X''/X$  and  $Y''/Y$  are constants.

**Lemma 1.16.**  $f(x) = g(y)$  implies that  $f(x) = g(y) = \text{const}$ ,

*Proof.*  $f(x) = g(y) \Rightarrow f'(x) = \partial_x(g(y)) = 0 \Rightarrow f(x) = \text{const}$ . □

Let  $\lambda$  be a constant and we write

$$X'' + \lambda X = 0, \quad Y'' - \lambda Y = 0.$$

We call  $\lambda$  the separation constant. At this moment  $\lambda$  is arbitrary. Thus the PDE was reduced to two ODEs.

### 1.5.1 Solving separated solutions

If  $\lambda = 0$ , then two ODEs have the following linearly independent solutions.

$$X = 1, x, \quad Y = 1, y. \tag{1.22}$$

If  $\lambda \neq 0$ , then two ODEs have the following linearly independent solutions.

$$X = e^{\sqrt{-\lambda}x}, e^{-\sqrt{-\lambda}x}, \quad Y = e^{\sqrt{\lambda}y}, e^{-\sqrt{\lambda}y}. \tag{1.23}$$

In either case, the solution is given by superpositions:

$$u = \begin{cases} (A_1x + A_2)(B_1y + B_2), & \lambda = 0 \\ (A_1e^{\sqrt{-\lambda}x} + A_2e^{-\sqrt{-\lambda}x})(B_1e^{\sqrt{\lambda}y} + B_2e^{-\sqrt{\lambda}y}), & \lambda \neq 0 \end{cases} \quad (1.24)$$

where  $A_1, A_2, B_1, B_2$  are constants.

For  $\lambda > 0$ , by writing  $\lambda = k^2 (k > 0)$  we have

$$u(x, y) = (A_1e^{ikx} + A_2e^{-ikx})(B_1e^{ky} + B_2e^{-ky}), \quad (1.25)$$

and for  $\lambda < 0$ , by writing  $\lambda = -l^2 (l > 0)$  we have

$$u(x, y) = (A_1e^{lx} + A_2e^{-lx})(B_1e^{ily} + B_2e^{-ily}). \quad (1.26)$$

Therefore, we get

$$u = \begin{cases} (A_1x + A_2)(B_1y + B_2), \\ (A_1e^{ikx} + A_2e^{-ikx})(B_1e^{ky} + B_2e^{-ky}), \\ (A_1e^{lx} + A_2e^{-lx})(B_1e^{ily} + B_2e^{-ily}). \end{cases} \quad (1.27)$$

Instead of (1.23), we can also choose

$$X = \cos(\sqrt{\lambda}x), \sin(\sqrt{\lambda}x), \quad Y = \cosh(\sqrt{\lambda}y), \sinh(\sqrt{\lambda}y). \quad (1.28)$$

In this case, we have

$$u(x, y) = (A_1 \cos(kx) + A_2 \sin(kx))(B_1 \cosh(ky) + B_2 \sinh(ky)) \quad (1.29)$$

$$u(x, y) = (A_1 \cosh(lx) + A_2 \sinh(lx))(B_1 \cos(ly) + B_2 \sin(ly)) \quad (1.30)$$

Note that (1.29) becomes (1.25) and (1.30) becomes (1.26) by redefining the coefficients. We call solutions such as (1.24) through (1.30) separated solutions because they are given in the form  $u(x, y) = X(x)Y(y)$ .

The final result is

$$u = \begin{cases} (A_1x + A_2)(B_1y + B_2), \\ (A_1 \cos(kx) + A_2 \sin(kx))(B_1 \cosh(ky) + B_2 \sinh(ky)), \\ (A_1 \cosh(lx) + A_2 \sinh(lx))(B_1 \cos(ly) + B_2 \sin(ly)). \end{cases} \quad (1.31)$$

### 1.5.2 Solving the boundary value problem

The separation constant  $\lambda$  and coefficients  $A_1, A_2, B_1, B_2$  are partially determined by boundary conditions that in the region  $0 < x < L, 0 < y < \infty$ ,  $u(x, y)$  satisfies that

$$u(0, y) = 0, \quad u(L, y) = 0, \quad u(x, 0) = 0, \quad u(x, L) = \varphi(x). \quad (1.32)$$

Let us only consider the first three conditions.

$$u(0, y) = 0, \quad u(L, y) = 0, \quad u(x, 0) = 0. \quad (1.33)$$

In (1.31), we have three cases.

1.  $u = (A_1 x + A_2)(B_1 y + B_2)$ . In this case,  $u$  satisfies boundary conditions  $u(0, y) = u(L, y) = 0$  in (1.33) if and only if  $u = 0$ .
2.  $u = (A_1 \cos(kx) + A_2 \sin(kx))(B_1 \cosh(ky) + B_2 \sinh(ky))$ . In this case,  $u$  satisfies boundary conditions  $u(0, y) = u(L, y) = 0$  only when  $A_1 = A_2 = 0$ . That is, only the solution  $u = 0$  satisfies the boundary conditions.
3.  $u = (A_1 \cosh(lx) + A_2 \sinh(lx))(B_1 \cos(ly) + B_2 \sin(ly))$ . In this case,  $u$  satisfies boundary conditions  $u(0, y) = u(L, y) = 0$  when  $A_1 = 0$  and  $k = n\pi/L$ , where  $n$  is an integer. Furthermore we find  $B_1 = 0$  by the condition  $u(x, 0) = 0$ . That is, the solution  $A_2 B_2 \sin(kx) \sinh(ky)$  with  $k = n\pi/L (n = 0, \pm 1, \pm 2, \dots)$  satisfies the boundary conditions.

Therefore we obtain the following separated solutions of Laplace's equation satisfying the boundary conditions.

$$u(x, y) = A \sin \frac{n\pi x}{L} \sinh \frac{n\pi y}{L}, \quad n = 1, 2, \dots, \quad (1.34)$$

### 1.5.3 Matching with the last boundary condition

Now we consider the last boundary condition  $u(x, L) = \varphi(x)$  in (1.33). For arbitrary  $\varphi(x)$ , our separated solution  $A \sin \frac{n\pi x}{L} \sinh \frac{n\pi y}{L}$  cannot match with this boundary condition. However, we can use a linear combination of this separated solution to generate more solutions,

$$u(x, y) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L} \sinh \frac{n\pi y}{L}. \quad (1.35)$$

To match with  $u(x, L) = \varphi(x)$ , we take  $y = L$  in (1.35),

$$\varphi(x) = u(x, L) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L} \sinh \frac{n\pi L}{L}. \quad (1.36)$$



Therefore, we get

$$\varphi(x) = \sum_{n=1}^{\infty} A_n \sinh n\pi \sin \frac{n\pi x}{L}. \quad (1.37)$$

Then  $A_n$  can be solved from the sine Fourier coefficients introduced in section 2.2.3.

The result is

$$A_n = \frac{2}{L \sinh n\pi} \int_0^L \varphi(x) \sin \frac{n\pi x}{L} dx \quad (1.38)$$

Substitute into (1.35), then we solve Laplace's equation (1.20) with boundary condition.

## 2 Introduction to Fourier series

Every function  $f(x)$  on domain  $-L < x < L$  can be represented by a series of the form

$$A_0 + \sum_{n=1}^{\infty} \left( A_n \cos \frac{n\pi x}{L} + B_n \sin \frac{n\pi x}{L} \right)$$

This is called the Fourier series. We will first discuss how to compute the coefficients  $A_0, A_1, B_1, \dots$  in terms of  $f(x)$ , then we discuss several properties and variants of the Fourier series.

### 2.1 Definition of Fourier series

The Fourier series is an infinite sum of trigonometric functions defined as the following

**Definition 2.1** (Fourier series). *Let  $A_0, A_1, B_1, \dots$  be constants. The series below is called a Fourier series.*

$$A_0 + \sum_{n=1}^{\infty} \left( A_n \cos \frac{n\pi x}{L} + B_n \sin \frac{n\pi x}{L} \right) \quad (2.1)$$

It turns out that every function  $f(x)$  can be written into an infinite sum of trigonometric functions. As suggested by the following theorem,

**Theorem 2.2.** *Every function  $f(x)$  on domain  $-L < x < L$  can be represented by a Fourier series. In other word, for any  $f(x)$  on domain  $-L < x < L$ , there exists coefficients  $A_0, A_1, B_1, \dots$  such that*

$$f(x) = A_0 + \sum_{n=1}^{\infty} \left( A_n \cos \frac{n\pi x}{L} + B_n \sin \frac{n\pi x}{L} \right). \quad (2.2)$$

## 2.2 Fourier series and orthogonality

### 2.2.1 Orthogonality of trigonometric function

We want to compute the coefficients  $A_0, A_1, B_1, \dots$  in terms of  $f(x)$ . The following observation may be helpful.

**Question.** Given several orthogonal vector  $e_1, e_2, \dots, e_n$ , assume that we have a decomposition

$$v = v_1 e_1 + v_2 e_2 + \dots + v_n e_n \quad (2.3)$$

How do we compute the coefficients  $v_1, v_2, \dots, v_n$  in terms of  $v$ ?

**Answer.** Assume that we want to solve the coefficient  $v_j$ , then we take inner product of (2.3) with  $e_j$ . If we do this, by orthogonality  $\langle e_i, e_j \rangle = 0$  (if  $i \neq j$ ) and  $\langle e_j, e_j \rangle = |e_j|^2$ , then we get

$$\begin{aligned} \langle v, e_j \rangle &= \langle v_1 e_1 + v_2 e_2 + \dots + v_n e_n, e_j \rangle \\ &= v_j |e_j|^2. \end{aligned} \quad (2.4)$$

Therefore, we solve the coefficient  $v_j = \frac{\langle v, e_j \rangle}{|e_j|^2}$ .

The following notation is useful.

**Definition 2.3** (Kronecker delta). *The Kronecker delta  $\delta_{mn}$  is defined as*

$$\delta_{mn} = \begin{cases} 1, & m = n \\ 0, & m \neq n \end{cases} \quad (2.5)$$

Using this notation orthogonality of  $e_1, e_2, \dots, e_n$  is equivalent to  $\langle e_m, e_n \rangle = |e_m|^2 \delta_{mn}$ .

The following theorem implies that sine and cosine functions are similar to orthogonal vectors.

**Theorem 2.4** (Orthogonality of trigonometric functions). *Let  $n, m \geq 0$  be integers. We assume  $L > 0$ . The following orthogonality relations hold.*

$$\begin{aligned} \int_{-L}^L \cos \frac{n\pi x}{L} \cos \frac{m\pi x}{L} dx &= \begin{cases} 2L & (n = m = 0), \\ L\delta_{nm} & (\text{otherwise}), \end{cases} \\ \int_{-L}^L \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} dx &= \begin{cases} 0 & (n = m = 0), \\ L\delta_{nm} & (\text{otherwise}), \end{cases} \\ \int_{-L}^L \sin \frac{n\pi x}{L} \cos \frac{m\pi x}{L} dx &= 0 \quad (\text{all } n, m). \end{aligned} \quad (2.6)$$

*Proof.* We only prove the orthogonality for the first equation which only involves cosine. The proof of other equations is left as homework.

Let us first start with the following observation.

*Claim.* If  $n \neq 0$ , then we have

$$\int_{-L}^L \cos \frac{n\pi x}{L} dx = 0 \quad (2.7)$$

This claim is true because

$$\begin{aligned} \int_{-L}^L \cos \frac{n\pi x}{L} dx &= \left[ \frac{L}{n\pi} \sin \frac{n\pi x}{L} \right]_{-L}^L \\ &= \frac{L}{n\pi} (\sin n\pi - \sin(-n\pi)) = 0 \end{aligned} \quad (2.8)$$

Now we start the proof of (2.6) for cosine.

**Case 1.** ( $n = m = 0$ ) In this case, we have

$$\int_{-L}^L \cos \frac{n\pi x}{L} \cos \frac{m\pi x}{L} dx = \int_{-L}^L 1 \cdot 1 dx = 2L$$

**Case 2.** (At least one of  $n, m$  is nonzero). We have

$$\int_{-L}^L \cos \frac{n\pi x}{L} \cos \frac{m\pi x}{L} dx = \frac{1}{2} \int_{-L}^L \left( \cos \frac{(n-m)\pi x}{L} + \cos \frac{(n+m)\pi x}{L} \right) dx$$

Since at least one of  $n, m$  is nonzero,  $n + m > 0$ . The integral over  $\cos \frac{(n+m)\pi x}{L}$  is 0 by (2.7).

Therefore,

$$\int_{-L}^L \cos \frac{n\pi x}{L} \cos \frac{m\pi x}{L} dx = \frac{1}{2} \int_{-L}^L \cos \frac{(n-m)\pi x}{L} dx$$

**Case 2.1.** ( $n \neq m$ ) In this case,  $\frac{1}{2} \int_{-L}^L \cos \frac{(n-m)\pi x}{L} dx = 0$  by (2.7). Therefore, we get

$$\int_{-L}^L \cos \frac{n\pi x}{L} \cos \frac{m\pi x}{L} dx = 0 = \delta_{mn}.$$

**Case 2.2.** ( $n = m$ ), then

$$\int_{-L}^L \cos \frac{n\pi x}{L} \cos \frac{m\pi x}{L} dx = \frac{1}{2} \int_{-L}^L \cos \frac{(n-m)\pi x}{L} dx = \frac{1}{2} \int_{-L}^L \cos \frac{0 \cdot \pi x}{L} dx = L.$$

Thus the orthogonality relations for  $\int_{-L}^L \cos \frac{n\pi x}{L} \cos \frac{m\pi x}{L} dx$  is proved. The orthogonality relations for  $\int_{-L}^L \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} dx$  and  $\int_{-L}^L \sin \frac{n\pi x}{L} \cos \frac{m\pi x}{L} dx$  are similarly proved and left as homework.  $\square$

### 2.2.2 Formula of Fourier coefficients

We can use the orthogonality relations (2.6) to compute  $A_0, A_1, B_1, \dots$

To determine  $A_0$  in (2.2), we multiply  $\cos \frac{0 \cdot \pi x}{L} = 1$  on both sides and integrate with respect to  $x$ :

$$\int_{-L}^L f(x) dx = \int_{-L}^L A_0 dx + \sum_{n=1}^{\infty} \int_{-L}^L \left( A_n \cos \frac{n\pi x}{L} + B_n \sin \frac{n\pi x}{L} \right) dx = \int_{-L}^L A_0 dx = 2A_0.$$

where we use the fact that  $\int_{-L}^L \cos \frac{n\pi x}{L} dx = \int_{-L}^L \sin \frac{n\pi x}{L} dx = 0$ . This is just the first equation in (2.6) with  $m = 0$ .

Therefore, we get an expression of  $A_0$ ,

$$A_0 = \frac{1}{2L} \int_{-L}^L f(x) dx.$$

To determine  $A_m$  ( $m = 1, 2, \dots$ ) in (2.2), we multiply  $\cos \frac{m\pi x}{L}$  on both sides and integrate with respect to  $x$ :

$$\begin{aligned} \int_{-L}^L f(x) \cos \frac{m\pi x}{L} dx &= \int_{-L}^L A_0 \cos \frac{m\pi x}{L} dx + \sum_{n=1}^{\infty} \int_{-L}^L \left( A_n \cos \frac{n\pi x}{L} + B_n \sin \frac{n\pi x}{L} \right) \cos \frac{m\pi x}{L} dx \\ &= \sum_{n=1}^{\infty} A_n \int_{-L}^L \cos \frac{n\pi x}{L} \cos \frac{m\pi x}{L} dx = \sum_{n=1}^{\infty} A_n L \delta_{nm} = LA_m. \end{aligned}$$

Therefore, we get an expression of  $A_m$ ,

$$A_m = \int_{-L}^L f(x) \cos \frac{m\pi x}{L} dx.$$

Similarly we can determine  $B_m$  ( $m = 1, 2, \dots$ ) in (2.2) by multiplying  $\sin \frac{m\pi x}{L}$  on both sides and integrate with respect to  $x$ :

$$\begin{aligned} \int_{-L}^L f(x) \sin \frac{m\pi x}{L} dx &= \int_{-L}^L A_0 \sin \frac{m\pi x}{L} dx + \sum_{n=1}^{\infty} \int_{-L}^L \left( A_n \cos \frac{n\pi x}{L} + B_n \sin \frac{n\pi x}{L} \right) \sin \frac{m\pi x}{L} dx \\ &= \sum_{n=1}^{\infty} B_n \int_{-L}^L \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} dx = \sum_{n=1}^{\infty} B_n L \delta_{nm} = LB_m. \end{aligned}$$

Therefore, we get an expression of  $B_m$ ,

$$B_m = \int_{-L}^L f(x) \sin \frac{m\pi x}{L} dx.$$

In summary, we have the following theorem.

**Theorem 2.5.** *The Fourier coefficients in  $A_0, A_1, B_1, \dots$  are given by*

$$\begin{aligned} A_0 &= \frac{1}{2L} \int_{-L}^L f(x) dx, \\ A_n &= \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx, \\ B_n &= \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx. \end{aligned} \tag{2.9}$$

*Example 2.1.* Let us calculate the Fourier series of  $f(x) = x, -L < x < L$ .

*Solution.* We have

$$\begin{aligned} A_0 &= \frac{1}{2L} \int_{-L}^L x dx = 0, \\ A_n &= \frac{1}{L} \int_{-L}^L x \cos \frac{n\pi x}{L} dx = 0 \quad (x \cos \frac{n\pi x}{L} \text{ is an odd function}), \\ B_n &= \frac{1}{L} \int_{-L}^L x \sin \frac{n\pi x}{L} dx = \frac{1}{L} \left[ -\frac{L}{n\pi} x \cos \frac{n\pi x}{L} \Big|_{-L}^L + \frac{L}{n\pi} \int_{-L}^L \cos \frac{n\pi x}{L} dx \right] = \frac{2L}{n\pi} (-1)^{n+1}. \end{aligned}$$

Therefore we obtain

$$x = \sum_{n=1}^{\infty} \frac{2L}{n\pi} (-1)^{n+1} \sin \frac{n\pi x}{L}, \quad -L < x < L. \tag{2.10}$$

□

We have two observations from the above example

**Proposition 2.6.** *An odd or even function only has cos or sin term in its Fourier series.*

*Proof.* We only consider the even case. The odd case can be proved similarly.

Since  $f(x)$  is even, we know that  $\frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx$  is odd. By (2.9), all the  $B_n$  vanishes. Therefore, we only have cos term in the Fourier series. □

**Proposition 2.7.** *Given a function  $f(x)$  defined on  $-L < x < L$ , the Fourier series of  $f(x)$  coincide with  $f(x)$  only in the domain  $-L < x < L$ , unless  $f(x)$  is a periodic function.*

*Proof.* We will not prove this proposition. Instead, we draw the graph of  $y = x$  and its Fourier series

The Fourier series coincides with  $y = x$  only in the domain  $-L < x < L$ . But the Fourier series is a periodic function since it is a sum of several periodic functions  $\cos / \sin$ . Since  $y = x$  is not a periodic function, Fourier series cannot agree with it for all  $x$ .

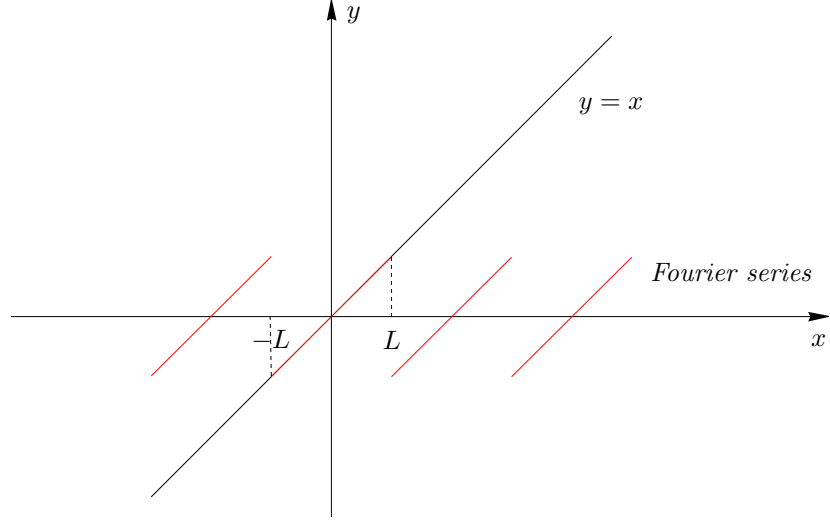


Figure 1: The graph of  $y = x$  and its Fourier series.

□

### 2.2.3 Solve the $A_n$ coefficients in (1.37)

Now we return to the separation of variable in section 1.5.3, where we derived

$$\varphi(x) = \sum_{n=1}^{\infty} A_n \sinh n\pi \sin \frac{n\pi x}{L}. \quad (2.11)$$

We want to solve  $A_n$  from  $\varphi(x)$ .

As explained in section 2.4, on  $0 < x < L$ , we also have the following orthogonality of  $\sin \frac{n\pi x}{L}$

$$\int_0^L \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} dx = \frac{L}{2} \delta_{nm}, \quad n, m \geq 1. \quad (2.12)$$

Therefore, if we want to solve  $A_m$ , we can multiply (2.11) by  $\sin \frac{n\pi x}{L}$  and then integrate over  $x$

$$\begin{aligned} \int_0^L \varphi(x) \sin \frac{m\pi x}{L} dx &= \sum_{n=1}^{\infty} \int_0^L A_n \sinh n\pi \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} dx \\ &= \sum_{n=1}^{\infty} A_n \sinh n\pi \int_0^L \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} dx = \sum_{n=1}^{\infty} A_n \sinh n\pi \frac{L}{2} \delta_{nm} = \frac{L}{2} A_m \sinh n\pi. \end{aligned}$$

From this, we can solve  $A_m$  and get

$$A_n = \frac{2}{L \sinh n\pi} \int_0^L \varphi(x) \sin \frac{n\pi x}{L} dx \quad (2.13)$$

This is exactly (1.38).

### 2.3 Complex form of Fourier series

We have seen that every function defined on  $-L < x < L$  can be rewritten as an infinite sum of  $\cos \frac{n\pi x}{L}$  and  $\sin \frac{n\pi x}{L}$ . By Euler's formula  $e^{ix} = \cos x + i \sin x$ , we know that every trigonometric function is a linear combination of  $e^{ix}$  and  $e^{-ix}$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}, \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i}. \quad (2.14)$$

Therefore, we have the following theorem.

**Theorem 2.8.** *In other word, for any  $f(x)$  on domain  $-L < x < L$ , there exists coefficients  $\dots, \alpha_{-1}, \alpha_0, \alpha_1, \alpha_2, \dots$  such that*

$$f(x) = \sum_{n=-\infty}^{\infty} \alpha_n e^{in\pi x/L}, \quad (2.15)$$

where  $\alpha_n$  satisfies the following properties.

1.  $\alpha_n$  is given by

$$\alpha_n = \frac{1}{2L} \int_{-L}^L f(x) e^{-in\pi x/L} dx \quad (2.16)$$

2. If  $f(x)$  is a real value function, then  $\alpha_n = \bar{\alpha}_{-n}$ , where  $\bar{\alpha}_{-n}$  is the complex conjugate of  $\alpha_{-n}$ .

*Proof.* Using Euler's formula (2.14), we can rewrite the Fourier series of  $f$  as follows.

$$\begin{aligned} f(x) &= A_0 + \sum_{n=1}^{\infty} \left( A_n \cos \frac{n\pi x}{L} + B_n \sin \frac{n\pi x}{L} \right) \\ &= A_0 + \sum_{n=1}^{\infty} \left( A_n \left( \frac{e^{in\pi x/L} + e^{-in\pi x/L}}{2} \right) + B_n \left( \frac{e^{in\pi x/L} - e^{-in\pi x/L}}{2i} \right) \right) \\ &= A_0 + \sum_{n=1}^{\infty} \left( \frac{A_n - iB_n}{2} e^{in\pi x/L} + \frac{A_n + iB_n}{2} e^{-in\pi x/L} \right). \end{aligned}$$

If we define

$$\alpha_0 = A_0, \quad \alpha_n = \frac{A_n - iB_n}{2}, \quad \alpha_{-n} = \frac{A_n + iB_n}{2}, \quad (n > 0) \quad (2.17)$$

then we obtain

$$f(x) = \alpha_0 + \sum_{n=1}^{\infty} \left( \alpha_n e^{in\pi x/L} + \alpha_{-n} e^{-in\pi x/L} \right) = \sum_{n=-\infty}^{\infty} \alpha_n e^{in\pi x/L}.$$

If  $f(x)$  is a real value function, then  $A_n$  and  $B_n$  are real numbers. From (2.17), we know that  $\alpha_n = \bar{\alpha}_{-n}$ .

The formula of  $\alpha_n$  is also a corollary of (2.17). When  $n > 0$ ,

$$\begin{aligned}\alpha_n &= \frac{A_n - iB_n}{2} = \frac{1}{2L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx - \frac{i}{2L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx \\ &= \frac{1}{2L} \int_{-L}^L f(x) e^{-in\pi x/L} dx\end{aligned}$$

When  $n = -m < 0$ ,

$$\begin{aligned}\alpha_n = \alpha_{-m} &= \frac{A_m + iB_m}{2} = \frac{1}{2L} \int_{-L}^L f(x) \cos \frac{m\pi x}{L} dx + \frac{i}{2L} \int_{-L}^L f(x) \sin \frac{m\pi x}{L} dx \\ &= \frac{1}{2L} \int_{-L}^L f(x) e^{im\pi x/L} dx = \frac{1}{2L} \int_{-L}^L f(x) e^{-in\pi x/L} dx\end{aligned}$$

When  $n = 0$ ,

$$\alpha_0 = A_0 = \frac{1}{2L} \int_{-L}^L f(x) dx = \frac{1}{2L} \int_{-L}^L f(x) e^{-i0\pi x/L} dx$$

Therefore, we have completed the proof.  $\square$

$e^{in\pi x/L}$  also have orthogonality relations.

**Theorem 2.9** (Orthogonality relations of complex Fourier series).

$$\int_{-L}^L e^{in\pi x/L} e^{-im\pi x/L} dx = \int_{-L}^L e^{i(n-m)\pi x/L} dx = 2L\delta_{mn}.$$

*Proof.* If  $n = m$ ,

$$\int_{-L}^L e^{i(n-m)\pi x/L} dx = \int_{-L}^L 1 dx = 2L = 2L\delta_{mn}.$$

If  $n \neq m$ ,

$$\int_{-L}^L e^{i(n-m)\pi x/L} dx = \frac{L}{i\pi(n-m)} \left[ e^{i(n-m)\pi x/L} \right]_{-L}^L = 0 = 2L\delta_{mn}.$$

We have completed the proof.  $\square$

The formula of  $\alpha_n$  can also be computed using orthogonality.

Assume that we want to compute  $\alpha_m$  in

$$f(x) = \sum_{n=-\infty}^{\infty} \alpha_n e^{in\pi x/L}.$$

Multiply both side by  $e^{-im\pi x/L}$  and then we get

$$\int_{-L}^L f(x) e^{-im\pi x/L} dx = \sum_{n=-\infty}^{\infty} \alpha_n \int_{-L}^L e^{in\pi x/L} e^{-im\pi x/L} dx = \sum_{n=-\infty}^{\infty} \alpha_n 2L\delta_{mn} = 2L\alpha_m.$$

Solve  $\alpha_m$ , then we prove (2.16) again.



## 2.4 Fourier cosine and sine series

A function defined on  $-L < x < L$  can be rewritten as an infinite sum of  $\cos \frac{n\pi x}{L}$  and  $\sin \frac{n\pi x}{L}$ . Notice that  $-L < x < L$  is the period of  $\cos \frac{n\pi x}{L}$  and  $\sin \frac{n\pi x}{L}$ . If a function  $f(x)$  is defined in the interval  $0 < x < L$ , which represents half of its period, then it is possible to express  $f(x)$  exclusively as a series of cosine terms,  $\cos \frac{n\pi x}{L}$ , or alternatively, only as a series of sine terms,  $\sin \frac{n\pi x}{L}$ . This is the Fourier cosine/sine series.

The idea is that, given a function  $f(x)$  defined on  $0 < x < L$ , we can extend this function as an even/odd function on  $-L < x < L$ . Then we compute the Fourier series of the extended function. The Fourier series only contains  $\cos \frac{n\pi x}{L}$  or  $\sin \frac{n\pi x}{L}$  terms by Proposition 2.6.

### 2.4.1 Fourier cosine series

We define the even extension  $f_E(x)$  of  $f(x)$  as

$$f_E(x) = \begin{cases} f(x), & 0 < x < L \\ 0, & x = 0, \\ f(-x), & -L < x < 0. \end{cases} \quad (2.18)$$

We note that  $f_E(x)$  is even. Indeed the value  $f_E(0)$  is arbitrary and not necessarily zero. The Fourier series is given by

$$f_E(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L}, \quad -L < x < L,$$

where

$$\begin{aligned} A_0 &= \frac{1}{2L} \int_{-L}^L f_E(x) dx = \frac{1}{L} \int_0^L f(x) dx, \\ A_n &= \frac{1}{2L} \int_{-L}^L f_E(x) \cos \frac{n\pi x}{L} dx = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx. \end{aligned}$$

On the interval  $0 < x < L$  we have

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L}, \quad 0 < x < L, \quad (2.19)$$

where

$$A_0 = \frac{1}{L} \int_0^L f(x) dx, \quad A_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx. \quad (2.20)$$

**Definition 2.10.** *The series (2.19) is called the Fourier cosine series.*

For Fourier cosine series, we also have orthogonality relations and (2.20) can be computed from these orthogonality relations.

**Theorem 2.11** (Orthogonality relations of cosine Fourier series).

$$\int_0^L \cos \frac{n\pi x}{L} \cos \frac{m\pi x}{L} dx = \begin{cases} L & (n = m = 0), \\ \frac{L}{2} \delta_{nm} & (otherwise), \end{cases} \quad (2.21)$$

*Example 2.2.* Let us compute the Fourier cosine series of  $f(x) = x$ ,  $0 < x < L$ .

*Solution.* We can directly apply (2.20). But let us try the even extension method.

We extend  $f$  as

$$f_E(x) = \begin{cases} x, & 0 < x < L, \\ 0, & x = 0, \\ -x, & -L < x < 0. \end{cases}$$

Indeed  $f_E(x) = |x|$ .

$$\begin{aligned} A_0 &= \frac{1}{2L} \int_{-L}^L f_E(x) dx = \frac{1}{L} \int_0^L x dx = \frac{L}{2}, \\ A_n &= \frac{1}{2L} \int_{-L}^L f_E(x) \cos \frac{n\pi x}{L} dx = \frac{2}{L} \int_0^L x \cos \frac{n\pi x}{L} dx \\ &= \frac{2}{L} \left[ \frac{L}{n\pi} x \sin \frac{n\pi x}{L} \Big|_0^L - \frac{L}{n\pi} \int_0^L \sin \frac{n\pi x}{L} dx \right] = \frac{2L}{(n\pi)^2} ((-1)^n - 1). \end{aligned}$$

Therefore,

$$x = \frac{L}{2} + \frac{2L}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} \cos \frac{n\pi x}{L}, \quad 0 < x < L.$$

□

## 2.4.2 Fourier sine series

We define the odd extension  $f_O(x)$  as

$$f_O(x) = \begin{cases} f(x), & 0 < x < L \\ 0, & x = 0, \\ -f(-x), & -L < x < 0. \end{cases} \quad (2.22)$$

We note that  $f_O(x)$  is odd. The Fourier series is given by

$$f_O(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}, \quad -L < x < L,$$

where

$$B_n = \frac{1}{L} \int_{-L}^L f_O(x) \sin \frac{n\pi x}{L} dx = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx.$$

On the interval  $0 < x < L$  we have

$$f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}, \quad 0 < x < L, \quad (2.23)$$

where

$$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx \quad (2.24)$$

**Definition 2.12.** *The series (2.23) is called the Fourier cosine series.*

For Fourier sine series, we also have orthogonality relations and (2.24) can be computed from these orthogonality relations.

**Theorem 2.13** (Orthogonality relations of sine Fourier series).

$$\int_0^L \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} dx = \begin{cases} 0 & (n = m = 0), \\ \frac{L}{2} \delta_{nm} & (\text{otherwise}), \end{cases} \quad (2.25)$$

*Example 2.3.* The Fourier sine series of  $f(x) = x$ ,  $0 < x < L$ , is obtained through the odd extension  $f_O(x)$ . The odd extension  $f_O(x)$  is again  $x$  and its Fourier series has been computed in (2.10).

## 2.5 Convergence of Fourier series

**TODO:** in this section we study pointwise convergence

**Definition 2.14** (Left and right limits). *For a given  $f(x)$ , let us write*

$$f(x+0) = \lim_{\varepsilon \rightarrow 0} f(x+\varepsilon), \quad f(x-0) = \lim_{\varepsilon \rightarrow 0} f(x-\varepsilon), \quad (2.26)$$

where  $\varepsilon > 0$ .

**Definition 2.15** (Piecewise continuous). *A function  $f(x)$  defined on  $a < x < b$ , is said to be piecewise continuous if there is a finite set of points  $a = x_0 < x_1 < \dots < x_p < x_{p+1} = b$  such that  $f(x)$  is continuous at  $x \neq x_i$  ( $i = 1, \dots, p$ ),  $f(x_i+0)$  ( $i = 0, \dots, p$ ) exists, and  $f(x_i-0)$  ( $i = 1, \dots, p+1$ ) exists.*

**Definition 2.16** (Piecewise smooth). *A function  $f(x)$ ,  $a < x < b$ , is said to be piecewise smooth if  $f(x)$  and all of its derivatives are piecewise continuous.*

*Example 2.4.* The function  $f(x) = |x|$ ,  $-L < x < L$ , is piecewise smooth. The function  $f(x) = x^2 \sin(1/x)$ ,  $-L < x < L$ , is piecewise continuous but is not piecewise smooth because  $\lim_{\varepsilon \rightarrow 0} f'(0 \pm \varepsilon)$  does not exist. The function  $f(x) = 1/(x^2 - L^2)$ ,  $-L < x < L$ , is not piecewise continuous because  $f(-L + 0)$  and  $f(L - 0)$  are not finite.

**Definition 2.17** (Convergence). *Given a Fourier series  $f(x) = A_0 + \sum_{n=1}^{\infty} [A_n \cos(n\pi x/L) + B_n \sin(n\pi x/L)]$ , we define*

1. **Partial sums.** The partial sum, denoted by  $f_N(x)$ , is defined to be

$$f_N(x) = A_0 + \sum_{n=1}^N [A_n \cos(n\pi x/L) + B_n \sin(n\pi x/L)]. \quad (2.27)$$

2. **Convergence at  $x$ .** We say the Fourier is convergent at  $x$  if

$$\lim_{N \rightarrow \infty} f_N(x) = f(x). \quad (2.28)$$

3. **Uniformly convergence.** We say the Fourier is uniformly convergent if

$$\lim_{N \rightarrow \infty} \max_{x \in [-L, L]} |f_N(x) - f(x)| = 0. \quad (2.29)$$

*This means that the convergence is equally well for all  $x$ .*

**Theorem 2.18** (Convergence theorem). *Let  $f(x)$ ,  $-L < x < L$ , be piecewise smooth. Then the Fourier series of  $f$  converges for all  $x$  to the value  $\frac{1}{2}[\bar{f}(x+0) + \bar{f}(x-0)]$ , where  $\bar{f}$  is the  $2L$ -periodic function which equals to  $f$  on  $-L < x < L$ .*

*If  $f(x)$  is continuous on  $[-L, L]$  and  $f(-L) = f(L)$  in addition to the conditions assumed in the above theorem, then the Fourier series uniformly converges.*

*Example 2.5.* Here are two examples of convergence.

1. The Fourier series of  $f(x) = |x|$ ,  $-L < x < L$ , (see Example 2.2) uniformly converges.
2. The Fourier series of  $f(x) = x$ ,  $-L < x < L$  does not uniformly converge but converges at any point to  $f(x)$  except for  $x = -L, L$ .

As in the following picture, the Fourier series of  $f(x) = x$ ,  $-L < x < L$  has a very bad convergence near  $x = -L, L$ , but series of  $f(x) = |x|$  has a much better convergence.

**TODO: add a picture**

## 2.6 Parseval's Theorem and Mean Square Error

### 2.6.1 The Parseval's Theorem for Fourier series

For orthogonal vectors  $e_1, \dots, e_n$ ,  $e_i \perp e_j$ , their linear combination  $v = v_1 e_1 + \dots + v_n e_n$  satisfies the Pythagorean theorem

$$|v|^2 = |v_1 e_1 + \dots + v_n e_n|^2 = |v_1 e_1|^2 + \dots + |v_n e_n|^2 \quad (2.30)$$

The proof of (2.30) is given by

$$|v|^2 = \left\langle \sum_{j=1}^n v_j e_j, \sum_{k=1}^n v_k e_k \right\rangle = \sum_{j=1}^n \sum_{k=1}^n v_j v_k \underbrace{\langle e_j, e_k \rangle}_{=0 \text{ if } j \neq k} = \sum_{j=1}^n v_j^2 |e_j|^2$$

For Fourier series, the following theorem claims that a similar identity is true.

**Theorem 2.19** (Parseval's theorem). *Let  $f(x)$  defined on  $-L < x < L$  be a piecewise smooth function with Fourier series*

$$f(x) = A_0 + \sum_{n=1}^{\infty} [A_n \cos(n\pi x/L) + B_n \sin(n\pi x/L)] \quad (2.31)$$

*Then, mean square of  $f(x) = \frac{1}{2L} \int_{-L}^L f(x)^2 dx$  satisfies the following identity*

$$\frac{1}{2L} \int_{-L}^L f(x)^2 dx = A_0^2 + \frac{1}{2} \sum_{n=1}^{\infty} (A_n^2 + B_n^2) \quad (2.32)$$

*Proof.* Since  $\cos(1\pi x/L) = 1$  and  $\sin(n\pi x/L) = 0$ , we know that (2.31) is equivalent to

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} [A_n \cos(n\pi x/L) + B_n \sin(n\pi x/L)] \\ &= \sum_{n=0}^{\infty} (A_n \cos_n + B_n \sin_n) \end{aligned} \quad (2.33)$$

where we introduce the notation

$$\cos_n = \cos(n\pi x/L), \quad \sin_n = \sin(n\pi x/L) \quad (2.34)$$

Similar to (2.30) and its proof, we have

$$\begin{aligned}
\frac{1}{2L} \int_{-L}^L f(x)^2 dx &= \frac{1}{2L} \int_{-L}^L \sum_{n=0}^{\infty} [A_n \cos_n + B_n \sin_n] \sum_{m=0}^{\infty} [A_m \cos_m + B_m \sin_m] dx \\
&= \frac{1}{2L} \int_{-L}^L \sum_{n,m=0}^{\infty} (A_n A_m \underbrace{\cos_n \cos_m}_{=0 \text{ if } n \neq m} + A_n B_m \underbrace{\cos_n \sin_m}_{=0} \\
&\quad + B_n A_m \underbrace{\sin_n \cos_m}_{=0} + B_n B_m \underbrace{\sin_n \sin_m}_{=0 \text{ if } n \neq m}) dx \\
&= A_0^2 + \frac{1}{2} \sum_{n=1}^{\infty} (A_n^2 + B_n^2)
\end{aligned} \tag{2.35}$$

where in the last line, we applied the orthogonality of trigonometric functions (2.6).  $\square$

### 2.6.2 The mean square error

**Definition 2.20** (Mean square error). We define the mean square error  $\sigma_N^2$  as

$$\sigma_N^2 = \frac{1}{2L} \int_{-L}^L [f(x) - f_N(x)]^2 dx \tag{2.36}$$

where  $f_N(x)$  is the partial sum

$$f_N(x) = A_0 + \sum_{n=1}^N [A_n \cos(n\pi x/L) + B_n \sin(n\pi x/L)]. \tag{2.37}$$

By Parseval's theorem, we obtain the following expression of the mean square error.

**Proposition 2.21.** Given a partial summation whose mean square error is defined by (2.36), then we have

$$\sigma_N^2 = \frac{1}{2} \sum_{n=N+1}^{\infty} (A_n^2 + B_n^2) \tag{2.38}$$

*Proof.* This is an easy corollary of the Parseval's identity (2.32), if we notice that

$$f(x) - f_N(x) = \sum_{n=N+1}^{\infty} [A_n \cos(n\pi x/L) + B_n \sin(n\pi x/L)]$$

Therefore, we finished the proof.  $\square$

*Example 2.6.* Let us find  $\sigma_N^2$  for  $f(x) = x$ ,  $-L < x < L$ .

*Solution.* From Example 2.1, we have  $A_0 = A_n = 0$  and

$$f_N(x) = \sum_{n=1}^N B_n \sin \frac{n\pi x}{L}, \quad B_n = \frac{2L}{n\pi} (-1)^{n+1}. \tag{2.39}$$

By (2.38), we obtain

$$\sigma_N^2 = \frac{1}{2} \sum_{n=N+1}^{\infty} \left( \frac{2L}{n\pi} (-1)^{n+1} \right)^2 = \frac{2L^2}{\pi^2} \sum_{n=N+1}^{\infty} \frac{1}{n^2}.$$

By Theorem 2.22, we get

$$\int_{N+1}^{\infty} \frac{1}{x^2} dx \leq \sum_{n=N+1}^{\infty} \frac{1}{n^2} \leq \int_{N+1}^{\infty} \frac{1}{(x-1)^2} dx = \int_N^{\infty} \frac{1}{x^2} dx.$$

We have

$$\int_{N+1}^{\infty} \frac{1}{x^2} dx = \frac{1}{N+1} \geq \frac{1}{N} - \frac{1}{N^2}, \quad \int_N^{\infty} \frac{1}{x^2} dx = \frac{1}{N}.$$

Let us introduce the symbol  $O$  (this is called “big O”) to express the order. For some  $f_N$ ,  $f_N = O(N^{-1})$  as  $N \rightarrow \infty$  means that there exist a constant  $C > 0$  such that  $|f_N| \leq CN^{-1}$ . Therefore we obtain

$$\sigma_N^2 = \frac{2L^2}{\pi^2} \frac{1}{N} \left[ 1 + O\left(\frac{1}{N}\right) \right] = O(N^{-1}), \quad N \rightarrow \infty. \quad (2.40)$$

We note that  $\sigma_N^2$  goes to zero as  $N \rightarrow \infty$  although we know that the sum in (2.39) does not converge uniformly. This happened because we considered the mean square and took the integral.

**TODO: add a picture for convergence** □

**Theorem 2.22** (Integral test). *Given a monotonic and positive function  $f(x)$ , we have*

$$\int_{N+1}^{\infty} f(x-1) dx \leq \sum_{n=N+1}^{\infty} f(n) \leq \int_{N+1}^{\infty} f(x) dx \quad (2.41)$$

*Proof.* **TODO: compare the area below the graph of  $y = f(x)$**

**TODO: add a picture** □

*Example 2.7.* Let us find  $\sigma_{2N}^2$  for  $f(x) = |x|$ ,  $-L < x < L$ .

*Solution.* From Example 2.2 we know that  $B_n = 0$ ,  $A_{2m} = 0$  ( $m = 1, 2, \dots$ ), and

$$f_{2N}(x) = A_0 + \sum_{m=1}^N A_{2m-1} \cos \frac{(2m-1)\pi x}{L}, \quad A_0 = \frac{L}{2}, \quad A_{2m-1} = -\frac{4L}{\pi^2(2m-1)^2}.$$

This is also the Fourier cosine series of  $x$ ,  $0 < x < L$ , in Example 2.2. Hence we obtain

$$\sigma_{2N}^2 = \frac{1}{2} \sum_{n=2N+1}^{\infty} A_n^2 = \frac{1}{2} \sum_{m=N+1}^{\infty} A_{2m-1}^2 = \frac{8L^2}{\pi^4} \sum_{m=N+1}^{\infty} \frac{1}{(2m-1)^4}.$$

Note that by Theorem 2.22

$$\int_{N+1}^{\infty} \frac{1}{(2x-3)^4} dx \leq \sum_{m=N+1}^{\infty} \frac{1}{(2m-1)^4} \leq \int_{N+1}^{\infty} \frac{1}{(2x-1)^4} dx,$$

and by Taylor expansion  $\frac{1}{(1+x)^3} = 1 - 3x + 6x^2 + \dots$ ,  $LHS = \frac{1}{8N^3} \frac{1}{(1+1/2N)^3} = \frac{1}{6(2N+1)^3} = \frac{1}{48N^3} + O(N^{-4})$  and  $RHS = \frac{1}{6(2N-1)^3} = \frac{1}{48N^3} + O(N^{-4})$ . Therefore we obtain

$$\sigma_{2N}^2 = \frac{L^2}{6\pi^4 N^3} + O(N^{-4}) = O(N^{-3}), \quad N \rightarrow \infty. \quad (2.42)$$

Thus the Fourier series of  $x$  converges as  $O(1/N)$  and the Fourier series of  $|x|$  converges as  $O(1/N^3)$ . Equations (2.40) and (2.42) explain the difference between figure ??.

### 2.6.3 Parseval's theorem for complex, cosine and sine Fourier series

**Theorem 2.23** (Parseval's theorem for complex Fourier series). *Let  $f(x)$  defined on  $-L < x < L$  be a piecewise smooth function with Fourier series*

$$f(x) = \sum_{n=-\infty}^{\infty} \alpha_n e^{in\pi x/L} \quad (2.43)$$

*Then, mean square of  $f(x) = \frac{1}{2L} \int_{-L}^L f(x)^2 dx$  satisfies the following identity*

$$\frac{1}{2L} \int_{-L}^L |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |\alpha_n|^2. \quad (2.44)$$

*Proof. Method 1.* We have

$$\begin{aligned} \frac{1}{2L} \int_{-L}^L |f(x)|^2 dx &= \frac{1}{2L} \int_{-L}^L \sum_{n=-\infty}^{\infty} \alpha_n e^{in\pi x/L} \sum_{m=-\infty}^{\infty} \bar{\alpha}_m e^{-im\pi x/L} dx \\ &= \sum_{n,m=0}^{\infty} \alpha_n \bar{\alpha}_m \underbrace{\frac{1}{2L} \int_{-L}^L e^{i(n-m)\pi x/L} dx}_{=0 \text{ if } n \neq m} \\ &= \sum_{n=-\infty}^{\infty} |\alpha_n|^2 \end{aligned} \quad (2.45)$$

*Method 2.* (2.44) is seen by the calculation below. By Parseval's identity for the Fourier series



(2.32),

$$\begin{aligned}
\frac{1}{2L} \int_{-L}^L f(x)^2 dx &= A_0^2 + \frac{1}{2} \sum_{n=1}^{\infty} (A_n^2 + B_n^2) \\
&= A_0^2 + 2 \sum_{n=1}^{\infty} \frac{A_n - iB_n}{2} \frac{A_n + iB_n}{2} \\
&= \alpha_0^2 + 2 \sum_{n=1}^{\infty} \alpha_n \alpha_{-n} = \alpha_0^2 + \sum_{n=1}^{\infty} |\alpha_n|^2 + \sum_{n=1}^{\infty} |\alpha_{-n}|^2 \\
&= \alpha_0^2 + \sum_{n=1}^{\infty} |\alpha_n|^2 + \sum_{n=-\infty}^{-1} |\alpha_n|^2 \\
&= \sum_{n=-\infty}^{\infty} |\alpha_n|^2
\end{aligned}$$

□

**Theorem 2.24** (Parseval's theorem for cosine Fourier series). *Let  $f(x)$  defined on  $-L < x < L$  be a piecewise smooth function with cosine Fourier series*

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L} \quad (2.46)$$

*Then, mean square of  $f(x) = \frac{1}{2L} \int_{-L}^L f(x)^2 dx$  satisfies the following identity*

$$\frac{1}{2L} \int_{-L}^L |f(x)|^2 dx = A_0^2 + \frac{1}{2} \sum_{n=1}^{\infty} A_n^2. \quad (2.47)$$

*Proof.* This is homework. □

**Theorem 2.25** (Parseval's theorem for sine Fourier series). *Let  $f(x)$  defined on  $-L < x < L$  be a piecewise smooth function with sine Fourier series*

$$f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} \quad (2.48)$$

*Then, mean square of  $f(x) = \frac{1}{2L} \int_{-L}^L f(x)^2 dx$  satisfies the following identity*

$$\frac{1}{L} \int_0^L f(x)^2 dx = \frac{1}{2} \sum_{n=1}^{\infty} B_n^2. \quad (2.49)$$

*Proof.* This is homework. □

### 3 PDEs in rectangular coordinates

In this section, we will consider separation of variable for more general equation in rectangular coordinates, possibly with variable coefficients and more general boundary conditions.

### 3.1 Boundary conditions for general PDEs

#### 3.1.1 Dirichlet, Neumann and Robin boundary conditions

In the ODE class, we have learned the following boundary value problem

$$\begin{aligned} u'' &= 1, & x &\in [0, 1] \\ u(0) &= 0, & u(1) &= 0. \end{aligned} \tag{3.1}$$

where we prescribe a condition for every point on the boundary of the domain  $[0, 1]$ . (The boundary is 0 and 1.) If we missing any of these conditions, we cannot get a unique solution.

For example, if we remove the condition  $u(1) = 0$ , then we get solution  $u(x) = \frac{1}{2}x^2 + Cx$ , which contains an undetermined constant  $C$ .

Given a PDE defined on a domain  $R$ , we must also prescribe the boundary condition on  $\partial R$  to obtain a unique solution.

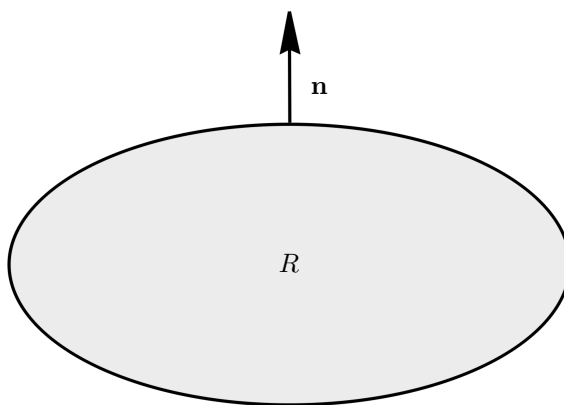


Figure 2: The region  $R$  and its normal vector.

Before we introduce the concept of boundary conditions, let us first explain the concept of normal vectors.

**Definition 3.1** (Normal vector). *Given a domain  $R$  and a point  $x \in \partial R$ , a normal vector  $\mathbf{n}$  at  $x$  is the vector as demonstrated by figure 2. In this class, normal vector is always pointing outwards.*

Now we introduce the boundary conditions which will be considered in this class.

**Definition 3.2** (Boundary conditions). *Consider a PDE  $F[u] = 0$  defined on the domain  $R$ . Here are the boundary conditions that we consider in this class.*

1. **Dirichlet boundary condition.** The value of  $u$  on the boundary is given.

$$u = g(x), \quad x \in \partial R. \quad (3.2)$$

2. **Neumann boundary condition.** The normal derivative on the boundary is given.

$$\mathbf{n} \cdot \nabla u = g(x), \quad x \in \partial R. \quad (3.3)$$

3. **Robin boundary condition.** A linear combination of the above two boundary conditions.

$$a(x)u + b(x)\mathbf{n} \cdot \nabla u = g(x), \quad x \in \partial R. \quad (3.4)$$

### 3.1.2 Heat equations as an example

We take the heat equation  $u_t = Ku_{zz}$ , defined on  $R = \{(z, t) : 0 \leq t < \infty, 0 \leq z \leq L\}$ , as an example to explain these boundary conditions.

The domain  $R$  of the heat equation is described by the following picture,

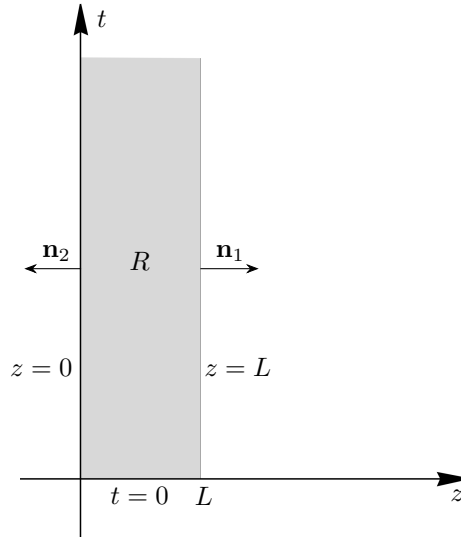


Figure 3: The domain for the heat equation.

There are three pieces of  $\partial R$ ,

$$\begin{aligned} z = 0, \quad t > 0 \\ z = L, \quad t > 0 \\ 0 < z < L, \quad t = 0 \end{aligned} \quad (3.5)$$

On the first two parts of  $\partial R$  corresponding to  $z = 0, L$ , we impose Robin boundary condition. On the last part corresponding to  $t = 0$ , we impose the Dirichlet boundary condition. Then we get the following system of equations,

$$\begin{cases} u_t = Ku_{zz}, & 0 < z < L, \quad t > 0, \\ a(z, t)u + b(z, t)\mathbf{n} \cdot \nabla u = g(z, t), & z = 0, L, \quad t > 0, \\ u(z, 0) = f(z), & 0 < z < L, \quad t = 0. \end{cases} \quad (3.6)$$

Here the choice of Dirichlet and Robin boundary condition comes from the physics of heat conduction. We will explain this in section ??.

**TODO: second order need one condition but first order need one**

Now we explain how to simplify the Robin boundary condition in (3.6).

On  $z = 0$ , from figure 3, we know that the normal vector  $\mathbf{n} = (-1, 0)$  and  $\nabla u = (u_z, u_t)$ , so we get  $\mathbf{n} \cdot \nabla u = -u_z$ . Therefore,  $a(z, t)u + b(z, t)\mathbf{n} \cdot \nabla u = g(z, t)$  simplifies to

$$a(0, t)u - b(0, t)u_z = g(0, t). \quad (3.7)$$

On  $z = L$ , from figure 3, we know that the normal vector  $\mathbf{n} = (1, 0)$  and  $\nabla u = (u_z, u_t)$ , so we get  $\mathbf{n} \cdot \nabla u = u_z$ . Therefore,  $a(z, t)u + b(z, t)\mathbf{n} \cdot \nabla u = g(z, t)$  simplifies to

$$a(L, t)u + b(L, t)u_z = g(L, t). \quad (3.8)$$

We introduce new functions  $a(t)$ ,  $\tilde{a}(t)$ ,  $b(t)$ ,  $\tilde{b}(t)$ ,  $g(t)$  and  $\tilde{g}(t)$  by the following equations

$$\begin{aligned} a(t) &= a(0, t), & b(t) &= b(0, t), & g(t) &= g(0, t) \\ \tilde{a}(t) &= a(L, t), & \tilde{b}(t) &= b(L, t), & \tilde{g}(t) &= g(L, t) \end{aligned} \quad (3.9)$$

Then the Robin boundary conditions becomes

$$\begin{aligned} a(t)u - b(t)u_z &= g(t), & z &= 0 \\ \tilde{a}(t)u + \tilde{b}(t)u_z &= \tilde{g}(t), & z &= L \end{aligned} \quad (3.10)$$

The heat equation becomes

$$\begin{cases} u_t = Ku_{zz}, & 0 < z < L, \quad t > 0, \\ a(t)u - b(t)u_z = g(t), & z = 0, \quad t > 0, \\ \tilde{a}(t)u + \tilde{b}(t)u_z = \tilde{g}(t), & z = L, \quad t > 0, \\ u = f(z), & 0 < z < L, \quad t = 0. \end{cases} \quad (3.11)$$

In order to make (3.11) solvable, we impose the homogeneous condition.

**Definition 3.3** (Homogeneous). *We say (3.11) is homogeneous if*

1.  $a(t), \tilde{a}(t), b(t), \tilde{b}(t), g(t)$  and  $\tilde{g}(t)$  are independent of  $t$ .
2.  $g(t) = \tilde{g}(t) = 0$ .

With homogeneous assumption, (3.11) becomes

$$\begin{cases} u_t = Ku_{zz}, & 0 < z < L, \quad t > 0, \\ au - bu_z = 0, & z = 0, \quad t > 0, \\ \tilde{a}u + \tilde{b}u_z = 0, & z = L, \quad t > 0, \\ u = f(z), & 0 < z < L, \quad t = 0. \end{cases} \quad (3.12)$$

To further simplify the above equation, we doing the change of variable

$$b \rightarrow bL, \quad \tilde{b} \rightarrow \tilde{b}L \quad (3.13)$$

followed by the change of variable

$$\begin{aligned} a &\rightarrow r \cos \alpha, & b &\rightarrow r \sin \alpha, \\ \tilde{a} &\rightarrow r \cos \beta, & \tilde{b} &\rightarrow r \sin \beta. \end{aligned} \quad (3.14)$$

Finally, the heat equation becomes

$$\begin{cases} u_t = Ku_{zz}, & 0 < z < L, \quad t > 0, \\ u \cos \alpha - Lu_z \sin \alpha = 0, & z = 0, \quad t > 0, \\ u \cos \beta + Lu_z \sin \beta = 0, & z = L, \quad t > 0, \\ u = f(z), & 0 < z < L, \quad t = 0, \end{cases} \quad (3.15)$$

### 3.1.3 Separation of variable in heat equation

Let us solve the heat equation in a simple case of  $\alpha = \beta = 0$  in (3.15). The separated solution is written as  $u(z, t) = \phi(z)T(t)$ . Thus we obtain

$$T'(t) + \lambda KT(t) = 0, \quad \phi''(z) + \lambda \phi(z) = 0.$$

The boundary conditions are written as

$$\phi(0) = \phi(L) = 0.$$

We obtain

$$T(t) = e^{-\lambda K t}, \quad \phi = A \sin(\sqrt{\lambda} z) + B \cos(\sqrt{\lambda} z), \quad \lambda > 0.$$

By plugging  $\phi = A \sin(\sqrt{\lambda} z) + B \cos(\sqrt{\lambda} z)$  into the boundary conditions, we find that  $B = 0$  and  $\sqrt{\lambda} L = n\pi$  where  $n$  is an integer. Therefore we obtain

$$\phi(z) = \phi_n(z) = \sin\left(\sqrt{\lambda_n} z\right), \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, \dots,$$

where we set the arbitrary constant in  $\phi_n(z)$  to be 1 (recall we will take a superposition). Thus the separated solutions are obtained as

$$u(z, t) = \phi_n(z) e^{-\lambda_n K t}, \quad n = 1, 2, \dots$$

If no initial condition  $u(z, 0) = f(z)$  is given, the above separated solutions are the solutions to the problem. However, they do not satisfies  $u(z, 0) = f(z)$ . Let us consider the linear combination of separated solution and match with the initial condition.

The linear combination is

$$u(z, t) = \sum_{n=1}^{\infty} C_n \phi_n(z) e^{-\lambda_n K t},$$

where  $C_n$  are constants.

By  $u(z, 0) = f(z)$ , we know that

$$f(z) = u(z, 0) = \sum_{n=1}^{\infty} C_n \phi_n(z) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi z}{L}.$$

From this we know that  $C_n$  is the  $B_n$  coefficients of the Fourier sine series. We thus obtain

$$u(z, t) = \sum_{n=1}^{\infty} B_n \phi_n(z) e^{-\lambda_n K t}, \quad 0 < z < L, \quad t > 0.$$

$$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx.$$

*Example 3.1.* The heat equation  $u_t = K u_{zz}$  for  $0 < z < L, t > 0$  with  $u(0, t) = u(L, t) = 0$  and  $u(z, 0) = 1$  is solved as

$$u(z, t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi z}{L} e^{-(n\pi/L)^2 K t},$$

where

$$B_n = \frac{2}{\pi} \frac{1 - (-1)^n}{n}.$$

Here as mentioned above,  $B_n$  can be directly computed as coefficients of the Fourier sine series of  $f(z) = 1$

### 3.1.4 Some linear algebra

## 3.2 The Sturm-Liouville eigenvalue problem

### 3.2.1 Orthogonal functions

**Definition 3.4** (Inner product). We extend dot product  $\varphi \cdot \psi$  and define inner product as

$$\langle \varphi, \psi \rangle = \int_a^b \varphi(x)\psi(x)dx. \quad (3.16)$$

Sometimes the inner product is defined as follows. We can have a weight function  $\rho(x)$ , and the weighted inner product is given by

$$\langle \varphi, \psi \rangle_\rho = \int_a^b \varphi(x)\psi(x)\rho(x)dx \quad (3.17)$$

where  $\rho(x) > 0$  is a weight function.

For complex functions, we can write the complex inner product as

$$\langle \varphi, \psi \rangle = \int_a^b \varphi(x)\bar{\psi}(x)\rho(x)dx. \quad (3.18)$$

Here  $\bar{\psi}$  is the complex conjugate of  $\psi$  ( $\bar{\psi}(x) = f(x) - ig(x)$  when  $\psi = f + ig$ ).

**Definition 3.5** (Orthogonality). Two functions  $\varphi, \psi$  are said to be orthogonal on  $[a, b]$  if  $\langle \varphi, \psi \rangle = 0$ .

*Example 3.2.* The functions  $\varphi(x) = \sin x$  and  $\psi(x) = \cos x$  are orthogonal on  $[0, \pi]$ .

*Example 3.3.* The set of functions  $\sin x, \sin 2x, \dots, \sin Nx$  is orthogonal on  $[0, \pi]$ .

*Example 3.4.* Which of the following pairs of functions are orthogonal on the interval  $0 \leq x \leq 1$  ?

$$\varphi_1 = \sin 2\pi x, \quad \varphi_2 = x, \quad \varphi_3 = \cos 2\pi x, \quad \varphi_4 = 1.$$

$\langle \varphi_1, \varphi_3 \rangle = 0, \langle \varphi_1, \varphi_4 \rangle = 0, \langle \varphi_2, \varphi_3 \rangle = 0, \langle \varphi_3, \varphi_4 \rangle = 0$ . All others are nonzero. Therefore the pairs  $(\varphi_1, \varphi_3), (\varphi_1, \varphi_4), (\varphi_2, \varphi_3)$ , and  $(\varphi_3, \varphi_4)$  are orthogonal.

**Definition 3.6** (Norm). As follows we define norm, which is the “length” of a function.

$$\|\varphi\| = \|\varphi\|_{L^2(a,b)} = \sqrt{\langle \varphi, \varphi \rangle}.$$

We note that the norm is always nonnegative. The norm  $\|\varphi - \psi\|$  is the distance between two functions  $\varphi$  and  $\psi$ .

**Definition 3.7** (Projection). Let  $(\varphi_1, \dots, \varphi_N)$  be a set of orthogonal functions with  $\|\varphi_i\| \neq 0$ . Let  $f(x)$  be a function. Then  $\hat{c}_1\varphi_1 + \dots + \hat{c}_N\varphi_N$  with  $\hat{c}_i = \langle f, \varphi_i \rangle / \|\varphi_i\|^2$  is the projection of  $f$  onto  $(\varphi_1, \dots, \varphi_N)$ .  $\hat{c}_i$  is called the  $i$  th Fourier coefficient of  $f$ .

**TODO: revise this**

Note that the minimum of  $\|f - (c_1\varphi_1 + \dots + c_N\varphi_N)\|$  is achieved when  $c_i = \hat{c}_i$ .

*Example 3.5.* Find the projection of  $f(x) = 1$  onto  $(\varphi_1, \varphi_2) = (\sin x, \sin 2x)$  on the interval  $0 \leq x \leq \pi$ . By  $\langle f, \varphi_1 \rangle = 2$ ,  $\langle f, \varphi_2 \rangle = 0$ , and  $\|\varphi_1\|^2 = \|\varphi_2\|^2 = \frac{\pi}{2}$ , we obtain  $\frac{4}{\pi} \sin x$ .

**Definition 3.8** (Orthonormal). The functions  $(\varphi_1, \dots, \varphi_N)$  are orthonormal if  $\langle \varphi_i, \varphi_j \rangle = \delta_{ij}$ . Here  $\delta_{ij}$  is the Kronecker delta ( $\delta_{ij} = 0$  if  $i \neq j$  and  $= 1$  if  $i = j$ ).

### 3.2.2 The Sturm-Liouville eigenvalue problem

### 3.2.3 Convergence of orthogonal series

## 3.3 The heat equation

### 3.3.1 Physics of heat conduction

### 3.3.2 The homogeneous case

### 3.3.3 The non-homogeneous case

## 3.4 The wave equation

### 3.4.1 Physics of string vibration

### 3.4.2 The homogeneous case

### 3.4.3 The non-homogeneous case

## 3.5 The Laplace's equation

### 3.5.1 Physics of static electricity

### 3.5.2 The homogeneous case

### 3.5.3 The non-homogeneous case