# Boundary Value Problems for Partial Differential Equations

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## 1 Introduction and preliminaries

### 1.1 Definitions of partial differential equations

**Definition 1.1** (Notations of partial derivatives). *For $f(x)$ with one variable $x$, we know $f'(x) = \frac{df}{dx}$. For $u(x, y)$, we introduce partial derivatives as*

$$\frac{\partial u}{\partial x} = \left.\frac{du}{dx}\right|_{y \text{ is fixed}} = \partial_x u = u_x. \tag{1.1}$$

*Similarly,*

$$\frac{\partial^2 u}{\partial x^2} = \partial_x^2 u = \partial_{xx} u = u_{xx} \tag{1.2}$$

*Example* 1.1. For $u(x, y) = xy^2$, we have

$$\frac{\partial u}{\partial x} = \partial_x u = u_x = y^2, \qquad \frac{\partial u}{\partial y} = \partial_y u = u_y = 2xy,$$

$$\frac{\partial^2 u}{\partial x^2} = \partial_x^2 u = \partial_{xx} u = u_{xx} = 0, \qquad \frac{\partial^2 u}{\partial y^2} = \partial_y^2 u = \partial_{yy} u = u_{yy} = 2x$$

**Definition 1.2** (Definition of general PDEs). *Given a function $u = u(x, y)$ of two variables, (similarly $u = u(x_1, \cdots, x_n)$ of $n$ variables) and an expression $F(u_{xx}, u_{xy}, u_{yy}, u_x, u_y, u, x, y)$ of partial derivatives of $u$, the following equation is a <u>partial differential equation</u>, abbreviated as <u>PDE</u>.*

$$F(u_{xx}, u_{xy}, u_{yy}, u_x, u_y, u, x, y) = 0 \tag{1.3}$$

*In the future, we may also use the notation $F[u]$ to represent $F(u_{xx}, u_{xy}, u_{yy}, u_x, u_y, u, x, y)$. And (1.3) can be rewritten as*

$$F[u] = 0. \tag{1.4}$$

*Remark* 1.3. $F[u]$ may also involve derivatives of order $\geq 2$, but we do not discuss it in this course.

*Example* 1.2 (Examples of PDEs). Here are some examples of PDEs.

$$u_{xx} - u_y = 0 \quad \text{(the heat equation)}$$
$$u_{xx} - u_{yy} = 0 \quad \text{(the wave equation)}$$
$$u_{xx} + u_{yy} = 0 \quad \text{(Laplace's equation)} \tag{1.5}$$
$$u_x + u_y = 0 \quad \text{(the transport equation)}$$
$$u_x + uu_y = 0 \quad \text{(the Burgers equation)}$$

**Definition 1.4** (Order of PDEs). *The __order__ of a PDE is the order of the highest-order derivative in the equation. In* (1.5)*, the first three PDEs are second order, and the last two are first order.*

**Definition 1.5** (Linear PDEs). *Given a PDE $F[u] = 0$, if it satisfies*

$$F[u + v] = F[u] + F[v] \text{ and } F[cu] = cF[u], \tag{1.6}$$

*then we say that $F[u] = 0$ is a __linear PDE__. In* (1.5)*, the first four PDEs are linear, while the last one is not.*

We have the following proposition which characterizes all second order linear PDEs,

**Proposition 1.6.** *The second-order linear PDEs can always be written as*

$$a(x, y)u_{xx} + b(x, y)u_{xy} + c(x, y)u_{yy} + d(x, y)u_x + e(x, y)u_y + f(x, y)u = g(x, y), \tag{1.7}$$

*We assume $a^2 + b^2 + c^2 \neq 0$ for any $x$, $y$ (at least one of $a, b, c$ is nonzero).*

*Proof.* This go beyond the scope of this course. $\qquad\square$

**Definition 1.7.** *We call $a, b, c, d, e, f$ __coefficients__ and $g$ __source term__.*

## 1.2  Classification of second-order PDEs

In this course, we will mainly consider second-order linear PDEs.

$$a(x, y)u_{xx} + b(x, y)u_{xy} + c(x, y)u_{yy} + d(x, y)u_x + e(x, y)u_y + f(x, y)u = g(x, y), \tag{1.8}$$

These equations are classified as follows by the coefficients $a, b, c$.

**Definition 1.8** (Classification of PDEs)**.** *The second-order linear PDEs* (1.8) *are classified as* elliptic, parabolic *and* hyperbolic *by the following,*

$$\begin{cases} 4ac - b^2 > 0 & elliptic \\ 4ac - b^2 = 0 & parabolic \\ 4ac - b^2 < 0 & hyperbolic \end{cases} \tag{1.9}$$

*where we note that* $a, b, c$ *are functions of* $x, y$ *and the inequalities in* (1.9) *is required to be true for any* $x, y$.

## 1.3 Review of ODEs

**Definition 1.9** (Separable ODEs)**.** *The following ODE is the* separable *ODE*

$$y' + p(x)y = 0. \tag{1.10}$$

**Theorem 1.10.** *Separable ODE can be solved in the following way*

$$y' + p(x)y = 0 \quad \Rightarrow \quad \frac{dy}{dx} + p(x)y = 0 \quad \Rightarrow \quad \frac{dy}{y} = -p(x)dx \quad \Rightarrow \quad \int \frac{dy}{y} = -\int p(x)dx$$

*and the solution is*

$$y(x) = Ce^{-\int p(x)dx} \tag{1.11}$$

*Proof.* There is nothing to prove. $\square$

*Example* 1.3. Let us solve the following ODEs

$$y' = -6xy$$

Apply the procedure in Theorem 1.10

$$y' = -6xy \quad \Rightarrow \quad \frac{dy}{y} = -6xdx \quad \Rightarrow \quad \int \frac{dy}{y} = -\int 6xdx \quad \Rightarrow \quad \ln|y| = -3x^2 + C'$$

Therefore, the solution is

$$y = Ce^{-3x^2}.$$

where $C(= \pm e^{C'})$ is an arbitrary constant.

**Definition 1.11** (Linear ODEs)**.** *The following ODE is the* linear *ODE*

$$y' + p(x)y = q(x). \tag{1.12}$$

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**Theorem 1.12.** *Linear ODE can be solved by the following procedure*

1. *Solve the corresponding separable equation $y' - p(x)y = 0$ to obtain a solution $\hat{y} = e^{\int p(x)dx}$.*

2. *Multiply the linear ODE by $\hat{y}$ and rewrite the ODE*

$$y' + p(x)y = q(x) \quad \Rightarrow \quad \hat{y}(y' + p(x)y) = \hat{y}q(x) \quad \Rightarrow \quad (\hat{y}y)' = \hat{y}q(x)$$

3. *Integrate the above equation*

$$\hat{y}y = \int \hat{y}q(x)dx + C \quad \Rightarrow \quad y = \frac{1}{\hat{y}}\left(\int \hat{y}q(x)dx + C\right)$$

$$\Rightarrow \quad y = e^{-\int p(x)dx}\left(\int q(x)e^{\int p(x)dx}dx + C\right)$$

*Proof.* There is nothing to prove. □

*Remark* 1.13. In (2), we applied the following equation

$$(\hat{y}y)' = \hat{y}(y' + p(x)y)$$

which is a corollary of the Leibniz rule.

$$(\hat{y}y)' = \hat{y}'y + \hat{y}y' = \hat{y}y' + p(x)\hat{y}y = \hat{y}(y' + p(x)y)$$

*Example* 1.4. $(x^2 + 1)y' + 3xy = 6x$, $y(0) = 3$ is solved as $y(x) = 2 + (x^2 + 1)^{-3/2}$ by the procedure in Theorem 1.12.

To apply Theorem 1.12, we divide both sides by $(x^2 + 1)$.

$$(x^2 + 1)y' + 3xy = 6x \quad \Rightarrow \quad y' + \frac{3x}{x^2 + 1}y = \frac{6x}{x^2 + 1}$$

then we apply the three steps in Theorem 1.12.

1. Solve the corresponding separable equation $y' - \frac{3x}{x^2+1}y = 0$ to obtain a solution $(x^2 + 1)^{\frac{3}{2}}$.

2. Multiply the linear ODE by $(x^2 + 1)^{\frac{3}{2}}$ and rewrite the ODE

$$y' + \frac{3x}{x^2 + 1}y = \frac{6x}{x^2 + 1} \quad \Rightarrow \quad ((x^2 + 1)^{\frac{3}{2}}y)' = 6x(x^2 + 1)^{\frac{1}{2}}$$

3. Integrate the above equation

$$((x^2 + 1)^{\frac{3}{2}}y)' = 6x(x^2 + 1)^{\frac{1}{2}} \quad \Rightarrow \quad y = (x^2 + 1)^{-\frac{3}{2}}\left(\int 6x(x^2 + 1)^{\frac{1}{2}} + C\right)$$

$$\Rightarrow \quad y = 2 + C(x^2 + 1)^{-\frac{3}{2}}$$

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We note that the solutions with undetermined constant $C$ are called general solutions. Finally, we apply the initial condition $y(0) = 3$ to obtain $C = 1$, so the solution of the initial value problem is

$$y = 2 + (x^2 + 1)^{-\frac{3}{2}}.$$

**Definition 1.14** (Second order ODEs)**.** *The* constant coefficient second order ODEs *are the following equations*

$$ay'' + by' + cy = 0 \ (a \neq 0). \tag{1.13}$$

**Theorem 1.15.** *Constant coefficient second order ODEs can be solved by the following procedure*

1. *Solve the* characteristic equation $a\lambda^2 + b\lambda + c = 0$ *to get two solutions $\lambda_1$ and $\lambda_2$.*

2. *If $\lambda_1 \neq \lambda_2$, the general solution is*

$$y(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} \tag{1.14}$$

3. *If $\lambda_1 = \lambda_2 = \lambda$, the general solution is*

$$y(x) = (C_1 + C_2 x)e^{\lambda x} \tag{1.15}$$

4. *If $\lambda_1, \lambda_2$ are complex roots $\alpha \pm i\beta$, apply the Euler's formula to rewrite (1.14)*

$$y(x) = C_1 e^{(\alpha + i\beta)x} + C_2 e^{(\alpha - i\beta)x} = e^{\alpha x} \left( c_1 \cos \beta x + c_2 \sin \beta x \right) \tag{1.16}$$

*where $c_1 = C_1 + C_2$, $c_2 = i(C_1 - C_2)$.*

*Proof.* This was explained in your ODE course. $\qquad\square$

*Example* 1.5. $y'' - 3y' + 2y = 0$ is solved as $y(x) = C_1 e^x + C_2 e^{2x}$ by the procedure in Theorem 1.15.

1. Solve the characteristic equation $\lambda^2 - 3\lambda + 2 = 0$ to get two solutions $\lambda_1 = 1$ and $\lambda_2 = 2$.

2. Since $\lambda_1 \neq \lambda_2$, by (1.14), the general solution is

$$y(x) = C_1 e^x + C_2 e^{2x} \tag{1.17}$$

*Example* 1.6. $y'' + y = 0$ is solved as $y(x) = C_1 \cos(x) + C_2 \sin(x)$ by the procedure in Theorem 1.15.

1. Solve the characteristic equation $\lambda^2 + 1 = 0$ to get two solutions $\lambda_1 = i$ and $\lambda_2 = -i$.

2. Since $\lambda_1, \lambda_2$ are complex, by (1.16), the general solution is

$$y(x) = C_1 \cos x + C_2 \sin x \tag{1.18}$$

*Example* 1.7. $y'' + 2y' + y = 0$ is solved as $y(x) = (C_1 + C_2 x)e^{-x}$ by the procedure in Theorem 1.15.

1. Solve the characteristic equation $\lambda^2 + 2\lambda + 1 = 0$ to get $\lambda_1 = \lambda_2 = -1$.

2. Since $\lambda_1, \lambda_2$ are equal, by (1.15), the general solution is

$$y(x) = (C_1 + C_2 x)e^{-x} \tag{1.19}$$

## 1.4  Separation of variables

Many linear PDEs can be reduced to linear ODEs with the method of separation of variables, described below.

We take the Laplace's equation

$$u_{xx} + u_{yy} = 0 \tag{1.20}$$

with boundary condition

$$u(0, y) = 0, \quad u(L, y) = 0, \quad u(x, 0) = 0, \quad u(x, L) = \varphi(x). \tag{1.21}$$

as an example.

We are looking for a separated solution. Substitute into (1.20), then we get

$$X''Y + XY'' = 0 \quad \Rightarrow \quad \frac{X''}{X} = -\frac{Y''}{Y}$$

The following lemma implies that $X''/X$ and $Y''/Y$ are constants.

**Lemma 1.16.** $f(x) = g(y)$ *implies that* $f(x) = g(y) = const,$

*Proof.* $f(x) = g(y) \Rightarrow f'(x) = \partial_x(g(y)) = 0 \Rightarrow f(x) = const.$ □

Let $\lambda$ be a constant and we write

$$X'' + \lambda X = 0, \quad Y'' - \lambda Y = 0.$$

We call $\lambda$ the separation constant. At this moment $\lambda$ is arbitrary. Thus the PDE was reduced to two ODEs.

### 1.4.1 Solving separated solutions

If $\lambda = 0$, then two ODEs have the following linearly independent solutions.

$$X = 1, x, \quad Y = 1, y. \tag{1.22}$$

If $\lambda \neq 0$, then two ODEs have the following linearly independent solutions.

$$X = e^{\sqrt{-\lambda}x}, e^{-\sqrt{-\lambda}x}, \quad Y = e^{\sqrt{\lambda}y}, e^{-\sqrt{\lambda}y}. \tag{1.23}$$

In either case, the solution is given by superpositions:

$$u = \begin{cases} (A_1 x + A_2)(B_1 y + B_2), & \lambda = 0 \\ \left(A_1 e^{\sqrt{-\lambda}x} + A_2 e^{-\sqrt{-\lambda}x}\right)\left(B_1 e^{\sqrt{\lambda}y} + B_2 e^{-\sqrt{\lambda}y}\right), & \lambda \neq 0 \end{cases} \tag{1.24}$$

where $A_1, A_2, B_1, B_2$ are constants.

For $\lambda > 0$, by writing $\lambda = k^2 (k > 0)$ we have

$$u(x, y) = \left(A_1 e^{ikx} + A_2 e^{-ikx}\right)\left(B_1 e^{ky} + B_2 e^{-ky}\right), \tag{1.25}$$

and for $\lambda < 0$, by writing $\lambda = -l^2 (l > 0)$ we have

$$u(x, y) = \left(A_1 e^{lx} + A_2 e^{-lx}\right)\left(B_1 e^{ily} + B_2 e^{-ily}\right). \tag{1.26}$$

Therefore, we get

$$u = \begin{cases} (A_1 x + A_2)(B_1 y + B_2), \\ \left(A_1 e^{ikx} + A_2 e^{-ikx}\right)\left(B_1 e^{ky} + B_2 e^{-ky}\right), \\ \left(A_1 e^{lx} + A_2 e^{-lx}\right)\left(B_1 e^{ily} + B_2 e^{-ily}\right). \end{cases} \tag{1.27}$$

Instead of (1.23), we can also choose

$$X = \cos(\sqrt{\lambda}x), \sin(\sqrt{\lambda}x), \quad Y = \cosh(\sqrt{\lambda}y), \sinh(\sqrt{\lambda}y). \tag{1.28}$$

In this case, we have

$$u(x, y) = (A_1 \cos(kx) + A_2 \sin(kx))(B_1 \cosh(ky) + B_2 \sinh(ky)) \tag{1.29}$$

$$u(x, y) = (A_1 \cosh(lx) + A_2 \sinh(lx))(B_1 \cos(ly) + B_2 \sin(ly)) \tag{1.30}$$

Note that (1.29) becomes (1.25) and (1.30) becomes (1.26) by redefining the coefficients. We call solutions such as (1.24) through (1.30) separated solutions because they are given in the form $u(x, y) = X(x)Y(y)$.

The final result is

$$u = \begin{cases} (A_1 x + A_2)(B_1 y + B_2), \\ (A_1 \cos(kx) + A_2 \sin(kx))(B_1 \cosh(ky) + B_2 \sinh(ky)), \\ (A_1 \cosh(lx) + A_2 \sinh(lx))(B_1 \cos(ly) + B_2 \sin(ly)). \end{cases} \qquad (1.31)$$

### 1.4.2 Solving the boundary value problem

The separation constant $\lambda$ and coefficients $A_1, A_2, B_1, B_2$ are partially determined by boundary conditions that in the region $0 < x < L, 0 < y < \infty$, $u(x, y)$ satisfies that

$$u(0, y) = 0, \quad u(L, y) = 0, \quad u(x, 0) = 0, \quad u(x, L) = \varphi(x). \qquad (1.32)$$

Let us only consider the first three conditions.

$$u(0, y) = 0, \quad u(L, y) = 0, \quad u(x, 0) = 0. \qquad (1.33)$$

In (1.31), we have three cases.

1. $u = (A_1 x + A_2)(B_1 y + B_2)$. In this case, $u$ satisfies boundary conditions $u(0, y) = u(L, y) = 0$ in (1.33) if and only if $u = 0$.

2. $u = (A_1 \cos(kx) + A_2 \sin(kx))(B_1 \cosh(ky) + B_2 \sinh(ky))$. In this case, $u$ satisfies boundary conditions $u(0, y) = u(L, y) = 0$ when $A_1 = 0$ and $k = n\pi/L$, where $n$ is an integer. Furthermore we find $B_1 = 0$ by the condition $u(x, 0) = 0$. That is, the solution $A_2 B_2 \sin(kx) \sinh(ky)$ with $k = n\pi/L$ ($n = 0, \pm 1, \pm 2, \ldots$) satisfies the boundary conditions.

3. $u = (A_1 \cosh(lx) + A_2 \sinh(lx))(B_1 \cos(ly) + B_2 \sin(ly))$. In this case, $u$ satisfies boundary conditions $u(0, y) = u(L, y) = 0$ only when $A_1 = A_2 = 0$. That is, only the solution $u = 0$ satisfies the boundary conditions.

Therefore we obtain the following separated solutions of Laplace's equation satisfying the boundary conditions.

$$u(x, y) = A \sin \frac{n\pi x}{L} \sinh \frac{n\pi y}{L}, \quad n = 1, 2, \ldots, \qquad (1.34)$$

### 1.4.3 Matching with the last boundary condition

Now we consider the last boundary condition $u(x, L) = \varphi(x)$ in (1.33). For arbitrary $\varphi(x)$, our separated solution $A \sin \frac{n\pi x}{L} \sinh \frac{n\pi y}{L}$ cannot match with this boundary condition. However, we can

use a linear combination of this separated solution to generate more solutions,

$$u(x, y) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L} \sinh \frac{n\pi y}{L}. \tag{1.35}$$

To match with $u(x, L) = \varphi(x)$, we take $y = L$ in (1.35),

$$\varphi(x) = u(x, L) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L} \sinh \frac{n\pi L}{L}. \tag{1.36}$$

Therefore, we get

$$\varphi(x) = \sum_{n=1}^{\infty} A_n \sinh n\pi \sin \frac{n\pi x}{L}. \tag{1.37}$$

Then $A_n$ can be solved from the sine Fourier coefficients introduced in section 2.2.3.

The result is

$$A_n = \frac{2}{L \sinh n\pi} \int_0^L \varphi(x) \sin \frac{n\pi x}{L} dx \tag{1.38}$$

Substitute into (1.35), then we solve Laplace's equation (1.20) with boundary condition.

# 2 Introduction to Fourier series

Every function $f(x)$ on domain $-L < x < L$ can be represented by a series of the form

$$A_0 + \sum_{n=1}^{\infty} \left( A_n \cos \frac{n\pi x}{L} + B_n \sin \frac{n\pi x}{L} \right)$$

This is called the Fourier series. We will first discuss how to compute the coefficients $A_0, A_1, B_1,$ ... in terms of $f(x)$, then we discuss several properties and variants of the Fourier series.

## 2.1 Definition of Fourier series

The Fourier series is an infinite sum of trigonometric functions defined as the following

**Definition 2.1** (Fourier series). *Let $A_0, A_1, B_1, \ldots$ be constants. The series below is called a Fourier series.*

$$A_0 + \sum_{n=1}^{\infty} \left( A_n \cos \frac{n\pi x}{L} + B_n \sin \frac{n\pi x}{L} \right) \tag{2.1}$$

It turns out that every function $f(x)$ can be written into an infinite sum of trigonometric functions. As suggested by the following theorem,

**Theorem 2.2.** *Every function $f(x)$ on domain $-L < x < L$ can be represented by a Fourier series. In other word, for any $f(x)$ on domain $-L < x < L$, there exists coefficients $A_0, A_1, B_1, \ldots$ such that*

$$f(x) = A_0 + \sum_{n=1}^{\infty} \left( A_n \cos \frac{n\pi x}{L} + B_n \sin \frac{n\pi x}{L} \right). \tag{2.2}$$

## 2.2 Fourier series and orthogonality

### 2.2.1 Orthogonality of trigonometric function

We want to compute the coefficients $A_0, A_1, B_1, \ldots$ in terms of $f(x)$. The following observation may be helpful.

**Question.** Given several orthogonal vector $e_1, e_2, \ldots, e_n$, assume that we have a decomposition

$$v = v_1 e_1 + v_2 e_2 + \cdots + v_n e_n \tag{2.3}$$

How do we compute the coefficients $v_1, v_2, \ldots, v_n$ in terms of $v$?

**Answer.** Assume that we want to solve the coefficient $v_j$, then we take inner product of (2.3) with $e_j$. If we do this, by orthogonality $\langle e_i, e_j \rangle = 0$ (if $i \neq j$) and $\langle e_j, e_j \rangle = |e_j|^2$, then we get

$$\begin{aligned}
\langle v, e_j \rangle &= \langle v_1 e_1 + v_2 e_2 + \cdots + v_n e_n, e_j \rangle \\
&= v_j |e_j|^2.
\end{aligned} \tag{2.4}$$

Therefore, we solve the coefficient $v_j = \frac{\langle v, e_j \rangle}{|e_j|^2}$.

The following notation is useful.

**Definition 2.3** (Kronecker delta). *The <u>Kronecker delta</u> $\delta_{mn}$ is defined as*

$$\delta_{mn} = \begin{cases} 1, & m = n \\ 0, & m \neq n \end{cases} \tag{2.5}$$

*Using this notation orthogonality of $e_1, e_2, \ldots, e_n$ is equivalent to $\langle e_m, e_n \rangle = |e_m|^2 \delta_{mn}$.*

The following theorem implies that sine and cosine functions are similar to orthogonal vectors.

**Theorem 2.4** (Orthogonality of trigonometric functions). *Let $n, m \geq 0$ be integers. We assume*

$L > 0$. *The following orthogonality relations hold.*

$$\int_{-L}^{L} \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx = \begin{cases} 2L & (n = m = 0), \\ L\delta_{nm} & (\textit{otherwise}), \end{cases}$$

$$\int_{-L}^{L} \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}dx = \begin{cases} 0 & (n = m = 0), \\ L\delta_{nm} & (\textit{otherwise}), \end{cases} \qquad (2.6)$$

$$\int_{-L}^{L} \sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx = 0 \quad (\textit{all } n, m).$$

*Proof.* We only prove the orthogonality for the first equation which only involves cosine. The proof of other equations is left as homework.

Let us first start with the following observation.

*Claim.* If $n \neq 0$, then we have

$$\int_{-L}^{L} \cos\frac{n\pi x}{L}dx = 0 \qquad (2.7)$$

This claim is true because

$$\begin{aligned} \int_{-L}^{L} \cos\frac{n\pi x}{L}dx &= \left[\frac{L}{n\pi}\sin\frac{n\pi x}{L}\right]_{-L}^{L} \\ &= \frac{L}{n\pi}(\sin n\pi - \sin(-n\pi)) = 0 \end{aligned} \qquad (2.8)$$

Now we start the proof of (2.6) for cosine.

**Case 1.** ($n = m = 0$) In this case, we have

$$\int_{-L}^{L} \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx = \int_{-L}^{L} 1 \cdot 1 dx = 2L$$

**Case 2.** (At least one of $n, m$ is nonzero). We have

$$\int_{-L}^{L} \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx = \frac{1}{2}\int_{-L}^{L}\left(\cos\frac{(n-m)\pi x}{L} + \cos\frac{(n+m)\pi x}{L}\right)dx$$

Since at least one of $n, m$ is nonzero, $n + m > 0$. The integral over $\cos\frac{(n+m)\pi x}{L}$ is 0 by (2.7). Therefore,

$$\int_{-L}^{L} \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx = \frac{1}{2}\int_{-L}^{L}\cos\frac{(n-m)\pi x}{L}dx$$

**Case 2.1.** ($n \neq m$) In this case, $\frac{1}{2}\int_{-L}^{L}\cos\frac{(n-m)\pi x}{L}dx = 0$ by (2.7). Therefore, we get

$$\int_{-L}^{L} \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx = 0 = \delta_{mn}.$$

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**Case 2.2.** $(n = m)$, then

$$\int_{-L}^{L} \cos \frac{n\pi x}{L} \cos \frac{m\pi x}{L} dx = \frac{1}{2} \int_{-L}^{L} \cos \frac{(n-m)\pi x}{L} dx = \frac{1}{2} \int_{-L}^{L} \cos \frac{0 \cdot \pi x}{L} dx = L.$$

Thus the orthogonality relations for $\int_{-L}^{L} \cos \frac{n\pi x}{L} \cos \frac{m\pi x}{L} dx$ is proved. The orthogonality relations for $\int_{-L}^{L} \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} dx$ and $\int_{-L}^{L} \sin \frac{n\pi x}{L} \cos \frac{m\pi x}{L} dx$ are similarly proved and left as homework. $\square$

### 2.2.2  Formula of Fourier coefficients

We can use the orthogonality relations (2.6) to compute $A_0, A_1, B_1, \ldots$

To determine $A_0$ in (2.2), we multiply $\cos \frac{0 \cdot \pi x}{L} = 1$ on both sides and integrate with respect to $x$:

$$\int_{-L}^{L} f(x) dx = \int_{-L}^{L} A_0 dx + \sum_{n=1}^{\infty} \int_{-L}^{L} \left( A_n \cos \frac{n\pi x}{L} + B_n \sin \frac{n\pi x}{L} \right) dx = \int_{-L}^{L} A_0 dx = 2A_0.$$

where we use the fact that $\int_{-L}^{L} \cos \frac{n\pi x}{L} dx = \int_{-L}^{L} \sin \frac{n\pi x}{L} dx = 0$. This is just the first equation in (2.6) with $m = 0$.

Therefore, we get an expression of $A_0$,

$$A_0 = \frac{1}{2L} \int_{-L}^{L} f(x) dx.$$

To determine $A_m$ $(m = 1, 2, \ldots)$ in (2.2), we multiply $\cos \frac{m\pi x}{L}$ on both sides and integrate with respect to $x$:

$$\int_{-L}^{L} f(x) \cos \frac{m\pi x}{L} dx = \int_{-L}^{L} A_0 \cos \frac{m\pi x}{L} dx + \sum_{n=1}^{\infty} \int_{-L}^{L} \left( A_n \cos \frac{n\pi x}{L} + B_n \sin \frac{n\pi x}{L} \right) \cos \frac{m\pi x}{L} dx$$

$$= \sum_{n=1}^{\infty} A_n \int_{-L}^{L} \cos \frac{n\pi x}{L} \cos \frac{m\pi x}{L} dx = \sum_{n=1}^{\infty} A_n L \delta_{nm} = L A_m.$$

Therefore, we get an expression of $A_m$,

$$A_m = \int_{-L}^{L} f(x) \cos \frac{m\pi x}{L} dx.$$

Similarly we can determine $B_m$ $(m = 1, 2, \ldots)$ in (2.2) by multiplying $\sin \frac{m\pi x}{L}$ on both sides and integrate with respect to $x$:

$$\int_{-L}^{L} f(x) \sin \frac{m\pi x}{L} dx = \int_{-L}^{L} A_0 \sin \frac{m\pi x}{L} dx + \sum_{n=1}^{\infty} \int_{-L}^{L} \left( A_n \cos \frac{n\pi x}{L} + B_n \sin \frac{n\pi x}{L} \right) \sin \frac{m\pi x}{L} dx$$

$$= \sum_{n=1}^{\infty} B_n \int_{-L}^{L} \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} dx = \sum_{n=1}^{\infty} B_n L \delta_{nm} = L B_m.$$

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Therefore, we get an expression of $B_m$,

$$B_m = \int_{-L}^{L} f(x) \sin \frac{m\pi x}{L} dx.$$

In summary, we have the following theorem.

**Theorem 2.5.** *The Fourier coefficients in $A_0, A_1, B_1, \ldots$ are given by*

$$A_0 = \frac{1}{2L} \int_{-L}^{L} f(x) dx,$$

$$A_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} dx, \qquad (2.9)$$

$$B_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} dx.$$

*Example* 2.1. Let us calculate the Fourier series of $f(x) = x, -L < x < L$.

*Solution.* We have

$$A_0 = \frac{1}{2L} \int_{-L}^{L} x dx = 0,$$

$$A_n = \frac{1}{L} \int_{-L}^{L} x \cos \frac{n\pi x}{L} dx = 0 \quad (x \cos \frac{n\pi x}{L} \text{ is an odd function}),$$

$$B_n = \frac{1}{L} \int_{-L}^{L} x \sin \frac{n\pi x}{L} dx = \frac{1}{L} \left[ -\frac{L}{n\pi} x \cos \frac{n\pi x}{L} \Big|_{-L}^{L} + \frac{L}{n\pi} \int_{-L}^{L} \cos \frac{n\pi x}{L} dx \right] = \frac{2L}{n\pi} (-1)^{n+1}.$$

Therefore we obtain

$$x = \sum_{n=1}^{\infty} \frac{2L}{n\pi} (-1)^{n+1} \sin \frac{n\pi x}{L}, \quad -L < x < L. \qquad (2.10)$$

□

We have two observations from the above example

**Proposition 2.6.** *An odd or even function only has* cos *or* sin *term in its Fourier series.*

*Proof.* We only consider the even case. The odd case can be proved similarly.

Since $f(x)$ is even, we know that $\frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} dx$ is odd. By (2.9), all the $B_n$ vanishes. Therefore, we only have the cos term in the Fourier series. □

**Proposition 2.7.** *Given a function $f(x)$ defined on $-L < x < L$, the Fourier series of $f(x)$ coincide with $f(x)$ only in the domain $-L < x < L$, unless $f(x)$ is a periodic function.*

*Proof.* We will not prove this proposition. Instead, we draw the graph of $y = x$ and its Fourier series

The Fourier series coincides with $y = x$ only in the domain $-L < x < L$. But the Fourier series is a periodic function since it is a sum of several periodic functions $\cos/\sin$. Since $y = x$ is not a periodic function, the Fourier series cannot agree with it for all $x$.
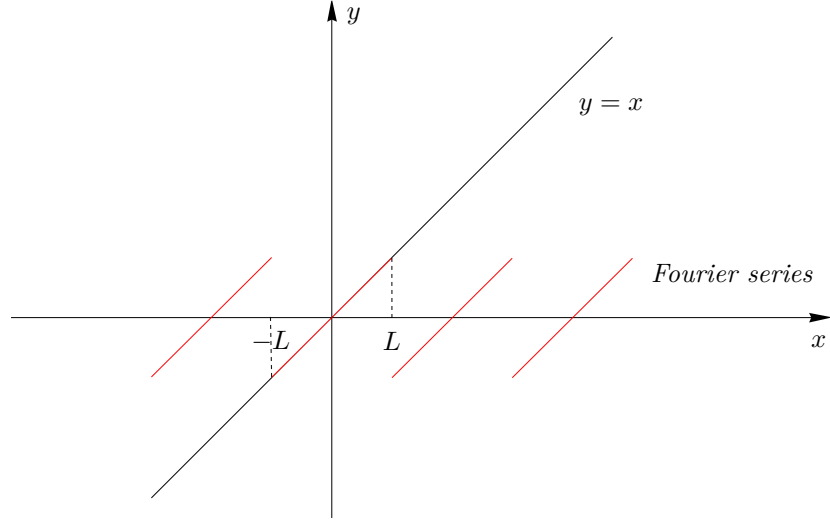


Figure 1: The graph of $y = x$ and its Fourier series.

$\square$

### 2.2.3 Solve the $A_n$ coefficients in (1.37)

Now we return to the separation of variable in section 1.4.3, where we derived

$$\varphi(x) = \sum_{n=1}^{\infty} A_n \sinh n\pi \sin \frac{n\pi x}{L}. \tag{2.11}$$

We want to solve $A_n$ from $\varphi(x)$.

As explained in section 2.4, on $0 < x < L$, we also have the following orthogonality of $\sin \frac{n\pi x}{L}$

$$\int_0^L \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} dx = \frac{L}{2} \delta_{nm}, \qquad n, m \geq 1. \tag{2.12}$$

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Therefore, if we want to solve $A_m$, we can multiply (2.11) by $\sin \frac{n\pi x}{L}$ and then integrate over $x$

$$\int_0^L \varphi(x) \sin \frac{m\pi x}{L} dx = \sum_{n=1}^{\infty} \int_0^L A_n \sinh n\pi \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} dx$$

$$= \sum_{n=1}^{\infty} A_n \sinh n\pi \int_0^L \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} dx = \sum_{n=1}^{\infty} A_n \sinh n\pi \frac{L}{2} \delta_{nm} = \frac{L}{2} A_m \sinh n\pi.$$

From this, we can solve $A_m$ and get

$$A_n = \frac{2}{L \sinh n\pi} \int_0^L \varphi(x) \sin \frac{n\pi x}{L} dx \qquad (2.13)$$

This is exactly (1.38).

## 2.3  Complex form of Fourier series

We have seen that every function defined on $-L < x < L$ can be rewritten as an infinite sum of $\cos \frac{n\pi x}{L}$ and $\sin \frac{n\pi x}{L}$. By Euler's formula $e^{ix} = \cos x + i \sin x$, we know that every trigonometric function is a linear combination of $e^{ix}$ and $e^{-ix}$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}, \qquad \sin x = \frac{e^{ix} - e^{-ix}}{2i}. \qquad (2.14)$$

Therefore, we have the following theorem.

**Theorem 2.8.** *In other word, for any $f(x)$ on domain $-L < x < L$, there exists coefficients $\dots, \alpha_{-1}, \alpha_0, \alpha_1, \alpha_2, \dots$ such that*

$$f(x) = \sum_{n=-\infty}^{\infty} \alpha_n e^{in\pi x/L}, \qquad (2.15)$$

*where $\alpha_n$ satisfies the following properties.*

*1. $\alpha_n$ is given by*

$$\alpha_n = \frac{1}{2L} \int_{-L}^L f(x) e^{-in\pi x/L} dx \qquad (2.16)$$

*2. If $f(x)$ is a real value function, then $\alpha_n = \bar{\alpha}_{-n}$, where $\bar{\alpha}_{-n}$ is the complex conjugate of $\alpha_{-n}$.*

*Proof.* Using Euler's formula (2.14), we can rewrite the Fourier series of $f$ as follows.

$$f(x) = A_0 + \sum_{n=1}^{\infty} \left( A_n \cos \frac{n\pi x}{L} + B_n \sin \frac{n\pi x}{L} \right)$$

$$= A_0 + \sum_{n=1}^{\infty} \left( A_n \left( \frac{e^{in\pi x/L} + e^{-in\pi x/L}}{2} \right) + B_n \left( \frac{e^{in\pi x/L} - e^{-in\pi x/L}}{2i} \right) \right)$$

$$= A_0 + \sum_{n=1}^{\infty} \left( \frac{A_n - iB_n}{2} e^{in\pi x/L} + \frac{A_n + iB_n}{2} e^{-in\pi x/L} \right).$$

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If we define
$$\alpha_0 = A_0, \quad \alpha_n = \frac{A_n - iB_n}{2}, \quad \alpha_{-n} = \frac{A_n + iB_n}{2}, \quad (n > 0) \tag{2.17}$$

then we obtain
$$f(x) = \alpha_0 + \sum_{n=1}^{\infty} \left( \alpha_n e^{in\pi x/L} + \alpha_{-n} e^{-in\pi x/L} \right) = \sum_{n=-\infty}^{\infty} \alpha_n e^{in\pi x/L}.$$

If $f(x)$ is a real value function, then $A_n$ and $B_n$ are real numbers. From (2.17), we know that $\alpha_n = \bar{\alpha}_{-n}$.

The formula of $\alpha_n$ is also a corollary of (2.17). When $n > 0$,

$$\alpha_n = \frac{A_n - iB_n}{2} = \frac{1}{2L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} dx - \frac{i}{2L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} dx$$
$$= \frac{1}{2L} \int_{-L}^{L} f(x) e^{-in\pi x/L} dx$$

When $n = -m < 0$,

$$\alpha_n = \alpha_{-m} = \frac{A_m + iB_m}{2} = \frac{1}{2L} \int_{-L}^{L} f(x) \cos \frac{m\pi x}{L} dx + \frac{i}{2L} \int_{-L}^{L} f(x) \sin \frac{m\pi x}{L} dx$$
$$= \frac{1}{2L} \int_{-L}^{L} f(x) e^{im\pi x/L} dx = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-in\pi x/L} dx$$

When $n = 0$,
$$\alpha_0 = A_0 = \frac{1}{2L} \int_{-L}^{L} f(x) dx = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-i0\pi x/L} dx$$

Therefore, we have completed the proof. $\qquad\square$

$e^{in\pi x/L}$ also have orthogonality relations.

**Theorem 2.9** (Orthogonality relations of complex Fourier series).

$$\int_{-L}^{L} e^{in\pi x/L} e^{-im\pi x/L} dx = \int_{-L}^{L} e^{i(n-m)\pi x/L} dx = 2L\delta_{mn}.$$

*Proof.* If $n = m$,
$$\int_{-L}^{L} e^{i(n-m)\pi x/L} dx = \int_{-L}^{L} 1 dx = 2L = 2L\delta_{mn}.$$

If $n \neq m$,
$$\int_{-L}^{L} e^{i(n-m)\pi x/L} dx = \frac{L}{i\pi(n-m)} \left[ e^{i(n-m)\pi x/L} \right]_{-L}^{L} = 0 = 2L\delta_{mn}.$$

We have completed the proof. $\qquad\square$

The formula of $\alpha_n$ can also be computed using orthogonality.

Assume that we want to compute $\alpha_m$ in

$$f(x) = \sum_{n=-\infty}^{\infty} \alpha_n e^{in\pi x/L}.$$

Multiply both side by $e^{-im\pi x/L}$ and then we get

$$\int_{-L}^{L} f(x) e^{-im\pi x/L} dx = \sum_{n=-\infty}^{\infty} \alpha_n \int_{-L}^{L} e^{in\pi x/L} e^{-im\pi x/L} dx \sum_{n=-\infty}^{\infty} \alpha_n 2L\delta_{mn} = 2L\alpha_m.$$

Solve $\alpha_m$, then we prove (2.16) again.

## 2.4 Fourier cosine and sine series

A function defined on $-L < x < L$ can be rewritten as an infinite sum of $\cos\frac{n\pi x}{L}$ and $\sin\frac{n\pi x}{L}$. Notice that $-L < x < L$ is the period of $\cos\frac{n\pi x}{L}$ and $\sin\frac{n\pi x}{L}$. If a function $f(x)$ is defined in the interval $0 < x < L$, which represents half of its period, then it is possible to express $f(x)$ exclusively as a series of cosine terms, $\cos\frac{n\pi x}{L}$, or alternatively, only as a series of sine terms, $\sin\frac{n\pi x}{L}$. This is the Fourier cosine/sine series.

The idea is that, given a function $f(x)$ defined on $0 < x < L$, we can extend this function as an even/odd function on $-L < x < L$. Then we compute the Fourier series of the extended function. The Fourier series only contains $\cos\frac{n\pi x}{L}$ or $\sin\frac{n\pi x}{L}$ terms by Proposition 2.6.

### 2.4.1 Fourier cosine series

We define the even extension $f_E(x)$ of $f(x)$ as

$$f_E(x) = \begin{cases} f(x), & 0 < x < L \\ 0, & x = 0, \\ f(-x), & -L < x < 0. \end{cases} \tag{2.18}$$

We note that $f_E(x)$ is even. Indeed the value $f_E(0)$ is arbitrary and not necessarily zero. The Fourier series is given by

$$f_E(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L}, \quad -L < x < L,$$

where

$$A_0 = \frac{1}{2L} \int_{-L}^{L} f_E(x) dx = \frac{1}{L} \int_0^L f(x) dx,$$

$$A_n = \frac{1}{2L} \int_{-L}^{L} f_E(x) \cos\frac{n\pi x}{L} dx = \frac{2}{L} \int_0^L f(x) \cos\frac{n\pi x}{L} dx.$$

On the interval $0 < x < L$ we have

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L}, \quad 0 < x < L, \tag{2.19}$$

where

$$A_0 = \frac{1}{L} \int_0^L f(x)dx, \quad A_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx. \tag{2.20}$$

**Definition 2.10.** *The series* (2.19) *is called the Fourier cosine series.*

For Fourier cosine series, we also have orthogonality relations and (2.20) can be computed from these orthogonality relations.

**Theorem 2.11** (Orthogonality relations of cosine Fourier series).

$$\int_0^L \cos \frac{n\pi x}{L} \cos \frac{m\pi x}{L} dx = \begin{cases} L & (n = m = 0), \\ \dfrac{L}{2} \delta_{nm} & (otherwise), \end{cases} \tag{2.21}$$

*Example* 2.2. Let us compute the Fourier cosine series of $f(x) = x$, $0 < x < L$.

*Solution.* We can directly apply (2.20). But let us try the even extension method.

We extend $f$ as

$$f_E(x) = \begin{cases} x, & 0 < x < L, \\ 0, & x = 0, \\ -x, & -L < x < 0. \end{cases}$$

Indeed $f_E(x) = |x|$.

$$A_0 = \frac{1}{2L} \int_{-L}^{L} f_E(x)dx = \frac{1}{L} \int_0^L xdx = \frac{L}{2},$$

$$A_n = \frac{1}{2L} \int_{-L}^{L} f_E(x) \cos \frac{n\pi x}{L} dx = \frac{2}{L} \int_0^L x \cos \frac{n\pi x}{L} dx$$

$$= \frac{2}{L} \left[ \frac{L}{n\pi} x \sin \frac{n\pi x}{L} \Big|_0^L - \frac{L}{n\pi} \int_0^L \sin \frac{n\pi x}{L} dx \right] = \frac{2L}{(n\pi)^2} \left( (-1)^n - 1 \right).$$

Therefore,

$$x = \frac{L}{2} + \frac{2L}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} \cos \frac{n\pi x}{L}, \quad 0 < x < L.$$

$\square$

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### 2.4.2 Fourier sine series

We define the odd extension $f_O(x)$ as

$$f_O(x) = \begin{cases} f(x), & 0 < x < L \\ 0, & x = 0, \\ -f(-x), & -L < x < 0. \end{cases} \tag{2.22}$$

We note that $f_O(x)$ is odd. The Fourier series is given by

$$f_O(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}, \quad -L < x < L,$$

where

$$B_n = \frac{1}{L} \int_{-L}^{L} f_O(x) \sin \frac{n\pi x}{L} dx = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx.$$

On the interval $0 < x < L$ we have

$$f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}, \quad 0 < x < L, \tag{2.23}$$

where

$$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx \tag{2.24}$$

**Definition 2.12.** *The series* (2.23) *is called the Fourier cosine series.*

For Fourier sine series, we also have orthogonality relations and (2.24) can be computed from these orthogonality relations.

**Theorem 2.13** (Orthogonality relations of sine Fourier series)**.**

$$\int_0^L \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} dx = \begin{cases} 0 & (n = m = 0), \\ \frac{L}{2}\delta_{nm} & (otherwise), \end{cases} \tag{2.25}$$

*Example* 2.3. The Fourier sine series of $f(x) = x$, $0 < x < L$, is obtained through the odd extension $f_O(x)$. The odd extension $f_O(x)$ is again $x$ and its Fourier series has been computed in (2.10).

## 2.5 Convergence of Fourier series

In this section, we study the convergence of Fourier series of piecewise continuous functions.

**Definition 2.14** (Left and right limits). *For a given $f(x)$, let us write*

$$f(x + 0) = \lim_{\varepsilon \to 0} f(x + \varepsilon), \quad f(x - 0) = \lim_{\varepsilon \to 0} f(x - \varepsilon), \tag{2.26}$$

*where $\varepsilon > 0$.*

**Definition 2.15** (Piecewise continuous). *A function $f(x)$ defined on $a < x < b$, is said to be piecewise continuous if there is a finite set of points $a = x_0 < x_1 < \cdots < x_p < x_{p+1} = b$ such that $f(x)$ is continuous at $x \neq x_i$ $(i = 1, \ldots, p)$, $f(x_i + 0)(i = 0, \ldots, p)$ exists, and $f(x_i - 0)$ $(i = 1, \ldots, p + 1)$ exists.*

**Definition 2.16** (Piecewise smooth). *A function $f(x), a < x < b$, is said to be piecewise smooth if $f(x)$ and all of its derivatives are piecewise continuous.*

*Example 2.4.* The function $f(x) = |x|, -L < x < L$, is piecewise smooth. The function $f(x) = x^2 \sin(1/x), -L < x < L$, is piecewise continuous but is not piecewise smooth because $\lim_{\varepsilon \to 0} f'(0 \pm \varepsilon)$ does not exist. The function $f(x) = 1/(x^2 - L^2)$, $-L < x < L$, is not piecewise continuous because $f(-L + 0)$ and $f(L - 0)$ are not finite.

**Definition 2.17** (Convergence). *Given a Fourier series $f(x) = A_0 + \sum_{n=1}^{\infty}[A_n \cos(n\pi x/L) + B_n \sin(n\pi x/L)]$, we define*

1. **Partial sums.** *The partial sum, denoted by $f_N(x)$, is defined to be*

$$f_N(x) = A_0 + \sum_{n=1}^{N} [A_n \cos(n\pi x/L) + B_n \sin(n\pi x/L)]. \tag{2.27}$$

2. **Convergence at $x$.** *We say the Fourier is convergent at $x$ if*

$$\lim_{N \to \infty} f_N(x) = f(x). \tag{2.28}$$

3. **Uniformly convergence.** *We say the Fourier is uniformly convergent if*

$$\lim_{N \to \infty} \max_{x \in [-L, L]} |f_N(x) - f(x)| = 0. \tag{2.29}$$

*This means that the convergence is equally well for all $x$.*

**Theorem 2.18** (Convergence theorem). *Let $f(x)$, $-L < x < L$, be piecewise smooth. Then the Fourier series of $f$ converges for all $x$ to the value $\frac{1}{2}[\bar{f}(x+0) + \bar{f}(x-0)]$, where $\bar{f}$ is the $2L$-periodic function which equals to $f$ on $-L < x < L$.*

*If $f(x)$ is continuous on $[-L, L]$ and $f(-L) = f(L)$ in addition to the conditions assumed in the above theorem, then the Fourier series uniformly converges.*

*Example* 2.5. Here are two examples of convergence.

1. The Fourier series of $f(x) = |x|$, $-L < x < L$, (see Example 2.2) uniformly converges.

2. The Fourier series of $f(x) = x$, $-L < x < L$ does not uniformly converge but converges at any point to $f(x)$ except for $x = -L$, $L$.

As in the following picture, the Fourier series of $f(x) = x$, $-L < x < L$ has a very bad convergence near $x = -L$, $L$, but the series of $f(x) = |x|$ has much better convergence.

**TODO: add a picture**

## 2.6  Parseval's Theorem and Mean Square Error

### 2.6.1  The Parseval's Theorem for Fourier series

For orthogonal vectors $e_1, \ldots, e_n$, $e_i \perp e_j$, their linear combination $v = v_1 e_1 + \cdots + v_n e_n$ satisfies the Pythagorean theorem

$$|v|^2 = |v_1 e_1 + \cdots + v_n e_n|^2 = |v_1 e_1|^2 + \cdots + |v_n e_n|^2 \tag{2.30}$$

The proof of (2.30) is given by

$$|v|^2 = \left\langle \sum_{j=1}^n v_j e_j, \sum_{k=1}^n v_k e_k \right\rangle = \sum_{j=1}^n \sum_{k=1}^n v_j v_k \underbrace{\langle e_j, e_k \rangle}_{=0 \ if \ j \neq k} = \sum_{j=1}^n v_j^2 |e_j|^2$$

For the Fourier series, the following theorem claims that a similar identity is true.

**Theorem 2.19** (Parseval's theorem)**.** *Let $f(x)$ defined on $-L < x < L$ be a piecewise smooth function with Fourier series*

$$f(x) = A_0 + \sum_{n=1}^\infty [A_n \cos(n\pi x/L) + B_n \sin(n\pi x/L)] \tag{2.31}$$

*Then, the mean square $\frac{1}{2L} \int_{-L}^L f(x)^2 dx$ of $f(x)$ satisfies the following identity*

$$\frac{1}{2L} \int_{-L}^L f(x)^2 dx = A_0^2 + \frac{1}{2} \sum_{n=1}^\infty \left( A_n^2 + B_n^2 \right) \tag{2.32}$$

*Proof.* Since $\cos(0\pi x/L) = 1$ and $\sin(0\pi x/L) = 0$, we know that (2.31) is equivalent to

$$\begin{aligned} f(x) &= \sum_{n=0}^\infty [A_n \cos(n\pi x/L) + B_n \sin(n\pi x/L)] \\ &= \sum_{n=0}^\infty (A_n \cos_n + B_n \sin_n) \end{aligned} \tag{2.33}$$

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where we introduce the notation

$$\cos_n = \cos(n\pi x/L), \quad \sin_n = \sin(n\pi x/L) \tag{2.34}$$

Similar to (2.30) and its proof, we have

$$
\begin{aligned}
\frac{1}{2L}\int_{-L}^{L} f(x)^2 dx &= \frac{1}{2L}\int_{-L}^{L} \sum_{n=0}^{\infty}[A_n \cos_n + B_n \sin_n] \sum_{m=0}^{\infty}[A_m \cos_m + B_m \sin_m]dx \\
&= \frac{1}{2L}\int_{-L}^{L} \sum_{n,m=0}^{\infty} (A_n A_m \underbrace{\cos_n \cos_m}_{=0 \ if \ n \neq m} + A_n B_m \underbrace{\cos_n \sin_m}_{=0} \\
&\qquad\qquad + B_n A_m \underbrace{\sin_n \cos_m}_{=0} + B_n B_m \underbrace{\sin_n \sin_m}_{=0 \ if \ n \neq m})dx \\
&= A_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(A_n^2 + B_n^2\right)
\end{aligned}
\tag{2.35}
$$

where in the last line, we applied the orthogonality of trigonometric functions (2.6). □

### 2.6.2   The mean square error

**Definition 2.20** (Mean square error). *We define the <u>mean square error</u> $\sigma_N^2$ as*

$$\sigma_N^2 = \frac{1}{2L}\int_{-L}^{L}[f(x) - f_N(x)]^2 dx \tag{2.36}$$

*where $f_N(x)$ is the partial sum*

$$f_N(x) = A_0 + \sum_{n=1}^{N}[A_n \cos(n\pi x/L) + B_n \sin(n\pi x/L)]. \tag{2.37}$$

By Parseval's theorem, we obtain the following expression of the mean square error.

**Proposition 2.21.** *Given a partial summation whose mean square error is defined by (2.36), then we have*

$$\sigma_N^2 = \frac{1}{2}\sum_{n=N+1}^{\infty}\left(A_n^2 + B_n^2\right) \tag{2.38}$$

*Proof.* This is an easy corollary of the Parseval's identity (2.32), if we notice that

$$f(x) - f_N(x) = \sum_{n=N+1}^{\infty}[A_n \cos(n\pi x/L) + B_n \sin(n\pi x/L)]$$

Therefore, we finished the proof. □

*Example* 2.6. Let us find $\sigma_N^2$ for $f(x) = x$, $-L < x < L$.

*Solution.* From Example 2.1, we have $A_0 = A_n = 0$ and

$$f_N(x) = \sum_{n=1}^{N} B_n \sin \frac{n\pi x}{L}, \quad B_n = \frac{2L}{n\pi}(-1)^{n+1}. \tag{2.39}$$

By (2.38), we obtain

$$\sigma_N^2 = \frac{1}{2} \sum_{n=N+1}^{\infty} \left( \frac{2L}{n\pi}(-1)^{n+1} \right)^2 = \frac{2L^2}{\pi^2} \sum_{n=N+1}^{\infty} \frac{1}{n^2}.$$

By Theorem 2.22, we get

$$\int_{N+1}^{\infty} \frac{1}{x^2} dx \leq \sum_{n=N+1}^{\infty} \frac{1}{n^2} \leq \int_{N+1}^{\infty} \frac{1}{(x-1)^2} dx = \int_{N}^{\infty} \frac{1}{x^2} dx.$$

We have

$$\int_{N+1}^{\infty} \frac{1}{x^2} dx = \frac{1}{N+1} \geq \frac{1}{N} - \frac{1}{N^2}, \quad \int_{N}^{\infty} \frac{1}{x^2} dx = \frac{1}{N}.$$

Let us introduce the symbol $O$ (this is called "big O") to express the order. For some $f_N$, $f_N = O\left(N^{-1}\right)$ as $N \to \infty$ means that there exist a constant $C > 0$ such that $|f_N| \leq CN^{-1}$. Therefore we obtain

$$\sigma_N^2 = \frac{2L^2}{\pi^2} \frac{1}{N} \left[ 1 + O\left(\frac{1}{N}\right) \right] = O\left(N^{-1}\right), \quad N \to \infty. \tag{2.40}$$

We note that $\sigma_N^2$ goes to zero as $N \to \infty$ although we know that the sum in (2.39) does not converge uniformly. This happened because we considered the mean square and took the integral. **TODO: add a picture for convergence** ☐

**Theorem 2.22** (Integral test). *Given a monotonic and positive function $f(x)$, we have*

$$\int_{N+1}^{\infty} f(x)dx \leq \sum_{n=N+1}^{\infty} f(n) \leq \int_{N}^{\infty} f(x)dx \tag{2.41}$$

*Proof.* **TODO: compare the area below the graph of** $y = f(x)$
　　**TODO: add a picture** ☐

*Example* 2.7. Let us find $\sigma_{2N}^2$ for $f(x) = |x|, -L < x < L$.

*Solution.* From Example 2.2 we know that $B_n = 0$, $A_{2m} = 0$ ($m = 1, 2, \ldots$), and

$$f_{2N}(x) = A_0 + \sum_{m=1}^{N} A_{2m-1} \cos \frac{(2m-1)\pi x}{L}, \quad A_0 = \frac{L}{2}, \quad A_{2m-1} = -\frac{4L}{\pi^2(2m-1)^2}.$$

This is also the Fourier cosine series of $x$, $0 < x < L$, in Example 2.2. Hence we obtain

$$\sigma_{2N}^2 = \frac{1}{2} \sum_{n=2N+1}^{\infty} A_n^2 = \frac{1}{2} \sum_{m=N+1}^{\infty} A_{2m-1}^2 = \frac{8L^2}{\pi^4} \sum_{m=N+1}^{\infty} \frac{1}{(2m-1)^4}.$$

Note that by Theorem 2.22

$$\int_{N+1}^{\infty} \frac{1}{(2x-3)^4} dx \leq \sum_{m=N+1}^{\infty} \frac{1}{(2m-1)^4} \leq \int_{N+1}^{\infty} \frac{1}{(2x-1)^4} dx,$$

and by Taylor expansion $\frac{1}{(1+x)^3} = 1 - 3x + 6x^2 + \cdots$, $LHS = \frac{1}{8N^3} \frac{1}{(1+1/2N)^3} = \frac{1}{6(2N+1)^3} = \frac{1}{48N^3} + O\left(N^{-4}\right)$ and $RHS = \frac{1}{6(2N-1)^3} = \frac{1}{48N^3} + O\left(N^{-4}\right)$. Therefore we obtain

$$\sigma_{2N}^2 = \frac{L^2}{6\pi^4 N^3} + O\left(N^{-4}\right) = O\left(N^{-3}\right), \quad N \to \infty. \tag{2.42}$$

Thus the Fourier series of $x$ converges as $O(1/N)$ and the Fourier series of $|x|$ converges as $O\left(1/N^3\right)$. Equations (2.40) and (2.42) explain the difference between figure ??. $\qquad \square$

### 2.6.3 Parseval's theorem for complex, cosine and sine Fourier series

**Theorem 2.23** (Parseval's theorem for complex Fourier series)**.** *Let $f(x)$ defined on $-L < x < L$ be a piecewise smooth function with Fourier series*

$$f(x) = \sum_{n=-\infty}^{\infty} \alpha_n e^{in\pi x/L} \tag{2.43}$$

*Then, the mean square $\frac{1}{2L} \int_{-L}^{L} f(x)^2 dx$ of $f(x)$ satisfies the following identity*

$$\frac{1}{2L} \int_{-L}^{L} |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |\alpha_n|^2. \tag{2.44}$$

*Proof. Method 1.* We have

$$\begin{aligned}
\frac{1}{2L} \int_{-L}^{L} |f(x)|^2 dx &= \frac{1}{2L} \int_{-L}^{L} \sum_{n=-\infty}^{\infty} \alpha_n e^{in\pi x/L} \sum_{m=-\infty}^{\infty} \bar{\alpha}_m e^{-im\pi x/L} dx \\
&= \sum_{n,m=0}^{\infty} \alpha_n \bar{\alpha}_m \underbrace{\frac{1}{2L} \int_{-L}^{L} e^{i(n-m)\pi x/L} dx}_{=0 \ if \ n \neq m} \\
&= \sum_{n=-\infty}^{\infty} |\alpha_n|^2
\end{aligned} \tag{2.45}$$

*Method 2.* (2.44) is seen by the calculation below. By Parseval's identity for the Fourier series (2.32),

$$\frac{1}{2L}\int_{-L}^{L}f(x)^2 dx = A_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(A_n^2 + B_n^2\right)$$

$$= A_0^2 + 2\sum_{n=1}^{\infty}\frac{A_n - iB_n}{2}\frac{A_n + iB_n}{2}$$

$$= \alpha_0^2 + 2\sum_{n=1}^{\infty}\alpha_n\alpha_{-n} = \alpha_0^2 + \sum_{n=1}^{\infty}|\alpha_n|^2 + \sum_{n=1}^{\infty}|\alpha_{-n}|^2$$

$$= \alpha_0^2 + \sum_{n=1}^{\infty}|\alpha_n|^2 + \sum_{n=-\infty}^{-1}|\alpha_n|^2$$

$$= \sum_{n=-\infty}^{\infty}|\alpha_n|^2$$

$\square$

**Theorem 2.24** (Parseval's theorem for cosine Fourier series). *Let $f(x)$ defined on $-L < x < L$ be a piecewise smooth function with cosine Fourier series*

$$f(x) = A_0 + \sum_{n=1}^{\infty}A_n\cos\frac{n\pi x}{L} \tag{2.46}$$

*Then, the mean square $\frac{1}{2L}\int_{-L}^{L}f(x)^2 dx$ of $f(x)$ satisfies the following identity*

$$\frac{1}{2L}\int_{-L}^{L}|f(x)|^2 dx = A_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}A_n^2. \tag{2.47}$$

*Proof.* This is homework. $\square$

**Theorem 2.25** (Parseval's theorem for sine Fourier series). *Let $f(x)$ defined on $-L < x < L$ be a piecewise smooth function with sine Fourier series*

$$f(x) = \sum_{n=1}^{\infty}B_n\sin\frac{n\pi x}{L} \tag{2.48}$$

*Then, the mean square $\frac{1}{2L}\int_{-L}^{L}f(x)^2 dx$ of $f(x)$ satisfies the following identity*

$$\frac{1}{L}\int_{0}^{L}f(x)^2 dx = \frac{1}{2}\sum_{n=1}^{\infty}B_n^2. \tag{2.49}$$

*Proof.* This is homework. $\square$

# 3  PDEs in rectangular coordinates

In this section, we will consider the separation of variables for more general equations in rectangular coordinates, possibly with variable coefficients and more general boundary conditions.

## 3.1  Boundary conditions for general PDEs

### 3.1.1  Dirichlet, Neumann and Robin boundary conditions

In the ODE class, we have learned the following boundary value problem

$$u'' = 1, \qquad x \in [0,1]$$
$$u(0) = 0, \quad u(1) = 0. \tag{3.1}$$

where we prescribe a condition for every point on the boundary of the domain $[0,1]$. (The boundary is 0 and 1.) If we miss any of these conditions, we cannot get a unique solution.

For example, if we remove the condition $u(1) = 0$, then we get solution $u(x) = \frac{1}{2}x^2 + Cx$, which contains an undetermined constant $C$.

Given a PDE defined on a domain $R$, we must also prescribe the boundary condition on $\partial R$ to obtain a unique solution.
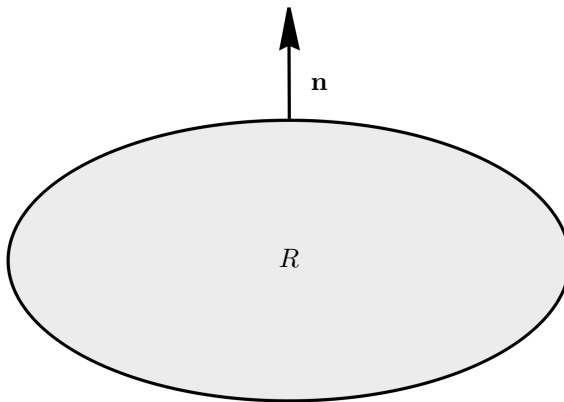


Figure 2: The region $R$ and its normal vector.

Before we introduce the concept of boundary conditions, let us first explain the concept of normal vectors.

**Definition 3.1** (Normal vector)**.** *Given a domain $R$ and a point $x \in \partial R$, a <u>normal vector</u> $\mathbf{n}$ at $x$ is the vector as demonstrated by figure 2. In this class, the normal vector is always pointing outwards.*

Now we introduce the boundary conditions that will be considered in this class.

**Definition 3.2** (Boundary conditions)**.** *Consider a PDE $F[u] = 0$ defined on the domain $R$. Here are the boundary conditions that we consider in this class.*

1. ***Dirichlet boundary condition.*** *The value of $u$ on the boundary is given.*

$$u = g(x), \quad x \in \partial R. \tag{3.2}$$

2. ***Neumann boundary condition.*** *The normal derivative on the boundary is given.*

$$\mathbf{n} \cdot \nabla u = g(x), \quad x \in \partial R. \tag{3.3}$$

3. ***Robin boundary condition.*** *A linear combination of the above two boundary conditions.*

$$a(x)u + b(x)\mathbf{n} \cdot \nabla u = g(x), \quad x \in \partial R. \tag{3.4}$$

### 3.1.2   Heat equations as an example

We take the heat equation $u_t = K u_{zz}$, defined on $R = \{(z,t) : 0 \leq t < \infty, \ 0 \leq x \leq L\}$, as an example to explain these boundary conditions.

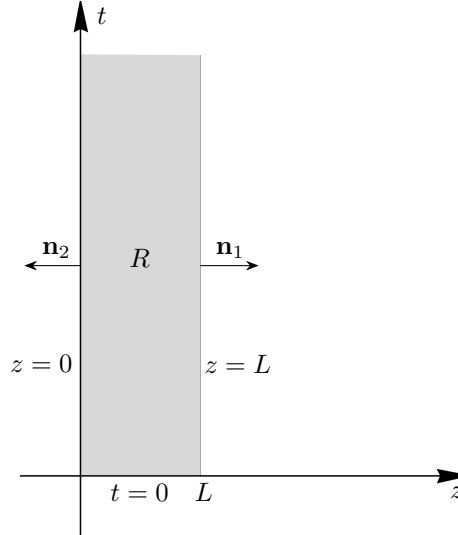The domain $R$ of the heat equation is described in the following picture,



Figure 3: The domain for the heat equation.

There are three pieces of $\partial R$,

$$
\begin{aligned}
z = 0, \quad & t > 0 \\
z = L, \quad & t > 0 \\
0 < z < L, \quad & t = 0
\end{aligned}
\tag{3.5}
$$

On the first two parts of $\partial R$ corresponding to $z = 0$, $L$, we impose the Robin boundary condition. On the last part corresponding to $t = 0$, we impose the Dirichlet boundary condition. Then we get the following system of equations,

$$
\begin{cases}
u_t = K u_{zz}, & 0 < z < L, \quad t > 0, \\
a(z,t)u + b(z,t)\mathbf{n} \cdot \nabla u = g(z,t), & z = 0,\, L, \quad t > 0, \\
u(z,0) = f(z), & 0 < z < L, \quad t = 0.
\end{cases}
\tag{3.6}
$$

Here the choice of Dirichlet and Robin boundary condition comes from the physics of heat conduction. We will explain this in section **??**.

**TODO: second order need one condition but first order need one**

Now we explain how to simplify the Robin boundary condition in (3.6).

On $z = 0$, from figure 3, we know that the normal vector $\mathbf{n} = (-1,0)$ and $\nabla u = (u_z, u_t)$, so we get $\mathbf{n} \cdot \nabla u = -u_z$. Therefore, $a(z,t)u + b(z,t)\mathbf{n} \cdot \nabla u = g(z,t)$ simplifies to

$$
a(0,t)u - b(0,t)u_z = g(0,t).
\tag{3.7}
$$

On $z = L$, from figure 3, we know that the normal vector $\mathbf{n} = (1,0)$ and $\nabla u = (u_z, u_t)$, so we get $\mathbf{n} \cdot \nabla u = u_z$. Therefore, $a(z,t)u + b(z,t)\mathbf{n} \cdot \nabla u = g(z,t)$ simplifies to

$$
a(L,t)u + b(L,t)u_z = g(L,t).
\tag{3.8}
$$

We introduce new functions $a(t)$, $\widetilde{a}(t)$, $b(t)$, $\widetilde{b}(t)$, $g(t)$ and $\widetilde{g}(t)$ by the following equations

$$
\begin{aligned}
a(t) &= a(0,t), \quad b(t) = b(0,t), \quad g(t) = g(0,t) \\
\widetilde{a}(t) &= a(L,t), \quad \widetilde{b}(t) = b(L,t), \quad \widetilde{g}(t) = g(L,t)
\end{aligned}
\tag{3.9}
$$

Then the Robin boundary condition becomes

$$
\begin{aligned}
a(t)u - b(t)u_z &= g(t), \quad z = 0 \\
\widetilde{a}(t)u + \widetilde{b}(t)u_z &= \widetilde{g}(t), \quad z = L
\end{aligned}
\tag{3.10}
$$

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The heat equation becomes

$$\begin{cases} u_t = Ku_{zz}, & 0 < z < L, \quad t > 0, \\ a(t)u - b(t)u_z = g(t), & z = 0, \quad t > 0, \\ \widetilde{a}(t)u + \widetilde{b}(t)u_z = \widetilde{g}(t), & z = L, \quad t > 0, \\ u = f(z), & 0 < z < L, \quad t = 0. \end{cases} \tag{3.11}$$

In order to make (3.11) solvable, we impose the homogeneous condition.

**Definition 3.3** (Homogeneous). *We say* (3.11) *is* <u>*homogeneous*</u> *if*

*1. $a(t)$, $\widetilde{a}(t)$, $b(t)$, $\widetilde{b}(t)$, $g(t)$ and $\widetilde{g}(t)$ are independent of $t$.*

*2. $g(t) = \widetilde{g}(t) = 0$.*

With homogeneous assumption, (3.11) becomes

$$\begin{cases} u_t = Ku_{zz}, & 0 < z < L, \quad t > 0, \\ au - bu_z = 0, & z = 0, \quad t > 0, \\ \widetilde{a}u + \widetilde{b}u_z = 0, & z = L, \quad t > 0, \\ u = f(z), & 0 < z < L, \quad t = 0. \end{cases} \tag{3.12}$$

To further simplify the above equation, we did the change of variable

$$b \to bL, \quad \widetilde{b} \to \widetilde{b}L \tag{3.13}$$

followed by the change of variable

$$\begin{aligned} a \to r\cos\alpha, \quad b \to r\sin\alpha, \\ \widetilde{a} \to r\cos\beta, \quad \widetilde{b} \to r\sin\beta. \end{aligned} \tag{3.14}$$

Finally, the heat equation becomes

$$\begin{cases} u_t = Ku_{zz}, & 0 < z < L, \quad t > 0, \\ u\cos\alpha - Lu_z\sin\alpha = 0, & z = 0, \quad t > 0, \\ u\cos\beta + Lu_z\sin\beta = 0, & z = L, \quad t > 0, \\ u = f(z), & 0 < z < L, \quad t = 0, \end{cases} \tag{3.15}$$

### 3.1.3 Separation of variable in heat equation

Let us solve the heat equation in a simple case of $\alpha = \beta = 0$ in (3.15). The separated solution is written as $u(z,t) = \phi(z)T(t)$. Thus we obtain

$$T'(t) + \lambda K T(t) = 0, \quad \phi''(z) + \lambda\phi(z) = 0.$$

The boundary conditions are written as

$$\phi(0) = \phi(L) = 0.$$

We obtain

$$T(t) = e^{-\lambda K t}, \quad \phi = A\sin(\sqrt{\lambda}z) + B\cos(\sqrt{\lambda}z), \quad \lambda > 0.$$

By plugging $\phi = A\sin(\sqrt{\lambda}z) + B\cos(\sqrt{\lambda}z)$ into the boundary conditions, we find that $B = 0$ and $\sqrt{\lambda}L = n\pi$ where $n$ is an integer. Therefore we obtain

$$\phi(z) = \phi_n(z) = \sin\left(\sqrt{\lambda_n}z\right), \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, \ldots,$$

where we set the arbitrary constant in $\phi_n(z)$ to be 1 (recall we will take a superposition). Thus the separated solutions are obtained as

$$u(z,t) = \phi_n(z)e^{-\lambda_n K t}, \quad n = 1, 2, \ldots$$

If no initial condition $u(z,0) = f(z)$ is given, the separated solutions are the solutions to the problem. However, they do not satisfies $u(z,0) = f(z)$. Let us consider the linear combination of separated solution and match with the initial condition.

The linear combination is

$$u(z,t) = \sum_{n=1}^{\infty} C_n\phi_n(z)e^{-\lambda_n K t},$$

where $C_n$ are constants.

By $u(z,0) = f(z)$, we know that

$$f(z) = u(z,0) = \sum_{n=1}^{\infty} C_n\phi_n(z) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi z}{L}.$$

From this, we know that $C_n$ is the $B_n$ coefficient of the Fourier sine series. We thus obtain

$$u(z,t) = \sum_{n=1}^{\infty} B_n\phi_n(z)e^{-\lambda_n K t}, \quad 0 < z < L, \quad t > 0.$$

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx.$$

*Example* 3.1. The heat equation $u_t = K u_{zz}$ for $0 < z < L, t > 0$ with $u(0,t) = u(L,t) = 0$ and $u(z,0) = 1$ is solved as

$$u(z,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi z}{L} e^{-(n\pi/L)^2 Kt},$$

where

$$B_n = \frac{2}{\pi} \frac{1 - (-1)^n}{n}.$$

Here as mentioned above, $B_n$ can be directly computed as coefficients of the Fourier sine series of $f(z) = 1$

### 3.1.4   Some linear algebra

After the separation of the variable in the heat equation, we obtain the following equation

$$\phi''(z) + \lambda\phi(z) = 0 \quad \Leftrightarrow \quad -\phi''(z) = \lambda\phi(z).$$

If we denote the negative second-order derivative operator by $A$, then we get

$$A\phi = \lambda\phi.$$

This is very similar to the eigenvalue and eigenvector equation for matrix $M$ in linear algebra.

$$Mv = \lambda v. \tag{3.16}$$

For a symmetric matrix, we have the following properties.

**Theorem 3.4.** *Assume that $M$ is a symmetric operator $(M = M^T)$, and $\langle x, y \rangle$ is the inner product of vectors, then we have*

1. *$\langle Mx, y \rangle = \langle x, My \rangle$.*

2. *The eigenvalues of $M$ are real numbers.*

3. *If $v$, $w$ are eigenvectors with different eigenvalues $\lambda$ and $\mu$ respectively, then $v \perp w$ ($\langle v, w \rangle = 0$).*

*Proof.* Notice that $\langle x, y \rangle = x^T y$, then we have $\langle Mx, y \rangle = (Mx)^T y = x^T M^T y = x^T My = \langle x, My \rangle$.

For complex vectors $\langle x, y \rangle = \bar{x}^T y$, $\langle x, Mx \rangle = \lambda\langle x, x \rangle$, $\langle Mx, x \rangle = \bar{\lambda}\langle x, x \rangle$ and $\langle Mx, x \rangle = \langle x, Mx \rangle$. Therefore, $\lambda = \bar{\lambda}$.

$\langle v, Mw \rangle = \mu\langle v, w \rangle$, $\langle Mv, w \rangle = \lambda\langle v, w \rangle$ and $\langle Mv, w \rangle = \langle v, Mw \rangle$. Therefore, $\mu\langle v, w \rangle = \lambda\langle v, w \rangle$, which implies that $\langle v, w \rangle$ due to $\lambda \neq \mu$. $\square$

It turns out that $A$ can be viewed as a symmetric matrix, and thus satisfies all properties described by Theorem 3.4.

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## 3.2 The Sturm-Liouville eigenvalue problem

Now we study the eigenvalue problems for differential operators. Let us first start with the concepts of orthogonal functions and symmetric operators. **TODO: we encounter so much orthogonality, which looks like a magic, explain the source of orthogonality**

### 3.2.1 Orthogonal functions and symmetric operators

**Definition 3.5** (Inner product)**.** *We extend dot product $\varphi \cdot \psi$ and define* <u>*inner product*</u> *as*

$$\langle \varphi, \psi \rangle = \int_a^b \varphi(x)\psi(x)dx. \tag{3.17}$$

*Sometimes the inner product is defined as follows. We can have a* <u>*weight function*</u> *$\rho(x)$, and the weighted inner product is given by*

$$\langle \varphi, \psi \rangle_\rho = \int_a^b \varphi(x)\psi(x)\rho(x)dx \tag{3.18}$$

*where $\rho(x) > 0$ is a weight function.*

*For complex functions, we can write the* <u>*complex inner product*</u> *as*

$$\langle \varphi, \psi \rangle = \int_a^b \varphi(x)\bar{\psi}(x)\rho(x)dx. \tag{3.19}$$

*Here $\bar{\psi}$ is the complex conjugate of $\psi$ ($\bar{\psi}(x) = f(x) - ig(x)$ when $\psi = f + ig$).*

**Definition 3.6** (Orthogonality)**.** *Two functions $\varphi, \psi$ are said to be* <u>*orthogonal*</u> *on $[a, b]$ if $\langle \varphi, \psi \rangle = 0$.*

*Example* 3.2. The functions $\varphi(x) = \sin x$ and $\psi(x) = \cos x$ are orthogonal on $[0, \pi]$.

*Example* 3.3. The set of functions $1, \cos \frac{\pi x}{L}, \sin \frac{\pi x}{L}, \ldots, \cos \frac{N \pi x}{L}, \sin \frac{N \pi x}{L}$ is orthogonal on $[-L, L]$.

*Example* 3.4. The set of functions $\{e^{\frac{in\pi x}{L}}\}_{n \in \mathbb{Z}}$ is orthogonal on $[-L, L]$.

*Example* 3.5. Which of the following pairs of functions are orthogonal on the interval $0 \le x \le 1$?

$$\varphi_1 = \sin 2\pi x, \quad \varphi_2 = x, \quad \varphi_3 = \cos 2\pi x, \quad \varphi_4 = 1.$$

$\langle \varphi_1, \varphi_3 \rangle = 0, \langle \varphi_1, \varphi_4 \rangle = 0, \langle \varphi_2, \varphi_3 \rangle = 0, \langle \varphi_3, \varphi_4 \rangle = 0$. All others are nonzero. Therefore the pairs $(\varphi_1, \varphi_3), (\varphi_1, \varphi_4), (\varphi_2, \varphi_3)$, and $(\varphi_3, \varphi_4)$ are orthogonal.

**Definition 3.7** (Norm)**.** *As follows we define* <u>*norm*</u>*, which is the "length" of a function.*

$$\|\varphi\| = \|\varphi\|_{L^2(a,b)} = \sqrt{\langle \varphi, \varphi \rangle}.$$

*We note that the norm is always nonnegative. The norm $\|\varphi - \psi\|$ is the distance between two functions $\varphi$ and $\psi$.*

**Definition 3.8** (Orthonormal). *The functions $(\varphi_1, \ldots, \varphi_N)$ are orthonormal if $\langle \varphi_i, \varphi_j \rangle = \delta_{ij}$. Here $\delta_{ij}$ is the Kronecker delta ($\delta_{ij} = 0$ if $i \neq j$ and $= 1$ if $i = j$).*

**Definition 3.9** (Normalization). *Let $\{\varphi_1, \varphi_2, \ldots, \varphi_N\}$ be a set of orthogonal functions satisfying $\varphi_i \perp \varphi_j$ for $i \neq j$. Each function $\varphi_i$ can be <u>normalized</u> to obtain an orthogonal sequence $\{\psi_i\}_{i=1,\ldots,N}$:*

$$\psi_i = \frac{\varphi_i}{\|\varphi_i\|}$$

*where $\|\varphi_i\|$ denotes the norm of the function $\varphi_i$.*

*Example* 3.6. After normalize the orthogonal functions $\cos x$, $\sin x$ on $[0, \pi]$, we obtain $\sqrt{\frac{2}{\pi}} \cos x$, $\sqrt{\frac{2}{\pi}} \sin x$.

**Definition 3.10** (Differential Operator). *A <u>differential operator</u> $A$ is an operator satisfying the following*

1. *$A$ is of the form $A = \sum_{i=0}^{n} a_i(x) \frac{d^i}{dx^i}$.*

2. *The operator $A$ has a domain $\mathrm{Dom}(A)$, which means that $A$ does accept all smooth functions as its input. (See Example 3.7 for an example of $\mathrm{Dom}(A)$)*

*where $n$ is a non-negative integer representing the <u>order</u> of the differential operator, $a_i(x)$ are the <u>coefficient</u> of $A$, and $\frac{d^i}{dx^i}$ represents the $i$-th derivative with respect to $x$, with $\frac{d^0}{dx^0}$ interpreted as the identity operator.*

**Definition 3.11** (Symmetric operators). *Given a differential operator $A$, we say $A$ is <u>symmetric</u>, if for any functions $\varphi, \psi \in \mathrm{Dom}(A)$,*

$$\langle A\varphi, \psi \rangle = \langle \varphi, A\psi \rangle. \tag{3.20}$$

*Example* 3.7. The differential operator $A$ defined by the following is a symmetric operator,

$$A = -\frac{d^2}{dx^2}, \qquad \mathrm{Dom}(A) = \{\varphi(x) : x \in [a, b], \varphi(a) = \varphi(b) = 0\} \tag{3.21}$$

To show that $A$ is symmetric, we need to verify that

$$\langle \varphi, A\psi \rangle = \langle A\varphi, \psi \rangle \tag{3.22}$$

for all functions $\varphi(x)$ and $\psi(x)$ satisfying $\varphi(a) = \varphi(b) = \psi(a) = \psi(b) = 0$. This condition translates into the following integral equality,

$$\int_a^b \varphi(x)\psi''(x)dx = \int_a^b \varphi''(x)\psi(x)dx \tag{3.23}$$

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Applying integration by parts and using the boundary conditions $\varphi(a) = \varphi(b) = \psi(a) = \psi(b) = 0$, we get

$$
\begin{aligned}
\int_a^b \varphi''(x)\psi(x)dx &= \underbrace{[\varphi'(x)\psi(x)]_a^b}_{=0} - \int_a^b \varphi'(x)\psi'(x)dx \\
&= -\int_a^b \varphi'(x)\psi'(x)dx = -\underbrace{[\varphi(x)\psi'(x)]_a^b}_{=0} + \int_a^b \varphi(x)\psi''(x)dx \qquad (3.24) \\
&= \int_a^b \varphi(x)\psi''(x)dx
\end{aligned}
$$

This equality confirms (3.22) and thus the fact that $A$ is symmetric on the given domain $\mathrm{Dom}(A)$.

*Example* 3.8. The definition of symmetric operator depends on domain. For example, if we define $A$ by

$$
A = -\frac{d^2}{dx^2}, \qquad \mathrm{Dom}(A) = \{\varphi(x) : x \in [a, b]\} \qquad (3.25)
$$

which is almost the same operator as the last example but with a different domain. This operator is NOT symmetric. If we take $\varphi(x) = 1$ and $\psi(x) = x$, then a straightforward computation gives $\langle \varphi, A\psi \rangle = 0$ and $\langle A\varphi, \psi \rangle \neq 0$. Therefore, $A$ is not symmetric.

### 3.2.2   The Sturm-Liouville eigenvalue problem

**Definition 3.12** (Sturm-Liouville problems). *Let $\phi$ be a function defined on the interval $(a, b)$, the* Sturm-Liouville problems *is the following boundary value problem,*

$$
\begin{aligned}
&[s(x)\phi'(x)]' + [\lambda\rho(x) - q(x)]\phi(x) = 0, \quad a < x < b, \\
&\phi(a)\cos\alpha - L\phi'(a)\sin\alpha = 0, \qquad\qquad\qquad\qquad\qquad (3.26) \\
&\phi(b)\cos\beta + L\phi'(b)\sin\beta = 0
\end{aligned}
$$

*where $\rho$ is a positive function $\rho(x) > 0$ and $L = b - a$, $\alpha, \beta \in [0, \pi)$ are some parameters.*

Define the operator $A$ by

$$
A\phi = -\frac{1}{\rho(x)}\left([s(x)\phi'(x)]' - q(x)\phi(x)\right) \qquad (3.27)
$$

with domain

$$
\left\{ \phi(x),\ x \in [a, b] \ \middle|\ \begin{array}{l} \phi(a)\cos\alpha - L\phi'(a)\sin\alpha = 0, \\ \phi(b)\cos\beta + L\phi'(b)\sin\beta = 0 \end{array} \right\} \qquad (3.28)
$$

then (3.26) is equivalent to

$$A\phi = \lambda\phi. \tag{3.29}$$

The following theorem claims that all symmetric second order differential operators are of the form given by (3.27) and (3.28).

**Theorem 3.13.** *A symmetric second order differential operator $A$ must be of the form given by (3.27) and (3.28).*

*Proof.* The proof go beyond the scope of this course. □

The following theorem explain the procedure of reducing an arbitrary second order differential operator to a Sturm-Liouville operator. Since it is abstract, it is good to first jump to the examples below the theorem.

**Theorem 3.14.** *Given an arbitrary second order linear differential equation $a(x)y''+b(x)y'+c(x)y = 0$, it can be reduced to the Sturm-Liouville equation (3.26) by the following procedure.*

1. *Divide both sides by $a(x)$ to obtain $y'' + p(x)y' + q(x)y = 0$.*

2. *Solve the equation $\widehat{y}' = p(x)\widehat{y}$.*

3. *Multiply both sides of $y'' + p(x)y' + q(x)y = 0$ by $\widehat{y}$*

$$\widehat{y}(y'' + p(x)y') + \widehat{y}q(x)y = 0 \quad \Rightarrow \quad (\widehat{y}y')' + \widehat{y}q(x)y = 0 \tag{3.30}$$

*Proof.* There is nothing to prove. □

*Example* 3.9 (Bessel functions). The <u>Bessel functions</u> are solutions of the following ODEs.

$$\phi'' + (d - 1)\frac{\phi'}{x} + \left(1 - \frac{\mu}{x^2}\right)\phi = 0. \tag{3.31}$$

We can apply the procedure in Theorem 3.14 to reduce the above equation to a Sturm-Liouville equation.

1. There is nothing to do with this step since $a(x) = 1$.

2. We first solve $\widehat{y}' = \frac{d-1}{x}\widehat{y}$ and obtain $\widehat{y}(x) = x^{d-1}$.

3. Multiply both side of $\phi'' + (d - 1)\frac{\phi'}{x} + \left(1 - \frac{\mu}{x^2}\right)\phi = 0$ by $x^{d-1}$, then we get

$$x^{d-1} \cdot \left(\phi'' + (d - 1)\frac{\phi'}{x} + \left(1 - \frac{\mu}{x^2}\right)\phi\right) = 0 \quad \Rightarrow \quad (x^{d-1}\phi')' + \left(x^{d-1} - \mu x^{d-3}\right)\phi = 0. \tag{3.32}$$

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Finally, we get

$$(x^{d-1}\phi'(x))' + \left(x^{d-1} - \mu x^{d-3}\right)\phi(x) = 0. \tag{3.33}$$

This is a Sturm-Liouville equation by setting $s(x) = \rho(x) = x^{d-1}$, $q(x) = \mu x^{d-3}$, and $\lambda = 1$.

In the case of $d = 2$ and $\mu = m^2$ with $m \in \mathbb{N}$, the function $\phi(x) = J_m(x)$ is called the standard Bessel function. In the case of $d = 3$ and $\mu = k(k+1)(k = 0, 1, 2, \ldots)$, the function $\phi(x) = j_k(x)$ is called the spherical Bessel function.

*Example* 3.10 (Legendre polynomials). The Legendre polynomials $P_k^m(x)$ are solutions of the following ODEs.

$$(1 - x^2)\phi'' - 2x\phi' + \left(k(k+1) - \frac{m^2}{1 - x^2}\right)\phi = 0. \tag{3.34}$$

We can apply the procedure in Theorem 3.14 to reduce the above equation to a Sturm-Liouville equation.

1. Divide both sides by $(1 - x^2)$, then we get

$$\phi'' - \frac{2x}{1 - x^2}\phi' + \left(\frac{k(k+1)}{1 - x^2} - \frac{m^2}{(1 - x^2)^2}\right)\phi = 0. \tag{3.35}$$

2. We first solve $\widehat{y}' = -\frac{2x}{1-x^2}\widehat{y}$ and obtain $\widehat{y}(x) = 1 - x^2$.

3. Multiply both side $1 - x^2$ (This step undo the operation of step 1 which sometimes appears), then we get

$$(1-x^2)\phi'' - 2x\phi' + \left(k(k+1) - \frac{m^2}{1 - x^2}\right)\phi = 0 \;\Rightarrow\; ((1-x^2)\phi')' + \left(k(k+1) - \frac{m^2}{1 - x^2}\right)\phi = 0. \tag{3.36}$$

Finally, we get

$$((1 - x^2)\phi')' + \left(k(k+1) - \frac{m^2}{1 - x^2}\right)\phi = 0. \tag{3.37}$$

This is a Sturm-Liouville equation by setting $s(x) = 1 - x^2$, $\rho(x) = 1$, $q(x) = m^2/(1 - x^2)$, and $\lambda = k(k+1)$ $(k = 0, 1, 2, \ldots)$ with $a = -1$ and $b = 1$.

*Example* 3.11 (Hermite polynomials). The Hermite polynomials $H_n(x)$ are solutions of the following ODEs.

$$\phi'' - x\phi' + n\phi = 0. \tag{3.38}$$

We can apply the procedure in Theorem 3.14 to reduce the above equation to a Sturm-Liouville equation.

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1. There is nothing to do with this step since $a(x) = 1$.

2. We first solve $\widehat{y}' = -x\widehat{y}$ and obtain $\widehat{y}(x) = e^{-\frac{x^2}{2}}$.

3. Multiply both side by $e^{-\frac{x^2}{2}}$, then we get

$$e^{-\frac{x^2}{2}}\left(\phi'' - x\phi'\right) + ne^{-\frac{x^2}{2}}\phi = 0 \quad \Rightarrow \quad \left(e^{-\frac{x^2}{2}}\phi'\right)' + ne^{-\frac{x^2}{2}}\phi = 0. \tag{3.39}$$

Finally, we get

$$\left(e^{-\frac{x^2}{2}}\phi'\right)' + ne^{-\frac{x^2}{2}}\phi = 0. \tag{3.40}$$

This is a Sturm-Liouville equation by setting $s(x) = \rho(x) = \exp\left(-x^2/2\right)$, $q(x) = 0$, $\lambda = n$ $(n = 0, 1, 2, \ldots)$ with $a = -\infty$ and $b = \infty$.

### 3.2.3 The Sturm-Liouville theorem

Recall that the Sturm-Liouville problem is equivalent to the eigenvalue problem $A\phi = \lambda\phi$ of the operator $A$ defined in (3.27) and (3.28), which are copied below.

$$A\phi = -\frac{1}{\rho(x)}\left([s(x)\phi'(x)]' - q(x)\phi(x)\right) \tag{3.41}$$

with domain

$$\left\{\phi(x),\ x \in [a, b]\ \middle|\ \begin{array}{l} \phi(a)\cos\alpha - L\phi'(a)\sin\alpha = 0, \\ \phi(b)\cos\beta + L\phi'(b)\sin\beta = 0 \end{array}\right\} \tag{3.42}$$

We have the following theorem, which is an analog of Theorem 3.4

**Theorem 3.15** (Sturm-Liouville theorem I). *Assume that $A$ is a differential operator defined by (3.27) and (3.28), and $\langle\varphi, \psi\rangle_\rho = \int_a^b \varphi(x)\psi(x)\rho(x)dx$ is the inner product defined by (3.18) , then we have*

1. *$\langle A\phi, \psi\rangle = \langle\phi, A\psi\rangle$.*

2. *The eigenvalues of $A$ are real numbers.*

3. *If $\varphi$, $\psi$ are eigenfunctions with different eigenvalues $\lambda$ and $\mu$ respectively, then $\langle\varphi, \psi\rangle_\rho = 0$.*

4. *Assume that $\{\lambda_1, \lambda_2, \ldots\}$ are the set of all eigenfunctions of $A$, then this set is infinite and discrete. **TODO: picture***

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*Proof.* 1. $A$ is symmetric. Consider $\langle A\phi, \psi \rangle$, and apply integration by parts

$$
\begin{aligned}
\langle A\phi, \psi \rangle &= \int_a^b \left( -\frac{1}{\rho(x)} [(s(x)\phi'(x))' - q(x)\phi(x)] \right) \psi(x)\rho(x)\, dx. \\
&= \int_a^b \left( -(s(x)\phi'(x))' + q(x)\phi(x) \right) \psi(x)\, dx. \\
&= -\left[ s(x)\phi'(x)\psi(x) \right]_a^b + \int_a^b s(x)\phi'(x)\psi'(x)\, dx - \int_a^b q(x)\phi(x)\psi(x)\, dx.
\end{aligned}
\tag{3.43}
$$

Similarly for $\langle \phi, A\psi \rangle = \langle A\psi, \phi \rangle$, we also have

$$
\begin{aligned}
\langle A\psi, \phi \rangle &= \int_a^b \left( -\frac{1}{\rho(x)} [(s(x)\psi'(x))' - q(x)\psi(x)] \right) \phi(x)\rho(x)\, dx. \\
&= \int_a^b \left( -(s(x)\psi'(x))' + q(x)\psi(x) \right) \phi(x)\, dx. \\
&= -\left[ s(x)\psi'(x)\phi(x) \right]_a^b + \int_a^b s(x)\psi'(x)\phi'(x)\, dx - \int_a^b q(x)\psi(x)\phi(x)\, dx.
\end{aligned}
\tag{3.44}
$$

It suffices to show that the boundary terms are equal

$$
[s(x)\phi'(x)\psi(x)]_a^b = [s(x)\psi'(x)\phi(x)]_a^b
\tag{3.45}
$$

which is equivalent to

$$
s(b)\Big( \phi'(b)\psi(b) - \psi'(b)\phi(b) \Big) = s(a)\Big( \phi'(a)\psi(a) - \psi'(a)\phi(a) \Big).
\tag{3.46}
$$

If we can show that $\phi'(a)\psi(a) = \psi'(a)\phi(a)$ and $\phi'(b)\psi(b) = \psi'(b)\phi(b)$, then (3.46) follows and the proof of conclusion 1 is completed. Since $\phi$ and $\psi$ belong to the domain of $A$, by (3.42), $\frac{\phi'(a)}{\phi(a)} = \frac{\cos\alpha}{L\sin\alpha} = \frac{\psi'(a)}{\psi(a)}$, which implies $\phi'(a)\psi(a) = \psi'(a)\phi(a)$. The second equation at $b$ can be proved similarly.

2. The eigenvalues of $A$ are real. Since the eigenvalues and eigenfunctions at this point may not be real, we consider the complex inner product. Let $\lambda$ be an eigenvalue with eigenfunction $\phi$,

$$
A\phi = \lambda\phi.
\tag{3.47}
$$

Consider the inner product $\langle \phi, A\phi \rangle$ and $\langle A\phi, \phi \rangle$,

$$
\langle A\phi, \phi \rangle = \langle \lambda\phi, \phi \rangle = \lambda \|\phi\|^2.
\tag{3.48}
$$

$$
\langle \phi, A\phi \rangle = \langle \phi, \lambda\phi \rangle = \overline{\lambda} \|\phi\|^2.
\tag{3.49}
$$

Since $A$ is symmetric, we have $\langle \phi, A\phi \rangle$ and $\langle A\phi, \phi \rangle$, which implies that

$$
\lambda \|\phi\|^2 = \overline{\lambda} \|\phi\|^2.
\tag{3.50}
$$

Hence, $\lambda = \overline{\lambda}$, implying that $\lambda$ is real.

3. Orthogonality of eigenfunctions with different eigenvalues. Now the eigenvalues and eigenfunctions at this point are real, we consider the real inner product.

Suppose $A\phi = \lambda\phi$ and $A\psi = \mu\psi$ with $\lambda \neq \mu$. Consider

$$\lambda\langle\phi, \psi\rangle = \langle A\phi, \psi\rangle = \langle\phi, A\psi\rangle = \mu\langle\phi, \psi\rangle. \tag{3.51}$$

Thus, $(\lambda - \mu)\langle\phi, \psi\rangle = 0$, implying $\langle\phi, \psi\rangle = 0$ if $\lambda \neq \mu$.

4. Discreteness of the eigenvalue set. **TODO: finish the proof** $\qquad\square$

The following property implies that the eigenvalues of all Sturm-Liouville problems are of multiplicity one. This property does not holds true for eigenvalues of matrices.

**Theorem 3.16** (Sturm-Liouville theorem II). *Assume that $A$ is a differential operator defined by (3.27) and $\phi_1$ and $\phi_2$ are the eigenfunctions of the same eigenvalue $\lambda$, then we exists a constant $C$ such that*

$$\phi_1(x) = C\phi_2(x). \tag{3.52}$$

*Proof.* We will use the following conclusion from the theory of ODE without proof.

$$\begin{cases} a(x)f'' + b(x)f' + c(x)f = 0 \\ f(x_0) = f'(x_0) = 0 \end{cases} \qquad \Rightarrow \qquad f(x) \equiv 0 \tag{3.53}$$

This is the uniqueness theorem of the initial value problem of second order ODEs.

We consider

$$\psi(x) = \begin{cases} \phi_2(a)\phi_1(x) - \phi_1(a)\phi_2(x), & \text{if } \alpha \neq 0 \\ \phi_2'(a)\phi_1(x) - \phi_1'(a)\phi_2(x), & \text{if } \alpha = 0. \end{cases}$$

This $\psi(x)$ obeys (3.26). If we can show that $\psi(a) = 0$ and $\psi'(a) = 0$, then by (3.53), we know that $\psi(x) \equiv 0$ for $a < x < b$. Therefore,

$$\phi_2(x) = C\phi_1(x), \quad C = \frac{\phi_2(a)}{\phi_1(a)} \quad \text{or} \quad \frac{\phi_2'(a)}{\phi_1'(a)}, \tag{3.54}$$

which proves (3.52)

Let us only consider the case of $\alpha \neq 0$, where $\phi(x) = \phi_2(a)\phi_1(x) - \phi_1(a)\phi_2(x)$. This obviously implies $\psi(a) = 0$. To prove $\psi'(a) = 0$, we notice that by (3.42), both $\phi_1$ and $\phi_2$ satisfy $\phi(a)\cos\alpha - L\phi'(a)\sin\alpha = 0$ as they belong to the domain of $A$. This implies that $\frac{\phi_1'(a)}{\phi_1(a)} = \frac{\cos\alpha}{L\sin\alpha} = \frac{\phi_2'(a)}{\phi_2(a)}$. Therefore, we conclude that $\psi'(a) = \phi_2(a)\phi_1'(a) - \phi_1(a)\phi_2'(a) = 0$. $\qquad\square$

**3.2.4   Convergence of orthogonal series**

## 3.3   The heat equation

**3.3.1   Physics of heat conduction**

**3.3.2   The homogeneous case**

**3.3.3   The non-homogeneous case**

## 3.4   The wave equation

**3.4.1   Physics of string vibration**

**3.4.2   The homogeneous case**

**3.4.3   The non-homogeneous case**

## 3.5   The Laplace's equation

**3.5.1   Physics of static electricity**

**3.5.2   The homogeneous case**

**3.5.3   The non-homogeneous case**