Boundary Value Problems for Partial Differential Equations

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1 Introduction and preliminaries

1.1 Definitions of partial differential equations

Definition 1.1 (Notations of partial derivatives)**.** *For $f(x)$ with one variable x, we know $f'(x) = \frac{df}{dx}$. For $u(x, y)$, we introduce partial derivatives as*

$$\frac{\partial u}{\partial x} = \left.\frac{du}{dx}\right|_{y \text{ is fixed}} = \partial_x u = u_x. \tag{1.1}$$

Similarly,

$$\frac{\partial^2 u}{\partial x^2} = \partial_x^2 u = \partial_{xx} u = u_{xx} \tag{1.2}$$

Example 1.1. For $u(x, y) = xy^2$, we have

$$\frac{\partial u}{\partial x} = \partial_x u = u_x = y^2, \qquad \frac{\partial u}{\partial y} = \partial_y u = u_y = 2xy,$$

$$\frac{\partial^2 u}{\partial x^2} = \partial_x^2 u = \partial_{xx} u = u_{xx} = 0, \qquad \frac{\partial^2 u}{\partial y^2} = \partial_y^2 u = \partial_{yy} u = u_{yy} = 2x$$

Definition 1.2 (Definition of general PDEs)**.** *Given a function $u = u(x, y)$ of two variables, (similarly $u = u(x_1, \cdots, x_n)$ of n variables) and an expression $F(u_{xx}, u_{xy}, u_{yy}, u_x, u_y, u, x, y)$ of partial derivatives of u, the following equation is a partial differential equation, abbreviated as PDE.*

$$F(u_{xx}, u_{xy}, u_{yy}, u_x, u_y, u, x, y) = 0 \tag{1.3}$$

In the future, we may also use the notation $F[u]$ to represent $F(u_{xx}, u_{xy}, u_{yy}, u_x, u_y, u, x, y)$. And (1.3) *can be rewritten as*

$$F[u] = 0. \tag{1.4}$$

Remark 1.3. $F[u]$ may also involve derivatives of order ≥ 2, but we do not discuss it in this course.

Example 1.2 (Examples of PDEs). Here are some examples of PDEs.

$$\begin{aligned} &u_{xx} - u_y = 0 \quad \text{(the heat equation)} \\ &u_{xx} - u_{yy} = 0 \quad \text{(the wave equation)} \\ &u_{xx} + u_{yy} = 0 \quad \text{(Laplace's equation)} \\ &u_x + u_y = 0 \quad \text{(the transport equation)} \\ &u_x + uu_y = 0 \quad \text{(the Burgers equation)} \end{aligned} \tag{1.5}$$

Definition 1.4 (Order of PDEs). *The order of a PDE is the order of the highest-order derivative in the equation. In* (1.5)*, the first three PDEs are second order, and the last two are first order.*

Definition 1.5 (Linear PDEs). *Given a PDE $F[u] = 0$, if it satisfies*

$$F[u+v] = F[u] + F[v] \text{ and } F[cu] = cF[u], \tag{1.6}$$

then we say that $F[u] = 0$ is a linear PDE. In (1.5)*, the first four PDEs are linear, while the last one is not.*

We have the following proposition which characterizes all second order linear PDEs,

Proposition 1.6. *The second-order linear PDEs can always be written as*

$$a(x,y)u_{xx} + b(x,y)u_{xy} + c(x,y)u_{yy} + d(x,y)u_x + e(x,y)u_y + f(x,y)u = g(x,y), \tag{1.7}$$

We assume $a^2 + b^2 + c^2 \neq 0$ for any x, y (at least one of a, b, c is nonzero).

Proof. This go beyond the scope of this course. □

Definition 1.7. *We call a, b, c, d, e, f coefficients and g source term.*

1.2 Classification of second-order PDEs

In this course, we will mainly consider second-order linear PDEs.

$$a(x,y)u_{xx} + b(x,y)u_{xy} + c(x,y)u_{yy} + d(x,y)u_x + e(x,y)u_y + f(x,y)u = g(x,y), \tag{1.8}$$

These equations are classified as follows by the coefficients a, b, c.

Definition 1.8 (Classification of PDEs)**.** *The second-order linear PDEs* (1.8) *are classified as* *elliptic*, *parabolic* *and* *hyperbolic* *by the following,*

$$\begin{cases} 4ac - b^2 > 0 & \textit{elliptic} \\ 4ac - b^2 = 0 & \textit{parabolic} \\ 4ac - b^2 < 0 & \textit{hyperbolic} \end{cases} \tag{1.9}$$

where we note that a, b, c *are functions of* x, y *and the inequalities in* (1.9) *is required to be true for any* x*,* y*.*

1.3 Review of ODEs

Definition 1.9 (Separable ODEs)**.** *The following ODE is the* *separable* *ODE*

$$y' + p(x)y = 0. \tag{1.10}$$

Theorem 1.10. *Separable ODE can be solved in the following way*

$$y' + p(x)y = 0 \quad \Rightarrow \quad \frac{dy}{dx} + p(x)y = 0 \quad \Rightarrow \quad \frac{dy}{y} = -p(x)dx \quad \Rightarrow \quad \int \frac{dy}{y} = -\int p(x)dx$$

and the solution is

$$y(x) = Ce^{-\int p(x)dx} \tag{1.11}$$

Proof. There is nothing to prove. $\square$

Example 1.3. Let us solve the following ODEs

$$y' = -6xy$$

Apply the procedure in Theorem 1.10

$$y' = -6xy \quad \Rightarrow \quad \frac{dy}{y} = -6xdx \quad \Rightarrow \quad \int \frac{dy}{y} = -\int 6xdx \quad \Rightarrow \quad \ln|y| = -3x^2 + C'$$

Therefore, the solution is

$$y = Ce^{-3x^2}.$$

where $C(= \pm e^{C'})$ is an arbitrary constant.

Definition 1.11 (Linear ODEs)**.** *The following ODE is the* *linear* *ODE*

$$y' + p(x)y = q(x). \tag{1.12}$$

Theorem 1.12. *Linear ODE can be solved by the following procedure*

1. *Solve the corresponding separable equation $y' - p(x)y = 0$ to obtain a solution $\hat{y} = e^{\int p(x)dx}$.*

2. *Multiply the linear ODE by $\hat{y}$ and rewrite the ODE*

$$y' + p(x)y = q(x) \quad \Rightarrow \quad \hat{y}(y' + p(x)y) = \hat{y}q(x) \quad \Rightarrow \quad (\hat{y}y)' = \hat{y}q(x)$$

3. *Integrate the above equation*

$$\begin{aligned} \hat{y}y = \int \hat{y}q(x)dx + C \quad &\Rightarrow \quad y = \frac{1}{\hat{y}}\left(\int \hat{y}q(x)dx + C\right) \\ &\Rightarrow \quad y = e^{-\int p(x)dx}\left(\int q(x)e^{\int p(x)dx}dx + C\right) \end{aligned}$$

Proof. There is nothing to prove. □

Remark 1.13. In (2), we applied the following equation

$$(\hat{y}y)' = \hat{y}(y' + p(x)y)$$

which is a corollary of the Leibniz rule.

$$(\hat{y}y)' = \hat{y}'y + \hat{y}y' = \hat{y}y' + p(x)\hat{y}y = \hat{y}(y' + p(x)y)$$

Example 1.4. $(x^2+1)y' + 3xy = 6x$, $y(0) = 3$ is solved as $y(x) = 2 + (x^2+1)^{-3/2}$ by the procedure in Theorem 1.12.

To apply Theorem 1.12, we divide both sides by (x^2+1).

$$(x^2+1)y' + 3xy = 6x \quad \Rightarrow \quad y' + \frac{3x}{x^2+1}y = \frac{6x}{x^2+1}$$

then we apply the three steps in Theorem 1.12.

1. Solve the corresponding separable equation $y' - \frac{3x}{x^2+1}y = 0$ to obtain a solution $(x^2+1)^{\frac{3}{2}}$.

2. Multiply the linear ODE by $(x^2+1)^{\frac{3}{2}}$ and rewrite the ODE

$$y' + \frac{3x}{x^2+1}y = \frac{6x}{x^2+1} \quad \Rightarrow \quad ((x^2+1)^{\frac{3}{2}}y)' = 6x(x^2+1)^{\frac{1}{2}}$$

3. Integrate the above equation

$$\begin{aligned} ((x^2+1)^{\frac{3}{2}}y)' = 6x(x^2+1)^{\frac{1}{2}} \quad &\Rightarrow \quad y = (x^2+1)^{-\frac{3}{2}}\left(\int 6x(x^2+1)^{\frac{1}{2}} + C\right) \\ &\Rightarrow \quad y = 2 + C(x^2+1)^{-\frac{3}{2}} \end{aligned}$$

We note that the solutions with undetermined constant C are called general solutions. Finally, we apply the initial condition $y(0) = 3$ to obtain $C = 1$, so the solution of the initial value problem is

$$y = 2 + (x^2 + 1)^{-\frac{3}{2}}.$$

Definition 1.14 (Second order ODEs)**.** *The constant coefficient second order ODEs are the following equations*

$$ay'' + by' + cy = 0 \ (a \neq 0). \tag{1.13}$$

Theorem 1.15. *Constant coefficient second order ODEs can be solved by the following procedure*

1. *Solve the characteristic equation* $a\lambda^2 + b\lambda + c = 0$ *to get two solutions* λ_1 *and* λ_2*.*

2. *If* $\lambda_1 \neq \lambda_2$*, the general solution is*

$$y(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} \tag{1.14}$$

3. *If* $\lambda_1 = \lambda_2 = \lambda$*, the general solution is*

$$y(x) = (C_1 + C_2 x)e^{\lambda x} \tag{1.15}$$

4. *If* λ_1, λ_2 *are complex roots* $\alpha \pm i\beta$*, apply the Euler's formula to rewrite* (1.14)

$$y(x) = C_1 e^{(\alpha+i\beta)x} + C_2 e^{(\alpha-i\beta)x} = e^{\alpha x}\left(c_1 \cos\beta x + c_2 \sin\beta x\right) \tag{1.16}$$

   where $c_1 = C_1 + C_2$*,* $c_2 = i(C_1 - C_2)$*.*

Proof. This was explained in your ODE course. □

Example 1.5. $y'' - 3y' + 2y = 0$ is solved as $y(x) = C_1 e^x + C_2 e^{2x}$ by the procedure in Theorem 1.15.

1. Solve the characteristic equation $\lambda^2 - 3\lambda + 2 = 0$ to get two solutions $\lambda_1 = 1$ and $\lambda_2 = 2$.

2. Since $\lambda_1 \neq \lambda_2$, by (1.14), the general solution is

$$y(x) = C_1 e^x + C_2 e^{2x} \tag{1.17}$$

Example 1.6. $y'' + y = 0$ is solved as $y(x) = C_1 \cos(x) + C_2 \sin(x)$ by the procedure in Theorem 1.15.

1. Solve the characteristic equation $\lambda^2 + 1 = 0$ to get two solutions $\lambda_1 = i$ and $\lambda_2 = -i$.

2. Since λ_1, λ_2 are complex, by (1.16), the general solution is

$$y(x) = C_1 \cos x + C_2 \sin x \tag{1.18}$$

Example 1.7. $y'' + 2y' + y = 0$ is solved as $y(x) = (C_1 + C_2 x)e^{-x}$ by the procedure in Theorem 1.15.

1. Solve the characteristic equation $\lambda^2 + 2\lambda + 1 = 0$ to get $\lambda_1 = \lambda_2 = -1$.

2. Since λ_1, λ_2 are equal, by (1.15), the general solution is

$$y(x) = (C_1 + C_2 x)e^{-x} \tag{1.19}$$

1.4 Separation of variables

Many linear PDEs can be reduced to linear ODEs with the method of separation of variables, described below.

We take the Laplace's equation

$$u_{xx} + u_{yy} = 0 \tag{1.20}$$

with boundary condition

$$u(0, y) = 0, \quad u(L, y) = 0, \quad u(x, 0) = 0, \quad u(x, L) = \varphi(x). \tag{1.21}$$

as an example.

We are looking for a separated solution. Substitute into (1.20), then we get

$$X''Y + XY'' = 0 \quad \Rightarrow \quad \frac{X''}{X} = -\frac{Y''}{Y}$$

The following lemma implies that X''/X and Y''/Y are constants.

Lemma 1.16. *$f(x) = g(y)$ implies that $f(x) = g(y) =$ const,*

Proof. $f(x) = g(y) \Rightarrow f'(x) = \partial_x(g(y)) = 0 \Rightarrow f(x) =$ *const.* □

Let λ be a constant and we write

$$X'' + \lambda X = 0, \quad Y'' - \lambda Y = 0.$$

We call λ the separation constant. At this moment λ is arbitrary. Thus the PDE was reduced to two ODEs.

1.4.1 Solving separated solutions

If $\lambda = 0$, then two ODEs have the following linearly independent solutions.

$$X = 1, x, \quad Y = 1, y. \tag{1.22}$$

If $\lambda \neq 0$, then two ODEs have the following linearly independent solutions.

$$X = e^{\sqrt{-\lambda}x}, e^{-\sqrt{-\lambda}x}, \quad Y = e^{\sqrt{\lambda}y}, e^{-\sqrt{\lambda}y}. \tag{1.23}$$

In either case, the solution is given by superpositions:

$$u = \begin{cases} (A_1 x + A_2)(B_1 y + B_2), & \lambda = 0 \\ \left(A_1 e^{\sqrt{-\lambda}x} + A_2 e^{-\sqrt{-\lambda}x}\right)\left(B_1 e^{\sqrt{\lambda}y} + B_2 e^{-\sqrt{\lambda}y}\right), & \lambda \neq 0 \end{cases} \tag{1.24}$$

where A_1, A_2, B_1, B_2 are constants.

For $\lambda > 0$, by writing $\lambda = k^2 (k > 0)$ we have

$$u(x, y) = \left(A_1 e^{ikx} + A_2 e^{-ikx}\right)\left(B_1 e^{ky} + B_2 e^{-ky}\right), \tag{1.25}$$

and for $\lambda < 0$, by writing $\lambda = -l^2 (l > 0)$ we have

$$u(x, y) = \left(A_1 e^{lx} + A_2 e^{-lx}\right)\left(B_1 e^{ily} + B_2 e^{-ily}\right). \tag{1.26}$$

Therefore, we get

$$u = \begin{cases} (A_1 x + A_2)(B_1 y + B_2), \\ \left(A_1 e^{ikx} + A_2 e^{-ikx}\right)\left(B_1 e^{ky} + B_2 e^{-ky}\right), \\ \left(A_1 e^{lx} + A_2 e^{-lx}\right)\left(B_1 e^{ily} + B_2 e^{-ily}\right). \end{cases} \tag{1.27}$$

Instead of (1.23), we can also choose

$$X = \cos(\sqrt{\lambda}x), \sin(\sqrt{\lambda}x), \quad Y = \cosh(\sqrt{\lambda}y), \sinh(\sqrt{\lambda}y). \tag{1.28}$$

In this case, we have

$$u(x, y) = (A_1 \cos(kx) + A_2 \sin(kx))(B_1 \cosh(ky) + B_2 \sinh(ky)) \tag{1.29}$$

$$u(x, y) = (A_1 \cosh(lx) + A_2 \sinh(lx))(B_1 \cos(ly) + B_2 \sin(ly)) \tag{1.30}$$

Note that (1.29) becomes (1.25) and (1.30) becomes (1.26) by redefining the coefficients. We call solutions such as (1.24) through (1.30) separated solutions because they are given in the form $u(x, y) = X(x)Y(y)$.

The final result is

$$u = \begin{cases} (A_1 x + A_2)(B_1 y + B_2), \\ (A_1 \cos(kx) + A_2 \sin(kx))(B_1 \cosh(ky) + B_2 \sinh(ky)), \\ (A_1 \cosh(lx) + A_2 \sinh(lx))(B_1 \cos(ly) + B_2 \sin(ly)). \end{cases} \tag{1.31}$$

1.4.2 Solving the boundary value problem

The separation constant λ and coefficients A_1, A_2, B_1, B_2 are partially determined by boundary conditions that in the region $0 < x < L, 0 < y < \infty$, $u(x, y)$ satisfies that

$$u(0, y) = 0, \quad u(L, y) = 0, \quad u(x, 0) = 0, \quad u(x, L) = \varphi(x). \tag{1.32}$$

Let us only consider the first three conditions.

$$u(0, y) = 0, \quad u(L, y) = 0, \quad u(x, 0) = 0. \tag{1.33}$$

In (1.31), we have three cases.

1. $u = (A_1 x + A_2)(B_1 y + B_2)$. In this case, u satisfies boundary conditions $u(0, y) = u(L, y) = 0$ in (1.33) if and only if $u = 0$.

2. $u = (A_1 \cos(kx) + A_2 \sin(kx))(B_1 \cosh(ky) + B_2 \sinh(ky))$. In this case, u satisfies boundary conditions $u(0, y) = u(L, y) = 0$ when $A_1 = 0$ and $k = n\pi/L$, where n is an integer. Furthermore we find $B_1 = 0$ by the condition $u(x, 0) = 0$. That is, the solution $A_2 B_2 \sin(kx) \sinh(ky)$ with $k = n\pi/L$ $(n = 0, \pm 1, \pm 2, \ldots)$ satisfies the boundary conditions.

3. $u = (A_1 \cosh(lx) + A_2 \sinh(lx))(B_1 \cos(ly) + B_2 \sin(ly))$. In this case, u satisfies boundary conditions $u(0, y) = u(L, y) = 0$ only when $A_1 = A_2 = 0$. That is, only the solution $u = 0$ satisfies the boundary conditions.

Therefore we obtain the following separated solutions of Laplace's equation satisfying the boundary conditions.

$$u(x, y) = A \sin \frac{n\pi x}{L} \sinh \frac{n\pi y}{L}, \quad n = 1, 2, \ldots, \tag{1.34}$$

1.4.3 Matching with the last boundary condition

Now we consider the last boundary condition $u(x, L) = \varphi(x)$ in (1.33). For arbitrary $\varphi(x)$, our separated solution $A \sin \frac{n\pi x}{L} \sinh \frac{n\pi y}{L}$ cannot match with this boundary condition. However, we can use a linear combination of this separated solution to generate more solutions,

$$u(x,y) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L}\sinh\frac{n\pi y}{L}. \tag{1.35}$$

To match with $u(x,L) = \varphi(x)$, we take $y = L$ in (1.35),

$$\varphi(x) = u(x,L) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L}\sinh\frac{n\pi L}{L}. \tag{1.36}$$

Therefore, we get

$$\varphi(x) = \sum_{n=1}^{\infty} A_n \sinh n\pi \sin\frac{n\pi x}{L}. \tag{1.37}$$

Then A_n can be solved from the sine Fourier coefficients introduced in section 2.2.3.

The result is

$$A_n = \frac{2}{L\sinh n\pi}\int_0^L \varphi(x)\sin\frac{n\pi x}{L}dx \tag{1.38}$$

Substitute into (1.35), then we solve Laplace's equation (1.20) with boundary condition.

2 Introduction to Fourier series

Every function $f(x)$ on domain $-L < x < L$ can be represented by a series of the form

$$A_0 + \sum_{n=1}^{\infty}\left(A_n\cos\frac{n\pi x}{L} + B_n\sin\frac{n\pi x}{L}\right)$$

This is called the Fourier series. We will first discuss how to compute the coefficients A_0, A_1, B_1, $\ldots$ in terms of $f(x)$, then we discuss several properties and variants of the Fourier series.

2.1 Definition of Fourier series

The Fourier series is an infinite sum of trigonometric functions defined as the following

Definition 2.1 (Fourier series)**.** *Let $A_0, A_1, B_1, \ldots$ be constants. The series below is called a Fourier series.*

$$A_0 + \sum_{n=1}^{\infty}\left(A_n\cos\frac{n\pi x}{L} + B_n\sin\frac{n\pi x}{L}\right) \tag{2.1}$$

It turns out that every function $f(x)$ can be written into an infinite sum of trigonometric functions. As suggested by the following theorem,

Theorem 2.2. *Every function $f(x)$ on domain $-L < x < L$ can be represented by a Fourier series. In other word, for any $f(x)$ on domain $-L < x < L$, there exists coefficients $A_0, A_1, B_1, \dots$ such that*

$$f(x) = A_0 + \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi x}{L} + B_n \sin \frac{n\pi x}{L} \right). \tag{2.2}$$

2.2 Fourier series and orthogonality

2.2.1 Orthogonality of trigonometric function

We want to compute the coefficients $A_0, A_1, B_1, \dots$ in terms of $f(x)$. The following observation may be helpful.

Question. Given several orthogonal vector $e_1, e_2, \dots, e_n$, assume that we have a decomposition

$$v = v_1 e_1 + v_2 e_2 + \cdots + v_n e_n \tag{2.3}$$

How do we compute the coefficients $v_1, v_2, \dots, v_n$ in terms of v?

Answer. Assume that we want to solve the coefficient v_j, then we take inner product of (2.3) with e_j. If we do this, by orthogonality $\langle e_i, e_j \rangle = 0$ (if $i \neq j$) and $\langle e_j, e_j \rangle = |e_j|^2$, then we get

$$\begin{aligned} \langle v, e_j \rangle =& \langle v_1 e_1 + v_2 e_2 + \cdots + v_n e_n, e_j \rangle \\ =& v_j |e_j|^2. \end{aligned} \tag{2.4}$$

Therefore, we solve the coefficient $v_j = \frac{\langle v, e_j \rangle}{|e_j|^2}$.

The following notation is useful.

Definition 2.3 (Kronecker delta)**.** *The Kronecker delta δ_{mn} is defined as*

$$\delta_{mn} = \begin{cases} 1, & m = n \\ 0, & m \neq n \end{cases} \tag{2.5}$$

Using this notation orthogonality of $e_1, e_2, \dots, e_n$ is equivalent to $\langle e_m, e_n \rangle = |e_m|^2 \delta_{mn}$.

The following theorem implies that sine and cosine functions are similar to orthogonal vectors.

Theorem 2.4 (Orthogonality of trigonometric functions)**.** *Let $n, m \geq 0$ be integers. We assume*

$L > 0$. The following orthogonality relations hold.

$$\begin{aligned}
&\int_{-L}^{L} \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx = \begin{cases} 2L & (n=m=0), \\ L\delta_{nm} & (\textit{otherwise}), \end{cases} \\
&\int_{-L}^{L} \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}dx = \begin{cases} 0 & (n=m=0), \\ L\delta_{nm} & (\textit{otherwise}), \end{cases} \\
&\int_{-L}^{L} \sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx = 0 \quad (\textit{all } n, m).
\end{aligned} \tag{2.6}$$

Proof. We only prove the orthogonality for the first equation which only involves cosine. The proof of other equations is left as homework.

Let us first start with the following observation.

Claim. If $n \neq 0$, then we have

$$\int_{-L}^{L} \cos\frac{n\pi x}{L}dx = 0 \tag{2.7}$$

This claim is true because

$$\begin{aligned}
\int_{-L}^{L} \cos\frac{n\pi x}{L}dx &= \left[\frac{L}{n\pi}\sin\frac{n\pi x}{L}\right]_{-L}^{L} \\
&= \frac{L}{n\pi}(\sin n\pi - \sin(-n\pi)) = 0
\end{aligned} \tag{2.8}$$

Now we start the proof of (2.6) for cosine.

Case 1. $(n = m = 0)$ In this case, we have

$$\int_{-L}^{L} \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx = \int_{-L}^{L} 1\cdot 1dx = 2L$$

Case 2. (At least one of n, m is nonzero). We have

$$\int_{-L}^{L} \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx = \frac{1}{2}\int_{-L}^{L}\left(\cos\frac{(n-m)\pi x}{L} + \cos\frac{(n+m)\pi x}{L}\right)dx$$

Since at least one of n, m is nonzero, $n + m > 0$. The integral over $\cos\frac{(n+m)\pi x}{L}$ is 0 by (2.7). Therefore,

$$\int_{-L}^{L} \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx = \frac{1}{2}\int_{-L}^{L}\cos\frac{(n-m)\pi x}{L}dx$$

Case 2.1. $(n \neq m)$ In this case, $\frac{1}{2}\int_{-L}^{L}\cos\frac{(n-m)\pi x}{L}dx = 0$ by (2.7). Therefore, we get

$$\int_{-L}^{L} \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx = 0 = \delta_{mn}.$$

Case 2.2. $(n = m)$, then

$$\int_{-L}^{L} \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx = \frac{1}{2}\int_{-L}^{L}\cos\frac{(n-m)\pi x}{L}dx = \frac{1}{2}\int_{-L}^{L}\cos\frac{0\cdot\pi x}{L}dx = L.$$

Thus the orthogonality relations for $\int_{-L}^{L}\cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx$ is proved. The orthogonality relations for $\int_{-L}^{L}\sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}dx$ and $\int_{-L}^{L}\sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx$ are similarly proved and left as homework. □

2.2.2 Formula of Fourier coefficients

We can use the orthogonality relations (2.6) to compute $A_0, A_1, B_1, \ldots$

To determine A_0 in (2.2), we multiply $\cos\frac{0\cdot\pi x}{L} = 1$ on both sides and integrate with respect to x:

$$\int_{-L}^{L} f(x)dx = \int_{-L}^{L} A_0 dx + \sum_{n=1}^{\infty}\int_{-L}^{L}\left(A_n\cos\frac{n\pi x}{L} + B_n\sin\frac{n\pi x}{L}\right)dx = \int_{-L}^{L} A_0 dx = 2A_0.$$

where we use the fact that $\int_{-L}^{L}\cos\frac{n\pi x}{L}dx = \int_{-L}^{L}\sin\frac{n\pi x}{L}dx = 0$. This is just the first equation in (2.6) with $m = 0$.

Therefore, we get an expression of A_0,

$$A_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx.$$

To determine A_m $(m = 1, 2, \ldots)$ in (2.2), we multiply $\cos\frac{m\pi x}{L}$ on both sides and integrate with respect to x:

$$\begin{aligned}\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}dx &= \int_{-L}^{L} A_0\cos\frac{m\pi x}{L}dx + \sum_{n=1}^{\infty}\int_{-L}^{L}\left(A_n\cos\frac{n\pi x}{L} + B_n\sin\frac{n\pi x}{L}\right)\cos\frac{m\pi x}{L}dx\\ &= \sum_{n=1}^{\infty} A_n\int_{-L}^{L}\cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx = \sum_{n=1}^{\infty} A_n L\delta_{nm} = LA_m.\end{aligned}$$

Therefore, we get an expression of A_m,

$$A_m = \int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}dx.$$

Similarly we can determine B_m $(m = 1, 2, \ldots)$ in (2.2) by multiplying $\sin\frac{m\pi x}{L}$ on both sides and integrate with respect to x:

$$\begin{aligned}\int_{-L}^{L} f(x)\sin\frac{m\pi x}{L}dx &= \int_{-L}^{L} A_0\sin\frac{m\pi x}{L}dx + \sum_{n=1}^{\infty}\int_{-L}^{L}\left(A_n\cos\frac{n\pi x}{L} + B_n\sin\frac{n\pi x}{L}\right)\sin\frac{m\pi x}{L}dx\\ &= \sum_{n=1}^{\infty} B_n\int_{-L}^{L}\sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}dx = \sum_{n=1}^{\infty} B_n L\delta_{nm} = LB_m.\end{aligned}$$

Therefore, we get an expression of B_m,

$$B_m = \int_{-L}^{L} f(x) \sin \frac{m\pi x}{L} dx.$$

In summary, we have the following theorem.

Theorem 2.5. *The Fourier coefficients in $A_0, A_1, B_1, \ldots$ are given by*

$$\begin{aligned}
A_0 &= \frac{1}{2L} \int_{-L}^{L} f(x) dx, \\
A_n &= \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} dx, \\
B_n &= \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} dx.
\end{aligned} \tag{2.9}$$

Example 2.1. Let us calculate the Fourier series of $f(x) = x, -L < x < L$.

Solution. We have

$$\begin{aligned}
A_0 &= \frac{1}{2L} \int_{-L}^{L} x dx = 0, \\
A_n &= \frac{1}{L} \int_{-L}^{L} x \cos \frac{n\pi x}{L} dx = 0 \quad (x \cos \frac{n\pi x}{L} \textit{ is an odd function}), \\
B_n &= \frac{1}{L} \int_{-L}^{L} x \sin \frac{n\pi x}{L} dx = \frac{1}{L} \left[- \frac{L}{n\pi} x \cos \frac{n\pi x}{L} \Big|_{-L}^{L} + \frac{L}{n\pi} \int_{-L}^{L} \cos \frac{n\pi x}{L} dx \right] = \frac{2L}{n\pi} (-1)^{n+1}.
\end{aligned}$$

Therefore we obtain

$$x = \sum_{n=1}^{\infty} \frac{2L}{n\pi} (-1)^{n+1} \sin \frac{n\pi x}{L}, \quad -L < x < L. \tag{2.10}$$

□

We have two observations from the above example

Proposition 2.6. *An odd or even function only has* cos *or* sin *term in its Fourier series.*

Proof. We only consider the even case. The odd case can be proved similarly.

Since $f(x)$ is even, we know that $\frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} dx$ is odd. By (2.9), all the B_n vanishes. Therefore, we only have the cos term in the Fourier series. □

Proposition 2.7. *Given a function $f(x)$ defined on $-L < x < L$, the Fourier series of $f(x)$ coincide with $f(x)$ only in the domain $-L < x < L$, unless $f(x)$ is a periodic function.*

Proof. We will not prove this proposition. Instead, we draw the graph of $y = x$ and its Fourier series

The Fourier series coincides with $y = x$ only in the domain $-L < x < L$. But the Fourier series is a periodic function since it is a sum of several periodic functions cos / sin. Since $y = x$ is not a periodic function, the Fourier series cannot agree with it for all x.

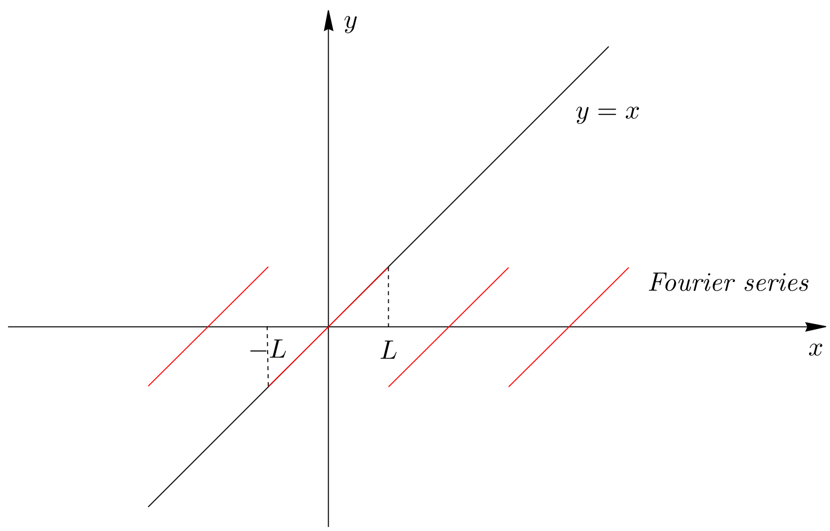

Figure 1: The graph of $y = x$ and its Fourier series.

□

2.2.3 Solve the A_n coefficients in (1.37)

Now we return to the separation of variable in section 1.4.3, where we derived

$$\varphi(x) = \sum_{n=1}^{\infty} A_n \sinh n\pi \sin \frac{n\pi x}{L}. \tag{2.11}$$

We want to solve A_n from $\varphi(x)$.

As explained in section 2.4, on $0 < x < L$, we also have the following orthogonality of $\sin \frac{n\pi x}{L}$

$$\int_0^L \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} dx = \frac{L}{2}\delta_{nm}, \qquad n, m \geq 1. \tag{2.12}$$

Therefore, if we want to solve A_m, we can multiply (2.11) by $\sin \frac{n\pi x}{L}$ and then integrate over x

$$\int_0^L \varphi(x) \sin \frac{m\pi x}{L} dx = \sum_{n=1}^{\infty} \int_0^L A_n \sinh n\pi \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} dx$$
$$= \sum_{n=1}^{\infty} A_n \sinh n\pi \int_0^L \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} dx = \sum_{n=1}^{\infty} A_n \sinh n\pi \frac{L}{2} \delta_{nm} = \frac{L}{2} A_m \sinh n\pi.$$

From this, we can solve A_m and get

$$A_n = \frac{2}{L \sinh n\pi} \int_0^L \varphi(x) \sin \frac{n\pi x}{L} dx \tag{2.13}$$

This is exactly (1.38).

2.3 Complex form of Fourier series

We have seen that every function defined on $-L < x < L$ can be rewritten as an infinite sum of $\cos \frac{n\pi x}{L}$ and $\sin \frac{n\pi x}{L}$. By Euler's formula $e^{ix} = \cos x + i \sin x$, we know that every trigonometric function is a linear combination of e^{ix} and e^{-ix}

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}, \qquad \sin x = \frac{e^{ix} - e^{-ix}}{2i}. \tag{2.14}$$

Therefore, we have the following theorem.

Theorem 2.8. *In other word, for any $f(x)$ on domain $-L < x < L$, there exists coefficients $\dots, \alpha_{-1}, \alpha_0, \alpha_1, \alpha_2, \dots$ such that*

$$f(x) = \sum_{n=-\infty}^{\infty} \alpha_n e^{in\pi x/L}, \tag{2.15}$$

where α_n satisfies the following properties.

1. *α_n is given by*
$$\alpha_n = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-in\pi x/L} dx \tag{2.16}$$

2. *If $f(x)$ is a real value function, then $\alpha_n = \bar{\alpha}_{-n}$, where $\bar{\alpha}_{-n}$ is the complex conjugate of α_{-n}.*

Proof. Using Euler's formula (2.14), we can rewrite the Fourier series of f as follows.

$$\begin{aligned} f(x) =& A_0 + \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi x}{L} + B_n \sin \frac{n\pi x}{L} \right) \\ =& A_0 + \sum_{n=1}^{\infty} \left(A_n \left(\frac{e^{in\pi x/L} + e^{-in\pi x/L}}{2} \right) + B_n \left(\frac{e^{in\pi x/L} - e^{-in\pi x/L}}{2i} \right) \right) \\ =& A_0 + \sum_{n=1}^{\infty} \left(\frac{A_n - iB_n}{2} e^{in\pi x/L} + \frac{A_n + iB_n}{2} e^{-in\pi x/L} \right). \end{aligned}$$

If we define

$$\alpha_0 = A_0, \quad \alpha_n = \frac{A_n - iB_n}{2}, \quad \alpha_{-n} = \frac{A_n + iB_n}{2}, \quad (n > 0) \tag{2.17}$$

then we obtain

$$f(x) = \alpha_0 + \sum_{n=1}^{\infty} \left(\alpha_n e^{in\pi x/L} + \alpha_{-n} e^{-in\pi x/L} \right) = \sum_{n=-\infty}^{\infty} \alpha_n e^{in\pi x/L}.$$

If $f(x)$ is a real value function, then A_n and B_n are real numbers. From (2.17), we know that $\alpha_n = \bar{\alpha}_{-n}$.

The formula of α_n is also a corollary of (2.17). When $n > 0$,

$$\begin{aligned}\alpha_n =& \frac{A_n - iB_n}{2} = \frac{1}{2L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} dx - \frac{i}{2L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} dx \\ =& \frac{1}{2L} \int_{-L}^{L} f(x) e^{-in\pi x/L} dx\end{aligned}$$

When $n = -m < 0$,

$$\begin{aligned}\alpha_n = \alpha_{-m} =& \frac{A_m + iB_m}{2} = \frac{1}{2L} \int_{-L}^{L} f(x) \cos \frac{m\pi x}{L} dx + \frac{i}{2L} \int_{-L}^{L} f(x) \sin \frac{m\pi x}{L} dx \\ =& \frac{1}{2L} \int_{-L}^{L} f(x) e^{im\pi x/L} dx = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-in\pi x/L} dx\end{aligned}$$

When $n = 0$,

$$\alpha_0 = A_0 = \frac{1}{2L} \int_{-L}^{L} f(x) dx = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-i0\pi x/L} dx$$

Therefore, we have completed the proof. □

$e^{in\pi x/L}$ also have orthogonality relations.

Theorem 2.9 (Orthogonality relations of complex Fourier series)**.**

$$\int_{-L}^{L} e^{in\pi x/L} e^{-im\pi x/L} dx = \int_{-L}^{L} e^{i(n-m)\pi x/L} dx = 2L\delta_{mn}.$$

Proof. If $n = m$,

$$\int_{-L}^{L} e^{i(n-m)\pi x/L} dx = \int_{-L}^{L} 1 dx = 2L = 2L\delta_{mn}.$$

If $n \neq m$,

$$\int_{-L}^{L} e^{i(n-m)\pi x/L} dx = \frac{L}{i\pi(n-m)} \left[e^{i(n-m)\pi x/L} \right]_{-L}^{L} = 0 = 2L\delta_{mn}.$$

We have completed the proof. □

The formula of α_n can also be computed using orthogonality.

Assume that we want to compute α_m in

$$f(x) = \sum_{n=-\infty}^{\infty} \alpha_n e^{in\pi x/L}.$$

Multiply both side by $e^{-im\pi x/L}$ and then we get

$$\int_{-L}^{L} f(x)e^{-im\pi x/L}dx = \sum_{n=-\infty}^{\infty} \alpha_n \int_{-L}^{L} e^{in\pi x/L}e^{-im\pi x/L}dx \sum_{n=-\infty}^{\infty} \alpha_n 2L\delta_{mn} = 2L\alpha_m.$$

Solve α_m, then we prove (2.16) again.

2.4 Fourier cosine and sine series

A function defined on $-L < x < L$ can be rewritten as an infinite sum of $\cos \frac{n\pi x}{L}$ and $\sin \frac{n\pi x}{L}$. Notice that $-L < x < L$ is the period of $\cos \frac{n\pi x}{L}$ and $\sin \frac{n\pi x}{L}$. If a function $f(x)$ is defined in the interval $0 < x < L$, which represents half of its period, then it is possible to express $f(x)$ exclusively as a series of cosine terms, $\cos \frac{n\pi x}{L}$, or alternatively, only as a series of sine terms, $\sin \frac{n\pi x}{L}$. This is the Fourier cosine/sine series.

The idea is that, given a function $f(x)$ defined on $0 < x < L$, we can extend this function as an even/odd function on $-L < x < L$. Then we compute the Fourier series of the extended function. The Fourier series only contains $\cos \frac{n\pi x}{L}$ or $\sin \frac{n\pi x}{L}$ terms by Proposition 2.6.

2.4.1 Fourier cosine series

We define the even extension $f_E(x)$ of $f(x)$ as

$$f_E(x) = \begin{cases} f(x), & 0 < x < L \\ 0, & x = 0, \\ f(-x), & -L < x < 0. \end{cases} \tag{2.18}$$

We note that $f_E(x)$ is even. Indeed the value $f_E(0)$ is arbitrary and not necessarily zero. The Fourier series is given by

$$f_E(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L}, \quad -L < x < L,$$

where

$$A_0 = \frac{1}{2L}\int_{-L}^{L} f_E(x)dx = \frac{1}{L}\int_0^L f(x)dx,$$
$$A_n = \frac{1}{2L}\int_{-L}^{L} f_E(x)\cos\frac{n\pi x}{L}dx = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx.$$

On the interval $0 < x < L$ we have

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L}, \quad 0 < x < L, \tag{2.19}$$

where

$$A_0 = \frac{1}{L} \int_0^L f(x)dx, \quad A_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx. \tag{2.20}$$

Definition 2.10. *The series* (2.19) *is called the Fourier cosine series.*

For Fourier cosine series, we also have orthogonality relations and (2.20) can be computed from these orthogonality relations.

Theorem 2.11 (Orthogonality relations of cosine Fourier series)**.**

$$\int_0^L \cos \frac{n\pi x}{L} \cos \frac{m\pi x}{L} dx = \begin{cases} L & (n = m = 0), \\ \dfrac{L}{2}\delta_{nm} & (\textit{otherwise}), \end{cases} \tag{2.21}$$

Example 2.2. Let us compute the Fourier cosine series of $f(x) = x$, $0 < x < L$.

Solution. We can directly apply (2.20). But let us try the even extension method.

We extend f as

$$f_E(x) = \begin{cases} x, & 0 < x < L, \\ 0, & x = 0, \\ -x, & -L < x < 0. \end{cases}$$

Indeed $f_E(x) = |x|$.

$$\begin{aligned} A_0 &= \frac{1}{2L} \int_{-L}^L f_E(x)dx = \frac{1}{L} \int_0^L xdx = \frac{L}{2}, \\ A_n &= \frac{1}{2L} \int_{-L}^L f_E(x) \cos \frac{n\pi x}{L} dx = \frac{2}{L} \int_0^L x \cos \frac{n\pi x}{L} dx \\ &= \frac{2}{L} \left[\frac{L}{n\pi} x \sin \frac{n\pi x}{L} \Big|_0^L - \frac{L}{n\pi} \int_0^L \sin \frac{n\pi x}{L} dx \right] = \frac{2L}{(n\pi)^2} \left((-1)^n - 1\right). \end{aligned}$$

Therefore,

$$x = \frac{L}{2} + \frac{2L}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} \cos \frac{n\pi x}{L}, \quad 0 < x < L.$$

□

2.4.2 Fourier sine series

We define the odd extension $f_O(x)$ as

$$f_O(x) = \begin{cases} f(x), & 0 < x < L \\ 0, & x = 0, \\ -f(-x), & -L < x < 0. \end{cases} \tag{2.22}$$

We note that $f_O(x)$ is odd. The Fourier series is given by

$$f_O(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}, \quad -L < x < L,$$

where

$$B_n = \frac{1}{L} \int_{-L}^{L} f_O(x) \sin \frac{n\pi x}{L} dx = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx.$$

On the interval $0 < x < L$ we have

$$f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}, \quad 0 < x < L, \tag{2.23}$$

where

$$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx \tag{2.24}$$

Definition 2.12. *The series* (2.23) *is called the Fourier cosine series.*

For Fourier sine series, we also have orthogonality relations and (2.24) can be computed from these orthogonality relations.

Theorem 2.13 (Orthogonality relations of sine Fourier series)**.**

$$\int_0^L \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} dx = \begin{cases} 0 & (n = m = 0), \\ \frac{L}{2}\delta_{nm} & (\textit{otherwise}), \end{cases} \tag{2.25}$$

Example 2.3. The Fourier sine series of $f(x) = x$, $0 < x < L$, is obtained through the odd extension $f_O(x)$. The odd extension $f_O(x)$ is again x and its Fourier series has been computed in (2.10).

2.5 Convergence of Fourier series

In this section, we study the convergence of Fourier series of piecewise continuous functions.

Definition 2.14 (Left and right limits). *For a given $f(x)$, let us write*

$$f(x+0) = \lim_{\varepsilon\to 0} f(x+\varepsilon), \quad f(x-0) = \lim_{\varepsilon\to 0} f(x-\varepsilon), \tag{2.26}$$

where $\varepsilon > 0$.

Definition 2.15 (Piecewise continuous). *A function $f(x)$ defined on $a < x < b$, is said to be piecewise continuous if there is a finite set of points $a = x_0 < x_1 < \cdots < x_p < x_{p+1} = b$ such that $f(x)$ is continuous at $x \neq x_i$ $(i = 1, \ldots, p)$, $f(x_i + 0)(i = 0, \ldots, p)$ exists, and $f(x_i - 0)$ $(i = 1, \ldots, p+1)$ exists.*

Definition 2.16 (Piecewise smooth). *A function $f(x), a < x < b$, is said to be piecewise smooth if $f(x)$ and all of its derivatives are piecewise continuous.*

Example 2.4. The function $f(x) = |x|, -L < x < L$, is piecewise smooth. The function $f(x) = x^2 \sin(1/x), -L < x < L$, is piecewise continuous but is not piecewise smooth because $\lim_{\varepsilon\to 0} f'(0\pm\varepsilon)$ does not exist. The function $f(x) = 1/\left(x^2 - L^2\right)$, $-L < x < L$, is not piecewise continuous because $f(-L+0)$ and $f(L-0)$ are not finite.

Definition 2.17 (Convergence). *Given a Fourier series $f(x) = A_0 + \sum_{n=1}^{\infty}[A_n \cos(n\pi x/L) + B_n \sin(n\pi x/L)]$, we define*

1. ***Partial sums.*** *The partial sum, denoted by $f_N(x)$, is defined to be*

$$f_N(x) = A_0 + \sum_{n=1}^{N} \left[A_n \cos(n\pi x/L) + B_n \sin(n\pi x/L)\right]. \tag{2.27}$$

2. ***Convergence at x.*** *We say the Fourier is convergent at x if*

$$\lim_{N\to\infty} f_N(x) = f(x). \tag{2.28}$$

3. ***Uniformly convergence.*** *We say the Fourier is uniformly convergent if*

$$\lim_{N\to\infty} \max_{x\in[-L,L]} |f_N(x) - f(x)| = 0. \tag{2.29}$$

   This means that the convergence is equally well for all x.

Theorem 2.18 (Convergence theorem). *Let $f(x)$, $-L < x < L$, be piecewise smooth. Then the Fourier series of f converges for all x to the value $\frac{1}{2}[\bar{f}(x+0) + \bar{f}(x-0)]$, where $\bar{f}$ is the $2L$-periodic function which equals to f on $-L < x < L$.*

If $f(x)$ is continuous on $[-L, L]$ and $f(-L) = f(L)$ in addition to the conditions assumed in the above theorem, then the Fourier series uniformly converges.

Example 2.5. Here are two examples of convergence.

1. The Fourier series of $f(x) = |x|$, $-L < x < L$, (see Example 2.2) uniformly converges.
2. The Fourier series of $f(x) = x$, $-L < x < L$ does not uniformly converge but converges at any point to $f(x)$ except for $x = -L$, L.

As in the following picture, the Fourier series of $f(x) = x$, $-L < x < L$ has a very bad convergence near $x = -L$, L, but the series of $f(x) = |x|$ has much better convergence.

TODO: add a picture

2.6 Parseval's Theorem and Mean Square Error

2.6.1 The Parseval's Theorem for Fourier series

For orthogonal vectors $e_1, \ldots, e_n$, $e_i \perp e_j$, their linear combination $v = v_1 e_1 + \cdots + v_n e_n$ satisfies the Pythagorean theorem

$$|v|^2 = |v_1 e_1 + \cdots + v_n e_n|^2 = |v_1 e_1|^2 + \cdots + |v_n e_n|^2 \tag{2.30}$$

The proof of (2.30) is given by

$$|v|^2 = \left\langle \sum_{j=1}^{n} v_j e_j, \sum_{k=1}^{n} v_k e_k \right\rangle = \sum_{j=1}^{n}\sum_{k=1}^{n} v_j v_k \underbrace{\langle e_j, e_k \rangle}_{=0 \ \mathit{if}\ j \neq k} = \sum_{j=1}^{n} v_j^2 |e_j|^2$$

For the Fourier series, the following theorem claims that a similar identity is true.

Theorem 2.19 (Parseval's theorem)**.** *Let $f(x)$ defined on $-L < x < L$ be a piecewise smooth function with Fourier series*

$$f(x) = A_0 + \sum_{n=1}^{\infty} [A_n \cos(n\pi x/L) + B_n \sin(n\pi x/L)] \tag{2.31}$$

Then, the mean square $\frac{1}{2L}\int_{-L}^{L} f(x)^2 dx$ of $f(x)$ satisfies the following identity

$$\frac{1}{2L}\int_{-L}^{L} f(x)^2 dx = A_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} \left(A_n^2 + B_n^2\right) \tag{2.32}$$

Proof. Since $\cos(0\pi x/L) = 1$ and $\sin(0\pi x/L) = 0$, we know that (2.31) is equivalent to

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} [A_n \cos(n\pi x/L) + B_n \sin(n\pi x/L)] \\ &= \sum_{n=0}^{\infty} (A_n \cos_n + B_n \sin_n) \end{aligned} \tag{2.33}$$

where we introduce the notation

$$\cos_n = \cos(n\pi x/L), \quad \sin_n = \sin(n\pi x/L) \tag{2.34}$$

Similar to (2.30) and its proof, we have

$$\begin{aligned}
\frac{1}{2L}\int_{-L}^{L} f(x)^2 dx =& \frac{1}{2L}\int_{-L}^{L}\sum_{n=0}^{\infty}[A_n \cos_n + B_n \sin_n]\sum_{m=0}^{\infty}[A_m \cos_m + B_m \sin_m]dx \\
=& \frac{1}{2L}\int_{-L}^{L}\sum_{n,m=0}^{\infty}(A_n A_m \underbrace{\cos_n \cos_m}_{=0 \ if \ n\neq m} + A_n B_m \underbrace{\cos_n \sin_m}_{=0} \\
& + B_n A_m \underbrace{\sin_n \cos_m}_{=0} + B_n B_m \underbrace{\sin_n \sin_m}_{=0 \ if \ n\neq m})dx \\
=& A_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(A_n^2 + B_n^2\right)
\end{aligned} \tag{2.35}$$

where in the last line, we applied the orthogonality of trigonometric functions (2.6). □

2.6.2 The mean square error

Definition 2.20 (Mean square error)**.** *We define the mean square error* σ_N^2 *as*

$$\sigma_N^2 = \frac{1}{2L}\int_{-L}^{L}[f(x) - f_N(x)]^2 dx \tag{2.36}$$

where $f_N(x)$ *is the partial sum*

$$f_N(x) = A_0 + \sum_{n=1}^{N}\left[A_n \cos(n\pi x/L) + B_n \sin(n\pi x/L)\right]. \tag{2.37}$$

By Parseval's theorem, we obtain the following expression of the mean square error.

Proposition 2.21. *Given a partial summation whose mean square error is defined by* (2.36)*, then we have*

$$\sigma_N^2 = \frac{1}{2}\sum_{n=N+1}^{\infty}\left(A_n^2 + B_n^2\right) \tag{2.38}$$

Proof. This is an easy corollary of the Parseval's identity (2.32), if we notice that

$$f(x) - f_N(x) = \sum_{n=N+1}^{\infty}\left[A_n \cos(n\pi x/L) + B_n \sin(n\pi x/L)\right]$$

Therefore, we finished the proof. □

Example 2.6. Let us find σ_N^2 for $f(x) = x$, $-L < x < L$.

Solution. From Example 2.1, we have $A_0 = A_n = 0$ and

$$f_N(x) = \sum_{n=1}^{N} B_n \sin \frac{n\pi x}{L}, \quad B_n = \frac{2L}{n\pi}(-1)^{n+1}. \tag{2.39}$$

By (2.38), we obtain

$$\sigma_N^2 = \frac{1}{2} \sum_{n=N+1}^{\infty} \left(\frac{2L}{n\pi}(-1)^{n+1} \right)^2 = \frac{2L^2}{\pi^2} \sum_{n=N+1}^{\infty} \frac{1}{n^2}.$$

By Theorem 2.22, we get

$$\int_{N+1}^{\infty} \frac{1}{x^2} dx \leq \sum_{n=N+1}^{\infty} \frac{1}{n^2} \leq \int_{N+1}^{\infty} \frac{1}{(x-1)^2} dx = \int_{N}^{\infty} \frac{1}{x^2} dx.$$

We have

$$\int_{N+1}^{\infty} \frac{1}{x^2} dx = \frac{1}{N+1} \geq \frac{1}{N} - \frac{1}{N^2}, \quad \int_{N}^{\infty} \frac{1}{x^2} dx = \frac{1}{N}.$$

Let us introduce the symbol O (this is called "big O") to express the order. For some f_N, $f_N = O\left(N^{-1}\right)$ as $N \to \infty$ means that there exist a constant $C > 0$ such that $|f_N| \leq CN^{-1}$. Therefore we obtain

$$\sigma_N^2 = \frac{2L^2}{\pi^2} \frac{1}{N} \left[1 + O\left(\frac{1}{N} \right) \right] = O\left(N^{-1}\right), \quad N \to \infty. \tag{2.40}$$

We note that σ_N^2 goes to zero as $N \to \infty$ although we know that the sum in (2.39) does not converge uniformly. This happened because we considered the mean square and took the integral.
TODO: add a picture for convergence □

Theorem 2.22 (Integral test). *Given a monotonic and positive function $f(x)$, we have*

$$\int_{N+1}^{\infty} f(x) dx \leq \sum_{n=N+1}^{\infty} f(n) \leq \int_{N}^{\infty} f(x) dx \tag{2.41}$$

Proof. **TODO: compare the area below the graph of $y = f(x)$**

TODO: add a picture □

Example 2.7. Let us find σ_{2N}^2 for $f(x) = |x|, -L < x < L$.

Solution. From Example 2.2 we know that $B_n = 0$, $A_{2m} = 0$ $(m = 1, 2, \ldots)$, and

$$f_{2N}(x) = A_0 + \sum_{m=1}^{N} A_{2m-1} \cos \frac{(2m-1)\pi x}{L}, \quad A_0 = \frac{L}{2}, \quad A_{2m-1} = -\frac{4L}{\pi^2(2m-1)^2}.$$

This is also the Fourier cosine series of x, $0 < x < L$, in Example 2.2. Hence we obtain

$$\sigma_{2N}^2 = \frac{1}{2} \sum_{n=2N+1}^{\infty} A_n^2 = \frac{1}{2} \sum_{m=N+1}^{\infty} A_{2m-1}^2 = \frac{8L^2}{\pi^4} \sum_{m=N+1}^{\infty} \frac{1}{(2m-1)^4}.$$

Note that by Theorem 2.22

$$\int_{N+1}^{\infty} \frac{1}{(2x-3)^4} dx \leq \sum_{m=N+1}^{\infty} \frac{1}{(2m-1)^4} \leq \int_{N+1}^{\infty} \frac{1}{(2x-1)^4} dx,$$

and by Taylor expansion $\frac{1}{(1+x)^3} = 1 - 3x + 6x^2 + \cdots$, $LHS = \frac{1}{8N^3} \frac{1}{(1+1/2N)^3} = \frac{1}{6(2N+1)^3} = \frac{1}{48N^3} + O\left(N^{-4}\right)$ and $RHS = \frac{1}{6(2N-1)^3} = \frac{1}{48N^3} + O\left(N^{-4}\right)$. Therefore we obtain

$$\sigma_{2N}^2 = \frac{L^2}{6\pi^4 N^3} + O\left(N^{-4}\right) = O\left(N^{-3}\right), \quad N \to \infty. \tag{2.42}$$

Thus the Fourier series of x converges as $O(1/N)$ and the Fourier series of $|x|$ converges as $O\left(1/N^3\right)$. Equations (2.40) and (2.42) explain the difference between figure **??**. □

2.6.3 Parseval's theorem for complex, cosine and sine Fourier series

Theorem 2.23 (Parseval's theorem for complex Fourier series)**.** *Let $f(x)$ defined on $-L < x < L$ be a piecewise smooth function with Fourier series*

$$f(x) = \sum_{n=-\infty}^{\infty} \alpha_n e^{in\pi x/L} \tag{2.43}$$

Then, the mean square $\frac{1}{2L} \int_{-L}^{L} f(x)^2 dx$ of $f(x)$ satisfies the following identity

$$\frac{1}{2L} \int_{-L}^{L} |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |\alpha_n|^2. \tag{2.44}$$

Proof. Method 1. We have

$$\begin{aligned} \frac{1}{2L} \int_{-L}^{L} |f(x)|^2 dx =& \frac{1}{2L} \int_{-L}^{L} \sum_{n=-\infty}^{\infty} \alpha_n e^{in\pi x/L} \sum_{m=-\infty}^{\infty} \bar{\alpha}_m e^{-im\pi x/L} dx \\ =& \sum_{n,m=0}^{\infty} \alpha_n \bar{\alpha}_m \underbrace{\frac{1}{2L} \int_{-L}^{L} e^{i(n-m)\pi x/L} dx}_{=0 \text{ if } n \neq m} \\ =& \sum_{n=-\infty}^{\infty} |\alpha_n|^2 \end{aligned} \tag{2.45}$$

Method 2. (2.44) is seen by the calculation below. By Parseval's identity for the Fourier series (2.32),

$$
\begin{aligned}
\frac{1}{2L}\int_{-L}^{L} f(x)^2dx &= A_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(A_n^2+B_n^2\right) \\
&= A_0^2 + 2\sum_{n=1}^{\infty}\frac{A_n-iB_n}{2}\frac{A_n+iB_n}{2} \\
&= \alpha_0^2 + 2\sum_{n=1}^{\infty}\alpha_n\alpha_{-n} = \alpha_0^2 + \sum_{n=1}^{\infty}|\alpha_n|^2 + \sum_{n=1}^{\infty}|\alpha_{-n}|^2 \\
&= \alpha_0^2 + \sum_{n=1}^{\infty}|\alpha_n|^2 + \sum_{n=-\infty}^{-1}|\alpha_n|^2 \\
&= \sum_{n=-\infty}^{\infty}|\alpha_n|^2
\end{aligned}
$$

□

Theorem 2.24 (Parseval's theorem for cosine Fourier series)**.** *Let $f(x)$ defined on $-L<x<L$ be a piecewise smooth function with cosine Fourier series*

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L} \tag{2.46}$$

Then, the mean square $\frac{1}{2L}\int_{-L}^{L} f(x)^2dx$ of $f(x)$ satisfies the following identity

$$\frac{1}{2L}\int_{-L}^{L}|f(x)|^2dx = A_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}A_n^2. \tag{2.47}$$

Proof. This is homework. □

Theorem 2.25 (Parseval's theorem for sine Fourier series)**.** *Let $f(x)$ defined on $-L<x<L$ be a piecewise smooth function with sine Fourier series*

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} \tag{2.48}$$

Then, the mean square $\frac{1}{2L}\int_{-L}^{L} f(x)^2dx$ of $f(x)$ satisfies the following identity

$$\frac{1}{L}\int_{0}^{L} f(x)^2dx = \frac{1}{2}\sum_{n=1}^{\infty}B_n^2. \tag{2.49}$$

Proof. This is homework. □

3 PDEs in rectangular coordinates

In this section, we will consider the separation of variables for more general equations in rectangular coordinates, possibly with variable coefficients and more general boundary conditions.

3.1 Boundary conditions for general PDEs

3.1.1 Dirichlet, Neumann and Robin boundary conditions

In the ODE class, we have learned the following boundary value problem

$$
\begin{aligned}
&u'' = 1, \qquad x \in [0,1] \\
&u(0) = 0, \quad u(1) = 0.
\end{aligned}
\tag{3.1}
$$

where we prescribe a condition for every point on the boundary of the domain $[0, 1]$. (The boundary is 0 and 1.) If we miss any of these conditions, we cannot get a unique solution.

For example, if we remove the condition $u(1) = 0$, then we get solution $u(x) = \frac{1}{2}x^2 + Cx$, which contains an undetermined constant C.

Given a PDE defined on a domain R, we must also prescribe the boundary condition on ∂R to obtain a unique solution.

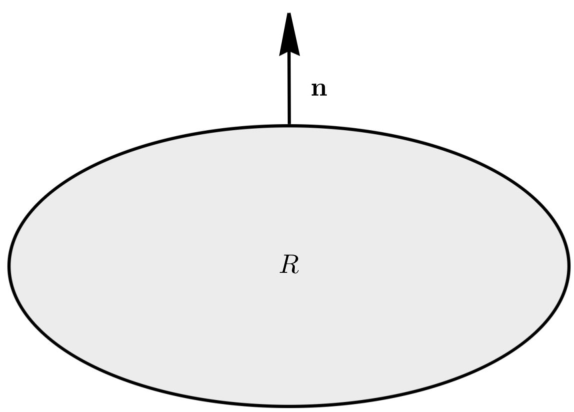

Figure 2: The region R and its normal vector.

Before we introduce the concept of boundary conditions, let us first explain the concept of normal vectors.

Definition 3.1 (Normal vector)**.** *Given a domain R and a point $x \in \partial R$, a* <u>*normal vector*</u> $\mathbf{n}$ *at x is the vector as demonstrated by figure 2. In this class, the normal vector is always pointing outwards.*

Now we introduce the boundary conditions that will be considered in this class.

Definition 3.2 (Boundary conditions)**.** *Consider a PDE $F[u] = 0$ defined on the domain R. Here are the boundary conditions that we consider in this class.*

1. ***Dirichlet boundary condition.*** *The value of u on the boundary is given.*

$$u = g(x), \quad x \in \partial R. \tag{3.2}$$

2. ***Neumann boundary condition.*** *The normal derivative on the boundary is given.*

$$\mathbf{n} \cdot \nabla u = g(x), \quad x \in \partial R. \tag{3.3}$$

3. ***Robin boundary condition.*** *A linear combination of the above two boundary conditions.*

$$a(x)u + b(x)\mathbf{n} \cdot \nabla u = g(x), \quad x \in \partial R. \tag{3.4}$$

3.1.2 Heat equations as an example

We take the heat equation $u_t = Ku_{zz}$, defined on $R = \{(z,t) : 0 \leq t < \infty,\ 0 \leq x \leq L\}$, as an example to explain these boundary conditions.

The domain R of the heat equation is described in the following picture,

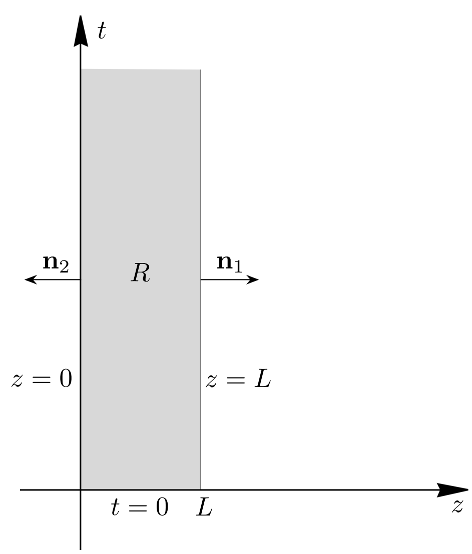

Figure 3: The domain for the heat equation.

There are three pieces of ∂R,

$$
\begin{aligned}
&z = 0, \quad t > 0 \\
&z = L, \quad t > 0 \\
&0 < z < L, \quad t = 0
\end{aligned}
\tag{3.5}
$$

On the first two parts of ∂R corresponding to $z = 0,\ L$, we impose the Robin boundary condition. On the last part corresponding to $t = 0$, we impose the Dirichlet boundary condition. Then we get the following system of equations,

$$
\begin{cases}
u_t = K u_{zz}, & 0 < z < L, \quad t > 0, \\
a(z,t)u + b(z,t)\mathbf{n} \cdot \nabla u = g(z,t), & z = 0,\ L, \quad t > 0, \\
u(z,0) = f(z), & 0 < z < L, \quad t = 0.
\end{cases}
\tag{3.6}
$$

Here the choice of Dirichlet and Robin boundary condition comes from the physics of heat conduction. We will explain this in section **??**.

TODO: second order need one condition but first order need one

Now we explain how to simplify the Robin boundary condition in (3.6).

On $z = 0$, from figure 3, we know that the normal vector $\mathbf{n} = (-1, 0)$ and $\nabla u = (u_z, u_t)$, so we get $\mathbf{n} \cdot \nabla u = -u_z$. Therefore, $a(z,t)u + b(z,t)\mathbf{n} \cdot \nabla u = g(z,t)$ simplifies to

$$
a(0,t)u - b(0,t)u_z = g(0,t). \tag{3.7}
$$

On $z = L$, from figure 3, we know that the normal vector $\mathbf{n} = (1, 0)$ and $\nabla u = (u_z, u_t)$, so we get $\mathbf{n} \cdot \nabla u = u_z$. Therefore, $a(z,t)u + b(z,t)\mathbf{n} \cdot \nabla u = g(z,t)$ simplifies to

$$
a(L,t)u + b(L,t)u_z = g(L,t). \tag{3.8}
$$

We introduce new functions $a(t)$, $\widetilde{a}(t)$, $b(t)$, $\widetilde{b}(t)$, $g(t)$ and $\widetilde{g}(t)$ by the following equations

$$
\begin{aligned}
a(t) = a(0,t), \quad b(t) = b(0,t), \quad g(t) = g(0,t) \\
\widetilde{a}(t) = a(L,t), \quad \widetilde{b}(t) = b(L,t), \quad \widetilde{g}(t) = g(L,t)
\end{aligned}
\tag{3.9}
$$

Then the Robin boundary condition becomes

$$
\begin{aligned}
a(t)u - b(t)u_z = g(t), \quad z = 0 \\
\widetilde{a}(t)u + \widetilde{b}(t)u_z = \widetilde{g}(t), \quad z = L
\end{aligned}
\tag{3.10}
$$

The heat equation becomes

$$\begin{cases} u_t = Ku_{zz}, & 0 < z < L, \quad t > 0, \\ a(t)u - b(t)u_z = g(t), & z = 0, \quad t > 0, \\ \widetilde{a}(t)u + \widetilde{b}(t)u_z = \widetilde{g}(t), & z = L, \quad t > 0, \\ u = f(z), & 0 < z < L, \quad t = 0. \end{cases} \tag{3.11}$$

In order to make (3.11) solvable, we impose the homogeneous condition.

Definition 3.3 (Homogeneous)**.** *We say* (3.11) *is* <u>*homogeneous*</u> *if*

1. *$a(t)$, $\widetilde{a}(t)$, $b(t)$, $\widetilde{b}(t)$, $g(t)$ and $\widetilde{g}(t)$ are independent of t.*
2. *$g(t) = \widetilde{g}(t) = 0$.*

With homogeneous assumption, (3.11) becomes

$$\begin{cases} u_t = Ku_{zz}, & 0 < z < L, \quad t > 0, \\ au - bu_z = 0, & z = 0, \quad t > 0, \\ \widetilde{a}u + \widetilde{b}u_z = 0, & z = L, \quad t > 0, \\ u = f(z), & 0 < z < L, \quad t = 0. \end{cases} \tag{3.12}$$

To further simplify the above equation, we did the change of variable

$$b \to bL, \quad \widetilde{b} \to \widetilde{b}L \tag{3.13}$$

followed by the change of variable

$$\begin{aligned} a \to r\cos\alpha, \quad b \to r\sin\alpha, \\ \widetilde{a} \to r\cos\beta, \quad \widetilde{b} \to r\sin\beta. \end{aligned} \tag{3.14}$$

Finally, the heat equation becomes

$$\begin{cases} u_t = Ku_{zz}, & 0 < z < L, \quad t > 0, \\ u\cos\alpha - Lu_z\sin\alpha = 0, & z = 0, \quad t > 0, \\ u\cos\beta + Lu_z\sin\beta = 0, & z = L, \quad t > 0, \\ u = f(z), & 0 < z < L, \quad t = 0, \end{cases} \tag{3.15}$$

3.1.3 Separation of variable in heat equation

Let us solve the heat equation in a simple case of $\alpha = \beta = 0$ in (3.15). The separated solution is written as $u(z,t) = \phi(z)T(t)$. Thus we obtain

$$T'(t) + \lambda K T(t) = 0, \quad \phi''(z) + \lambda\phi(z) = 0.$$

The boundary conditions are written as

$$\phi(0) = \phi(L) = 0.$$

We obtain

$$T(t) = e^{-\lambda K t}, \quad \phi = A\sin(\sqrt{\lambda}z) + B\cos(\sqrt{\lambda}z), \quad \lambda > 0.$$

By plugging $\phi = A\sin(\sqrt{\lambda}z) + B\cos(\sqrt{\lambda}z)$ into the boundary conditions, we find that $B = 0$ and $\sqrt{\lambda}L = n\pi$ where n is an integer. Therefore we obtain

$$\phi(z) = \phi_n(z) = \sin\left(\sqrt{\lambda_n}z\right), \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, \ldots,$$

where we set the arbitrary constant in $\phi_n(z)$ to be 1 (recall we will take a superposition). Thus the separated solutions are obtained as

$$u(z,t) = \phi_n(z)e^{-\lambda_n K t}, \quad n = 1, 2, \ldots$$

If no initial condition $u(z,0) = f(z)$ is given, the separated solutions are the solutions to the problem. However, they do not satisfies $u(z,0) = f(z)$. Let us consider the linear combination of separated solution and match with the initial condition.

The linear combination is

$$u(z,t) = \sum_{n=1}^{\infty} C_n\phi_n(z)e^{-\lambda_n K t},$$

where C_n are constants.

By $u(z,0) = f(z)$, we know that

$$f(z) = u(z,0) = \sum_{n=1}^{\infty} C_n\phi_n(z) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi z}{L}.$$

From this, we know that C_n is the B_n coefficient of the Fourier sine series. We thus obtain

$$u(z,t) = \sum_{n=1}^{\infty} B_n\phi_n(z)e^{-\lambda_n K t}, \quad 0 < z < L, \quad t > 0.$$

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx.$$

Example 3.1. The heat equation $u_t = Ku_{zz}$ for $0 < z < L, t > 0$ with $u(0,t) = u(L,t) = 0$ and $u(z,0) = 1$ is solved as

$$u(z,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi z}{L} e^{-(n\pi/L)^2 Kt},$$

where

$$B_n = \frac{2}{\pi} \frac{1-(-1)^n}{n}.$$

Here as mentioned above, B_n can be directly computed as coefficients of the Fourier sine series of $f(z) = 1$

3.1.4 Some linear algebra

The orthogonality play an important role in computing the Fourier coefficients. Next, we'll explore the origin of orthogonality in the separation of variables method. Let us explain. After separation of the variable in the heat equation, we obtain the following equation

$$\phi''(z) + \lambda\phi(z) = 0 \quad \Leftrightarrow \quad -\phi''(z) = \lambda\phi(z).$$

If we denote the negative second-order derivative operator by A, then we get

$$A\phi = \lambda\phi.$$

This is very similar to the eigenvalue and eigenvector equation for matrix M in linear algebra.

$$Mv = \lambda v. \tag{3.16}$$

For a symmetric matrix, we have the following properties.

Theorem 3.4. *Assume that M is a symmetric operator ($M = M^T$), and $\langle x, y\rangle$ is the inner product of vectors, then we have*

1. $\langle Mx, y\rangle = \langle x, My\rangle$.

2. *The eigenvalues of M are real numbers.*

3. *If v, w are eigenvectors with different eigenvalues λ and μ respectively, then $v \perp w$ ($\langle v, w\rangle = 0$).*

Proof. Notice that $\langle x, y\rangle = x^T y$, then we have $\langle Mx, y\rangle = (Mx)^T y = x^T M^T y = x^T My = \langle x, My\rangle$.

For complex vectors $\langle x, y\rangle = \bar{x}^T y$, $\langle x, Mx\rangle = \lambda\langle x, x\rangle$, $\langle Mx, x\rangle = \bar{\lambda}\langle x, x\rangle$ and $\langle Mx, x\rangle = \langle x, Mx\rangle$. Therefore, $\lambda = \bar{\lambda}$.

$\langle v, Mw\rangle = \mu\langle v, w\rangle$, $\langle Mv, w\rangle = \lambda\langle v, w\rangle$ and $\langle Mv, w\rangle = \langle v, Mw\rangle$. Therefore, $\mu\langle v, w\rangle = \lambda\langle v, w\rangle$, which implies that $\langle v, w\rangle$ due to $\lambda \neq \mu$. $\square$

It turns out that A can be viewed as a symmetric matrix, and thus satisfies all properties described by Theorem 3.4.

3.2 The Sturm-Liouville eigenvalue problem

Now we study the eigenvalue problems for differential operators. Let us first start with the concepts of orthogonal functions and symmetric operators.

3.2.1 Orthogonal functions and symmetric operators

Definition 3.5 (Inner product)**.** *We extend dot product $\varphi \cdot \psi$ and define inner product as*

$$\langle \varphi, \psi \rangle = \int_a^b \varphi(x)\psi(x)dx. \tag{3.17}$$

Sometimes the inner product is defined as follows. We can have a weight function $\rho(x)$, and the weighted inner product is given by

$$\langle \varphi, \psi \rangle_\rho = \int_a^b \varphi(x)\psi(x)\rho(x)dx \tag{3.18}$$

where $\rho(x) > 0$ is a weight function.

For complex functions, we can write the complex inner product as

$$\langle \varphi, \psi \rangle = \int_a^b \varphi(x)\bar{\psi}(x)\rho(x)dx. \tag{3.19}$$

Here $\bar{\psi}$ is the complex conjugate of ψ ($\bar{\psi}(x) = f(x) - ig(x)$ when $\psi = f + ig$).

Definition 3.6 (Orthogonality)**.** *Two functions φ, ψ are said to be orthogonal on $[a, b]$ if $\langle \varphi, \psi \rangle = 0$.*

Example 3.2. The functions $\varphi(x) = \sin x$ and $\psi(x) = \cos x$ are orthogonal on $[0, \pi]$.

Example 3.3. The set of functions $1, \cos\frac{\pi x}{L}, \sin\frac{\pi x}{L}, \dots, \cos\frac{N\pi x}{L}, \sin\frac{N\pi x}{L}$ is orthogonal on $[-L, L]$.

Example 3.4. The set of functions $\{e^{\frac{in\pi x}{L}}\}_{n\in\mathbb{Z}}$ is orthogonal on $[-L, L]$.

Example 3.5. Which of the following pairs of functions are orthogonal on the interval $0 \le x \le 1$?

$$\varphi_1 = \sin 2\pi x, \quad \varphi_2 = x, \quad \varphi_3 = \cos 2\pi x, \quad \varphi_4 = 1.$$

$\langle \varphi_1, \varphi_3 \rangle = 0, \langle \varphi_1, \varphi_4 \rangle = 0, \langle \varphi_2, \varphi_3 \rangle = 0, \langle \varphi_3, \varphi_4 \rangle = 0$. All others are nonzero. Therefore the pairs $(\varphi_1, \varphi_3), (\varphi_1, \varphi_4), (\varphi_2, \varphi_3)$, and (φ_3, φ_4) are orthogonal.

Definition 3.7 (Norm)**.** *As follows we define <u>norm</u>, which is the "length" of a function.*

$$\|\varphi\| = \|\varphi\|_{L^2(a,b)} = \sqrt{\langle \varphi, \varphi \rangle}.$$

We note that the norm is always nonnegative. The norm $\|\varphi - \psi\|$ is the distance between two functions φ and ψ.

Definition 3.8 (Orthonormal)**.** *The functions $(\varphi_1, \ldots, \varphi_N)$ are orthonormal if $\langle \varphi_i, \varphi_j \rangle = \delta_{ij}$. Here δ_{ij} is the Kronecker delta ($\delta_{ij} = 0$ if $i \neq j$ and $= 1$ if $i = j$).*

Definition 3.9 (Normalization)**.** *Let $\{\varphi_1, \varphi_2, \ldots, \varphi_N\}$ be a set of orthogonal functions satisfying $\varphi_i \perp \varphi_j$ for $i \neq j$. Each function φ_i can be <u>normalized</u> to obtain an orthogonal sequence $\{\psi_i\}_{i=1,\ldots,N}$:*

$$\psi_i = \frac{\varphi_i}{\|\varphi_i\|}$$

where $\|\varphi_i\|$ denotes the norm of the function φ_i.

Example 3.6. After normalize the orthogonal functions $\cos x$, $\sin x$ on $[0, \pi]$, we obtain $\sqrt{\frac{2}{\pi}} \cos x$, $\sqrt{\frac{2}{\pi}} \sin x$.

Definition 3.10 (Differential Operator)**.** *A <u>differential operator</u> A is an operator satisfying the following*

1. *A is of the form $A = \sum_{i=0}^{n} a_i(x) \frac{d^i}{dx^i}$.*
2. *The operator A has a domain $\mathrm{Dom}(A)$, which means that A does accept all smooth functions as its input. (See Example 3.7 for an example of $\mathrm{Dom}(A)$)*

where n is a non-negative integer representing the <u>order</u> of the differential operator, $a_i(x)$ are the <u>coefficient</u> of A, and $\frac{d^i}{dx^i}$ represents the i-th derivative with respect to x, with $\frac{d^0}{dx^0}$ interpreted as the identity operator.

Definition 3.11 (Symmetric operators)**.** *Given a differential operator A, we say A is <u>symmetric</u>, if for any functions $\varphi, \psi \in \mathrm{Dom}(A)$,*

$$\langle A\varphi, \psi \rangle = \langle \varphi, A\psi \rangle. \tag{3.20}$$

Example 3.7. The differential operator A defined by the following is a symmetric operator,

$$A = -\frac{d^2}{dx^2}, \qquad \mathrm{Dom}(A) = \{\varphi(x) : x \in [a, b],\ \varphi(a) = \varphi(b) = 0\} \tag{3.21}$$

To show that A is symmetric, we need to verify that

$$\langle \varphi, A\psi \rangle = \langle A\varphi, \psi \rangle \tag{3.22}$$

for all functions $\varphi(x)$ and $\psi(x)$ satisfying $\varphi(a) = \varphi(b) = \psi(a) = \psi(b) = 0$. This condition translates into the following integral equality,

$$\int_a^b \varphi(x)\psi''(x)dx = \int_a^b \varphi''(x)\psi(x)dx \tag{3.23}$$

Applying integration by parts and using the boundary conditions $\varphi(a) = \varphi(b) = \psi(a) = \psi(b) = 0$, we get

$$\begin{aligned}
\int_a^b \varphi''(x)\psi(x)dx &= \underbrace{[\varphi'(x)\psi(x)]_a^b}_{=0} - \int_a^b \varphi'(x)\psi'(x)dx \\
&= -\int_a^b \varphi'(x)\psi'(x)dx = -\underbrace{[\varphi(x)\psi'(x)]_a^b}_{=0} + \int_a^b \varphi(x)\psi''(x)dx \\
&= \int_a^b \varphi(x)\psi''(x)dx
\end{aligned} \tag{3.24}$$

This equality confirms (3.22) and thus the fact that A is symmetric on the given domain $\mathrm{Dom}(A)$.

Example 3.8. The definition of symmetric operator depends on domain. For example, if we define A by

$$A = -\frac{d^2}{dx^2}, \qquad \mathrm{Dom}(A) = \{\varphi(x) : x \in [a, b]\} \tag{3.25}$$

which is almost the same operator as the last example but with a different domain. This operator is NOT symmetric. If we take $\varphi(x) = 1$ and $\psi(x) = x$, then a straightforward computation gives $\langle \varphi, A\psi \rangle = 0$ and $\langle A\varphi, \psi \rangle \neq 0$. Therefore, A is not symmetric.

3.2.2 The Sturm-Liouville eigenvalue problem

Definition 3.12 (Sturm-Liouville problems)**.** *Let ϕ be a function defined on the interval (a, b), the* <u>*Sturm-Liouville problems*</u> *is the following boundary value problem,*

$$\begin{aligned}
&[s(x)\phi'(x)]' + [\lambda\rho(x) - q(x)]\phi(x) = 0, \quad a < x < b, \\
&\phi(a)\cos\alpha - L\phi'(a)\sin\alpha = 0, \\
&\phi(b)\cos\beta + L\phi'(b)\sin\beta = 0
\end{aligned} \tag{3.26}$$

where ρ is a positive function $\rho(x) > 0$ and $L = b - a$, $\alpha, \beta \in [0, \pi)$ are some parameters.

Define the operator A by

$$A\phi = -\frac{1}{\rho(x)}\Big([s(x)\phi'(x)]' - q(x)\phi(x)\Big) \tag{3.27}$$

with domain

$$\left\{\phi(x),\ x \in [a,b] \,\middle|\, \begin{array}{l} \phi(a)\cos\alpha - L\phi'(a)\sin\alpha = 0, \\ \phi(b)\cos\beta + L\phi'(b)\sin\beta = 0 \end{array}\right\} \tag{3.28}$$

then (3.26) is equivalent to

$$A\phi = \lambda\phi. \tag{3.29}$$

The following theorem claims that all symmetric second order differential operators are of the form given by (3.27) and (3.28).

Theorem 3.13. *A symmetric second order differential operator A must be of the form given by* (3.27) *and* (3.28).

Proof. The proof go beyond the scope of this course. □

The following theorem explain the procedure of reducing an arbitrary second order differential operator to a Sturm-Liouville operator. Since it is abstract, it is good to first jump to the examples below the theorem.

Theorem 3.14. *Given an arbitrary second order linear differential equation $a(x)y''+b(x)y'+c(x)y = 0$, it can be reduced to the Sturm-Liouville equation* (3.26) *by the following procedure.*

1. *Divide both sides by $a(x)$ to obtain $y'' + p(x)y' + q(x)y = 0$.*

2. *Solve the equation $\widehat{y}' = p(x)\widehat{y}$.*

3. *Multiply both sides of $y'' + p(x)y' + q(x)y = 0$ by $\widehat{y}$*

$$\widehat{y}(y'' + p(x)y') + \widehat{y}q(x)y = 0 \quad \Rightarrow \quad (\widehat{y}y')' + \widehat{y}q(x)y = 0 \tag{3.30}$$

Proof. There is nothing to prove. □

Example 3.9 (Bessel functions). The Bessel functions are solutions of the following ODEs.

$$\phi'' + (d-1)\frac{\phi'}{x} + \left(1 - \frac{\mu}{x^2}\right)\phi = 0. \tag{3.31}$$

We can apply the procedure in Theorem 3.14 to reduce the above equation to a Sturm-Liouville equation.

1. There is nothing to do with this step since $a(x) = 1$.

2. We first solve $\widehat{y}' = \frac{d-1}{x}\widehat{y}$ and obtain $\widehat{y}(x) = x^{d-1}$.

3. Multiply both side of $\phi'' + (d-1)\frac{\phi'}{x} + \left(1 - \frac{\mu}{x^2}\right)\phi = 0$ by x^{d-1}, then we get

$$x^{d-1} \cdot \left(\phi'' + (d-1)\frac{\phi'}{x} + \left(1 - \frac{\mu}{x^2}\right)\phi\right) = 0 \quad \Rightarrow \quad (x^{d-1}\phi')' + \left(x^{d-1} - \mu x^{d-3}\right)\phi = 0. \tag{3.32}$$

Finally, we get

$$(x^{d-1}\phi'(x))' + \left(x^{d-1} - \mu x^{d-3}\right)\phi(x) = 0. \tag{3.33}$$

This is a Sturm-Liouville equation by setting $s(x) = \rho(x) = x^{d-1}$, $q(x) = \mu x^{d-3}$, and $\lambda = 1$.

In the case of $d = 2$ and $\mu = m^2$ with $m \in \mathbb{N}$, the function $\phi(x) = J_m(x)$ is called the standard Bessel function. In the case of $d = 3$ and $\mu = k(k+1)(k = 0, 1, 2, \ldots)$, the function $\phi(x) = j_k(x)$ is called the spherical Bessel function.

Example 3.10 (Legendre polynomials). The Legendre polynomials $P_k^m(x)$ are solutions of the following ODEs.

$$(1-x^2)\phi'' - 2x\phi' + \left(k(k+1) - \frac{m^2}{1-x^2}\right)\phi = 0. \tag{3.34}$$

We can apply the procedure in Theorem 3.14 to reduce the above equation to a Sturm-Liouville equation.

1. Divide both sides by $(1-x^2)$, then we get

$$\phi'' - \frac{2x}{1-x^2}\phi' + \left(\frac{k(k+1)}{1-x^2} - \frac{m^2}{(1-x^2)^2}\right)\phi = 0. \tag{3.35}$$

2. We first solve $\widehat{y}' = -\frac{2x}{1-x^2}\widehat{y}$ and obtain $\widehat{y}(x) = 1 - x^2$.

3. Multiply both side $1 - x^2$ (This step undo the operation of step 1 which sometimes appears), then we get

$$(1-x^2)\phi'' - 2x\phi' + \left(k(k+1) - \frac{m^2}{1-x^2}\right)\phi = 0 \;\Rightarrow\; ((1-x^2)\phi')' + \left(k(k+1) - \frac{m^2}{1-x^2}\right)\phi = 0. \tag{3.36}$$

Finally, we get

$$((1-x^2)\phi')' + \left(k(k+1) - \frac{m^2}{1-x^2}\right)\phi = 0. \tag{3.37}$$

This is a Sturm-Liouville equation by setting $s(x) = 1 - x^2$, $\rho(x) = 1$, $q(x) = m^2/(1-x^2)$, and $\lambda = k(k+1)$ $(k = 0, 1, 2, \ldots)$ with $a = -1$ and $b = 1$.

Example 3.11 (Hermite polynomials). The Hermite polynomials $H_n(x)$ are solutions of the following ODEs.

$$\phi'' - x\phi' + n\phi = 0. \tag{3.38}$$

We can apply the procedure in Theorem 3.14 to reduce the above equation to a Sturm-Liouville equation.

1. There is nothing to do with this step since $a(x) = 1$.
2. We first solve $\widehat{y}' = -x\widehat{y}$ and obtain $\widehat{y}(x) = e^{-\frac{x^2}{2}}$.
3. Multiply both side by $e^{-\frac{x^2}{2}}$, then we get

$$e^{-\frac{x^2}{2}}(\phi'' - x\phi') + ne^{-\frac{x^2}{2}}\phi = 0 \quad \Rightarrow \quad \left(e^{-\frac{x^2}{2}}\phi'\right)' + ne^{-\frac{x^2}{2}}\phi = 0. \tag{3.39}$$

Finally, we get

$$\left(e^{-\frac{x^2}{2}}\phi'\right)' + ne^{-\frac{x^2}{2}}\phi = 0. \tag{3.40}$$

This is a Sturm-Liouville equation by setting $s(x) = \rho(x) = \exp\left(-x^2/2\right)$, $q(x) = 0$, $\lambda = n$ ($n = 0, 1, 2, \ldots$) with $a = -\infty$ and $b = \infty$.

3.2.3 The Sturm-Liouville theorem

Recall that the Sturm-Liouville problem is equivalent to the eigenvalue problem $A\phi = \lambda\phi$ of the operator A defined in (3.27) and (3.28), which are copied below.

$$A\phi = -\frac{1}{\rho(x)}\left([s(x)\phi'(x)]' - q(x)\phi(x)\right) \tag{3.41}$$

with domain

$$\left\{\phi(x),\ x \in [a, b] \,\middle|\, \begin{array}{l} \phi(a)\cos\alpha - L\phi'(a)\sin\alpha = 0, \\ \phi(b)\cos\beta + L\phi'(b)\sin\beta = 0 \end{array}\right\} \tag{3.42}$$

We have the following theorem, which is an analog of Theorem 3.4

Theorem 3.15 (Sturm-Liouville theorem I)**.** *Assume that A is a differential operator defined by* (3.27) *and* (3.28), *and $\langle\varphi, \psi\rangle_\rho = \int_a^b \varphi(x)\psi(x)\rho(x)dx$ is the inner product defined by* (3.18) , *then we have*

1. $\langle A\phi, \psi\rangle = \langle\phi, A\psi\rangle$.
2. *The eigenvalues of A are real numbers.*

3. *If φ, ψ are eigenfunctions with different eigenvalues λ and μ respectively, then $\langle \varphi, \psi \rangle_\rho = 0$.*

4. *Assume that $\{\lambda_1, \lambda_2, \dots\}$ are the set of all eigenfunctions of A, then this set is infinite and discrete.* ***TODO: picture***

Proof. 1. A is symmetric. Consider $\langle A\phi, \psi \rangle$, and apply integration by parts

$$\begin{aligned} \langle A\phi, \psi \rangle &= \int_a^b \left(-\frac{1}{\rho(x)} [(s(x)\phi'(x))' - q(x)\phi(x)] \right) \psi(x)\rho(x)\, dx. \\ &= \int_a^b \left(-(s(x)\phi'(x))' + q(x)\phi(x) \right) \psi(x)\, dx. \\ &= -\left[s(x)\phi'(x)\psi(x) \right]_a^b + \int_a^b s(x)\phi'(x)\psi'(x)\, dx - \int_a^b q(x)\phi(x)\psi(x)\, dx. \end{aligned} \tag{3.43}$$

Similarly for $\langle \phi, A\psi \rangle = \langle A\psi, \phi \rangle$, we also have

$$\begin{aligned} \langle A\psi, \phi \rangle &= \int_a^b \left(-\frac{1}{\rho(x)} [(s(x)\psi'(x))' - q(x)\psi(x)] \right) \phi(x)\rho(x)\, dx. \\ &= \int_a^b \left(-(s(x)\psi'(x))' + q(x)\psi(x) \right) \phi(x)\, dx. \\ &= -\left[s(x)\psi'(x)\phi(x) \right]_a^b + \int_a^b s(x)\psi'(x)\phi'(x)\, dx - \int_a^b q(x)\psi(x)\phi(x)\, dx. \end{aligned} \tag{3.44}$$

It suffices to show that the boundary terms are equal

$$[s(x)\phi'(x)\psi(x)]_a^b = [s(x)\psi'(x)\phi(x)]_a^b \tag{3.45}$$

which is equivalent to

$$s(b)\Big(\phi'(b)\psi(b) - \psi'(b)\phi(b)\Big) = s(a)\Big(\phi'(a)\psi(a) - \psi'(a)\phi(a)\Big). \tag{3.46}$$

If we can show that $\phi'(a)\psi(a) = \psi'(a)\phi(a)$ and $\phi'(b)\psi(b) = \psi'(b)\phi(b)$, then (3.46) follows and the proof of conclusion 1 is completed. Since ϕ and ψ belong to the domain of A, by (3.42), $\frac{\phi'(a)}{\phi(a)} = \frac{\cos\alpha}{L\sin\alpha} = \frac{\psi'(a)}{\psi(a)}$, which implies $\phi'(a)\psi(a) = \psi'(a)\phi(a)$. The second equation at b can be proved similarly.

2. The eigenvalues of A are real. Since the eigenvalues and eigenfunctions at this point may not be real, we consider the complex inner product. Let λ be an eigenvalue with eigenfunction ϕ,

$$A\phi = \lambda\phi. \tag{3.47}$$

Consider the inner product $\langle \phi, A\phi \rangle$ and $\langle A\phi, \phi \rangle$,

$$\langle A\phi, \phi \rangle = \langle \lambda\phi, \phi \rangle = \lambda \|\phi\|^2. \tag{3.48}$$

$$\langle \phi, A\phi \rangle = \langle \phi, \lambda\phi \rangle = \overline{\lambda} \|\phi\|^2. \tag{3.49}$$

Since A is symmetric, we have $\langle \phi, A\phi \rangle$ and $\langle A\phi, \phi \rangle$, which implies that

$$\lambda \|\phi\|^2 = \overline{\lambda} \|\phi\|^2. \tag{3.50}$$

Hence, $\lambda = \overline{\lambda}$, implying that λ is real.

3. Orthogonality of eigenfunctions with different eigenvalues. Now the eigenvalues and eigenfunctions at this point are real, we consider the real inner product.

Suppose $A\phi = \lambda\phi$ and $A\psi = \mu\psi$ with $\lambda \neq \mu$. Consider

$$\lambda \langle \phi, \psi \rangle = \langle A\phi, \psi \rangle = \langle \phi, A\psi \rangle = \mu \langle \phi, \psi \rangle. \tag{3.51}$$

Thus, $(\lambda - \mu)\langle \phi, \psi \rangle = 0$, implying $\langle \phi, \psi \rangle = 0$ if $\lambda \neq \mu$.

4. Discreteness of the eigenvalue set. **TODO: finish the proof** □

The following property implies that the eigenvalues of all Sturm-Liouville problems are of multiplicity one. This property does not holds true for eigenvalues of matrices.

Theorem 3.16 (Sturm-Liouville theorem II)**.** *Assume that A is a differential operator defined by* (3.27) *and ϕ_1 and ϕ_2 are the eigenfunctions of the same eigenvalue λ, then we exists a constant C such that*

$$\phi_1(x) = C\phi_2(x). \tag{3.52}$$

Proof. We will use the following conclusion from the theory of ODE without proof.

$$\begin{cases} a(x)f'' + b(x)f' + c(x)f = 0 \\ f(x_0) = f'(x_0) = 0 \end{cases} \qquad \Rightarrow \qquad f(x) \equiv 0 \tag{3.53}$$

This is the uniqueness theorem of the initial value problem of second order ODEs.

We consider

$$\psi(x) = \begin{cases} \phi_2(a)\phi_1(x) - \phi_1(a)\phi_2(x), & \text{if } \alpha \neq 0 \\ \phi_2'(a)\phi_1(x) - \phi_1'(a)\phi_2(x), & \text{if } \alpha = 0. \end{cases}$$

This $\psi(x)$ obeys (3.26). If we can show that $\psi(a) = 0$ and $\psi'(a) = 0$, then by (3.53), we know that $\psi(x) \equiv 0$ for $a < x < b$. Therefore,

$$\phi_2(x) = C\phi_1(x), \quad C = \frac{\phi_2(a)}{\phi_1(a)} \quad \text{or} \quad \frac{\phi_2'(a)}{\phi_1'(a)}, \tag{3.54}$$

which proves (3.52)

Let us only consider the case of $\alpha \neq 0$, where $\phi(x) = \phi_2(a)\phi_1(x) - \phi_1(a)\phi_2(x)$. This obviously implies $\psi(a) = 0$. To prove $\psi'(a) = 0$, we notice that by (3.42), both ϕ_1 and ϕ_2 satisfy $\phi(a)\cos\alpha - L\phi'(a)\sin\alpha = 0$ as they belong to the domain of A. This implies that $\frac{\phi_1'(a)}{\phi_1(a)} = \frac{\cos\alpha}{L\sin\alpha} = \frac{\phi_2'(a)}{\phi_2(a)}$. Therefore, we conclude that $\psi'(a) = \phi_2(a)\phi_1'(a) - \phi_1(a)\phi_2'(a) = 0$. $\square$

3.2.4 Generalized Fourier series

Just like the usual Fourier series, we can express a function $f(x)$ in terms of orthogonal eigenfunctions of a Sturm-Liouville operator,

Theorem 3.17 (Generalized Fourier series)**.** *Given a Sturm-Liouville operator A of the form* (3.27) *with coefficient $\rho(x)$, let $\{\phi_n\}_{n\in\mathbb{N}} \subseteq Dom(A)$ be its eigenfunctions and $f(x) \in Dom(A)$ be an arbitrary piecewise smooth function. Then we have the following expansion*

$$f(x) = \sum_{n=1}^{\infty} c_n\phi_n(x), \tag{3.55}$$

where the coefficients $\{c_n\}_{n\in\mathbb{N}}$ can be computed by

$$c_n = \frac{\langle f, \phi_n\rangle_\rho}{\|\phi_n\|^2} \tag{3.56}$$

We have the following convergence theorem of general orthogonal expansion, which is similar to Theorem 2.18.

Theorem 3.18 (Convergence)**.** *Given a Sturm-Liouville operator A and a function $f(x) \in Dom(A)$, define the partial summation by $f_N(x) = \sum_{n=1}^{N} c_n\phi_n(x)$. Then we have*

1. *Let $f(x)$ be a piecewise smooth function, then we have*

$$f_N(x) \to \frac{1}{2}[f(x+0) + f(x-0)] \quad on \quad x \in (a,b) \quad as \quad N \to \infty \tag{3.57}$$

2. *If additionally assume that $f(x)$ is continuous,then $f_N(x)$ converges uniformly to $f(x)$ on $[a,b]$.*

3. *Let $\|f\|^2 = \int_a^b f(x)^2\rho(x)dx$ and $\sigma_N = \|f - f_N\|$, then we have the Parseval's identity*

$$\|f\|^2 = \sum_{n=1}^{\infty} |c_n|^2 \|\phi_n\|^2 \tag{3.58}$$

and

$$\sigma_N^2 = \sum_{n=N+1}^{\infty} |c_n|^2 \|\phi_n\|^2 . \tag{3.59}$$

3.3 The heat equation

3.3.1 Physics of heat conduction

3.3.2 The homogeneous case

3.3.3 The non-homogeneous case

3.4 The wave equation

3.4.1 Physics of string vibration

3.4.2 The homogeneous case

3.4.3 The non-homogeneous case

3.5 The Laplace's equation

3.5.1 Physics of static electricity

3.5.2 The homogeneous case

3.5.3 The non-homogeneous case