Boundary Value Problems for Partial Differential Equations

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1 Introduction and preliminaries

1.1 Definitions of partial differential equations

Definition 1.1 (Notations of partial derivatives)**.** *For $f(x)$ with one variable x, we know $f'(x) = \frac{df}{dx}$. For $u(x, y)$, we introduce partial derivatives as*

$$\frac{\partial u}{\partial x} = \frac{du}{dx}\bigg|_{y \text{ is fixed}} = \partial_x u = u_x. \tag{1.1}$$

Similarly,

$$\frac{\partial^2 u}{\partial x^2} = \partial_x^2 u = \partial_{xx} u = u_{xx} \tag{1.2}$$

Example 1. For $u(x, y) = xy^2$, we have

$$\frac{\partial u}{\partial x} = \partial_x u = u_x = y^2, \qquad \frac{\partial u}{\partial y} = \partial_y u = u_y = 2xy,$$

$$\frac{\partial^2 u}{\partial x^2} = \partial_x^2 u = \partial_{xx} u = u_{xx} = 0, \qquad \frac{\partial^2 u}{\partial y^2} = \partial_y^2 u = \partial_{yy} u = u_{yy} = 2x$$

Definition 1.2 (Definition of general PDEs)**.** *Given a function $u = u(x, y)$ of two variables, (similarly $u = u(x_1, \cdots, x_n)$ of n variables) and an expression $F(u_{xx}, u_{xy}, u_{yy}, u_x, u_y, u, x, y)$ of partial derivatives of u, the following equation is a partial differential equation, abbreviated as PDE.*

$$F(u_{xx}, u_{xy}, u_{yy}, u_x, u_y, u, x, y) = 0 \tag{1.3}$$

In the future, we may also use the notation $F[u]$ to represent $F(u_{xx}, u_{xy}, u_{yy}, u_x, u_y, u, x, y)$. And (1.3) *can be rewritten as*

$$F[u] = 0. \tag{1.4}$$

Remark 1.3. **TODO: higher order is possible but we do not discuss it in this course**

Example 2 (Examples of PDEs). Here are some examples of PDEs.

$$
\begin{aligned}
&u_{xx} - u_y = 0 \quad \text{(the heat equation)} \\
&u_{xx} - u_{yy} = 0 \quad \text{(the wave equation)} \\
&u_{xx} + u_{yy} = 0 \quad \text{(Laplace's equation)} \\
&u_x + u_y = 0 \quad \text{(the transport equation)}
\end{aligned}
\tag{1.5}
$$

Definition 1.4 (Order of PDEs). *The order of a PDE is the order of the highest-order derivative in the equation. In* (1.5)*, the first three PDEs are second order, and the last one is first order.*

Definition 1.5 (Linear PDEs). ***TODO: def of linear pdes***

Proposition 1.6. ***TODO: general form of second order linear pdes***

We have the following proposition,

Proposition 1.7. *The second-order PDEs can always be written as*

$$a(x,y)u_{xx} + b(x,y)u_{xy} + c(x,y)u_{yy} + d(x,y)u_x + e(x,y)u_y + f(x,y)u = g(x,y), \tag{1.6}$$

We assume $a^2 + b^2 + c^2 \neq 0$ for any x, y (at least one of a, b, c is nonzero).

Definition 1.8. *We call a, b, c, d, e, f coefficients and g source term.*

1.2 Classification of second-order PDEs

In this course, we will mainly consider second-order linear PDEs.

$$a(x,y)u_{xx} + b(x,y)u_{xy} + c(x,y)u_{yy} + d(x,y)u_x + e(x,y)u_y + f(x,y)u = g(x,y), \tag{1.7}$$

These equations are classified as follows by the coefficients a, b, c.

Definition 1.9 (Classification of PDEs). *The second-order linear PDEs* (1.7) *are classified as elliptic, parabolic and hyperbolic by the following,*

$$
\begin{cases}
4ac - b^2 > 0 & \textit{elliptic} \\
4ac - b^2 = 0 & \textit{parabolic} \\
4ac - b^2 < 0 & \textit{hyperbolic}
\end{cases}
\tag{1.8}
$$

where we note that a, b, c are functions of x, y and the inequalities in (1.8) *is required to be true for any x, y.*

1.3 Review of ODEs

Definition 1.10 (Separable ODEs)**.** *The following ODE is the* <u>*separable*</u> *ODE*

$$y' + p(x)y = 0. \tag{1.9}$$

Theorem 1.11. *Separable ODE can be solved in the following way*

$$y' + p(x)y = 0 \quad \Rightarrow \quad \frac{dy}{dx} + p(x)y = 0 \quad \Rightarrow \quad \frac{dy}{y} = -p(x)dx \quad \Rightarrow \quad \int \frac{dy}{y} = -\int p(x)dx$$

and the solution is

$$y(x) = Ce^{-\int p(x)dx} \tag{1.10}$$

Example 3. Let us solve the following ODEs

$$y' = -6xy$$

Apply the procedure in Theorem 1.11

$$y' = -6xy \quad \Rightarrow \quad \frac{dy}{y} = -6xdx \quad \Rightarrow \quad \int \frac{dy}{y} = -\int 6xdx \quad \Rightarrow \quad \ln|y| = -3x^2 + C'$$

Therefore, the solution is

$$y = Ce^{-3x^2}.$$

where $C(= \pm e^{C'})$ is an arbitrary constant.

Definition 1.12 (Linear ODEs)**.** *The following ODE is the* <u>*linear*</u> *ODE*

$$y' + p(x)y = q(x). \tag{1.11}$$

Theorem 1.13. *Linear ODE can be solved by the following procedure*

1. *Solve the corresponding separable equation $y' - p(x)y = 0$ to obtain a solution $\hat{y} = e^{\int p(x)dx}$.*

2. *Multiply the linear ODE by $\hat{y}$ and rewrite the ODE*

$$y' + p(x)y = q(x) \quad \Rightarrow \quad \hat{y}(y' + p(x)y) = \hat{y}q(x) \quad \Rightarrow \quad (\hat{y}y)' = \hat{y}q(x)$$

3. *Integrate the above equation*

$$\hat{y}y = \int \hat{y}q(x)dx + C \quad \Rightarrow \quad y = \frac{1}{\hat{y}}\left(\int \hat{y}q(x)dx + C\right)$$
$$\Rightarrow \quad y = e^{-\int p(x)dx}\left(\int q(x)e^{\int p(x)dx}dx + C\right)$$

Remark 1.14. In (2), we applied the following equation

$$(\hat{y}y)' = \hat{y}(y' + p(x)y)$$

which is a corollary of the Leibniz rule.

$$(\hat{y}y)' = \hat{y}'y + \hat{y}y' = \hat{y}y' + p(x)\hat{y}y = \hat{y}(y' + p(x)y)$$

Example 4. $(x^2+1)y' + 3xy = 6x, y(0) = 3$ is solved as $y(x) = 2 + (x^2+1)^{-3/2}$ by the procedure in Theorem 1.13.

1. Divide both sides by (x^2+1).

$$(x^2+1)y' + 3xy = 6x \quad \Rightarrow \quad y' + \frac{3x}{x^2+1}y = \frac{6x}{x^2+1}$$

2. Solve the corresponding separable equation $y' - \frac{3x}{x^2+1}y = 0$ to obtain a solution $(x^2+1)^{\frac{3}{2}}$.

3. Multiply the linear ODE by $(x^2+1)^{\frac{3}{2}}$ and rewrite the ODE

$$y' + \frac{3x}{x^2+1}y = \frac{6x}{x^2+1} \quad \Rightarrow \quad ((x^2+1)^{\frac{3}{2}}y)' = 6x(x^2+1)^{\frac{1}{2}}$$

4. Integrate the above equation

$$\begin{aligned} ((x^2+1)^{\frac{3}{2}}y)' = 6x(x^2+1)^{\frac{1}{2}} \quad &\Rightarrow \quad y = (x^2+1)^{-\frac{3}{2}}\left(\int 6x(x^2+1)^{\frac{1}{2}} + C\right) \\ &\Rightarrow \quad y = 2 + C(x^2+1)^{-\frac{3}{2}} \end{aligned}$$

Definition 1.15 (Second order ODEs)**.** *The constant coefficient second order ODEs are the following equations*

$$ay'' + by' + cy = 0 \ (a \neq 0). \tag{1.12}$$

Theorem 1.16. *Constant coefficient second order ODEs can be solved by the following procedure*

1. *Solve the characteristic equation* $a\lambda^2 + b\lambda + c = 0$ *to get two solutions* λ_1 *and* λ_2*.*

2. *If* $\lambda_1 \neq \lambda_2$*, the general solution is*

$$y(x) = C_1e^{\lambda_1 x} + C_2e^{\lambda_2 x} \tag{1.13}$$

3. *If* $\lambda_1 = \lambda_2 = \lambda$*, the general solution is*

$$y(x) = (C_1 + C_2x)e^{\lambda x} \tag{1.14}$$

4. If λ_1, λ_2 are complex roots $\alpha \pm i\beta$, apply the Euler's formula to rewrite (1.13)

$$y(x) = C_1 e^{(\alpha+i\beta)x} + C_2 e^{(\alpha-i\beta)x} = e^{\alpha x}\left(c_1 \cos\beta x + c_2 \sin\beta x\right)$$

where $c_1 = C_1 + C_2$, $c_2 = i(C_1 - C_2)$.

Example 5. $y'' - 3y' + 2y = 0$ is solved as $y(x) = C_1 e^x + C_2 e^{2x}$ by the procedure in Theorem 1.16. **TODO:**

Example 6. $y'' + y = 0$ is solved as $y(x) = C_1 \cos(x) + C_2 \sin(x)$ by the procedure in Theorem 1.16. **TODO:**

Example 7. $y'' + 2y' + y = 0$ is solved as $y(x) = (C_1 + C_2 x)e^x$ by the procedure in Theorem 1.16. **TODO:**

1.4 Separation of variables

Many linear PDEs can be reduced to linear ODEs with the method of separation of variables, described below.

We take the Laplace's equation

$$u_{xx} + u_{yy} = 0 \tag{1.15}$$

as an example.

TODO: Boundary conditions

We are looking for a separated solution. Substitute into (1.15), then we get

$$X''Y + XY'' = 0 \quad \Rightarrow \quad \frac{X''}{X} = -\frac{Y''}{Y}$$

The following lemma implies that X''/X and Y''/Y are constants.

Lemma 1.17. ***TODO:*** *$f(x) = g(y)$ implies that $f(x) = g(y) = const$,*

Proof. $f(x) = g(y) \Rightarrow f'(x) = 0 \Rightarrow f(x) = const.$ □

Let λ be a constant and we write

$$X'' + \lambda X = 0, \quad Y'' - \lambda Y = 0.$$

We call λ the separation constant. At this moment λ is arbitrary. Thus the PDE was reduced to two ODEs.

1.4.1 Solving separated solutions

If $\lambda = 0$, then two ODEs have the following linearly independent solutions.

$$X = 1, x, \quad Y = 1, y. \tag{1.16}$$

If $\lambda \neq 0$, then two ODEs have the following linearly independent solutions.

$$X = e^{\sqrt{-\lambda}x}, e^{-\sqrt{-\lambda}x}, \quad Y = e^{\sqrt{\lambda}y}, e^{-\sqrt{\lambda}y}. \tag{1.17}$$

In either case, the solution is given by superpositions:

$$u = \begin{cases} (A_1 x + A_2)(B_1 y + B_2), & \lambda = 0 \\ \left(A_1 e^{\sqrt{-\lambda}x} + A_2 e^{-\sqrt{-\lambda}x}\right)\left(B_1 e^{\sqrt{\lambda}y} + B_2 e^{-\sqrt{\lambda}y}\right), & \lambda \neq 0 \end{cases} \tag{1.18}$$

where A_1, A_2, B_1, B_2 are constants.

For $\lambda > 0$, by writing $\lambda = k^2 (k > 0)$ we have

$$u(x, y) = \left(A_1 e^{ikx} + A_2 e^{-ikx}\right)\left(B_1 e^{ky} + B_2 e^{-ky}\right), \tag{1.19}$$

and for $\lambda < 0$, by writing $\lambda = -l^2 (l > 0)$ we have

$$u(x, y) = \left(A_1 e^{lx} + A_2 e^{-lx}\right)\left(B_1 e^{ily} + B_2 e^{-ily}\right). \tag{1.20}$$

Instead of (1.17) we can also choose

$$X = \cos(\sqrt{\lambda}x), \sin(\sqrt{\lambda}x), \quad Y = \cosh(\sqrt{\lambda}y), \sinh(\sqrt{\lambda}y). \tag{1.21}$$

In this case, we have

$$u(x, y) = (A_1 \cos(kx) + A_2 \sin(kx))(B_1 \cosh(ky) + B_2 \sinh(ky)) \tag{1.22}$$

$$u(x, y) = (A_1 \cosh(lx) + A_2 \sinh(lx))(B_1 \cos(ly) + B_2 \sin(ly)) \tag{1.23}$$

Note that (1.22) becomes (1.19) and (1.23) becomes (1.20) by redefining the coefficients. We call solutions such as (1.18) through (1.23) separated solutions because they are given in the form $u(x, y) = X(x)Y(y)$.

1.4.2 Solving the boundary value problem

TODO: revise the section The separation constant λ and coefficients A_1, A_2, B_1, B_2 are partially determined by boundary conditions. Suppose that our Laplace's equation is considered in the region $0 < x < L, 0 < y < \infty$ with boundary conditions

$$u(0, y) = 0, \quad u(L, y) = 0, \quad u(x, 0) = 0.$$

We find that $u = (A_1 x + A_2)(B_1 y + B_2)$ in (1.3) satisfies the boundary conditions only when $u = 0$.

We then find that (1.5) and (1.7) satisfy the conditions $u(0, y) = u(L, y) = 0$ only when $A_1 = A_2 = 0$. That is, only the solution $u = 0$ satisfies the boundary conditions.

Finally (1.4) and (1.6) satisfy $u(0, y) = u(L, y) = 0$ when $A_1 = 0$ and $k = n\pi/L$, where n is an integer. Furthermore we find $B_1 = 0$ by the condition $u(x, 0) = 0$. That is, the solution $A_2 B_2 \sin(kx) \sinh(ky)$ with $k = n\pi/L (n = 0, \pm 1, \pm 2, \ldots)$ satisfies the boundary conditions.

Therefore we obtain the following separated solutions of Laplace's equation satisfying the boundary conditions.

$$u(x, y) = A \sin \frac{n\pi x}{L} \sinh \frac{n\pi y}{L}, \quad n = 1, 2, \ldots,$$

TODO: linear combination and Fourier series