Boundary Value Problems for Partial Differential Equations

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1 Introduction and preliminaries

1.1 Definitions of partial differential equations

Definition 1.1 (Notations of partial derivatives). *For $f(x)$ with one variable x, we know $f'(x) = \frac{df}{dx}$. For $u(x, y)$, we introduce partial derivatives as*

$$\frac{\partial u}{\partial x} = \frac{du}{dx}\bigg|_{y \text{ is fixed}} = \partial_x u = u_x. \tag{1.1}$$

Similarly,

$$\frac{\partial^2 u}{\partial x^2} = \partial_x^2 u = \partial_{xx} u = u_{xx} \tag{1.2}$$

Example 1.1. For $u(x, y) = xy^2$, we have

$$\frac{\partial u}{\partial x} = \partial_x u = u_x = y^2, \qquad \frac{\partial u}{\partial y} = \partial_y u = u_y = 2xy,$$

$$\frac{\partial^2 u}{\partial x^2} = \partial_x^2 u = \partial_{xx} u = u_{xx} = 0, \qquad \frac{\partial^2 u}{\partial y^2} = \partial_y^2 u = \partial_{yy} u = u_{yy} = 2x$$

Definition 1.2 (Definition of general PDEs). *Given a function $u = u(x, y)$ of two variables, (similarly $u = u(x_1, \cdots, x_n)$ of n variables) and an expression $F(u_{xx}, u_{xy}, u_{yy}, u_x, u_y, u, x, y)$ of partial derivatives of u, the following equation is a partial differential equation, abbreviated as PDE.*

$$F(u_{xx}, u_{xy}, u_{yy}, u_x, u_y, u, x, y) = 0 \tag{1.3}$$

In the future, we may also use the notation $F[u]$ to represent $F(u_{xx}, u_{xy}, u_{yy}, u_x, u_y, u, x, y)$. And (1.3) *can be rewritten as*

$$F[u] = 0. \tag{1.4}$$

Remark 1.3. $F[u]$ may also involve derivatives of order ≥ 2, but we do not discuss it in this course.

Example 1.2 (Examples of PDEs). Here are some examples of PDEs.

$$\begin{aligned}
&u_{xx} - u_y = 0 \quad \text{(the heat equation)} \\
&u_{xx} - u_{yy} = 0 \quad \text{(the wave equation)} \\
&u_{xx} + u_{yy} = 0 \quad \text{(Laplace's equation)} \\
&u_x + u_y = 0 \quad \text{(the transport equation)} \\
&u_x + uu_y = 0 \quad \text{(the Burgers equation)}
\end{aligned} \tag{1.5}$$

Definition 1.4 (Order of PDEs)**.** *The* <u>*order*</u> *of a PDE is the order of the highest-order derivative in the equation. In* (1.5)*, the first three PDEs are second order, and the last two are first order.*

Definition 1.5 (Linear PDEs)**.** *Given a PDE* $F[u] = 0$*, if it satisfies*

$$F[u+v] = F[u] + F[v] \text{ and } F[cu] = cF[u], \tag{1.6}$$

then we say that $F[u] = 0$ *is a* <u>*linear PDE*</u>*. In* (1.5)*, the first four PDEs are linear, while the last one is not.*

We have the following proposition which characterizes all second order linear PDEs,

Proposition 1.6. *The second-order linear PDEs can always be written as*

$$a(x,y)u_{xx} + b(x,y)u_{xy} + c(x,y)u_{yy} + d(x,y)u_x + e(x,y)u_y + f(x,y)u = g(x,y), \tag{1.7}$$

We assume $a^2 + b^2 + c^2 \neq 0$ *for any* x, y *(at least one of* a, b, c *is nonzero).*

Proof. This go beyond the scope of this course. □

Definition 1.7. *We call* a, b, c, d, e, f <u>*coefficients*</u> *and* g <u>*source term*</u>*.*

1.2 Classification of second-order PDEs

In this course, we will mainly consider second-order linear PDEs.

$$a(x,y)u_{xx} + b(x,y)u_{xy} + c(x,y)u_{yy} + d(x,y)u_x + e(x,y)u_y + f(x,y)u = g(x,y), \tag{1.8}$$

These equations are classified as follows by the coefficients a, b, c.

Definition 1.8 (Classification of PDEs)**.** *The second-order linear PDEs* (1.8) *are classified as* *elliptic*, *parabolic* *and* *hyperbolic* *by the following,*

$$\begin{cases} 4ac - b^2 > 0 & \textit{elliptic} \\ 4ac - b^2 = 0 & \textit{parabolic} \\ 4ac - b^2 < 0 & \textit{hyperbolic} \end{cases} \tag{1.9}$$

where we note that a, b, c are functions of x, y and the inequalities in (1.9) *is required to be true for any x, y.*

1.3 Review of ODEs

Definition 1.9 (Separable ODEs)**.** *The following ODE is the* *separable* *ODE*

$$y' + p(x)y = 0. \tag{1.10}$$

Theorem 1.10. *Separable ODE can be solved in the following way*

$$y' + p(x)y = 0 \quad \Rightarrow \quad \frac{dy}{dx} + p(x)y = 0 \quad \Rightarrow \quad \frac{dy}{y} = -p(x)dx \quad \Rightarrow \quad \int \frac{dy}{y} = -\int p(x)dx$$

and the solution is

$$y(x) = Ce^{-\int p(x)dx} \tag{1.11}$$

Proof. There is nothing to prove. $\square$

Example 1.3. Let us solve the following ODEs

$$y' = -6xy$$

Apply the procedure in Theorem 1.10

$$y' = -6xy \quad \Rightarrow \quad \frac{dy}{y} = -6xdx \quad \Rightarrow \quad \int \frac{dy}{y} = -\int 6xdx \quad \Rightarrow \quad \ln|y| = -3x^2 + C'$$

Therefore, the solution is

$$y = Ce^{-3x^2}.$$

where $C(= \pm e^{C'})$ is an arbitrary constant.

Definition 1.11 (Linear ODEs)**.** *The following ODE is the* *linear* *ODE*

$$y' + p(x)y = q(x). \tag{1.12}$$

Theorem 1.12. *Linear ODE can be solved by the following procedure*

1. *Solve the corresponding separable equation $y' - p(x)y = 0$ to obtain a solution $\hat{y} = e^{\int p(x)dx}$.*

2. *Multiply the linear ODE by $\hat{y}$ and rewrite the ODE*

$$y' + p(x)y = q(x) \quad \Rightarrow \quad \hat{y}(y' + p(x)y) = \hat{y}q(x) \quad \Rightarrow \quad (\hat{y}y)' = \hat{y}q(x)$$

3. *Integrate the above equation*

$$\begin{aligned} \hat{y}y = \int \hat{y}q(x)dx + C \quad &\Rightarrow \quad y = \frac{1}{\hat{y}}\left(\int \hat{y}q(x)dx + C\right) \\ \Rightarrow \quad &y = e^{-\int p(x)dx}\left(\int q(x)e^{\int p(x)dx}dx + C\right) \end{aligned}$$

Proof. There is nothing to prove. □

Remark 1.13. In (2), we applied the following equation

$$(\hat{y}y)' = \hat{y}(y' + p(x)y)$$

which is a corollary of the Leibniz rule.

$$(\hat{y}y)' = \hat{y}'y + \hat{y}y' = \hat{y}y' + p(x)\hat{y}y = \hat{y}(y' + p(x)y)$$

Example 1.4. $(x^2+1)y' + 3xy = 6x$, $y(0) = 3$ is solved as $y(x) = 2 + (x^2+1)^{-3/2}$ by the procedure in Theorem 1.12.

To apply Theorem 1.12, we divide both sides by (x^2+1).

$$(x^2+1)y' + 3xy = 6x \quad \Rightarrow \quad y' + \frac{3x}{x^2+1}y = \frac{6x}{x^2+1}$$

then we apply the three steps in Theorem 1.12.

1. Solve the corresponding separable equation $y' - \frac{3x}{x^2+1}y = 0$ to obtain a solution $(x^2+1)^{\frac{3}{2}}$.

2. Multiply the linear ODE by $(x^2+1)^{\frac{3}{2}}$ and rewrite the ODE

$$y' + \frac{3x}{x^2+1}y = \frac{6x}{x^2+1} \quad \Rightarrow \quad ((x^2+1)^{\frac{3}{2}}y)' = 6x(x^2+1)^{\frac{1}{2}}$$

3. Integrate the above equation

$$\begin{aligned} ((x^2+1)^{\frac{3}{2}}y)' = 6x(x^2+1)^{\frac{1}{2}} \quad &\Rightarrow \quad y = (x^2+1)^{-\frac{3}{2}}\left(\int 6x(x^2+1)^{\frac{1}{2}} + C\right) \\ &\Rightarrow \quad y = 2 + C(x^2+1)^{-\frac{3}{2}} \end{aligned}$$

We note that the solutions with undetermined constant C are called general solutions. Finally, we apply the initial condition $y(0) = 3$ to obtain $C = 1$, so the solution of the initial value problem is

$$y = 2 + (x^2 + 1)^{-\frac{3}{2}}.$$

Definition 1.14 (Second order ODEs)**.** *The constant coefficient second order ODEs are the following equations*

$$ay'' + by' + cy = 0 \ (a \neq 0). \tag{1.13}$$

Theorem 1.15. *Constant coefficient second order ODEs can be solved by the following procedure*

1. *Solve the characteristic equation* $a\lambda^2 + b\lambda + c = 0$ *to get two solutions* λ_1 *and* λ_2*.*

2. *If* $\lambda_1 \neq \lambda_2$*, the general solution is*

$$y(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} \tag{1.14}$$

3. *If* $\lambda_1 = \lambda_2 = \lambda$*, the general solution is*

$$y(x) = (C_1 + C_2 x)e^{\lambda x} \tag{1.15}$$

4. *If* λ_1, λ_2 *are complex roots* $\alpha \pm i\beta$*, apply the Euler's formula to rewrite* (1.14)

$$y(x) = C_1 e^{(\alpha+i\beta)x} + C_2 e^{(\alpha-i\beta)x} = e^{\alpha x} \left(c_1 \cos \beta x + c_2 \sin \beta x\right) \tag{1.16}$$

   where $c_1 = C_1 + C_2$*,* $c_2 = i(C_1 - C_2)$*.*

Proof. This was explained in your ODE course. □

Example 1.5. $y'' - 3y' + 2y = 0$ is solved as $y(x) = C_1 e^x + C_2 e^{2x}$ by the procedure in Theorem 1.15.

1. Solve the characteristic equation $\lambda^2 - 3\lambda + 2 = 0$ to get two solutions $\lambda_1 = 1$ and $\lambda_2 = 2$.

2. Since $\lambda_1 \neq \lambda_2$, by (1.14), the general solution is

$$y(x) = C_1 e^x + C_2 e^{2x} \tag{1.17}$$

Example 1.6. $y'' + y = 0$ is solved as $y(x) = C_1 \cos(x) + C_2 \sin(x)$ by the procedure in Theorem 1.15.

1. Solve the characteristic equation $\lambda^2 + 1 = 0$ to get two solutions $\lambda_1 = i$ and $\lambda_2 = -i$.

2. Since λ_1, λ_2 are complex, by (1.16), the general solution is

$$y(x) = C_1 \cos x + C_2 \sin x \tag{1.18}$$

Example 1.7. $y'' + 2y' + y = 0$ is solved as $y(x) = (C_1 + C_2 x)e^{-x}$ by the procedure in Theorem 1.15.

1. Solve the characteristic equation $\lambda^2 + 2\lambda + 1 = 0$ to get $\lambda_1 = \lambda_2 = -1$.

2. Since λ_1, λ_2 are equal, by (1.15), the general solution is

$$y(x) = (C_1 + C_2 x)e^{-x} \tag{1.19}$$

1.4 Separation of variables

Many linear PDEs can be reduced to linear ODEs with the method of separation of variables, described below.

We take the Laplace's equation

$$u_{xx} + u_{yy} = 0 \tag{1.20}$$

with boundary condition

$$u(0, y) = 0, \quad u(L, y) = 0, \quad u(x, 0) = 0, \quad u(x, L) = \varphi(x). \tag{1.21}$$

as an example.

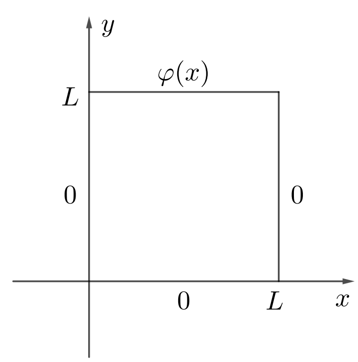

Figure 1: The boundary value problem.

We are looking for a separated solution of the form, $u(x, y) = X(x)Y(y)$. Substitute into (1.20), then we get

$$X''Y + XY'' = 0 \quad \Rightarrow \quad \frac{X''}{X} = -\frac{Y''}{Y}$$

The following lemma implies that X''/X and Y''/Y are constants.

Lemma 1.16. *$f(x) = g(y)$ implies that $f(x) = g(y) = const,$*

Proof. $f(x) = g(y) \Rightarrow f'(x) = \partial_x(g(y)) = 0 \Rightarrow f(x) = const.$ $\square$

Let λ be the constant and we write

$$X'' + \lambda X = 0, \quad Y'' - \lambda Y = 0.$$

We call λ the separation constant. At this moment λ is arbitrary. Thus the PDE was reduced to two ODEs.

1.4.1 Solving the boundary value problem

The function $u(x, y)$ satisfies the following boundary conditions

$$u(0, y) = 0, \quad u(L, y) = 0, \quad u(x, 0) = 0, \quad u(x, L) = \varphi(x). \tag{1.22}$$

Let us only consider the first three conditions.

$$u(0, y) = 0, \quad u(L, y) = 0, \quad u(x, 0) = 0. \tag{1.23}$$

The first two boundary conditions $u(0, y) = u(L, y) = 0$ imply that $X(0)Y(y) = X(L)Y(y) = 0$. Since taking $Y(y) = 0$ would force the separated solution $u(x, y) = X(x)Y(y) = 0$, yielding only the trivial solution, we must have $X(0) = X(L) = 0$. Similarly, the condition $u(x, 0) = 0$ gives $Y(0) = 0$.

Combining these boundary conditions with the ODEs derived previously, we obtain the following boundary value problems:

$$\begin{cases} X'' + \lambda X = 0, & X(0) = X(L) = 0, \\ Y'' - \lambda Y = 0, & Y(0) = 0. \end{cases} \tag{1.24}$$

To solve the boundary value problem, we distinguish 3 different cases $\lambda = 0$, $\lambda = -k^2 < 0$ and $\lambda = k^2 > 0$.

Case 1: $\lambda = 0$. The X-equation becomes $X'' = 0$, so $X(x) = Ax + B$. The boundary conditions $X(0) = X(L) = 0$ force $A = B = 0$, giving only the trivial solution $X \equiv 0$.

Case 2: $\lambda = -k^2 < 0$. The X-equation becomes $X'' - k^2 X = 0$, with general solution $X(x) = A\cosh(kx) + B\sinh(kx)$. From $X(0) = 0$ we get $A = 0$, and from $X(L) = 0$ we get $B\sinh(kL) = 0$. Since $k \neq 0$, $\sinh(kL) \neq 0$, so $B = 0$. Again only the trivial solution.

Case 3: $\lambda = k^2 > 0$. The X-equation becomes $X'' + k^2 X = 0$, with general solution $X(x) = A\cos(kx) + B\sin(kx)$. From $X(0) = 0$ we get $A = 0$. From $X(L) = 0$ we get $B\sin(kL) = 0$. For a nontrivial solution we need $\sin(kL) = 0$, which gives $kL = n\pi$ for $n = 1, 2, \ldots$. Thus

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \qquad X_n(x) = \sin\frac{n\pi x}{L}, \quad n = 1, 2, \ldots.$$

With $\lambda = \lambda_n$, the Y-equation becomes $Y'' - \left(\frac{n\pi}{L}\right)^2 Y = 0$. Let $\mu = \frac{n\pi}{L}$. The characteristic equation is $r^2 - \mu^2 = 0$, giving $r = \pm\mu$, so the general solution is $Y(y) = C_1 e^{\mu y} + C_2 e^{-\mu y}$. The condition $Y(0) = 0$ gives $C_1 + C_2 = 0$, i.e. $C_2 = -C_1$, so $Y(y) = C_1\left(e^{\mu y} - e^{-\mu y}\right)$. Recall that the hyperbolic sine and hyperbolic cosine functions are defined by

$$\sinh(z) = \frac{e^z - e^{-z}}{2}, \qquad \cosh(z) = \frac{e^z + e^{-z}}{2}.$$

Using this notation, we can write

$$Y(y) = D\sinh\frac{n\pi y}{L}.$$

Therefore, by $u(x,y) = X(x)Y(y)$, we obtain the following separated solutions of Laplace's equation.

$$u(x,y) = A\sin\frac{n\pi x}{L}\sinh\frac{n\pi y}{L}, \quad n = 1, 2, \ldots, \tag{1.25}$$

1.4.2 Matching with the last boundary condition

Now we consider the last boundary condition $u(x,L) = \varphi(x)$ in (1.23). For arbitrary $\varphi(x)$, our separated solution $A\sin\frac{n\pi x}{L}\sinh\frac{n\pi y}{L}$ cannot match with this boundary condition. However, we can use a linear combination of this separated solution to generate more solutions,

$$u(x,y) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L}\sinh\frac{n\pi y}{L}. \tag{1.26}$$

To match with $u(x,L) = \varphi(x)$, we take $y = L$ in (1.26),

$$\varphi(x) = u(x,L) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L}\sinh\frac{n\pi L}{L}. \tag{1.27}$$

Therefore, we get

$$\varphi(x) = \sum_{n=1}^{\infty} A_n \sinh n\pi \sin\frac{n\pi x}{L}. \tag{1.28}$$

Then A_n can be solved from the sine Fourier coefficients introduced in section 2.2.3.

The result is

$$A_n = \frac{2}{L\sinh n\pi}\int_0^L \varphi(x)\sin\frac{n\pi x}{L}dx \tag{1.29}$$

Substitute into (1.26), then we solve Laplace's equation (1.20) with boundary condition.

2 Introduction to Fourier series

Every function $f(x)$ on domain $-L < x < L$ can be represented by a series of the form

$$A_0 + \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi x}{L} + B_n \sin \frac{n\pi x}{L} \right)$$

This is called the Fourier series. We will first discuss how to compute the coefficients A_0, A_1, B_1, $\ldots$ in terms of $f(x)$, then we discuss several properties and variants of the Fourier series.

2.1 Definition of Fourier series

The Fourier series is an infinite sum of trigonometric functions defined as the following

Definition 2.1 (Fourier series)**.** *Let $A_0, A_1, B_1, \ldots$ be constants. The series below is called a Fourier series.*

$$A_0 + \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi x}{L} + B_n \sin \frac{n\pi x}{L} \right) \tag{2.1}$$

It turns out that every function $f(x)$ can be written into an infinite sum of trigonometric functions. As suggested by the following theorem,

Theorem 2.2. *Every function $f(x)$ on domain $-L < x < L$ can be represented by a Fourier series. In other word, for any $f(x)$ on domain $-L < x < L$, there exists coefficients $A_0, A_1, B_1, \ldots$ such that*

$$f(x) = A_0 + \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi x}{L} + B_n \sin \frac{n\pi x}{L} \right). \tag{2.2}$$

2.2 Fourier series and orthogonality

2.2.1 Orthogonality of trigonometric function

We want to compute the coefficients $A_0, A_1, B_1, \ldots$ in terms of $f(x)$. The following observation may be helpful.

Question. Given several orthogonal vector $e_1, e_2, \ldots, e_n$, assume that we have a decomposition

$$v = v_1 e_1 + v_2 e_2 + \cdots + v_n e_n \tag{2.3}$$

How do we compute the coefficients $v_1, v_2, \ldots, v_n$ in terms of v?

Answer. Assume that we want to solve the coefficient v_j, then we take inner product of (2.3) with e_j. If we do this, by orthogonality $\langle e_i, e_j\rangle = 0$ (if $i \neq j$) and $\langle e_j, e_j\rangle = |e_j|^2$, then we get

$$\begin{aligned}\langle v, e_j\rangle =&\langle v_1e_1 + v_2e_2 + \cdots + v_ne_n, e_j\rangle \\ =&v_j|e_j|^2.\end{aligned} \tag{2.4}$$

Therefore, we solve the coefficient $v_j = \frac{\langle v,e_j\rangle}{|e_j|^2}$.

The following notation is useful.

Definition 2.3 (Kronecker delta)**.** *The* <u>*Kronecker delta*</u> δ_{mn} *is defined as*

$$\delta_{mn} = \begin{cases} 1, & m = n \\ 0, & m \neq n \end{cases} \tag{2.5}$$

Using this notation orthogonality of $e_1, e_2, \dots, e_n$ *is equivalent to* $\langle e_m, e_n\rangle = |e_m|^2\delta_{mn}$.

The following theorem implies that sine and cosine functions are similar to orthogonal vectors.

Theorem 2.4 (Orthogonality of trigonometric functions)**.** *Let* $n, m \geq 0$ *be integers. We assume* $L > 0$*. The following orthogonality relations hold.*

$$\begin{aligned}&\int_{-L}^{L} \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx = \begin{cases} 2L & (n = m = 0), \\ L\delta_{nm} & (\textit{otherwise}), \end{cases} \\ &\int_{-L}^{L} \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}dx = \begin{cases} 0 & (n = m = 0), \\ L\delta_{nm} & (\textit{otherwise}), \end{cases} \\ &\int_{-L}^{L} \sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx = 0 \quad (\textit{all } n, m).\end{aligned} \tag{2.6}$$

Proof. We only prove the orthogonality for the first equation which only involves cosine. The proof of other equations is left as homework.

Let us first start with the following observation.

Claim. If $n \neq 0$, then we have

$$\int_{-L}^{L} \cos\frac{n\pi x}{L}dx = 0 \tag{2.7}$$

This claim is true because

$$\begin{aligned}\int_{-L}^{L} \cos\frac{n\pi x}{L}dx =& \left[\frac{L}{n\pi}\sin\frac{n\pi x}{L}\right]_{-L}^{L} \\ =&\frac{L}{n\pi}(\sin n\pi - \sin(-n\pi)) = 0\end{aligned} \tag{2.8}$$

Now we start the proof of (2.6) for cosine.

Case 1. ($n = m = 0$) In this case, we have

$$\int_{-L}^{L} \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx = \int_{-L}^{L} 1\cdot 1 dx = 2L$$

Case 2. (At least one of n, m is nonzero). We have

$$\int_{-L}^{L} \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx = \frac{1}{2}\int_{-L}^{L}\left(\cos\frac{(n-m)\pi x}{L} + \cos\frac{(n+m)\pi x}{L}\right)dx$$

Since at least one of n, m is nonzero, $n+m>0$. The integral over $\cos\frac{(n+m)\pi x}{L}$ is 0 by (2.7). Therefore,

$$\int_{-L}^{L} \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx = \frac{1}{2}\int_{-L}^{L}\cos\frac{(n-m)\pi x}{L}dx$$

Case 2.1. ($n \neq m$) In this case, $\frac{1}{2}\int_{-L}^{L}\cos\frac{(n-m)\pi x}{L}dx = 0$ by (2.7). Therefore, we get

$$\int_{-L}^{L} \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx = 0 = \delta_{mn}.$$

Case 2.2. ($n = m$), then

$$\int_{-L}^{L} \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx = \frac{1}{2}\int_{-L}^{L}\cos\frac{(n-m)\pi x}{L}dx = \frac{1}{2}\int_{-L}^{L}\cos\frac{0\cdot\pi x}{L}dx = L.$$

Thus the orthogonality relations for $\int_{-L}^{L}\cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx$ is proved. The orthogonality relations for $\int_{-L}^{L}\sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}dx$ and $\int_{-L}^{L}\sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx$ are similarly proved and left as homework. □

2.2.2 Formula of Fourier coefficients

We can use the orthogonality relations (2.6) to compute $A_0, A_1, B_1, \ldots$

To determine A_0 in (2.2), we multiply $\cos\frac{0\cdot\pi x}{L} = 1$ on both sides and integrate with respect to x:

$$\int_{-L}^{L} f(x)dx = \int_{-L}^{L} A_0 dx + \sum_{n=1}^{\infty}\int_{-L}^{L}\left(A_n\cos\frac{n\pi x}{L} + B_n\sin\frac{n\pi x}{L}\right)dx = \int_{-L}^{L} A_0 dx = 2A_0.$$

where we use the fact that $\int_{-L}^{L}\cos\frac{n\pi x}{L}dx = \int_{-L}^{L}\sin\frac{n\pi x}{L}dx = 0$. This is just the first equation in (2.6) with $m = 0$.

Therefore, we get an expression of A_0,

$$A_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx.$$

To determine A_m $(m = 1, 2, \dots)$ in (2.2), we multiply $\cos\frac{m\pi x}{L}$ on both sides and integrate with respect to x:

$$\begin{aligned}\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}dx &= \int_{-L}^{L} A_0\cos\frac{m\pi x}{L}dx + \sum_{n=1}^{\infty}\int_{-L}^{L}\left(A_n\cos\frac{n\pi x}{L} + B_n\sin\frac{n\pi x}{L}\right)\cos\frac{m\pi x}{L}dx \\ &= \sum_{n=1}^{\infty} A_n\int_{-L}^{L}\cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx = \sum_{n=1}^{\infty} A_n L\delta_{nm} = LA_m.\end{aligned}$$

Therefore, we get an expression of A_m,

$$A_m = \int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}dx.$$

Similarly we can determine B_m $(m = 1, 2, \dots)$ in (2.2) by multiplying $\sin\frac{m\pi x}{L}$ on both sides and integrate with respect to x:

$$\begin{aligned}\int_{-L}^{L} f(x)\sin\frac{m\pi x}{L}dx &= \int_{-L}^{L} A_0\sin\frac{m\pi x}{L}dx + \sum_{n=1}^{\infty}\int_{-L}^{L}\left(A_n\cos\frac{n\pi x}{L} + B_n\sin\frac{n\pi x}{L}\right)\sin\frac{m\pi x}{L}dx \\ &= \sum_{n=1}^{\infty} B_n\int_{-L}^{L}\sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}dx = \sum_{n=1}^{\infty} B_n L\delta_{nm} = LB_m.\end{aligned}$$

Therefore, we get an expression of B_m,

$$B_m = \int_{-L}^{L} f(x)\sin\frac{m\pi x}{L}dx.$$

In summary, we have the following theorem.

Theorem 2.5. *The Fourier coefficients in $A_0, A_1, B_1, \dots$ are given by*

$$\begin{aligned}A_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)dx, \\ A_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \\ B_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx.\end{aligned} \tag{2.9}$$

Example 2.1. Let us calculate the Fourier series of $f(x) = x, -L < x < L$.

Solution. We have

$$\begin{aligned}A_0 &= \frac{1}{2L}\int_{-L}^{L} x dx = 0, \\ A_n &= \frac{1}{L}\int_{-L}^{L} x\cos\frac{n\pi x}{L}dx = 0 \quad (x\cos\frac{n\pi x}{L} \textit{ is an odd function}), \\ B_n &= \frac{1}{L}\int_{-L}^{L} x\sin\frac{n\pi x}{L}dx = \frac{1}{L}\left[-\frac{L}{n\pi}x\cos\frac{n\pi x}{L}\Big|_{-L}^{L} + \frac{L}{n\pi}\int_{-L}^{L}\cos\frac{n\pi x}{L}dx\right] = \frac{2L}{n\pi}(-1)^{n+1}.\end{aligned}$$

Therefore we obtain

$$x = \sum_{n=1}^{\infty} \frac{2L}{n\pi}(-1)^{n+1} \sin \frac{n\pi x}{L}, \quad -L < x < L. \tag{2.10}$$

□

We have two observations from the above example

Proposition 2.6. *An odd or even function only has* cos *or* sin *term in its Fourier series.*

Proof. We only consider the even case. The odd case can be proved similarly.

Since $f(x)$ is even, we know that $\frac{1}{L}\int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} dx$ is odd. By (2.9), all the B_n vanishes. Therefore, we only have the cos term in the Fourier series. □

Proposition 2.7. *Given a function $f(x)$ defined on $-L < x < L$, the Fourier series of $f(x)$ coincide with $f(x)$ only in the domain $-L < x < L$, unless $f(x)$ is a periodic function.*

Proof. We will not prove this proposition. Instead, we draw the graph of $y = x$ and its Fourier series

The Fourier series coincides with $y = x$ only in the domain $-L < x < L$. But the Fourier series is a periodic function since it is a sum of several periodic functions cos / sin. Since $y = x$ is not a periodic function, the Fourier series cannot agree with it for all x.

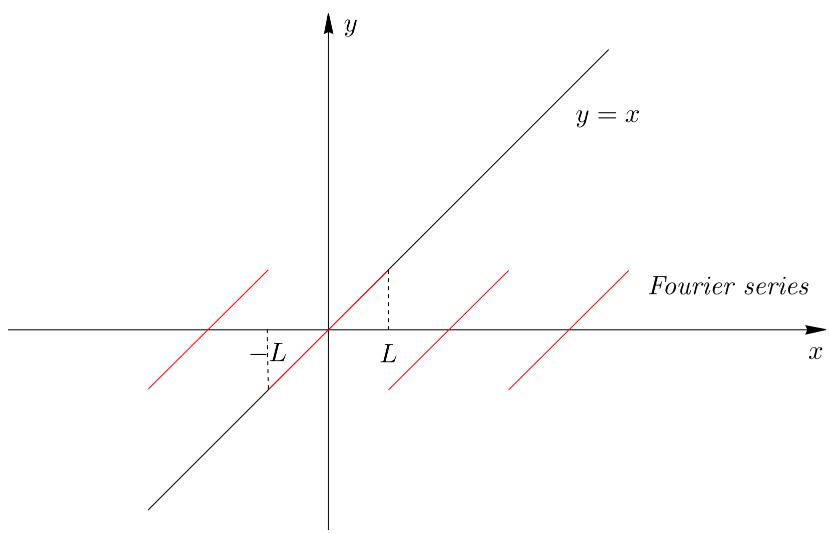

Figure 2: The graph of $y = x$ and its Fourier series.

□

2.2.3 Solve the A_n coefficients in (1.28)

Now we return to the separation of variable in section 1.4.2, where we derived

$$\varphi(x) = \sum_{n=1}^{\infty} A_n \sinh n\pi \sin \frac{n\pi x}{L}. \tag{2.11}$$

We want to solve A_n from $\varphi(x)$.

As explained in section 2.4, on $0 < x < L$, we also have the following orthogonality of $\sin \frac{n\pi x}{L}$

$$\int_0^L \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} dx = \frac{L}{2}\delta_{nm}, \qquad n, m \geq 1. \tag{2.12}$$

Therefore, if we want to solve A_m, we can multiply (2.11) by $\sin \frac{n\pi x}{L}$ and then integrate over x

$$\int_0^L \varphi(x) \sin \frac{m\pi x}{L} dx = \sum_{n=1}^{\infty} \int_0^L A_n \sinh n\pi \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} dx$$
$$= \sum_{n=1}^{\infty} A_n \sinh n\pi \int_0^L \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} dx = \sum_{n=1}^{\infty} A_n \sinh n\pi \frac{L}{2}\delta_{nm} = \frac{L}{2} A_m \sinh n\pi.$$

From this, we can solve A_m and get

$$A_n = \frac{2}{L \sinh n\pi} \int_0^L \varphi(x) \sin \frac{n\pi x}{L} dx \tag{2.13}$$

This is exactly (1.29).

2.3 Complex form of Fourier series

We have seen that every function defined on $-L < x < L$ can be rewritten as an infinite sum of $\cos \frac{n\pi x}{L}$ and $\sin \frac{n\pi x}{L}$. By Euler's formula $e^{ix} = \cos x + i \sin x$, we know that every trigonometric function is a linear combination of e^{ix} and e^{-ix}

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}, \qquad \sin x = \frac{e^{ix} - e^{-ix}}{2i}. \tag{2.14}$$

Therefore, we have the following theorem.

Theorem 2.8. *In other word, for any $f(x)$ on domain $-L < x < L$, there exists coefficients $\dots, \alpha_{-1}, \alpha_0, \alpha_1, \alpha_2, \dots$ such that*

$$f(x) = \sum_{n=-\infty}^{\infty} \alpha_n e^{in\pi x/L}, \tag{2.15}$$

where α_n satisfies the following properties.

1. α_n is given by

$$\alpha_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-in\pi x/L}dx \tag{2.16}$$

2. If $f(x)$ is a real value function, then $\alpha_n = \bar{\alpha}_{-n}$, where $\bar{\alpha}_{-n}$ is the complex conjugate of α_{-n}.

Proof. Using Euler's formula (2.14), we can rewrite the Fourier series of f as follows.

$$\begin{aligned}
f(x) =& A_0 + \sum_{n=1}^{\infty}\left(A_n \cos\frac{n\pi x}{L} + B_n \sin\frac{n\pi x}{L}\right)\\
=& A_0 + \sum_{n=1}^{\infty}\left(A_n\left(\frac{e^{in\pi x/L} + e^{-in\pi x/L}}{2}\right) + B_n\left(\frac{e^{in\pi x/L} - e^{-in\pi x/L}}{2i}\right)\right)\\
=& A_0 + \sum_{n=1}^{\infty}\left(\frac{A_n - iB_n}{2}e^{in\pi x/L} + \frac{A_n + iB_n}{2}e^{-in\pi x/L}\right).
\end{aligned}$$

If we define

$$\alpha_0 = A_0, \quad \alpha_n = \frac{A_n - iB_n}{2}, \quad \alpha_{-n} = \frac{A_n + iB_n}{2}, \quad (n > 0) \tag{2.17}$$

then we obtain

$$f(x) = \alpha_0 + \sum_{n=1}^{\infty}\left(\alpha_n e^{in\pi x/L} + \alpha_{-n}e^{-in\pi x/L}\right) = \sum_{n=-\infty}^{\infty}\alpha_n e^{in\pi x/L}.$$

If $f(x)$ is a real value function, then A_n and B_n are real numbers. From (2.17), we know that $\alpha_n = \bar{\alpha}_{-n}$.

The formula of α_n is also a corollary of (2.17). When $n > 0$,

$$\begin{aligned}
\alpha_n =& \frac{A_n - iB_n}{2} = \frac{1}{2L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx - \frac{i}{2L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx\\
=& \frac{1}{2L}\int_{-L}^{L} f(x)e^{-in\pi x/L}dx
\end{aligned}$$

When $n = -m < 0$,

$$\begin{aligned}
\alpha_n = \alpha_{-m} =& \frac{A_m + iB_m}{2} = \frac{1}{2L}\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}dx + \frac{i}{2L}\int_{-L}^{L} f(x)\sin\frac{m\pi x}{L}dx\\
=& \frac{1}{2L}\int_{-L}^{L} f(x)e^{im\pi x/L}dx = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-in\pi x/L}dx
\end{aligned}$$

When $n = 0$,

$$\alpha_0 = A_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-i0\pi x/L}dx$$

Therefore, we have completed the proof. ∎

$e^{in\pi x/L}$ also have orthogonality relations.

Theorem 2.9 (Orthogonality relations of complex Fourier series)**.**

$$\int_{-L}^{L} e^{in\pi x/L} e^{-im\pi x/L} dx = \int_{-L}^{L} e^{i(n-m)\pi x/L} dx = 2L\delta_{mn}. \tag{2.18}$$

Proof. If $n = m$,

$$\int_{-L}^{L} e^{i(n-m)\pi x/L} dx = \int_{-L}^{L} 1 dx = 2L = 2L\delta_{mn}.$$

If $n \neq m$,

$$\int_{-L}^{L} e^{i(n-m)\pi x/L} dx = \frac{L}{i\pi(n-m)} \left[e^{i(n-m)\pi x/L} \right]_{-L}^{L} = 0 = 2L\delta_{mn}.$$

We have completed the proof. □

The formula of α_n can also be computed using orthogonality.

Assume that we want to compute α_m in

$$f(x) = \sum_{n=-\infty}^{\infty} \alpha_n e^{in\pi x/L}.$$

Multiply both side by $e^{-im\pi x/L}$ and then we get

$$\int_{-L}^{L} f(x) e^{-im\pi x/L} dx = \sum_{n=-\infty}^{\infty} \alpha_n \int_{-L}^{L} e^{in\pi x/L} e^{-im\pi x/L} dx \sum_{n=-\infty}^{\infty} \alpha_n 2L\delta_{mn} = 2L\alpha_m.$$

Solve α_m, then we prove (2.16) again.

2.4 Fourier cosine and sine series

A function defined on $-L < x < L$ can be rewritten as an infinite sum of $\cos \frac{n\pi x}{L}$ and $\sin \frac{n\pi x}{L}$. Notice that $-L < x < L$ is the period of $\cos \frac{n\pi x}{L}$ and $\sin \frac{n\pi x}{L}$. If a function $f(x)$ is defined in the interval $0 < x < L$, which represents half of its period, then it is possible to express $f(x)$ exclusively as a series of cosine terms, $\cos \frac{n\pi x}{L}$, or alternatively, only as a series of sine terms, $\sin \frac{n\pi x}{L}$. This is the Fourier cosine/sine series.

The idea is that, given a function $f(x)$ defined on $0 < x < L$, we can extend this function as an even/odd function on $-L < x < L$. Then we compute the Fourier series of the extended function. The Fourier series only contains $\cos \frac{n\pi x}{L}$ or $\sin \frac{n\pi x}{L}$ terms by Proposition 2.6.

2.4.1 Fourier cosine series

We define the even extension $f_E(x)$ of $f(x)$ as

$$f_E(x) = \begin{cases} f(x), & 0 < x < L \\ 0, & x = 0, \\ f(-x), & -L < x < 0. \end{cases} \tag{2.19}$$

We note that $f_E(x)$ is even. Indeed the value $f_E(0)$ is arbitrary and not necessarily zero. The Fourier series is given by

$$f_E(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L}, \quad -L < x < L,$$

where

$$\begin{aligned} A_0 &= \frac{1}{2L}\int_{-L}^{L} f_E(x)dx = \frac{1}{L}\int_0^L f(x)dx, \\ A_n &= \frac{1}{2L}\int_{-L}^{L} f_E(x)\cos\frac{n\pi x}{L}dx = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx. \end{aligned}$$

On the interval $0 < x < L$ we have

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L}, \quad 0 < x < L, \tag{2.20}$$

where

$$A_0 = \frac{1}{L}\int_0^L f(x)dx, \quad A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx. \tag{2.21}$$

Definition 2.10. *The series* (2.20) *is called the Fourier cosine series.*

For Fourier cosine series, we also have orthogonality relations and (2.21) can be computed from these orthogonality relations.

Theorem 2.11 (Orthogonality relations of Fourier cosine series)**.**

$$\int_0^L \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx = \begin{cases} L & (n = m = 0), \\ \frac{L}{2}\delta_{nm} & (\textit{otherwise}), \end{cases} \tag{2.22}$$

Example 2.2. Let us compute the Fourier cosine series of $f(x) = x$, $0 < x < L$.

Solution. We can directly apply (2.21). But let us try the even extension method.

We extend f as

$$f_E(x) = \begin{cases} x, & 0 < x < L, \\ 0, & x = 0, \\ -x, & -L < x < 0. \end{cases}$$

Indeed $f_E(x) = |x|$.

$$\begin{aligned} A_0 &= \frac{1}{2L} \int_{-L}^{L} f_E(x) dx = \frac{1}{L} \int_0^L x dx = \frac{L}{2}, \\ A_n &= \frac{1}{2L} \int_{-L}^{L} f_E(x) \cos \frac{n\pi x}{L} dx = \frac{2}{L} \int_0^L x \cos \frac{n\pi x}{L} dx \\ &= \frac{2}{L} \left[\frac{L}{n\pi} x \sin \frac{n\pi x}{L} \bigg|_0^L - \frac{L}{n\pi} \int_0^L \sin \frac{n\pi x}{L} dx \right] = \frac{2L}{(n\pi)^2} \left((-1)^n - 1 \right). \end{aligned}$$

Therefore,

$$x = \frac{L}{2} + \frac{2L}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} \cos \frac{n\pi x}{L}, \quad 0 < x < L.$$

□

2.4.2 Fourier sine series

We define the odd extension $f_O(x)$ as

$$f_O(x) = \begin{cases} f(x), & 0 < x < L \\ 0, & x = 0, \\ -f(-x), & -L < x < 0. \end{cases} \tag{2.23}$$

We note that $f_O(x)$ is odd. The Fourier series is given by

$$f_O(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}, \quad -L < x < L,$$

where

$$B_n = \frac{1}{L} \int_{-L}^{L} f_O(x) \sin \frac{n\pi x}{L} dx = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx.$$

On the interval $0 < x < L$ we have

$$f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}, \quad 0 < x < L, \tag{2.24}$$

where

$$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx \tag{2.25}$$

Definition 2.12. *The series* (2.24) *is called the Fourier cosine series.*

For Fourier sine series, we also have orthogonality relations and (2.25) can be computed from these orthogonality relations.

Theorem 2.13 (Orthogonality relations of Fourier sine series)**.**

$$\int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}dx = \begin{cases} 0 & (n=m=0), \\ \dfrac{L}{2}\delta_{nm} & (\textit{otherwise}), \end{cases} \tag{2.26}$$

Example 2.3. The Fourier sine series of $f(x)=x$, $0<x<L$, is obtained through the odd extension $f_O(x)$. The odd extension $f_O(x)$ is again x and its Fourier series has been computed in (2.10).

2.5 Convergence of Fourier series

In this section, we study the convergence of Fourier series of piecewise continuous functions.

Definition 2.14 (Left and right limits)**.** *For a given $f(x)$, let us write*

$$f(x+0)=\lim_{\varepsilon\to 0}f(x+\varepsilon), \quad f(x-0)=\lim_{\varepsilon\to 0}f(x-\varepsilon), \tag{2.27}$$

where $\varepsilon>0$.

Definition 2.15 (Piecewise continuous)**.** *A function $f(x)$ defined on $a<x<b$, is said to be piecewise continuous if there is a finite set of points $a=x_0<x_1<\cdots<x_p<x_{p+1}=b$ such that $f(x)$ is continuous at $x\neq x_i$ $(i=1,\ldots,p)$, $f(x_i+0)(i=0,\ldots,p)$ exists, and $f(x_i-0)$ $(i=1,\ldots,p+1)$ exists.*

Definition 2.16 (Piecewise smooth)**.** *A function $f(x), a<x<b$, is said to be piecewise smooth if $f(x)$ and all of its derivatives are piecewise continuous.*

Example 2.4. The function $f(x)=|x|,-L<x<L$, is piecewise smooth. The function $f(x)=x^2\sin(1/x)$,$-L<x<L$, is piecewise continuous but is not piecewise smooth because $\lim_{\varepsilon\to 0}f'(0\pm\varepsilon)$ does not exist. The function $f(x)=1/\left(x^2-L^2\right)$, $-L<x<L$, is not piecewise continuous because $f(-L+0)$ and $f(L-0)$ are not finite.

Definition 2.17 (Convergence)**.** *Given a Fourier series $f(x)=A_0+\sum_{n=1}^{\infty}[A_n\cos(n\pi x/L)+B_n\sin(n\pi x/L)]$, we define*

1. ***Partial sums.*** *The partial sum, denoted by* $f_N(x)$*, is defined to be*

$$f_N(x) = A_0 + \sum_{n=1}^{N} \left[A_n \cos(n\pi x/L) + B_n \sin(n\pi x/L) \right]. \tag{2.28}$$

2. ***Convergence at*** x***.*** *We say the Fourier is convergent at* x *if*

$$\lim_{N\to\infty} f_N(x) = f(x). \tag{2.29}$$

3. ***Uniformly convergence.*** *We say the Fourier is uniformly convergent if*

$$\lim_{N\to\infty} \max_{x\in[-L,L]} |f_N(x) - f(x)| = 0. \tag{2.30}$$

   This means that the convergence is equally well for all x*.*

Theorem 2.18 (Convergence theorem)**.** *Let* $f(x)$*,* $-L < x < L$*, be piecewise smooth. Then the Fourier series of* f *converges for all* x *to the value* $\frac{1}{2}[\bar{f}(x+0) + \bar{f}(x-0)]$*, where* $\bar{f}$ *is the* $2L$*-periodic function which equals to* f *on* $-L < x < L$*.*

If $f(x)$ *is continuous on* $[-L, L]$ *and* $f(-L) = f(L)$ *in addition to the conditions assumed in the above theorem, then the Fourier series uniformly converges.*

Example 2.5. Here are two examples of convergence.

1. The Fourier series of $f(x) = |x|$, $-L < x < L$, (see Example 2.2) uniformly converges.

2. The Fourier series of $f(x) = x$, $-L < x < L$ does not uniformly converge but converges at any point to $f(x)$ except for $x = -L, L$.

As in the following picture, the Fourier series of $f(x) = x$, $-L < x < L$ has a very bad convergence near $x = -L$, L, but the series of $f(x) = |x|$ has much better convergence.

TODO: add a picture

2.6 Parseval's Theorem and Mean Square Error

2.6.1 The Parseval's Theorem for Fourier series

For orthogonal vectors $e_1, \ldots, e_n$, $e_i \perp e_j$, their linear combination $v = v_1e_1 + \cdots + v_ne_n$ satisfies the Pythagorean theorem

$$|v|^2 = |v_1e_1 + \cdots + v_ne_n|^2 = |v_1e_1|^2 + \cdots + |v_ne_n|^2 \tag{2.31}$$

The proof of (2.31) is given by

$$|v|^2 = \left\langle \sum_{j=1}^{n} v_j e_j, \sum_{k=1}^{n} v_k e_k \right\rangle = \sum_{j=1}^{n}\sum_{k=1}^{n} v_j v_k \underbrace{\langle e_j, e_k \rangle}_{=0 \ \text{if}\ j\neq k} = \sum_{j=1}^{n} v_j^2 |e_j|^2$$

For the Fourier series, the following theorem claims that a similar identity is true.

Theorem 2.19 (Parseval's theorem)**.** *Let $f(x)$ defined on $-L < x < L$ be a piecewise smooth function with Fourier series*

$$f(x) = A_0 + \sum_{n=1}^{\infty} [A_n \cos(n\pi x/L) + B_n \sin(n\pi x/L)] \tag{2.32}$$

Then, the mean square $\frac{1}{2L}\int_{-L}^{L} f(x)^2 dx$ of $f(x)$ satisfies the following identity

$$\frac{1}{2L}\int_{-L}^{L} f(x)^2 dx = A_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(A_n^2 + B_n^2\right) \tag{2.33}$$

Proof. Since $\cos(0\pi x/L) = 1$ and $\sin(0\pi x/L) = 0$, we know that (2.32) is equivalent to

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} [A_n \cos(n\pi x/L) + B_n \sin(n\pi x/L)] \\ &= \sum_{n=0}^{\infty} (A_n \cos_n + B_n \sin_n) \end{aligned} \tag{2.34}$$

where we introduce the notation

$$\cos_n = \cos(n\pi x/L), \quad \sin_n = \sin(n\pi x/L) \tag{2.35}$$

Similar to (2.31) and its proof, we have

$$\begin{aligned} \frac{1}{2L}\int_{-L}^{L} f(x)^2 dx =& \frac{1}{2L}\int_{-L}^{L} \sum_{n=0}^{\infty}[A_n \cos_n + B_n \sin_n] \sum_{m=0}^{\infty}[A_m \cos_m + B_m \sin_m] dx \\ =& \frac{1}{2L}\int_{-L}^{L} \sum_{n,m=0}^{\infty} (A_n A_m \underbrace{\cos_n \cos_m}_{=0 \ \text{if}\ n\neq m} + A_n B_m \underbrace{\cos_n \sin_m}_{=0} \\ & + B_n A_m \underbrace{\sin_n \cos_m}_{=0} + B_n B_m \underbrace{\sin_n \sin_m}_{=0 \ \text{if}\ n\neq m}) dx \\ =& A_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(A_n^2 + B_n^2\right) \end{aligned} \tag{2.36}$$

where in the last line, we applied the orthogonality of trigonometric functions (2.6). □

2.6.2 The mean square error

Definition 2.20 (Mean square error)**.** *We define the* <u>*mean square error*</u> σ_N^2 *as*

$$\sigma_N^2 = \frac{1}{2L}\int_{-L}^{L}[f(x) - f_N(x)]^2 dx \tag{2.37}$$

where $f_N(x)$ *is the partial sum*

$$f_N(x) = A_0 + \sum_{n=1}^{N}\left[A_n \cos(n\pi x/L) + B_n \sin(n\pi x/L)\right]. \tag{2.38}$$

By Parseval's theorem, we obtain the following expression of the mean square error.

Proposition 2.21. *Given a partial summation whose mean square error is defined by* (2.37)*, then we have*

$$\sigma_N^2 = \frac{1}{2}\sum_{n=N+1}^{\infty}\left(A_n^2 + B_n^2\right) \tag{2.39}$$

Proof. This is an easy corollary of the Parseval's identity (2.33), if we notice that

$$f(x) - f_N(x) = \sum_{n=N+1}^{\infty}\left[A_n \cos(n\pi x/L) + B_n \sin(n\pi x/L)\right]$$

Therefore, we finished the proof. $\square$

Example 2.6. Let us find σ_N^2 for $f(x) = x$, $-L < x < L$.

Solution. From Example 2.1, we have $A_0 = A_n = 0$ and

$$f_N(x) = \sum_{n=1}^{N} B_n \sin\frac{n\pi x}{L}, \quad B_n = \frac{2L}{n\pi}(-1)^{n+1}. \tag{2.40}$$

By (2.39), we obtain

$$\sigma_N^2 = \frac{1}{2}\sum_{n=N+1}^{\infty}\left(\frac{2L}{n\pi}(-1)^{n+1}\right)^2 = \frac{2L^2}{\pi^2}\sum_{n=N+1}^{\infty}\frac{1}{n^2}.$$

By Theorem 2.22, we get

$$\int_{N+1}^{\infty}\frac{1}{x^2}dx \le \sum_{n=N+1}^{\infty}\frac{1}{n^2} \le \int_{N+1}^{\infty}\frac{1}{(x-1)^2}dx = \int_{N}^{\infty}\frac{1}{x^2}dx.$$

We have

$$\int_{N+1}^{\infty}\frac{1}{x^2}dx = \frac{1}{N+1} \ge \frac{1}{N} - \frac{1}{N^2}, \quad \int_{N}^{\infty}\frac{1}{x^2}dx = \frac{1}{N}.$$

Let us introduce the symbol O (this is called "big O") to express the order. For some f_N, $f_N = O\left(N^{-1}\right)$ as $N \to \infty$ means that there exist a constant $C > 0$ such that $|f_N| \le CN^{-1}$. Therefore we obtain

$$\sigma_N^2 = \frac{2L^2}{\pi^2}\frac{1}{N}\left[1 + O\left(\frac{1}{N}\right)\right] = O\left(N^{-1}\right), \quad N \to \infty. \tag{2.41}$$

We note that σ_N^2 goes to zero as $N \to \infty$ although we know that the sum in (2.40) does not converge uniformly. This happened because we considered the mean square and took the integral. **TODO: add a picture for convergence** □

Theorem 2.22 (Integral test)**.** *Given a monotonic and positive function $f(x)$, we have*

$$\int_{N+1}^{\infty} f(x)dx \le \sum_{n=N+1}^{\infty} f(n) \le \int_{N}^{\infty} f(x)dx \tag{2.42}$$

Proof. **TODO: compare the area below the graph of** $y = f(x)$

TODO: add a picture □

Example 2.7. Let us find σ_{2N}^2 for $f(x) = |x|, -L < x < L$.

Solution. From Example 2.2 we know that $B_n = 0$, $A_{2m} = 0$ ($m = 1, 2, \ldots$), and

$$f_{2N}(x) = A_0 + \sum_{m=1}^{N} A_{2m-1}\cos\frac{(2m-1)\pi x}{L}, \quad A_0 = \frac{L}{2}, \quad A_{2m-1} = -\frac{4L}{\pi^2(2m-1)^2}.$$

This is also the Fourier cosine series of x, $0 < x < L$, in Example 2.2. Hence we obtain

$$\sigma_{2N}^2 = \frac{1}{2}\sum_{n=2N+1}^{\infty} A_n^2 = \frac{1}{2}\sum_{m=N+1}^{\infty} A_{2m-1}^2 = \frac{8L^2}{\pi^4}\sum_{m=N+1}^{\infty}\frac{1}{(2m-1)^4}.$$

Note that by Theorem 2.22

$$\int_{N+1}^{\infty}\frac{1}{(2x-3)^4}dx \le \sum_{m=N+1}^{\infty}\frac{1}{(2m-1)^4} \le \int_{N+1}^{\infty}\frac{1}{(2x-1)^4}dx,$$

and by Taylor expansion $\frac{1}{(1+x)^3} = 1 - 3x + 6x^2 + \cdots$, $LHS = \frac{1}{8N^3}\frac{1}{(1+1/2N)^3} = \frac{1}{6(2N+1)^3} = \frac{1}{48N^3} + O\left(N^{-4}\right)$ and $RHS = \frac{1}{6(2N-1)^3} = \frac{1}{48N^3} + O\left(N^{-4}\right)$. Therefore we obtain

$$\sigma_{2N}^2 = \frac{L^2}{6\pi^4 N^3} + O\left(N^{-4}\right) = O\left(N^{-3}\right), \quad N \to \infty. \tag{2.43}$$

Thus the Fourier series of x converges as $O(1/N)$ and the Fourier series of $|x|$ converges as $O\left(1/N^3\right)$. Equations (2.41) and (2.43) explain the difference between figure **??**. □

2.6.3 Parseval's theorem for complex, cosine and Fourier sine series

Theorem 2.23 (Parseval's theorem for complex Fourier series)**.** *Let $f(x)$ defined on $-L < x < L$ be a piecewise smooth function with Fourier series*

$$f(x) = \sum_{n=-\infty}^{\infty} \alpha_n e^{in\pi x/L} \tag{2.44}$$

Then, the mean square $\frac{1}{2L}\int_{-L}^{L} f(x)^2 dx$ of $f(x)$ satisfies the following identity

$$\frac{1}{2L}\int_{-L}^{L} |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |\alpha_n|^2 . \tag{2.45}$$

Proof. Method 1. We have

$$\begin{aligned}\frac{1}{2L}\int_{-L}^{L} |f(x)|^2 dx =& \frac{1}{2L}\int_{-L}^{L} \sum_{n=-\infty}^{\infty} \alpha_n e^{in\pi x/L} \sum_{m=-\infty}^{\infty} \bar{\alpha}_m e^{-im\pi x/L} dx \\ =& \sum_{n,m=-\infty}^{\infty} \alpha_n \bar{\alpha}_m \underbrace{\frac{1}{2L}\int_{-L}^{L} e^{i(n-m)\pi x/L} dx}_{=0 \text{ if } n\neq m} \\ =& \sum_{n=-\infty}^{\infty} |\alpha_n|^2\end{aligned} \tag{2.46}$$

Method 2. (2.45) is seen by the calculation below. By Parseval's identity for the Fourier series (2.33),

$$\begin{aligned}\frac{1}{2L}\int_{-L}^{L} f(x)^2 dx &= A_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(A_n^2 + B_n^2\right) \\ &= A_0^2 + 2\sum_{n=1}^{\infty} \frac{A_n - iB_n}{2}\frac{A_n + iB_n}{2} \\ &= \alpha_0^2 + 2\sum_{n=1}^{\infty} \alpha_n \alpha_{-n} = \alpha_0^2 + \sum_{n=1}^{\infty} |\alpha_n|^2 + \sum_{n=1}^{\infty} |\alpha_{-n}|^2 \\ &= \alpha_0^2 + \sum_{n=1}^{\infty} |\alpha_n|^2 + \sum_{n=-\infty}^{-1} |\alpha_n|^2 \\ &= \sum_{n=-\infty}^{\infty} |\alpha_n|^2\end{aligned}$$

□

Theorem 2.24 (Parseval's theorem for Fourier cosine series)**.** *Let $f(x)$ defined on $0 < x < L$ be a piecewise smooth function with Fourier cosine series*

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L} \tag{2.47}$$

Then, the mean square $\frac{1}{L}\int_0^L |f(x)|^2 dx$ *of* $f(x)$ *satisfies the following identity*

$$\frac{1}{L}\int_0^L |f(x)|^2 dx = A_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} A_n^2. \tag{2.48}$$

Proof. This is homework. □

Theorem 2.25 (Parseval's theorem for Fourier sine series)**.** *Let* $f(x)$ *defined on* $0 < x < L$ *be a piecewise smooth function with Fourier sine series*

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} \tag{2.49}$$

Then, the mean square $\frac{1}{2L}\int_{-L}^L f(x)^2 dx$ *of* $f(x)$ *satisfies the following identity*

$$\frac{1}{L}\int_0^L f(x)^2 dx = \frac{1}{2}\sum_{n=1}^{\infty} B_n^2. \tag{2.50}$$

Proof. This is homework. □

3 Boundary value problem and the Sturm-Liouville theory

In this section, we will consider the separation of variables for more general equations, which leads to the Sturm-Liouville eigenvalue problems. Then we introduce the notion of orthogonality, symmetric differential operators and prove the important Sturm-Liouville theorem.

3.1 Boundary conditions for general PDEs

3.1.1 Dirichlet, Neumann and Robin boundary conditions

In the ODE class, we have learned the following boundary value problem

$$\begin{aligned} &u'' = 1, \qquad x \in [0,1] \\ &u(0) = 0, \quad u(1) = 0. \end{aligned} \tag{3.1}$$

where we prescribe a condition for every point on the boundary of the domain $[0,1]$. (The boundary is 0 and 1.) If we miss any of these conditions, we cannot get a unique solution.

For example, if we remove the condition $u(1) = 0$, then we get solution $u(x) = \frac{1}{2}x^2 + Cx$, which contains an undetermined constant C.

Given a PDE defined on a domain R, we must also prescribe the boundary condition on ∂R to obtain a unique solution.

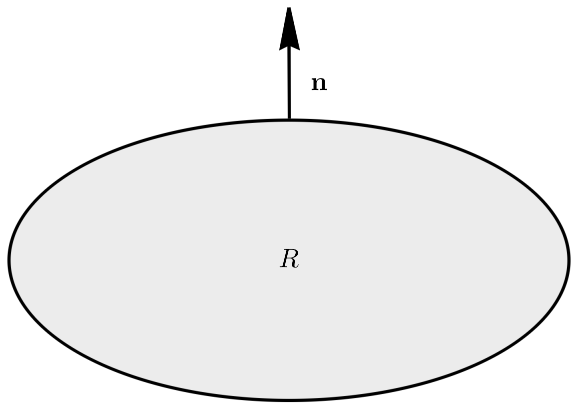

Figure 3: The region R and its normal vector.

Before we introduce the concept of boundary conditions, let us first explain the concept of normal vectors.

Definition 3.1 (Normal vector)**.** *Given a domain R and a point $x \in \partial R$, a normal vector $\mathbf{n}$ at x is the vector as demonstrated by figure 3. In this class, the normal vector is always pointing outwards.*

Now we introduce the boundary conditions that will be considered in this class.

Definition 3.2 (Boundary conditions)**.** *Consider a PDE $F[u] = 0$ defined on the domain R. Here are the boundary conditions that we consider in this class.*

1. ***Dirichlet boundary condition.*** *The value of u on the boundary is given.*

$$u = g(x), \quad x \in \partial R. \tag{3.2}$$

2. ***Neumann boundary condition.*** *The normal derivative on the boundary is given.*

$$\mathbf{n} \cdot \nabla u = g(x), \quad x \in \partial R. \tag{3.3}$$

3. ***Robin boundary condition.*** *A linear combination of the above two boundary conditions.*

$$a(x)u + b(x)\mathbf{n} \cdot \nabla u = g(x), \quad x \in \partial R. \tag{3.4}$$

3.1.2 Heat equations as an example

We take the heat equation $u_t = Ku_{zz}$, defined on $R = \{(z,t) : 0 \leq t < \infty,\ 0 \leq x \leq L\}$, as an example to explain these boundary conditions.

The domain R of the heat equation is described in the following picture,

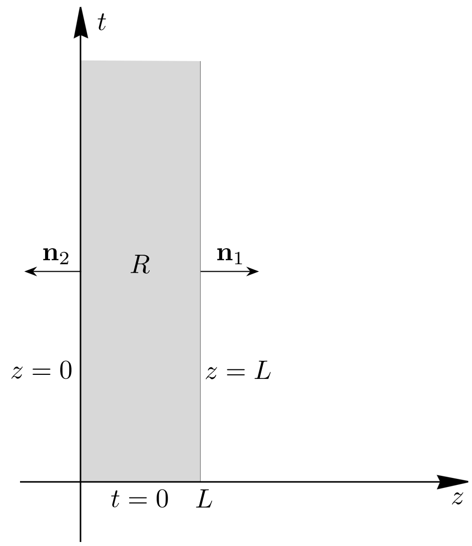

Figure 4: The domain for the heat equation.

There are three pieces of ∂R,

$$\begin{aligned} &z = 0, \quad t > 0 \\ &z = L, \quad t > 0 \\ &0 < z < L, \quad t = 0 \end{aligned} \tag{3.5}$$

On the first two parts of ∂R corresponding to $z = 0,\ L$, we impose the Robin boundary condition. On the last part corresponding to $t = 0$, we impose the Dirichlet boundary condition. Then we get the following system of equations,

$$\begin{cases} u_t = Ku_{zz}, & 0 < z < L, \quad t > 0, \\ a(z,t)u + b(z,t)\mathbf{n} \cdot \nabla u = g(z,t), & z = 0,\ L, \quad t > 0, \\ u(z,0) = f(z), & 0 < z < L, \quad t = 0. \end{cases} \tag{3.6}$$

Here the choice of Dirichlet and Robin boundary condition comes from the physics of heat conduction. We will explain this in section **??**.

TODO: second order need one condition but first order need one

Now we explain how to simplify the Robin boundary condition in (3.6).

On $z = 0$, from figure 4, we know that the normal vector $\mathbf{n} = (-1, 0)$ and $\nabla u = (u_z, u_t)$, so we get $\mathbf{n} \cdot \nabla u = -u_z$. Therefore, $a(z,t)u + b(z,t)\mathbf{n} \cdot \nabla u = g(z,t)$ simplifies to

$$a(0,t)u - b(0,t)u_z = g(0,t). \tag{3.7}$$

On $z = L$, from figure 4, we know that the normal vector $\mathbf{n} = (1, 0)$ and $\nabla u = (u_z, u_t)$, so we get $\mathbf{n} \cdot \nabla u = u_z$. Therefore, $a(z,t)u + b(z,t)\mathbf{n} \cdot \nabla u = g(z,t)$ simplifies to

$$a(L,t)u + b(L,t)u_z = g(L,t). \tag{3.8}$$

We introduce new functions $a(t)$, $\widetilde{a}(t)$, $b(t)$, $\widetilde{b}(t)$, $g(t)$ and $\widetilde{g}(t)$ by the following equations

$$\begin{aligned} a(t) &= a(0,t), \quad b(t) = b(0,t), \quad g(t) = g(0,t) \\ \widetilde{a}(t) &= a(L,t), \quad \widetilde{b}(t) = b(L,t), \quad \widetilde{g}(t) = g(L,t) \end{aligned} \tag{3.9}$$

Then the Robin boundary condition becomes

$$\begin{aligned} a(t)u - b(t)u_z &= g(t), \quad z = 0 \\ \widetilde{a}(t)u + \widetilde{b}(t)u_z &= \widetilde{g}(t), \quad z = L \end{aligned} \tag{3.10}$$

The heat equation becomes

$$\begin{cases} u_t = Ku_{zz}, & 0 < z < L, \quad t > 0, \\ a(t)u - b(t)u_z = g(t), & z = 0, \quad t > 0, \\ \widetilde{a}(t)u + \widetilde{b}(t)u_z = \widetilde{g}(t), & z = L, \quad t > 0, \\ u = f(z), & 0 < z < L, \quad t = 0. \end{cases} \tag{3.11}$$

In order to make (3.11) solvable, we impose the homogeneous condition.

Definition 3.3 (Homogeneous)**.** *We say* (3.11) *is* <u>*homogeneous*</u> *if*

1. $a(t)$, $\widetilde{a}(t)$, $b(t)$, $\widetilde{b}(t)$, $g(t)$ *and* $\widetilde{g}(t)$ *are independent of* t.
2. $g(t) = \widetilde{g}(t) = 0$.

With homogeneous assumption, (3.11) becomes

$$\begin{cases} u_t = Ku_{zz}, & 0 < z < L, \quad t > 0, \\ au - bu_z = 0, & z = 0, \quad t > 0, \\ \widetilde{a}u + \widetilde{b}u_z = 0, & z = L, \quad t > 0, \\ u = f(z), & 0 < z < L, \quad t = 0. \end{cases} \tag{3.12}$$

To further simplify the above equation, we did the change of variable

$$b \to bL, \quad \widetilde{b} \to \widetilde{b}L \tag{3.13}$$

followed by the change of variable

$$\begin{aligned} a &\to r\cos\alpha, \quad b \to r\sin\alpha, \\ \widetilde{a} &\to r\cos\beta, \quad \widetilde{b} \to r\sin\beta. \end{aligned} \tag{3.14}$$

Finally, the heat equation becomes

$$\begin{cases} u_t = Ku_{zz}, & 0 < z < L, \quad t > 0, \\ u\cos\alpha - Lu_z\sin\alpha = 0, & z = 0, \quad t > 0, \\ u\cos\beta + Lu_z\sin\beta = 0, & z = L, \quad t > 0, \\ u = f(z), & 0 < z < L, \quad t = 0, \end{cases} \tag{3.15}$$

3.1.3 Separation of variable in heat equation

Let us solve the heat equation in a simple case of $\alpha = \beta = 0$ in (3.15). The separated solution is written as $u(z,t) = \phi(z)T(t)$. Thus we obtain

$$T'(t) + \lambda KT(t) = 0, \quad \phi''(z) + \lambda\phi(z) = 0.$$

The boundary conditions are written as

$$\phi(0) = \phi(L) = 0.$$

We obtain

$$T(t) = e^{-\lambda Kt}, \quad \phi = A\sin(\sqrt{\lambda}z) + B\cos(\sqrt{\lambda}z), \quad \lambda > 0.$$

By plugging $\phi = A\sin(\sqrt{\lambda}z) + B\cos(\sqrt{\lambda}z)$ into the boundary conditions, we find that $B = 0$ and $\sqrt{\lambda}L = n\pi$ where n is an integer. Therefore we obtain

$$\phi(z) = \phi_n(z) = \sin\left(\sqrt{\lambda_n}z\right), \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, \ldots,$$

where we set the arbitrary constant in $\phi_n(z)$ to be 1 (recall we will take a superposition). Thus the separated solutions are obtained as

$$u(z,t) = \phi_n(z)e^{-\lambda_n Kt}, \quad n = 1, 2, \ldots$$

If no initial condition $u(z, 0) = f(z)$ is given, the separated solutions are the solutions to the problem. However, they do not satisfies $u(z, 0) = f(z)$. Let us consider the linear combination of separated solution and match with the initial condition.

The linear combination is

$$u(z, t) = \sum_{n=1}^{\infty} B_n \phi_n(z) e^{-\lambda_n K t},$$

where B_n are constants.

By $u(z, 0) = f(z)$, we know that

$$f(z) = u(z, 0) = \sum_{n=1}^{\infty} B_n \phi_n(z) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi z}{L}.$$

From this, we know that B_n is the coefficient of the Fourier sine series. We thus obtain

$$u(z, t) = \sum_{n=1}^{\infty} B_n \phi_n(z) e^{-\lambda_n K t}, \quad 0 < z < L, \quad t > 0.$$

$$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx.$$

Example 3.1. The heat equation $u_t = K u_{zz}$ for $0 < z < L, t > 0$ with $u(0, t) = u(L, t) = 0$ and $u(z, 0) = 1$ is solved as

$$u(z, t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi z}{L} e^{-(n\pi/L)^2 K t},$$

where

$$B_n = \frac{2}{\pi} \frac{1 - (-1)^n}{n}.$$

Here as mentioned above, B_n can be directly computed as coefficients of the Fourier sine series of $f(z) = 1$

3.1.4 Some linear algebra

The orthogonality play an important role in computing the Fourier coefficients. Next, we'll explore the origin of orthogonality in the separation of variables method. After separation of the variable in the heat equation, we obtain the following equation

$$\phi''(z) + \lambda \phi(z) = 0 \quad \Leftrightarrow \quad -\phi''(z) = \lambda \phi(z).$$

If we denote the negative second-order derivative operator by A, then we get

$$A\phi = \lambda \phi.$$

This is very similar to the eigenvalue and eigenvector equation for matrix M in linear algebra.

$$Mv = \lambda v. \tag{3.16}$$

For a symmetric matrix, we have the following properties.

Theorem 3.4. *Assume that M is a symmetric operator ($M = M^T$), and $\langle x, y\rangle$ is the inner product of vectors, then we have*

1. $\langle Mx, y\rangle = \langle x, My\rangle$.

2. *The eigenvalues of M are real numbers.*

3. *If v, w are eigenvectors with different eigenvalues λ and μ respectively, then $v \perp w$ ($\langle v, w\rangle = 0$).*

Proof. Notice that $\langle x, y\rangle = x^T y$, then we have $\langle Mx, y\rangle = (Mx)^T y = x^T M^T y = x^T My = \langle x, My\rangle$.

For complex vectors $\langle x, y\rangle = \bar{x}^T y$, $\langle x, Mx\rangle = \lambda\langle x, x\rangle$, $\langle Mx, x\rangle = \bar{\lambda}\langle x, x\rangle$ and $\langle Mx, x\rangle = \langle x, Mx\rangle$. Therefore, $\lambda = \bar{\lambda}$.

$\langle v, Mw\rangle = \mu\langle v, w\rangle$, $\langle Mv, w\rangle = \lambda\langle v, w\rangle$ and $\langle Mv, w\rangle = \langle v, Mw\rangle$. Therefore, $\mu\langle v, w\rangle = \lambda\langle v, w\rangle$, which implies that $\langle v, w\rangle$ due to $\lambda \neq \mu$. □

It turns out that A can be viewed as a symmetric matrix, and thus satisfies all properties described by Theorem 3.4.

3.2 The Sturm-Liouville eigenvalue problem

Now we study the eigenvalue problems for differential operators. Let us first start with the concepts of orthogonal functions and symmetric operators.

3.2.1 Orthogonal functions and symmetric operators

Definition 3.5 (Inner product)**.** *We extend dot product $\varphi \cdot \psi$ and define inner product as*

$$\langle \varphi, \psi\rangle = \int_a^b \varphi(x)\psi(x)dx. \tag{3.17}$$

Sometimes the inner product is defined as follows. We can have a weight function $\rho(x)$, and the weighted inner product is given by

$$\langle \varphi, \psi\rangle_\rho = \int_a^b \varphi(x)\psi(x)\rho(x)dx \tag{3.18}$$

where $\rho(x) > 0$ is a weight function.

For complex functions, we can write the complex inner product as

$$\langle \varphi, \psi \rangle = \int_a^b \varphi(x)\bar{\psi}(x)\rho(x)dx. \tag{3.19}$$

Here $\bar{\psi}$ is the complex conjugate of ψ ($\bar{\psi}(x) = f(x) - ig(x)$ when $\psi = f + ig$).

Definition 3.6 (Orthogonality)**.** *Two functions φ, ψ are said to be orthogonal on $[a, b]$ if $\langle \varphi, \psi \rangle = 0$.*

Example 3.2. The functions $\varphi(x) = \sin x$ and $\psi(x) = \cos x$ are orthogonal on $[0, \pi]$.

Example 3.3. The set of functions $1, \cos \frac{\pi x}{L}, \sin \frac{\pi x}{L}, \ldots, \cos \frac{N\pi x}{L}, \sin \frac{N\pi x}{L}$ is orthogonal on $[-L, L]$.

Example 3.4. The set of functions $\{e^{\frac{in\pi x}{L}}\}_{n\in\mathbb{Z}}$ is orthogonal on $[-L, L]$.

Example 3.5. Which of the following pairs of functions are orthogonal on the interval $0 \le x \le 1$?

$$\varphi_1 = \sin 2\pi x, \quad \varphi_2 = x, \quad \varphi_3 = \cos 2\pi x, \quad \varphi_4 = 1.$$

$\langle \varphi_1, \varphi_3 \rangle = 0, \langle \varphi_1, \varphi_4 \rangle = 0, \langle \varphi_2, \varphi_3 \rangle = 0, \langle \varphi_3, \varphi_4 \rangle = 0$. All others are nonzero. Therefore the pairs $(\varphi_1, \varphi_3), (\varphi_1, \varphi_4), (\varphi_2, \varphi_3)$, and (φ_3, φ_4) are orthogonal.

Definition 3.7 (Norm)**.** *As follows we define norm, which is the "length" of a function.*

$$\|\varphi\| = \|\varphi\|_{L^2(a,b)} = \sqrt{\langle \varphi, \varphi \rangle}.$$

We note that the norm is always nonnegative. The norm $\|\varphi - \psi\|$ is the distance between two functions φ and ψ.

Definition 3.8 (Orthonormal)**.** *The functions $(\varphi_1, \ldots, \varphi_N)$ are orthonormal if $\langle \varphi_i, \varphi_j \rangle = \delta_{ij}$. Here δ_{ij} is the Kronecker delta ($\delta_{ij} = 0$ if $i \neq j$ and $= 1$ if $i = j$).*

Definition 3.9 (Normalization)**.** *Let $\{\varphi_1, \varphi_2, \ldots, \varphi_N\}$ be a set of orthogonal functions satisfying $\varphi_i \perp \varphi_j$ for $i \neq j$. Each function φ_i can be normalized to obtain an orthogonal sequence $\{\psi_i\}_{i=1,\ldots,N}$:*

$$\psi_i = \frac{\varphi_i}{\|\varphi_i\|}$$

where $\|\varphi_i\|$ denotes the norm of the function φ_i.

Example 3.6. After normalize the orthogonal functions $\cos x$, $\sin x$ on $[0, \pi]$, we obtain $\sqrt{\frac{2}{\pi}} \cos x$, $\sqrt{\frac{2}{\pi}} \sin x$.

Definition 3.10 (Differential Operator). *A* <u>*differential operator*</u> *A is an operator satisfying the following*

1. *A is of the form $A = \sum_{i=0}^{n} a_i(x) \frac{d^i}{dx^i}$.*

2. *The operator A has a domain $\mathrm{Dom}(A)$, which means that A does accept all smooth functions as its input. (See Example 3.7 for an example of $\mathrm{Dom}(A)$)*

where n is a non-negative integer representing the <u>*order*</u> *of the differential operator, $a_i(x)$ are the* <u>*coefficient*</u> *of A, and $\frac{d^i}{dx^i}$ represents the i-th derivative with respect to x, with $\frac{d^0}{dx^0}$ interpreted as the identity operator.*

Definition 3.11 (Symmetric operators). *Given a differential operator A, we say A is* <u>*symmetric*</u>*, if for any functions $\varphi, \psi \in \mathrm{Dom}(A)$,*

$$\langle A\varphi, \psi \rangle = \langle \varphi, A\psi \rangle. \tag{3.20}$$

Example 3.7. The differential operator A defined by the following is a symmetric operator,

$$A = -\frac{d^2}{dx^2}, \qquad \mathrm{Dom}(A) = \{\varphi(x) : x \in [a,b],\, \varphi(a) = \varphi(b) = 0\} \tag{3.21}$$

To show that A is symmetric, we need to verify that

$$\langle \varphi, A\psi \rangle = \langle A\varphi, \psi \rangle \tag{3.22}$$

for all functions $\varphi(x)$ and $\psi(x)$ satisfying $\varphi(a) = \varphi(b) = \psi(a) = \psi(b) = 0$. This condition translates into the following integral equality,

$$\int_a^b \varphi(x)\psi''(x)dx = \int_a^b \varphi''(x)\psi(x)dx \tag{3.23}$$

Applying integration by parts and using the boundary conditions $\varphi(a) = \varphi(b) = \psi(a) = \psi(b) = 0$, we get

$$\begin{aligned} \int_a^b \varphi''(x)\psi(x)dx &= \underbrace{[\varphi'(x)\psi(x)]_a^b}_{=0} - \int_a^b \varphi'(x)\psi'(x)dx \\ &= -\int_a^b \varphi'(x)\psi'(x)dx = -\underbrace{[\varphi(x)\psi'(x)]_a^b}_{=0} + \int_a^b \varphi(x)\psi''(x)dx \\ &= \int_a^b \varphi(x)\psi''(x)dx \end{aligned} \tag{3.24}$$

This equality confirms (3.22) and thus the fact that A is symmetric on the given domain $\mathrm{Dom}(A)$.

Example 3.8. The definition of symmetric operator depends on domain. For example, if we define A by

$$A = -\frac{d^2}{dx^2}, \qquad \mathrm{Dom}(A) = \{\varphi(x) : x \in [a, b]\} \tag{3.25}$$

which is almost the same operator as the last example but with a different domain. This operator is NOT symmetric. If we take $\varphi(x) = 1$ and $\psi(x) = x^2$, then a straightforward computation gives $\langle\varphi, A\psi\rangle = 0$ and $\langle A\varphi, \psi\rangle \neq 0$. Therefore, A is not symmetric.

3.2.2 Separation of variables of second-order PDEs

TODO: how does this leads to Sturm Liouville

3.2.3 The Sturm-Liouville eigenvalue problem

Definition 3.12 (Sturm-Liouville problems)**.** *Let ϕ be a function defined on the interval (a, b), the* *Sturm-Liouville problems* *is the following boundary value problem,*

$$\begin{aligned} &[s(x)\phi'(x)]' + [\lambda\rho(x) - q(x)]\phi(x) = 0, \quad a < x < b, \\ &\phi(a)\cos\alpha - L\phi'(a)\sin\alpha = 0, \\ &\phi(b)\cos\beta + L\phi'(b)\sin\beta = 0 \end{aligned} \tag{3.26}$$

where ρ is a positive function $\rho(x) > 0$ and $L = b - a$, $\alpha, \beta \in [0, \pi)$ are some parameters.

Define the operator A by

$$A\phi = -\frac{1}{\rho(x)}\Big([s(x)\phi'(x)]' - q(x)\phi(x)\Big) \tag{3.27}$$

with domain

$$\left\{\phi(x),\ x \in [a, b] \;\middle|\; \begin{aligned} &\phi(a)\cos\alpha - L\phi'(a)\sin\alpha = 0, \\ &\phi(b)\cos\beta + L\phi'(b)\sin\beta = 0 \end{aligned}\right\} \tag{3.28}$$

then (3.26) is equivalent to

$$A\phi = \lambda\phi. \tag{3.29}$$

The following theorem claims that all symmetric second order differential operators are of the form given by (3.27) and (3.28).

Theorem 3.13. *A symmetric second order differential operator A must be of the form given by* (3.27) *and* (3.28)*.*

Proof. The proof go beyond the scope of this course. $\square$

The following theorem explain the procedure of reducing an arbitrary second order differential operator to a Sturm-Liouville operator. Since it is abstract, it is good to first jump to the examples below the theorem.

Theorem 3.14. *Given an arbitrary second order linear differential equation* $a(x)y''+b(x)y'+c(x)y = 0$*, it can be reduced to the Sturm-Liouville equation* (3.26) *by the following procedure.*

1. *Divide both sides by* $a(x)$ *to obtain* $y'' + p(x)y' + q(x)y = 0$.

2. *Solve the equation* $\widehat{y}' = p(x)\widehat{y}$.

3. *Multiply both sides of* $y'' + p(x)y' + q(x)y = 0$ *by* $\widehat{y}$

$$\widehat{y}(y'' + p(x)y') + \widehat{y}q(x)y = 0 \quad \Rightarrow \quad (\widehat{y}y')' + \widehat{y}q(x)y = 0 \tag{3.30}$$

Proof. There is nothing to prove. $\square$

Example 3.9 (Bessel functions). The Bessel functions are solutions of the following ODEs.

$$\phi'' + (d-1)\frac{\phi'}{x} + \left(1 - \frac{\mu}{x^2}\right)\phi = 0. \tag{3.31}$$

We can apply the procedure in Theorem 3.14 to reduce the above equation to a Sturm-Liouville equation.

1. There is nothing to do with this step since $a(x) = 1$.

2. We first solve $\widehat{y}' = \frac{d-1}{x}\widehat{y}$ and obtain $\widehat{y}(x) = x^{d-1}$.

3. Multiply both side of $\phi'' + (d-1)\frac{\phi'}{x} + \left(1 - \frac{\mu}{x^2}\right)\phi = 0$ by x^{d-1}, then we get

$$x^{d-1} \cdot \left(\phi'' + (d-1)\frac{\phi'}{x} + \left(1 - \frac{\mu}{x^2}\right)\phi\right) = 0 \quad \Rightarrow \quad (x^{d-1}\phi')' + \left(x^{d-1} - \mu x^{d-3}\right)\phi = 0. \tag{3.32}$$

Finally, we get

$$(x^{d-1}\phi'(x))' + \left(x^{d-1} - \mu x^{d-3}\right)\phi(x) = 0. \tag{3.33}$$

This is a Sturm-Liouville equation by setting $s(x) = \rho(x) = x^{d-1}$, $q(x) = \mu x^{d-3}$, and $\lambda = 1$.

In the case of $d = 2$ and $\mu = m^2$ with $m \in \mathbb{N}$, the function $\phi(x) = J_m(x)$ is called the standard Bessel function. In the case of $d = 3$ and $\mu = k(k+1)(k = 0, 1, 2, \ldots)$, the function $\phi(x) = j_k(x)$ is called the spherical Bessel function.

Example 3.10 (Legendre polynomials). The Legendre polynomials $P_k^m(x)$ are solutions of the following ODEs.

$$(1-x^2)\phi'' - 2x\phi' + \left(k(k+1) - \frac{m^2}{1-x^2}\right)\phi = 0. \tag{3.34}$$

We can apply the procedure in Theorem 3.14 to reduce the above equation to a Sturm-Liouville equation.

1. Divide both sides by $(1-x^2)$, then we get
$$\phi'' - \frac{2x}{1-x^2}\phi' + \left(\frac{k(k+1)}{1-x^2} - \frac{m^2}{(1-x^2)^2}\right)\phi = 0. \tag{3.35}$$

2. We first solve $\widehat{y}' = -\frac{2x}{1-x^2}\widehat{y}$ and obtain $\widehat{y}(x) = 1-x^2$.

3. Multiply both side $1-x^2$ (This step undo the operation of step 1 which sometimes appears), then we get
$$(1-x^2)\phi'' - 2x\phi' + \left(k(k+1) - \frac{m^2}{1-x^2}\right)\phi = 0 \;\Rightarrow\; ((1-x^2)\phi')' + \left(k(k+1) - \frac{m^2}{1-x^2}\right)\phi = 0. \tag{3.36}$$

Finally, we get

$$((1-x^2)\phi')' + \left(k(k+1) - \frac{m^2}{1-x^2}\right)\phi = 0. \tag{3.37}$$

This is a Sturm-Liouville equation by setting $s(x) = 1-x^2$, $\rho(x) = 1$, $q(x) = m^2/(1-x^2)$, and $\lambda = k(k+1)$ $(k = 0, 1, 2, \ldots)$ with $a = -1$ and $b = 1$.

Example 3.11 (Hermite polynomials). The Hermite polynomials $H_n(x)$ are solutions of the following ODEs.

$$\phi'' - x\phi' + n\phi = 0. \tag{3.38}$$

We can apply the procedure in Theorem 3.14 to reduce the above equation to a Sturm-Liouville equation.

1. There is nothing to do with this step since $a(x) = 1$.

2. We first solve $\widehat{y}' = -x\widehat{y}$ and obtain $\widehat{y}(x) = e^{-\frac{x^2}{2}}$.

3. Multiply both side by $e^{-\frac{x^2}{2}}$, then we get
$$e^{-\frac{x^2}{2}}(\phi'' - x\phi') + ne^{-\frac{x^2}{2}}\phi = 0 \quad\Rightarrow\quad \left(e^{-\frac{x^2}{2}}\phi'\right)' + ne^{-\frac{x^2}{2}}\phi = 0. \tag{3.39}$$

Finally, we get

$$\left(e^{-\frac{x^2}{2}}\phi'\right)' + ne^{-\frac{x^2}{2}}\phi = 0. \tag{3.40}$$

This is a Sturm-Liouville equation by setting $s(x) = \rho(x) = \exp\left(-x^2/2\right)$, $q(x) = 0$, $\lambda = n$ ($n = 0, 1, 2, \ldots$) with $a = -\infty$ and $b = \infty$.

3.2.4 The Sturm-Liouville theorem

Recall that the Sturm-Liouville problem is equivalent to the eigenvalue problem $A\phi = \lambda\phi$ of the operator A defined in (3.27) and (3.28), which are copied below.

$$A\phi = -\frac{1}{\rho(x)}\Big([s(x)\phi'(x)]' - q(x)\phi(x)\Big) \tag{3.41}$$

with domain

$$\left\{\phi(x),\ x \in [a,b] \;\middle|\; \begin{array}{l} \phi(a)\cos\alpha - L\phi'(a)\sin\alpha = 0, \\ \phi(b)\cos\beta + L\phi'(b)\sin\beta = 0 \end{array}\right\} \tag{3.42}$$

We have the following theorem, which is an analog of Theorem 3.4

Theorem 3.15 (Sturm-Liouville theorem I)**.** *Assume that A is a differential operator defined by* (3.27) *and* (3.28)*, and $\langle \varphi, \psi\rangle_\rho = \int_a^b \varphi(x)\psi(x)\rho(x)dx$ is the inner product defined by* (3.18) *, then we have*

1. $\langle A\phi, \psi\rangle = \langle \phi, A\psi\rangle$.
2. *The eigenvalues of A are real numbers.*
3. *If φ, ψ are eigenfunctions with different eigenvalues λ and μ respectively, then $\langle \varphi, \psi\rangle_\rho = 0$.*
4. *Assume that $\{\lambda_1, \lambda_2, \ldots\}$ are the set of all eigenfunctions of A, then this set is infinite and discrete.* ***TODO: picture***

Proof. 1. A is symmetric. Consider $\langle A\phi, \psi\rangle$, and apply integration by parts

$$\begin{aligned} \langle A\phi, \psi\rangle &= \int_a^b \left(-\frac{1}{\rho(x)}[(s(x)\phi'(x))' - q(x)\phi(x)]\right)\psi(x)\rho(x)\,dx. \\ &= \int_a^b \left(-(s(x)\phi'(x))' + q(x)\phi(x)\right)\psi(x)\,dx. \\ &= -\left[s(x)\phi'(x)\psi(x)\right]_a^b + \int_a^b s(x)\phi'(x)\psi'(x)\,dx - \int_a^b q(x)\phi(x)\psi(x)\,dx. \end{aligned} \tag{3.43}$$

Similarly for $\langle \phi, A\psi\rangle = \langle A\psi, \phi\rangle$, we also have

$$\begin{aligned}
\langle A\psi, \phi\rangle &= \int_a^b \left(-\frac{1}{\rho(x)}[(s(x)\psi'(x))' - q(x)\psi(x)]\right)\phi(x)\rho(x)\,dx. \\
&= \int_a^b \left(-(s(x)\psi'(x))' + q(x)\psi(x)\right)\phi(x)\,dx. \\
&= -\left[s(x)\psi'(x)\phi(x)\right]_a^b + \int_a^b s(x)\psi'(x)\phi'(x)\,dx - \int_a^b q(x)\psi(x)\phi(x)\,dx.
\end{aligned} \tag{3.44}$$

It suffices to show that the boundary terms are equal

$$[s(x)\phi'(x)\psi(x)]_a^b = [s(x)\psi'(x)\phi(x)]_a^b \tag{3.45}$$

which is equivalent to

$$s(b)\Big(\phi'(b)\psi(b) - \psi'(b)\phi(b)\Big) = s(a)\Big(\phi'(a)\psi(a) - \psi'(a)\phi(a)\Big). \tag{3.46}$$

If we can show that $\phi'(a)\psi(a) = \psi'(a)\phi(a)$ and $\phi'(b)\psi(b) = \psi'(b)\phi(b)$, then (3.46) follows and the proof of conclusion 1 is completed. Since ϕ and ψ belong to the domain of A, by (3.42), $\frac{\phi'(a)}{\phi(a)} = \frac{\cos\alpha}{L\sin\alpha} = \frac{\psi'(a)}{\psi(a)}$, which implies $\phi'(a)\psi(a) = \psi'(a)\phi(a)$. The second equation at b can be proved similarly.

2. The eigenvalues of A are real. Since the eigenvalues and eigenfunctions at this point may not be real, we consider the complex inner product. Let λ be an eigenvalue with eigenfunction ϕ,

$$A\phi = \lambda\phi. \tag{3.47}$$

Consider the inner product $\langle \phi, A\phi\rangle$ and $\langle A\phi, \phi\rangle$,

$$\langle A\phi, \phi\rangle = \langle \lambda\phi, \phi\rangle = \lambda\|\phi\|^2. \tag{3.48}$$

$$\langle \phi, A\phi\rangle = \langle \phi, \lambda\phi\rangle = \overline{\lambda}\|\phi\|^2. \tag{3.49}$$

Since A is symmetric, we have $\langle \phi, A\phi\rangle$ and $\langle A\phi, \phi\rangle$, which implies that

$$\lambda\|\phi\|^2 = \overline{\lambda}\|\phi\|^2. \tag{3.50}$$

Hence, $\lambda = \overline{\lambda}$, implying that λ is real.

3. Orthogonality of eigenfunctions with different eigenvalues. Now the eigenvalues and eigenfunctions at this point are real, we consider the real inner product.

Suppose $A\phi = \lambda\phi$ and $A\psi = \mu\psi$ with $\lambda \neq \mu$. Consider

$$\lambda\langle \phi, \psi\rangle = \langle A\phi, \psi\rangle = \langle \phi, A\psi\rangle = \mu\langle \phi, \psi\rangle. \tag{3.51}$$

Thus, $(\lambda - \mu)\langle \phi, \psi \rangle = 0$, implying $\langle \phi, \psi \rangle = 0$ if $\lambda \neq \mu$.

4. Discreteness of the eigenvalue set. **TODO: finish the proof TODO: add the proof of infinity** $\square$

The following property implies that the eigenvalues of all Sturm-Liouville problems are of multiplicity one. This property does not holds true for eigenvalues of matrices.

Theorem 3.16 (Sturm-Liouville theorem II)**.** *Assume that A is a differential operator defined by (3.27) and ϕ_1 and ϕ_2 are the eigenfunctions of the same eigenvalue λ, then we exists a constant C such that*

$$\phi_1(x) = C\phi_2(x). \tag{3.52}$$

Proof. We will use the following conclusion from the theory of ODE without proof.

$$\begin{cases} a(x)f'' + b(x)f' + c(x)f = 0 \\ f(x_0) = f'(x_0) = 0 \end{cases} \quad \Rightarrow \quad f(x) \equiv 0 \tag{3.53}$$

This is the uniqueness theorem of the initial value problem of second order ODEs.

We consider

$$\psi(x) = \begin{cases} \phi_2(a)\phi_1(x) - \phi_1(a)\phi_2(x), & \text{if } \alpha \neq 0 \\ \phi_2'(a)\phi_1(x) - \phi_1'(a)\phi_2(x), & \text{if } \alpha = 0. \end{cases}$$

This $\psi(x)$ obeys (3.26). If we can show that $\psi(a) = 0$ and $\psi'(a) = 0$, then by (3.53), we know that $\psi(x) \equiv 0$ for $a < x < b$. Therefore,

$$\phi_2(x) = C\phi_1(x), \quad C = \frac{\phi_2(a)}{\phi_1(a)} \quad \text{or} \quad \frac{\phi_2'(a)}{\phi_1'(a)}, \tag{3.54}$$

which proves (3.52)

Let us only consider the case of $\alpha \neq 0$, where $\phi(x) = \phi_2(a)\phi_1(x) - \phi_1(a)\phi_2(x)$. This obviously implies $\psi(a) = 0$. To prove $\psi'(a) = 0$, we notice that by (3.42), both ϕ_1 and ϕ_2 satisfy $\phi(a)\cos\alpha - L\phi'(a)\sin\alpha = 0$ as they belong to the domain of A. This implies that $\frac{\phi_1'(a)}{\phi_1(a)} = \frac{\cos\alpha}{L\sin\alpha} = \frac{\phi_2'(a)}{\phi_2(a)}$. Therefore, we conclude that $\psi'(a) = \phi_2(a)\phi_1'(a) - \phi_1(a)\phi_2'(a) = 0$. $\square$

3.2.5 Generalized Fourier series

Just like the usual Fourier series, we can express a function $f(x)$ in terms of orthogonal eigenfunctions of a Sturm-Liouville operator,

Theorem 3.17 (Generalized Fourier series)**.** *Given a Sturm-Liouville operator A of the form* (3.27) *with coefficient $\rho(x)$, let $\{\phi_n\}_{n\in\mathbb{N}} \subseteq Dom(A)$ be its eigenfunctions and $f(x) \in Dom(A)$ be an arbitrary piecewise smooth function. Then we have the following expansion*

$$f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x), \tag{3.55}$$

where the coefficients $\{c_n\}_{n\in\mathbb{N}}$ can be computed by

$$c_n = \frac{\langle f, \phi_n \rangle_\rho}{\|\phi_n\|^2} \tag{3.56}$$

Proof. The proof is beyond the scope of this course. **TODO: add the proof of coefficients** □

We have the following convergence theorem of general orthogonal expansion, which is similar to Theorem 2.18.

Theorem 3.18 (Convergence)**.** *Given a Sturm-Liouville operator A and a function $f(x) \in Dom(A)$, define the partial summation by $f_N(x) = \sum_{n=1}^{N} c_n \phi_n(x)$. Then we have*

1. *Let $f(x)$ be a piecewise smooth function, then we have*

$$f_N(x) \to \frac{1}{2}[f(x+0) + f(x-0)] \quad on \quad x \in (a,b) \quad as \quad N \to \infty \tag{3.57}$$

2. *If additionally assume that $f(x)$ is continuous,then $f_N(x)$ converges uniformly to $f(x)$ on $[a,b]$.*

3. *Let $\|f\|^2 = \int_a^b f(x)^2 \rho(x) dx$ and $\sigma_N = \|f - f_N\|$, then we have the Parseval's identity*

$$\|f\|^2 = \sum_{n=1}^{\infty} |c_n|^2 \|\phi_n\|^2 \tag{3.58}$$

and

$$\sigma_N^2 = \sum_{n=N+1}^{\infty} |c_n|^2 \|\phi_n\|^2 . \tag{3.59}$$

Proof. The proof of 1 and 2 is beyond the scope of this course. The proof of 3 is similar to that of Theorem 2.19 □

4 PDEs in rectangular coordinates

In this section, we will consider the separation of variables for more general equations in rectangular coordinates, possibly with variable coefficients and more general boundary conditions.

4.1 The Heat Equation

The general form of the heat equation is given by $u_t = Ku_{zz} + r(z,t)$ where $r(z,t)$ is the source term. Depending on the nature of the source term, the heat equation can be classified into three distinct cases: homogeneous where $r(z,t) = 0$, inhomogeneous with a time-independent source where $r(z)$ depends only on the spatial variable, inhomogeneous with a time-dependent source where $r(z,t)$ varies with both space and time. In the following sections, we will examine the solutions corresponding to each of these cases.

4.1.1 The homogeneous case

We begin with the homogeneous case of the heat equation. The most general form of the homogeneous heat equation is given by:

$$\begin{cases} u_t = Ku_{zz}, & 0 < z < L, \quad t > 0, \\ u\cos\alpha - Lu_z \sin\alpha = 0, & z = 0, \quad t > 0, \\ u\cos\beta + Lu_z \sin\beta = 0, & z = L, \quad t > 0, \\ u(z,0) = f(z), & 0 < z < L, \quad t = 0. \end{cases} \tag{4.1}$$

where $f(z), 0 < z < L$, is a piecewise smooth function, and $K > 0$ and $\alpha, \beta \in [0, \pi)$ are constants.

The homogeneous heat equation can be solved using the method of separation of variables, as detailed in the following theorem.

Theorem 4.1. *The solution to* (4.1) *can be obtained by the following steps:*

1. *Assume a separable solution of the form $u(z,t) = Z(z)T(t)$, and substitute into the PDE to obtain two ordinary differential equations (ODEs): one for $Z(z)$ and one for $T(t)$.*

$$T'(t) + \lambda KT(t) = 0, \qquad Z''(z) + \lambda Z(z) = 0.$$

   The spatial equation for $Z(z)$ is a Sturm-Liouville problem with equation $Z''(z) + \lambda Z(z) = 0$ subject to the boundary conditions

$$Z(0)\cos\alpha - LZ'(0)\sin\alpha = 0, \quad Z(L)\cos\beta + LZ'(L)\sin\beta = 0.$$

2. *Solving this Sturm-Liouville problem yields a set of eigenvalues λ_n and corresponding eigenfunctions $Z_n(z)$. For each λ_n, solve the time-dependent ODE $T'(t) + K\lambda_n T(t) = 0$ and obtain*

$T_n(t) = c_n e^{-K\lambda_n t}$. The general solution is given by an infinite sum of the separated solutions:

$$u(z,t) = \sum_{n=1}^{\infty} c_n Z_n(z) e^{-K\lambda_n t}.$$

3. *To match the initial condition $u(z,0) = f(z)$, express $f(z)$ as a generalized Fourier series:*

$$f(z) = \sum_{n=1}^{\infty} c_n Z_n(z).$$

The coefficients c_n are determined by the orthogonality of the eigenfunctions $Z_n(z)$, using the formula

$$c_n = \frac{\int_0^L f(z) Z_n(z)\, dz}{\int_0^L Z_n^2(z)\, dz}. \tag{4.2}$$

which is a corollary of the (3.56) *with* $\rho(x) = 1$.

Let us use an example to demonstrate the execution of the procedure in the above theorem.

Example 4.1. Let us solve the following heat equation in a slab.

$$\begin{cases} u_t = K u_{zz}, & t > 0, \quad 0 < z < L \\ u(0,t) = u_z(L,t) = 0 & t > 0 \\ u(z,0) = 1, & 0 < z < L \end{cases} \tag{4.3}$$

Solution. We solve the equation following the steps listed in Theorem 4.1.

Step 1. The separated solution is written as $u(z,t) = Z(z)T(t)$. Thus we obtain

$$T'(t) + \lambda K T(t) = 0, \quad Z''(z) + \lambda Z(z) = 0.$$

The boundary conditions are written as

$$Z(0) = Z'(L) = 0.$$

Step 2. Solving the ODE, we obtain

$$T(t) = e^{-\lambda K t}, \quad Z = A\sin(\sqrt{\lambda} z) + B\cos(\sqrt{\lambda} z), \quad \lambda > 0.$$

By plugging $Z = A\sin(\sqrt{\lambda} z) + B\cos(\sqrt{\lambda} z)$ into the boundary conditions, we find that $B = 0$ and $\sqrt{\lambda} L = (n + \frac{1}{2})\pi$ where n is an integer. Therefore we obtain

$$Z(z) = Z_n(z) = \sin\left(\sqrt{\lambda_n} z\right), \quad \lambda_n = \left(\frac{(n+\frac{1}{2})\pi}{L}\right)^2, \quad n = 1, 2, \ldots,$$

where we set the arbitrary constant in $Z_n(z)$ to be 1. Thus the general solutions are obtained as

$$u(z,t) = \sum_{n=1}^{\infty} C_n e^{-\lambda_n K t} \sin \frac{(n+\frac{1}{2})\pi z}{L},$$

where C_n are constants.

Step 3. By $u(z,0) = f(z) = 1$, we know that

$$1 = u(z,0) = \sum_{n=1}^{\infty} C_n Z_n(z) = \sum_{n=1}^{\infty} C_n \sin \frac{(n+\frac{1}{2})\pi z}{L}.$$

C_n can be computed by (4.2)

$$C_n = \frac{2}{L}\int_0^L 1 \cdot \sin \frac{(n+\frac{1}{2})\pi z}{L} dx = \frac{4}{(2n+1)\pi}.$$

Therefore, we obtain

$$u(z,t) = \sum_{n=1}^{\infty} \frac{4}{(2n+1)\pi} e^{-\left(\frac{(n+\frac{1}{2})\pi}{L}\right)^2 Kt} \sin \frac{(n+\frac{1}{2})\pi z}{L}, \quad 0 < z < L, \quad t > 0.$$

This provides the complete solution to the homogeneous heat equation. □

4.1.2 The non-homogeneous case

Now we consider the time-independent non-homogeneous heat equation on the slab $0 < z < L$,

$$\begin{cases} u_t = K u_{zz} + r(z), & 0 < z < L, \quad t > 0, \\ u \cos\alpha - L u_z \sin\alpha = T_1, & z = 0, \quad t > 0, \\ u \cos\beta + L u_z \sin\beta = T_2, & z = L, \quad t > 0, \\ u = f(z), & 0 < z < L, \quad t = 0, \end{cases} \tag{4.4}$$

where $f(z), 0 < z < L$, is a piecewise smooth function, and $K > 0$ and $\alpha, \beta \in [0, \pi)$ are constants. The equation is non-homogeneous because there is an internal source $r(z)$. The boundary conditions are non-homogeneous because T_1 and T_2 are nonzero.

The approach to solving this equation is analogous to solving the ODE $f'' + f = e^x$. First, a particular solution $\hat{f} = \frac{1}{2}e^x$ is found by assuming $\hat{f} = Ce^x$ and determining $C = \frac{1}{2}$. Then, the general solution is obtained by considering $g = f - \hat{f}$, where g satisfies the homogeneous equation associated with the original ODE.

For the PDE, the method is similar. We first find a particular solution by considering time-independent solutions, which leads to the following definition.

Definition 4.2. *A solution $U(z,t)$ to the first three equations of* (4.4) *satisfying $\partial_t U(z,t) = 0$ is called a* <u>*time-independent solution*</u> *or* <u>*stationary solution*</u>. *This implies that $U(z,t) = U(z)$ satisfies the following ODE,*

$$\begin{cases} Ku_{zz} + r(z) = 0, & 0 < z < L \\ u\cos\alpha - Lu_z \sin\alpha = T_1, & z = 0, \\ u\cos\beta + Lu_z \sin\beta = T_2, & z = L \end{cases} \tag{4.5}$$

The PDE can be solved using the following theorem:

Theorem 4.3. *The solution to* (4.4) *can be obtained by the following steps.*

1. *Solve for the time-independent solution $U(z)$ using the following ODE.*

$$\begin{cases} KU_{zz}(z) + r(z) = 0, & 0 < z < L \\ U\cos\alpha - LU_z \sin\alpha = T_1, & z = 0, \\ U\cos\beta + LU_z \sin\beta = T_2, & z = L \end{cases} \tag{4.6}$$

2. *Define $v(z,t) = u(z,t) - U(z)$. One can compute that $v(z,t)$ satisfies the homogeneous PDE,*

$$\begin{cases} v_t = Kv_{zz}, & 0 < z < L, \quad t > 0, \\ v\cos\alpha - Lv_z \sin\alpha = 0, & z = 0, \quad t > 0, \\ v\cos\beta + Lv_z \sin\beta = 0, & z = L, \quad t > 0, \\ v(z,0) = f(z) - U(z), & 0 < z < L, \quad t = 0, \end{cases} \tag{4.7}$$

   Then solve $v(z,t)$ from the above equation.

3. *Finally, compute $u(z,t)$ as $u(z,t) = v(z,t) + U(z)$.*

Proof. The only non-trivial step is showing that $v(z,t)$ satisfies (4.7), which can be verified by a straightforward computation. □

Let us explain the above procedure using the following examples.

Example 4.2. Let us solve the following heat equation in a slab.

$$\begin{cases} u_t = Ku_{zz}, & t > 0, \quad 0 < z < L \\ u(0,t) = T_1, \ u(L,t) = T_2 & t > 0 \\ u(z,0) = 1, & 0 < z < L \end{cases} \tag{4.8}$$

Solution. We solve the equation following the steps listed in Theorem 4.3.

Step 1. For this problem, (4.6) can be simplified to

$$U''(z) = 0, \quad 0 < z < L, \quad U(0) = T_1, \quad U(L) = T_2$$

The general solution is given by $U(z) = A + Bz$, and the coefficients A, B are determined by the boundary conditions as $A = T_1, B = (T_2 - T_1)/L$.

Step 2. We then consider $v(z,t) = u(z,t) - U(z)$ which satisfies

$$\begin{cases} v_t = Kv_{zz}, \quad t > 0, \quad 0 < z < L \\ v(0,t) = v(L,t) = 0, \quad t > 0, \\ v(z,0) = 1 - U(z), \quad 0 < z < L \end{cases}$$

By separation of variable, we write $v(z,t) = Z(z)T(t)$, where $T'' + \lambda KT = 0$ and $Z'' + \lambda Z = 0$. The solutions of the first equation is $T(t) = e^{-\lambda Kt}$. Combining the second equation with the Boundary conditions in the PDE, we get the Sturm-Liouville problem, $Z'' + \lambda Z = 0$, $0 < z < L$, $Z(0) = Z(L) = 0$, for which we obtain

$$Z(z) = \sin\frac{n\pi z}{L}, \quad \lambda = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, \ldots$$

Therefore, the separated solutions are $v_n(z,t) = \sin\frac{n\pi z}{L} e^{-(n\pi/L)^2 Kt}$ and the general solution $v(z,t)$ is

$$v(z,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi z}{L} e^{-(n\pi/L)^2 Kt}$$

The coefficients B_n are determined by the initial condition as

$$\sum_{n=1}^{\infty} B_n \sin\frac{n\pi z}{L} = v(z,0) = 1 - U(z) = 1 - T_1 - \frac{T_2 - T_1}{L} z$$

Using the formula for the coefficients of sine Fourier series, $B_n = \frac{2}{L}\int_0^L (1 - U(z)) \sin(n\pi z/L) dz$. Combining with the integrals, $\int_0^L \sin(n\pi z/L) dz = L\left(1 - (-1)^n\right)/(n\pi)$ and $\int_0^L z \sin(n\pi z/L) dz = L^2(-1)^{n+1}/(n\pi)$, we can compute B_n and obtain

$$v(z,t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1 - T_1 - (-1)^n (1 - T_2)}{n} \sin\frac{n\pi z}{L} e^{-(n\pi/L)^2 Kt}$$

Step 3. By $u(z,t) = v(z,t) + U(z)$, we get

$$u(z,t) = T_1 + \frac{T_2 - T_1}{L} z + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1 - T_1 - (-1)^n (1 - T_2)}{n} \sin\frac{n\pi z}{L} e^{-(n\pi/L)^2 Kt}$$

This completes the solution. $\square$

Example 4.3. Let us solve the following heat equation in a slab.

$$\begin{cases} u_t = K u_{zz}, & t > 0, \quad 0 < z < L \\ u_z(0,t) = \Phi, \ u_z(L,t) = \Phi & t > 0 \\ u(z,0) = 1, & 0 < z < L \end{cases} \tag{4.9}$$

Solution. We solve the equation following the steps listed in Theorem 4.3.

Step 1. For this problem, (4.6) can be simplified to

$$U''(z) = 0, \quad 0 < z < L, \quad U'(0) = U'(L) = \Phi$$

The general solution is given by $U(z) = A + Bz$, and the coefficients B are determined by the boundary conditions as $B = \Phi$. For arbitrary choices of A, $U(z) = A + \Phi z$ is a time-independent solution. For the purposes of the computation below, we only need a specific time-independent solution, so we choose $A = 0$, giving $U(z) = \Phi z$.

Step 2. We then consider $v(z,t) = u(z,t) - U(z)$ which satisfies

$$\begin{cases} v_t = K v_{zz}, \quad t > 0, \quad 0 < z < L \\ v_z(0,t) = v_z(L,t) = 0, \quad t > 0, \\ v(z,0) = 1 - U(z), \quad 0 < z < L \end{cases}$$

By separation of variable, we write $v(z,t) = Z(z)T(t)$, where $T'' + \lambda K T = 0$ and $Z'' + \lambda Z = 0$. The solutions of the first equation is $T(t) = e^{-\lambda K t}$. Combining the second equation with the Boundary conditions in the PDE, we get the Sturm-Liouville problem, $Z'' + \lambda Z = 0$, $0 < z < L$, $Z'(0) = Z'(L) = 0$, for which we obtain

$$Z(z) = \cos\frac{n\pi z}{L}, \quad \lambda = \left(\frac{n\pi}{L}\right)^2, \quad n = 0, 1, 2, \dots$$

Therefore, the separated solutions are $v_n(z,t) = \cos\frac{n\pi z}{L} e^{-(n\pi/L)^2 K t}$ and the general solution $v(z,t)$ is

$$v(z,t) = \sum_{n=1}^{\infty} A_n \cos\frac{n\pi z}{L} e^{-(n\pi/L)^2 K t}$$

The coefficients A_n are determined by the initial condition as

$$\sum_{n=1}^{\infty} A_n \cos\frac{n\pi z}{L} = v(z,0) = 1 - U(z) = 1 - \Phi z$$

Using the formula for the coefficients of cosine Fourier series, $A_n = \frac{2}{L}\int_0^L (1 - U(z))\cos(n\pi z/L)dz$. Combining with the integrals, $\int_0^L z\cos(n\pi z/L)dz = ((-1)^n - 1)\, L^2/(n\pi)^2$ and $\int_0^L \cos^2(n\pi z/L)dz =$

$L/2$ for $n \neq 0$, we can compute A_n and obtain

$$A_0 = 1 - A - \frac{L\Phi}{2}, \quad A_n = 2L\Phi\frac{1-(-1)^n}{(n\pi)^2}$$

and

$$v(z,t) = 1 - \frac{L}{2}\Phi + \frac{2L\Phi}{\pi^2}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^2}\cos\frac{n\pi z}{L}e^{-(n\pi/L)^2Kt}.$$

Step 3. By $u(z,t) = v(z,t) + U(z)$, we get

$$u(z,t) = 1 + \left(z - \frac{L}{2}\right)\Phi + \frac{2L\Phi}{\pi^2}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^2}\cos\frac{n\pi z}{L}e^{-(n\pi/L)^2Kt}.$$

This completes the solution. □

4.1.3 Uniqueness theorem of heat equation

It is not immediately obvious that every solution to the heat equation can be expressed as a linear combination of separated solutions. The logic behind the method of separation of variables is that we first solve the equation using this method and then show that this solution is unique. Therefore, the solution obtained from separation of variables must be the only possible solution to the problem.

Theorem 4.4 (Uniqueness)**.** *The heat equation below has a unique solution.*

$$\begin{cases} u_t = Ku_{zz} + r(z), & 0 < z < L, \quad t > 0, \\ u(0,t) = T_1, & z = 0, \quad t > 0, \\ u(L,t) = T_2, & z = L, \quad t > 0, \\ u = f(z), & 0 < z < L, \quad t = 0, \end{cases} \tag{4.10}$$

Remark 4.5. The uniqueness theorem applies to a much broader class of PDEs, but for simplicity, we consider only this specific case.

Proof. Suppose that u_1 and u_2 are solutions. Define their difference $u = u_1 - u_2$. Then we have $u_t = Ku_{zz}$, $u(0,t) = u(L,t) = 0$, and $u(z,0) = 0$. By multiplying u on both sides, we have

$$u_t = Ku_{zz} \quad \Rightarrow \quad uu_t = Kuu_{zz}$$

By integrating both sides, we obtain

$$\frac{1}{2}\partial_t\int_0^L u^2dz = K\left(\frac{1}{2}\partial_z(u^2)\Big|_0^L - \int_0^L u_z^2dz\right) = Kuu_z|_0^L - K\int_0^L u_z^2dz$$

We introduce

$$w(t) = \frac{1}{2}\int_0^L u(z,t)^2 dz$$

We have

$$w'(t) = -K\int_0^L u_z(z,t)^2 dz$$

Thus we have

$$w(t) \geq 0, \quad w'(t) \leq 0$$

The initial condition $u(z,0) = 0$ implies

$$w(0) = 0.$$

Therefore $w(t) = 0$ $(t \geq 0)$. This then implies $u_1 = u_2$ $(t \geq 0,\ 0 \leq z \leq L)$. □

4.1.4 Physics of heat conduction

4.1.5 The case of time-dependent source

Now we consider the time-dependent non-homogeneous heat equation on the slab $0 < z < L$,

$$\begin{cases} u_t = Ku_{zz} + r(z,t), & 0 < z < L, \quad t > 0, \\ u(0,t)\cos\alpha - Lu_z(0,t)\sin\alpha = T_1(t), & z = 0, \quad t > 0, \\ u(L,t)\cos\beta + Lu_z(L,t)\sin\beta = T_2(t), & z = L, \quad t > 0, \\ u(z,0) = f(z), & 0 < z < L, \quad t = 0, \end{cases} \tag{4.11}$$

where $f(z), 0 < z < L$, is a piecewise smooth function, and $K > 0$ and $\alpha, \beta \in [0, \pi)$ are constants. The equation is non-homogeneous because there is an internal source $r(z)$. The boundary conditions are non-homogeneous because $T_1(t)$ and $T_2(t)$ are nonzero.

Now, let's assume temporarily that $T_1(t) = T_2(t) = 0$. Define the operator A by

$$A\phi = -\partial_{zz}\phi \tag{4.12}$$

with the domain defined as,

$$\left\{ \phi(x),\ x \in [a,b] \,\middle|\, \begin{array}{l} \phi(a)\cos\alpha - L\phi'(a)\sin\alpha = 0, \\ \phi(b)\cos\beta + L\phi'(b)\sin\beta = 0 \end{array} \right\} \tag{4.13}$$

With this, equation (4.11) becomes

$$\begin{cases} u_t = -KAu + r(\cdot,t), \\ u|_{t=0} = f. \end{cases} \tag{4.14}$$

Now, if we view $-KA$ as a matrix M, this equation resembles the vector-valued ODE,

$$\begin{cases} \dfrac{\mathrm{d}\vec{u}}{\mathrm{d}t} = M\vec{u} + \vec{r}(t), \\ \vec{u}|_{t=0} = \vec{f}. \end{cases} \tag{4.15}$$

A common technique to solve this is using an eigenvector expansion. Consider the eigenvalues and eigenfunctions $\{(\lambda_1, \vec{v}_1), \ldots, (\lambda_n, \vec{v}_n)\}$ of M. We can expand $\vec{u}(t)$, $\vec{r}(t)$ and $\vec{f}$ in terms of these eigenvectors. Then we get $\vec{u}(t) = \sum_{i=1}^{n} u_i(t)\, \vec{v}_i$ and similar equalities for $\vec{r}(t)$ and $\vec{f}$. Inserting this expansion of $\vec{u}(t)$ to the ODE (4.15), we get

$$\begin{cases} \displaystyle\sum_{i=1}^{n} (u_i'(t) - \lambda_i u_i(t) - r_i(t))\, \vec{v}_i = 0, \\ u_i(0) = \vec{f}_i. \end{cases} \tag{4.16}$$

Thus, we obtain the system of equations

$$\begin{cases} u_i'(t) = \lambda_i u_i(t) + r_i(t), \\ u_i(0) = f_i. \end{cases} \tag{4.17}$$

This is a scalar linear ODE and can be solved using standard ODE techniques.

For the PDE, the similar works method is similar in the case when $T_1(t) = T_2(t) = 0$. For the general case, we need to reduce to the case of $T_1(t) = T_2(t) = 0$ by subtracting a suitable function, as explained by the following theorem.

Theorem 4.6. *The solution to* (4.11) *can be obtained by the following steps.*

1. *Solve for the solution $U(z,t)$ of the following ODE. Note that this ODE is analogous to the equation for the time-independent solution,*

$$\begin{cases} U_{zz}(z,t) = 0, & 0 < z < L \\ U(0,t)\cos\alpha - LU_z(0,t)\sin\alpha = T_1(t), & z = 0, \\ U(L,t)\cos\beta + LU_z(L,t)\sin\beta = T_2(t), & z = L \end{cases} \tag{4.18}$$

2. *Define $v(z,t) = u(z,t) - U(z,t)$ and compute $R(z,t) = r(z,t) - U_t(z,t)$ and $F(z) = f(z) - U(z,0)$. One can compute that $v(z,t)$ satisfies the homogeneous PDE,*

$$\begin{cases} v_t = Kv_{zz} + R(z,t), & 0 < z < L, \quad t > 0, \\ v(0,t)\cos\alpha - Lv_z(0,t)\sin\alpha = 0, & z = 0, \quad t > 0, \\ v(L,t)\cos\beta + Lv_z(L,t)\sin\beta = 0, & z = L, \quad t > 0, \\ v(z,0) = F(z), & 0 < z < L, \quad t = 0, \end{cases} \tag{4.19}$$

3. *Compute the eigenvalues and eigenfunctions* $\{(\lambda_n, \phi_n)\}_n$ *of the following operator*

$$A\phi = -\partial_{zz}\phi, \tag{4.20}$$

$$Dom(A) = \left\{ \phi(x),\, x \in [a,b] \;\middle|\; \begin{array}{l} \phi(a)\cos\alpha - L\phi'(a)\sin\alpha = 0, \\ \phi(b)\cos\beta + L\phi'(b)\sin\beta = 0 \end{array} \right\}. \tag{4.21}$$

4. *Compute the expansion coefficients of* $R(z,t) = \sum_{n=1}^{\infty} R_n(t)\,\phi_n(z)$ *and* $F(z) = \sum_{n=1}^{\infty} F_n\,\phi_n$. *Insert these expansions together with* $v(z,t) = \sum_{n=1}^{\infty} v_n(t)\,\phi_n$ *into* (4.19). *Then we get the following equation for* $v_n(t)$

$$\begin{cases} v_n'(t) = \lambda_n v_n(t) + R_n(t), \\ v_n(0) = F_n. \end{cases} \tag{4.22}$$

   Solve $v_n(z,t)$ *from the above equation.*

5. *Finally, compute* $v(z,t)$ *as* $v(z,t) = \sum_{n=1}^{\infty} v_n(t)\,\phi_n$ *and compute* $u(z,t)$ *as* $u(z,t) = v(z,t) + U(z,t)$.

Proof. The only non-trivial step is showing that $v(z,t)$ satisfies (4.19) and $v_n(z,t)$ satisfies (4.22), which can be verified by a straightforward computation. **TODO: add the proof of the second** □

Let us explain the above procedure using the following examples.

Example 4.4. Let us solve the following heat equation in a slab.

$$\begin{cases} u_t = K u_{zz}, & t > 0, \quad 0 < z < L \\ u(0,t) = 0,\; u(L,t) = t & t > 0 \\ u(z,0) = 0, & 0 < z < L \end{cases} \tag{4.23}$$

Solution. We solve the equation following the steps listed in Theorem 4.6.

Step 1. For this problem, (4.18) can be simplified to

$$U''(z) = 0, \quad 0 < z < L, \quad U(0) = 0, \quad U(L) = t$$

The general solution is given by $U(z,t) = A(t) + B(t)z$, and the coefficients A, B are determined by the boundary conditions as $A(t) = 0, B(t) = t/L$. Therefore, $U(z,t) = \frac{zt}{L}$

Step 2. We then consider $v(z,t) = u(z,t) - U(z,t)$, which satisfies $v_t = Kv_{zz} - U_t$. v is a solution of the following boundary value problem

$$\begin{cases} v_t = K v_{zz} - \dfrac{z}{L}, \quad t > 0, \quad 0 < z < L \\ v(0,t) = v(L,t) = 0, \quad t > 0, \\ v(z,0) = 0, \quad 0 < z < L \end{cases} \tag{4.24}$$

Step 3. We now compute the eigenvalues and eigenfunctions of the operator A defined by

$$A\phi = -\partial_{zz}\phi, \qquad Dom(A) = \{\phi(z) : \phi(0) = 0, \quad \phi(L) = 0\}.$$

The eigenfunctions $\phi_n(z)$ and eigenvalues λ_n of this operator are given by

$$\phi_n(z) = \sin\left(\frac{n\pi z}{L}\right), \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, 3, \dots$$

Step 4. Next, we expand $R = -\frac{z}{L}$ and $F = 0$ in terms of the eigenfunctions $\phi_n(z)$.

$$R(z,t) = -\frac{z}{L} = \sum_{n=1}^{\infty} R_n(t)\phi_n(z), \qquad F(z) = 0 = \sum_{n=1}^{\infty} F_n\phi_n(z) \tag{4.25}$$

We now compute the coefficients $R_n(t)$ of $R(z,t)$ in terms of the eigenfunctions:

$$R_n(t) = -\frac{2}{L}\int_0^L \frac{z}{L}\sin\left(\frac{n\pi z}{L}\right) dz = \frac{2(-1)^n}{n\pi}.$$

Similarly, the coefficients of F_n can also be evaluated as

$$F_n = \frac{2}{L}\int_0^L 0 \cdot \sin\left(\frac{n\pi z}{L}\right) dz = 0.$$

We can also write the expansion of $v(z,t)$ in terms of these eigenfunctions,

$$v(z,t) = \sum_{n=1}^{\infty} v_n(t)\phi_n(z). \tag{4.26}$$

Inserting (4.25) and (4.26) into (4.24), we get the equation for $v_n(t)$

$$\begin{cases} v_n'(t) = -\lambda_n v_n(t) + R_n(t), \\ v_n(0) = F_n \end{cases}$$

This is a standard linear ODE, and its solution is given by (notice that $R_n(t) = \frac{2(-1)^n}{n\pi}$)

$$v_n(t) = \frac{2(-1)^n}{n\pi}\frac{1 - e^{-\lambda_n K t}}{\lambda_n K}.$$

Step 6. Finally, we compute $v(z,t)$ by summing over the eigenfunctions:

$$v(z,t) = \sum_{n=1}^{\infty} v_n(t)\sin\left(\frac{n\pi z}{L}\right) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\frac{1 - e^{-\lambda_n K t}}{\lambda_n K}\sin\frac{n\pi z}{L}.$$

The full solution for $u(z,t)$ is then given by

$$u(z,t) = v(z,t) + U(z,t) = \frac{tz}{L} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\frac{1 - e^{-\lambda_n K t}}{\lambda_n K}\sin\frac{n\pi z}{L}.$$

This is the complete solution to the given heat equation. $\square$

4.2 The wave equation

4.2.1 Physics of string vibration

4.2.2 The 1D case

TODO: add some explanation

Let c be a positive constant and $f(x), 0 < x < L$, be a piecewise smooth function. We consider the wave equation below.

$$\begin{cases} y_{tt} = c^2 y_{xx}, \quad t > 0, \quad 0 < x < L \\ y(0,t) = y(L,t) = 0 \\ y(x,0) = f(x) \\ y_t(x,0) = 0 \end{cases} \tag{4.27}$$

We present two approaches to solving this equation.

Separation of variables. By separation of variables $y(x,t) = \phi(x)T(t)$, we obtain

$$\phi'' + \lambda\phi = 0, \quad \phi(0) = \phi(L) = 0$$
$$T'' + \lambda c^2 T = 0, \quad T'(0) = 0$$

We can solve these equations as $\phi(x) = \phi_n(x)$, $T(t) = T_n(t)$, $n = 1, 2, \ldots$, where

$$\phi_n(x) = \sin\left(\sqrt{\lambda_n} x\right), \quad T_n(t) = \cos\left(\sqrt{\lambda_n} ct\right), \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2$$

Thus the solution is written as

$$y(x,t) = \sum_{n=1}^{\infty} B_n \cos\frac{n\pi ct}{L} \sin\frac{n\pi x}{L}$$

The coefficients B_n are determined by $y(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} = f(x)$,

$$B_n = \frac{2}{L}\int_0^L f(x)\phi_n(x)dx$$

Eigenfunction expansion. We compute the eigenvalues and eigenfunctions of the operator A defined by

$$A\phi = -\partial_{xx}\phi, \qquad Dom(A) = \{\phi(x) : \phi(0) = 0, \quad \phi(L) = 0\}.$$

The eigenfunctions $\phi_n(z)$ and eigenvalues λ_n of this operator are given by

$$\phi_n(x) = \sin\left(\frac{n\pi x}{L}\right), \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, 3, \ldots$$

Expanding $f(x)$ and $y(x,t)$ in terms of these eigenfunctions, we get

$$y(x,t) = \sum_{n=1}^{\infty} y_n(t) \sin\left(\frac{n\pi x}{L}\right), \qquad f(x) = \sum_{n=1}^{\infty} f_n \sin\left(\frac{n\pi x}{L}\right). \tag{4.28}$$

Inserting into (4.27), we gathered

$$y_n''(t) = -\lambda_n y_n(t), \quad y_n(0) = f_n.$$

Solving this ODE and inserting $y_n(t)$ into (4.28), we get $y(x,t)$.

4.2.3 The 2D case

Just like the Fourier series for 1D functions, we can express a 2D function $f(x,y)$ in terms of trigonometric functions. For the ease of notations, we will only consider the Fourier sine series,

Theorem 4.7 (2D Fourier series)**.** *Taking sine series as an example, given an arbitrary piecewise smooth function $f(x,y)$ on $(x,y) \in [0,L_1] \times [0,L_2]$. Then we have the following expansion*

$$f(x,y) = \sum_{n=1}^{\infty} B_{mn} \sin\frac{m\pi x}{L_1} \sin\frac{n\pi y}{L_2}, \tag{4.29}$$

Proof. The proof is beyond the scope of this course. □

We also have the orthogonality relation and formula of coefficients for 2D Fourier serise

Theorem 4.8 (Orthogonality relations)**.** *For $m,n = 1,2,\cdots$, we have*

$$\int_0^{L_2}\int_0^{L_1} \sin\frac{m\pi x}{L_1} \sin\frac{n\pi y}{L_2} \sin\frac{m'\pi x}{L_1} \sin\frac{n'\pi y}{L_2} dxdy = \frac{L_1L_2}{4}\delta_{mm'}\delta_{nn'}$$

and the B_{mn} coefficients in (4.29) can be computed using the following formula

$$B_{mn} = \frac{4}{L_1L_2}\int_0^{L_2}\int_0^{L_1} \sin\frac{m\pi x}{L_1} \sin\frac{n\pi y}{L_2} f(x,y)dxdy. \tag{4.30}$$

Proof. Recall the orthogonality relation of 1D Fourier sine series (2.26)

$$\int_0^L \sin\frac{n\pi x}{L} \sin\frac{m\pi x}{L} dx = \frac{L}{2}\delta_{nm}$$

This property tells us that sine functions with different indices are orthogonal over the interval $[0,L]$, and the result is zero unless $n = m$, in which case the integral evaluates to $\frac{L}{2}$.

To complete the proof of the theorem, we'll use the orthogonality properties of the sine functions in each dimension and then compute the 2D Fourier coefficients B_{mn}.

Consider the double integral,

$$I = \int_0^{L_2} \int_0^{L_1} \sin\frac{m\pi x}{L_1} \sin\frac{n\pi y}{L_2} \sin\frac{m'\pi x}{L_1} \sin\frac{n'\pi y}{L_2}\, dx\, dy.$$

This can be separated into two integrals, one for x and one for y,

$$I = \left(\int_0^{L_1} \sin\frac{m\pi x}{L_1} \sin\frac{m'\pi x}{L_1} dx\right)\left(\int_0^{L_2} \sin\frac{n\pi y}{L_2} \sin\frac{n'\pi y}{L_2} dy\right).$$

Using the 1D orthogonality relation for both integrals, we have,

$$\int_0^{L_1} \sin\frac{m\pi x}{L_1} \sin\frac{m'\pi x}{L_1} dx = \frac{L_1}{2}\delta_{mm'}, \qquad \int_0^{L_2} \sin\frac{n\pi y}{L_2} \sin\frac{n'\pi y}{L_2} dy = \frac{L_2}{2}\delta_{nn'}.$$

Thus, the original double integral becomes

$$I = \frac{L_1}{2}\delta_{mm'} \cdot \frac{L_2}{2}\delta_{nn'} = \frac{L_1 L_2}{4}\delta_{mm'}\delta_{nn'}.$$

which proves the orthogonality relation.

The formula for the coefficients B_{mn} follows from the orthogonality relation by the computation below

$$\begin{aligned}
&\int_0^{L_2} \int_0^{L_1} \sin\frac{m\pi x}{L_1} \sin\frac{n\pi y}{L_2} dxdy \\
&= \int_0^{L_2} \int_0^{L_1} \sin\frac{m\pi x}{L_1} \sin\frac{n\pi y}{L_2} \sum_{m'=1}^{\infty}\sum_{n'=1}^{\infty} B_{m'n'}\omega_{m'n'} \sin\frac{m'\pi x}{L_1} \sin\frac{n'\pi y}{L_2} dxdy \\
&= \sum_{m'=1}^{\infty}\sum_{n'=1}^{\infty} B_{m'n'}\omega_{m'n'} \int_0^{L_2} \int_0^{L_1} \sin\frac{m\pi x}{L_1} \sin\frac{n\pi y}{L_2} \sin\frac{m'\pi x}{L_1} \sin\frac{n'\pi y}{L_2} dxdy \\
&= \sum_{m'=1}^{\infty}\sum_{n'=1}^{\infty} B_{m'n'}\omega_{m'n'} \frac{L_1 L_2}{4}\delta_{mm'}\delta_{nn'} = B_{mn}\omega_{mn}\frac{L_1 L_2}{4}
\end{aligned}$$

which implies the formula of the coefficients. □

We will use separation of variable to solve the 2D wave equation. The following lemma (which is an analog of Lemma 1.16) is important

Lemma 4.9. *$f(x) = g(y) + h(z)$ implies that $f(x) = c$, $g(y) = c_1$ and $h(z) = c_2$ and the constants satisfy $c = c_1 + c_2$.*

Proof. $f(x) = g(y) + h(z) \Rightarrow f'(x) = \partial_x(g(y)) = 0 \Rightarrow f(x) = const.$ □

Example 4.5. Solve the following wave equation.

$$\begin{cases}
u_{tt} = c^2\left(u_{xx} + u_{yy}\right), & 0 < x < L_1, \quad 0 < y < L_2, \quad t > 0, \\
u = 0, & x = 0, \quad x = L_1, \quad y = 0, \quad y = L_2, \quad t > 0, \\
u(x,y,0) = 0, \quad u_t(x,y,0) = 1, & 0 < x < L_1, \quad 0 < y < L_2.
\end{cases}$$

Solution. We assume the following form

$$u(x,y,t) = X(x)Y(y)T(t)$$

We obtain $T''/T(t) = c^2\left[X''/X(x) + Y''/Y(y)\right]$. Lemma 4.9 implies that $T''/T(t)$, $X''/X(x)$ and $Y''/Y(y)$ are all constants. By introducing separation constants λ, μ_1, and μ_2, we obtain

$$T'' + \lambda c^2 T = 0, \quad X'' + \mu_1 X = 0, \quad Y'' + \mu_2 Y = 0$$

where $T''/T = -\lambda c^2$, $X''/X = -\mu_1$, and $Y''/Y = -\mu_2$. Note that

$$\lambda = \mu_1 + \mu_2$$

There are many cases: $\lambda > 0$, $\lambda = 0$, $\lambda < 0$, $\mu_1 > 0$, $\mu_1 = 0$, $\mu_1 < 0$, $\mu_2 > 0$, $\mu_2 = 0$, and $\mu_2 < 0$. Let us consider the boundary conditions. For $X(x)$, we have

$$X'' + \mu_1 X = 0, \quad X(0) = X\left(L_1\right) = 0$$

Nontrivial solutions are possible only when $\mu_1 > 0$. We obtain

$$X(x) = \sin\frac{m\pi x}{L_1}, \quad \mu_1 = \left(\frac{m\pi}{L_1}\right)^2, \quad m = 1, 2, \ldots$$

Note that we omitted a constant factor.

Similarly for $Y(y)$, we have

$$Y(y) = \sin\frac{n\pi x}{L_2}, \quad \mu_2 = \left(\frac{n\pi}{L_2}\right)^2, \quad n = 1, 2, \ldots$$

Because $\lambda > 0$ ($\lambda = \mu_1 + \mu_2$), $T(t)$ is obtained as

$$T(t) = A\cos(\sqrt{\lambda}ct) + B\sin(\sqrt{\lambda}ct)$$

($u(x,y,0) = 0$ implies $T(0) = 0$ and we can easily obtain $A = 0$, but let's keep both terms here, so that we can see how A and B are determined in general.) The general solution is thus given by

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \sin\frac{m\pi x}{L_1} \sin\frac{n\pi y}{L_2}\left(A_{mn}\cos\left(\omega_{mn}t\right) + B_{mn}\sin\left(\omega_{mn}t\right)\right)$$

where

$$\omega_{mn} = c\sqrt{\left(\frac{m\pi}{L_1}\right)^2 + \left(\frac{n\pi}{L_2}\right)^2}$$

Now we consider the initial conditions. $u(x, y, 0) = 0$ implies $A_{mn} = 0$, and $u_t(x, y, 0) = 1$ implies

$$1 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn} \omega_{mn} \sin \frac{m\pi x}{L_1} \sin \frac{n\pi y}{L_2}$$

Using the formula of coefficients (4.30), we obtain

$$\begin{aligned} B_{mn}\omega_{mn} &= \frac{4}{L_1 L_2} \int_0^{L_1} \sin \frac{m\pi x}{L_1} dx \int_0^{L_2} \sin \frac{n\pi y}{L_2} dy \\ &= \frac{4}{L_1 L_2} \frac{L_1}{m\pi} (1 - \cos(m\pi)) \frac{L_2}{n\pi} (1 - \cos(n\pi)) \\ &= \frac{4}{\pi^2} \frac{[1 - (-1)^m] [1 - (-1)^n]}{mn} \end{aligned}$$

Finally, the solution is

$$u(x, y, t) = \frac{4}{\pi^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{[1 - (-1)^m] [1 - (-1)^n]}{mn\omega_{mn}} \sin \frac{m\pi x}{L_1} \sin \frac{n\pi y}{L_2} \sin(\omega_{mn} t)$$

The computation is thus finished. □

5 PDEs in cylindrical coordinates

5.1 Laplace equation in cylindrical coordinates

In this section, we will discuss how to solve the Laplace equation $\Delta u = 0$ in cylindrical coordinates. The operator Δ, often refered to as Laplacian, is defined by the following

$$\Delta = \partial_{xx} + \partial_{yy} \quad \text{(in 2D)}, \qquad \Delta = \partial_{xx} + \partial_{yy} + \partial_{zz} \quad \text{(in 3D)} \tag{5.1}$$

We consider the cylindrical domain $\{(x, y, z) : R_1^2 \le x^2 + y^2 \le R_2^2, , z \in [-L, L]\}$, for which cylindrical coordinates (ρ, φ, z) are particularly useful. The transformation from cartesian coordinates (x, y, z) to cylindrical coordinates can simplify the domain to $\{(x, y, z) : R_1 \le \rho \le R_2, \varphi \in [0, 2\pi], z \in [-L, L]\}$.

The relationship between Cartesian coordinates (x, y, z) and cylindrical coordinates (ρ, φ, z) is given by

$$x = \rho \cos(\varphi), \quad y = \rho \sin(\varphi), \quad z = z. \tag{5.2}$$

Additionally, we have the inverse relationships,

$$\rho = \sqrt{x^2 + y^2}, \quad \varphi = \arctan\left(\frac{y}{x}\right), \quad z = z. \tag{5.3}$$

Here is a picture of spherical coordinates. **TODO:**

To convert PDEs from Cartesian to cylindrical coordinates, we need to express the partial derivatives with respect to x, y and z in terms of ρ, φ, and z.

Proposition 5.1. *The first-order partial derivatives transform as,*

$$\begin{aligned}\partial_x &= \cos(\varphi)\partial_\rho - \frac{\sin(\varphi)}{\rho}\partial_\varphi \\ \partial_y &= \sin(\varphi)\partial_\rho + \frac{\cos(\varphi)}{\rho}\partial_\varphi\end{aligned} \tag{5.4}$$

Proof. Using the chain rule, we have

$$\begin{aligned}f_x &= f_\rho \rho_x - f_\varphi \varphi_x \\ f_y &= f_\rho \rho_y + f_\varphi \varphi_y\end{aligned} \tag{5.5}$$

By (5.3), we can compute that

$$\rho_x = \cos(\varphi), \quad \rho_y = \sin(\varphi), \quad \varphi_x = -\frac{\sin(\varphi)}{\rho}, \quad \rho_x = \frac{\cos(\varphi)}{\rho} \tag{5.6}$$

Inserting into (5.5), we get

$$\begin{aligned}f_x &= \cos(\varphi) f_\rho - \frac{\sin(\varphi)}{\rho} f_\varphi \\ f_y &= \sin(\varphi) f_\rho + \frac{\cos(\varphi)}{\rho} f_\varphi \\ \frac{\partial f}{\partial z} &= \frac{\partial f}{\partial z}\end{aligned} \tag{5.7}$$

This implies (5.4). □

Proposition 5.2. *The Laplacian transforms as*

$$\Delta f = f_{xx} + f_{yy} = \frac{1}{\rho}(\rho f_\rho)_\rho + \frac{1}{\rho^2} f_{\varphi\varphi}. \tag{5.8}$$

Proof. By (5.4), we get We can compute that

$$\begin{aligned}f_{xx} =& \left(\cos(\varphi)\partial_\rho - \frac{\sin(\varphi)}{\rho}\partial_\varphi\right)\left(\cos(\varphi)\partial_\rho - \frac{\sin(\varphi)}{\rho}\partial_\varphi\right) f, \\ =& \cos^2\varphi\, f_{\rho\rho} + \frac{2\cos\varphi\sin\varphi}{\rho^2} f_\varphi - \frac{2\sin\varphi\cos\varphi}{\rho} f_{\rho\varphi} + \frac{\sin^2\varphi}{\rho} f_\rho + \frac{\sin^2\varphi}{\rho^2} f_{\varphi\varphi}.\end{aligned} \tag{5.9}$$

$$\begin{aligned}f_{yy} =& \left(\cos(\varphi)\partial_\rho - \frac{\sin(\varphi)}{\rho}\partial_\varphi\right)\left(\cos(\varphi)\partial_\rho - \frac{\sin(\varphi)}{\rho}\partial_\varphi\right) f, \\ =& \sin^2\varphi\, f_{\rho\rho} - \frac{2\sin\varphi\cos\varphi}{\rho^2} f_\varphi + \frac{2\sin\varphi\cos\varphi}{\rho} f_{\rho\varphi} + \frac{\cos^2\varphi}{\rho} f_\rho + \frac{\cos^2\varphi}{\rho^2} f_{\varphi\varphi}.\end{aligned} \tag{5.10}$$

Summing the above equations, we get

$$\Delta f = \partial_{xx} f + \partial_{yy} f = \frac{1}{\rho}(\rho f_\rho)_\rho + \frac{1}{\rho^2} f_{\varphi\varphi}. \tag{5.11}$$

This completes the proof. $\square$

This expression is crucial for solving PDEs such as the Laplace equation or the wave equation in cylindrical coordinates.

5.2 Separation of Variables for the 2D Laplace equation

Let's consider solving the equation on a cylindrical domain

$$\Delta u = 0, \quad R_1 \leq \sqrt{x^2 + y^2} \leq R_2 \tag{5.12}$$

where Δ is the Laplacian operator.

5.2.1 The case of $R_1 = 0$ and $R_2 = R$

Let us find separated solutions of Laplace's equation $\Delta u = 0$ on the domain $0 \leq \sqrt{x^2 + y^2} \leq R$. In terms of cylindrical coordinates, the domain becomes $0 \leq \rho \leq R$, $-\pi \leq \varphi \leq \pi$ and the Laplacian becomes $\frac{1}{\rho}(\rho u_\rho)_\rho + \frac{1}{\rho^2} u_{\varphi\varphi}$. Assume that u is smooth and is independent of z. The boundary value problem should be written as

$$\begin{cases} \Delta u = \dfrac{1}{\rho}(\rho u_\rho)_\rho + \dfrac{1}{\rho^2} u_{\varphi\varphi} = 0, & 0 \leq \rho \leq R, \quad -\pi \leq \varphi \leq \pi \\ u(R, \varphi) = u_2(\varphi), & -\pi \leq \varphi \leq \pi. \end{cases} \tag{5.13}$$

By plugging $u(\rho, \varphi) = R(\rho)\Phi(\varphi)$ into $\Delta u = 0$, we obtain

$$0 = \frac{1}{\rho}(\rho u_\rho)_\rho + \frac{1}{\rho^2} u_{\varphi\varphi} = \frac{1}{\rho}(\rho R')'\Phi + \frac{1}{\rho^2} R\Phi''$$

Dividing by $R\Phi$ and multiplying by ρ^2, we have

$$\rho \frac{(\rho R')'}{R} = -\frac{\Phi''}{\Phi}$$

By introducing the separation constant λ, we have

$$\begin{cases} \Phi'' + \lambda \Phi = 0, \quad \Phi(-\pi) = \Phi(\pi), \quad \Phi'(-\pi) = \Phi'(\pi), \\ \rho(\rho R')' - \lambda R = 0. \end{cases} \tag{5.14}$$

Solve the eigenvalue and eigenfunction λ and Φ. We get that $\lambda = m^2$ and $\Phi_m(\varphi) = A_m \cos m\varphi + B_m \sin m\varphi$ $(m = 0, 1, 2, \ldots)$.

Given the expression $\lambda = m^2$ for λ, we can also solve $R(\rho)$. First inserting $\lambda = m^2$ and the change of variable $\rho = e^s$, the second equation becomes $R''(s) - m^2 R(s) = 0$. The general solution is given by $R(s) = Ae^{ms} + Be^{-ms}$. Transforming back to ρ variable, we get that $R_m(\rho) = C_m\rho^m + D_m\rho^{-m}$. The case of $m = 0$ is special, where the solution can be easily solved as $R_0(\rho) = C_0 + D_0 \ln \rho$.

Therefore, the separated solutions are obtained as

$$u(\rho, \varphi) = \begin{cases} \left(C_m\rho^m + D_m\rho^{-m}\right)\left(A_m \cos m\varphi + B_m \sin m\varphi\right), & m = 1, 2, \ldots, \\ C_0 + D_0 \ln \rho, & m = 0 \end{cases} \tag{5.15}$$

If $D_m \neq 0$ for some $m \geq 0$, we have $|u| \to \infty$ as $\rho \to 0$ and u is not smooth. Therefore,

$$u(\rho, \varphi) = \rho^m \left(A_m \cos m\varphi + B_m \sin m\varphi\right), \quad m = 0, 1, 2, \ldots$$

where we take $C_m = 1$ as we can merge C_m with A_m and B_m. For the case of $m = 0$, $u(\rho, \varphi)$ is constant according to the above expression.

Therefore, we can write the general solution

$$u(\rho, \varphi) = \sum_{m=0}^{\infty} \rho^m \left(A_m \cos m\varphi + B_m \sin m\varphi\right), \quad m = 0, 1, 2, \ldots$$

Finally, to decide the coefficients A_m and B_m, we use the boundary condition in (5.13).

$$u_2(\varphi) = u(R, \varphi) = \sum_{m=0}^{\infty} R^m \left(A_m \cos m\varphi + B_m \sin m\varphi\right), \quad m = 0, 1, 2, \ldots$$

Therefore, $A_m R^m$ and $B_m R^m$ are Fourier coefficients of the function $u_2(\varphi)$, which can be computed using the formula of coefficients (2.9).

5.2.2 The general case

Let us find separated solutions of Laplace's equation $\Delta u = 0$ on the domain $R_1 \leq \sqrt{x^2 + y^2} \leq R_2$. In terms of cylindrical coordinates, the domain becomes $R_1 \leq \rho \leq R_2$, $-\pi \leq \varphi \leq \pi$ and the Laplacian becomes $\frac{1}{\rho}(\rho u_\rho)_\rho + \frac{1}{\rho^2} u_{\varphi\varphi}$. Assume that u is smooth and is independent of z. The boundary value problem should be written as

$$\begin{cases} \Delta u = \dfrac{1}{\rho}(\rho u_\rho)_\rho + \dfrac{1}{\rho^2} u_{\varphi\varphi} = 0, & R_1 \leq \rho \leq R_2, \quad -\pi \leq \varphi \leq \pi \\ u(R_2, \varphi) = u_2(\varphi), \quad u(R_2, \varphi) = u_2(\varphi), & -\pi \leq \varphi \leq \pi. \end{cases} \tag{5.16}$$

By plugging $u(\rho,\varphi) = R(\rho)\Phi(\varphi)$ into $\Delta u = 0$, we obtain

$$0 = \frac{1}{\rho}(\rho u_\rho)_\rho + \frac{1}{\rho^2}u_{\varphi\varphi} = \frac{1}{\rho}(\rho R')'\Phi + \frac{1}{\rho^2}R\Phi''$$

Dividing by $R\Phi$ and multiplying by ρ^2, we have

$$\rho\frac{(\rho R')'}{R} = -\frac{\Phi''}{\Phi}$$

By introducing the separation constant λ, we have

$$\begin{cases} \Phi'' + \lambda\Phi = 0, \quad \Phi(-\pi) = \Phi(\pi), \quad \Phi'(-\pi) = \Phi'(\pi), \\ \rho(\rho R')' - \lambda R = 0. \end{cases} \tag{5.17}$$

Solve the eigenvalue and eigenfunction λ and Φ. We get that $\lambda = m^2$ and $\Phi_m(\varphi) = A_m \cos m\varphi + B_m \sin m\varphi$ $(m = 0, 1, 2, \ldots)$.

Given the expression $\lambda = m^2$ for λ, we can also solve $R(\rho)$. First inserting $\lambda = m^2$ and the change of variable $\rho = e^s$, the second equation becomes $R''(s) - m^2R(s) = 0$. The general solution is given by $R(s) = Ae^{ms} + Be^{-ms}$. Transforming back to ρ variable, we get that $R_m(\rho) = C_m\rho^m + D_m\rho^{-m}$. The case of $m = 0$ is special, where the solution can be easily solved as $R_0(\rho) = C_0 + D_0 \ln \rho$.

Therefore, the separated solutions are obtained as

$$u(\rho,\varphi) = \begin{cases} \left(C_m\rho^m + D_m\rho^{-m}\right)\left(A_m \cos m\varphi + B_m \sin m\varphi\right), & m = 1, 2, \ldots, \\ C_0 + D_0 \ln \rho, & m = 0 \end{cases} \tag{5.18}$$

Everything up to this point is the same as the case of $R_1 = 0$, $R_2 = R$. For general annulus domain, we do not have the condition that $|u| < \infty$ as $\rho \to 0$. Therefore, D_m are not necessarily vanishing and the general solution can be written as

$$\begin{aligned} u(\rho,\varphi) &= \sum_{m=0}^{\infty} \left(C_m\rho^m + D_m\rho^{-m}\right)\left(A_m \cos m\varphi + B_m \sin m\varphi\right) \\ &= \sum_{m=0}^{\infty} \left(A_m^{(1)}\rho^m + A_m^{(2)}\rho^{-m}\right)\cos m\varphi + \left(B_m^{(1)}\rho^m + B_m^{(2)}\rho^{-m}\right)\sin m\varphi \end{aligned}$$

where in the second line, we introduced the new constants $A_m^{(1)} = C_mA_m$, $A_m^{(2)} = D_mA_m$, $B_m^{(1)} = C_mB_m$ and $B_m^{(2)} = D_mB_m$.

Finally, to decide the coefficients $A_m^{(1)}$, $B_m^{(1)}$, $A_m^{(2)}$ and $B_m^{(2)}$, we use the boundary condition in (5.13).

$$u_1(\varphi) = u(R_1,\varphi) = \sum_{m=0}^{\infty} \left(A_m^{(1)}R_1^m + A_m^{(2)}R_1^{-m}\right)\cos m\varphi + \left(B_m^{(1)}R_1^m + B_m^{(2)}R_1^{-m}\right)\sin m\varphi$$

$$u_2(\varphi) = u(R_2, \varphi) = \sum_{m=0}^{\infty} \left(A_m^{(1)} R_2^m + A_m^{(2)} R_2^{-m}\right) \cos m\varphi + \left(B_m^{(1)} R_2^m + B_m^{(2)} R_2^{-m}\right) \sin m\varphi$$

Therefore, $\alpha_m^{(1)} = A_m^{(1)} R_1^m + A_m^{(2)} R_1^{-m}$ and $\beta_m^{(1)} = B_m^{(1)} R_1^m + B_m^{(2)} R_1^{-m}$ are Fourier coefficients of the function $u_1(\varphi)$; $\alpha_m^{(2)} = A_m^{(1)} R_2^m + A_m^{(2)} R_2^{-m}$ and $\beta_m^{(2)} = B_m^{(1)} R_2^m + B_m^{(2)} R_2^{-m}$ are Fourier coefficients of the function $u_2(\varphi)$, which can be computed using the formula of coefficients (2.9). After solving $\alpha_m^{(1)}$, $\beta_m^{(1)}$, $\alpha_m^{(2)}$, $\beta_m^{(2)}$, we can solve $A_m^{(1)}$, $B_m^{(1)}$, $A_m^{(2)}$ and $B_m^{(2)}$ from them.

5.3 The Bessel function

When applying separation of variables to more general PDEs involving the Laplacian operator, a new class of functions, known as Bessel functions, will emerge.

5.3.1 Eigenfunctions of the Laplacian

Let us find separated solutions more general PDE on a disk $-\Delta u = \lambda u$ on the domain $0 \le \sqrt{x^2 + y^2} \le R$. In terms of cylindrical coordinates, the domain becomes $0 \le \rho \le R$, $-\pi \le \varphi \le \pi$ and the Laplacian becomes $\frac{1}{\rho}(\rho u_\rho)_\rho + \frac{1}{\rho^2} u_{\varphi\varphi}$. Assume that u is smooth and is independent of z. The boundary value problem should be written as

$$\begin{cases} \Delta u = \dfrac{1}{\rho}(\rho u_\rho)_\rho + \dfrac{1}{\rho^2} u_{\varphi\varphi} = -\lambda u, & 0 \le \rho \le R, \quad -\pi \le \varphi \le \pi \\ u(R, \varphi) = u_2(\varphi), & -\pi \le \varphi \le \pi. \end{cases} \tag{5.19}$$

By plugging $u(\rho, \varphi) = R(\rho)\,\Phi(\varphi)$ into $\Delta u = -\lambda u$, we obtain

$$-\lambda R(\rho)\,\Phi(\varphi) = \frac{1}{\rho}(\rho R')'\Phi + \frac{1}{\rho^2} R\Phi''$$

Dividing by $R\Phi$ and multiplying by ρ^2, we have

$$\frac{\rho(\rho R')' + \lambda \rho^2 R}{R} = -\frac{\Phi''}{\Phi} = m^2$$

where we introduced the separation constant m^2, we have

$$\begin{cases} \Phi'' + m^2 \Phi = 0, \quad \Phi(-\pi) = \Phi(\pi), \quad \Phi'(-\pi) = \Phi'(\pi), \\ \rho(\rho R')' + (\lambda \rho^2 - m^2) R = 0. \end{cases} \tag{5.20}$$

Solve the eigenvalue and eigenfunction λ and Φ. We get that $\lambda = m^2$ and $\Phi_m(\varphi) = A_m \cos m\varphi + B_m \sin m\varphi$ $(m = 0, 1, 2, \ldots)$.

There is no elementary expression for the solution of the equation of $R(\rho)$. However, if we take $R(\rho) = J_m(\sqrt{\lambda}\rho)$, then the equation can be simplified to

$$\rho(\rho J_m')' + (\rho^2 - m^2)J_m = 0. \tag{5.21}$$

The solution $J_m(\rho)$ to the above equation satisfying $J_m(0) < \infty$ is refered to as the Bessel function.

Therefore, the separated solutions are obtained as

$$u(\rho, \varphi) = J_m(\sqrt{\lambda}\rho)\left(A_m \cos m\varphi + B_m \sin m\varphi\right), \quad m = 1, 2, \ldots. \tag{5.22}$$

The general solution can be written as

$$u(\rho, \varphi) = \sum_{m=0}^{\infty} J_m(\sqrt{\lambda}\rho)\left(A_m \cos m\varphi + B_m \sin m\varphi\right) \tag{5.23}$$

Finally, to decide the coefficients A_m and B_m, we use the boundary condition in (5.13).

$$u_2(\varphi) = u(R, \varphi) = \sum_{m=0}^{\infty} J_m(\sqrt{\lambda}R)\left(A_m \cos m\varphi + B_m \sin m\varphi\right), \quad m = 0, 1, 2, \ldots \tag{5.24}$$

Therefore, $A_m J_m(\sqrt{\lambda}R)$ and $B_m J_m(\sqrt{\lambda}R)$ are Fourier coefficients of the function $u_2(\varphi)$, which can be computed using the formula of coefficients (2.9).

5.3.2 The Bessel function

Definition 5.3 (The Bessel function)**.** *The following ODE is refered to as the Bessel equation*

$$x(xJ_m'(x))' + (x^2 - m^2)J_m(x) = 0. \tag{5.25}$$

The solution $J_m(x)$ to the above equation satisfying $J_m(0) < \infty$ is refered to as the Bessel function.

Remark 5.4. The Bessel function defined in this manner has an ambiguity in constants, as any solution of the form $CJ_m(x)$ also satisfies the above equation. This ambiguity is resolved by defining $J_m(x)$ using (5.26) below.

Lemma 5.5. *The Bessel equation is equivalent to the following equation.*

$$J_m''(x) + \frac{1}{x}J_m'(x) + \left(1 - \frac{m^2}{x^2}\right)J_m(x) = 0.$$

Proof. This follows from a simple computation. □

Proposition 5.6 (Integral formula for the Bessel function)**.** *We have (up to a constant)*

$$J_m(x) = \frac{1}{2\pi i^m}\int_{-\pi}^{\pi} e^{ix\cos\theta}e^{-im\theta}d\theta, \quad m = 0, 1, 2, \ldots. \tag{5.26}$$

Proof. Since $e^{iy} = e^{i\rho\cos\varphi}$ is a solution to the equation $-\Delta u = u$, by the expression for the general solution (5.23) (taking $\lambda = 1$), we know that

$$e^{i\rho\cos\varphi} = \sum_{m=0}^{\infty} J_m(\rho)\left(A_m \cos m\varphi + B_m \sin m\varphi\right) \tag{5.27}$$

Transforming to complex Fourier series, we get

$$e^{i\rho\cos\varphi} = \sum_{m=0}^{\infty} J_m(\rho)\left(\alpha_m e^{im\varphi} + \beta_m e^{-im\varphi}\right) \tag{5.28}$$

Therefore, $\alpha_m J_m(\rho)$ is the complex Fourier coefficients of the function $e^{i\rho\cos\varphi}$. Using the formula of coefficients, we know that (2.16)

$$\alpha_m J_m(\rho) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\rho\cos\varphi} e^{-im\varphi} d\varphi \tag{5.29}$$

Therefore, $J_m(\rho)$ is proportional to $\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\rho\cos\varphi}e^{-im\varphi}d\varphi$. If we choose the constant $\alpha_m = i^m$, then $J_m(\rho)$ is a real value function **TODO: add a proof to this**. Under this choice of constants, we prove (5.26). $\square$

TODO: add a picture

Lemma 5.7. *We have*

$$J_m(0) = \begin{cases} 1, & \text{if } m = 0, \\ 0, & \text{if } m \neq 0. \end{cases} \tag{5.30}$$

Proof. Let $x = 0$ in (5.26).

$$\begin{aligned} J_m(0) =& \frac{1}{2\pi i^m}\int_{-\pi}^{\pi} e^{i0\cdot\cos\varphi} e^{-im\varphi}\, d\varphi \\ =& \frac{1}{2\pi i^m}\int_{-\pi}^{\pi} e^{-im\varphi}\, d\varphi \end{aligned} \tag{5.31}$$

Evaluating this integral gives

$$J_m(0) = \frac{1}{2\pi i^m}\int_{-\pi}^{\pi} e^{-im\varphi}\, d\varphi = \begin{cases} 1 & \text{if } m = 0 \\ 0 & \text{if } m \neq 0 \end{cases} \tag{5.32}$$

This proof uses the orthogonality of Fourier series (2.18) by taking $n = 0$ and $L = 2\pi$. $\square$

5.3.3 The recurrence formula

Proposition 5.8. *The following recurrence formula holds.*

$$J_m(x) = \frac{x}{2m}\left[J_{m-1}(x) + J_{m+1}(x)\right], \quad m = 1, 2, \ldots. \tag{5.33}$$

$$J_m'(x) = \frac{1}{2}\left[J_{m-1}(x) - J_{m+1}(x)\right], \quad m = 0, 1, 2, \ldots. \tag{5.34}$$

$$\frac{d}{dx}\left[x^m J_m(x)\right] = x^m J_{m-1}(x), \quad m = 1, 2, \cdots \tag{5.35}$$

$$\frac{d}{dx}\left[x^{-m} J_m(x)\right] = -x^{-m} J_{m+1}(x), \quad m = 0, 1, 2, \cdots \tag{5.36}$$

Proof. Let us start with the proof of (5.33)

$$\begin{aligned}
J_{m-1}(x) + J_{m+1}(x) =& \frac{1}{2\pi i^m}\left(i\int_{-\pi}^{\pi} e^{ix\cos\varphi} e^{-i(m-1)\varphi} d\varphi - i\int_{-\pi}^{\pi} e^{ix\cos\varphi} e^{-i(m+1)\varphi} d\varphi\right)\\
=& \frac{1}{2\pi i^m}\int_{-\pi}^{\pi} e^{ix\cos\varphi} e^{-im\varphi} \underbrace{\left(ie^{i\varphi} - ie^{-i\varphi}\right)}_{2(\cos\varphi)'} d\varphi\\
=& \frac{2}{2\pi i^m}\int_{-\pi}^{\pi} \underbrace{e^{ix\cos\varphi}(\cos\varphi)'}_{\frac{1}{ix}(e^{ix\cos\varphi})'} e^{-im\varphi} d\varphi\\
=& \frac{2}{ix}\frac{1}{2\pi i^m}\int_{-\pi}^{\pi} \left(e^{ix\cos\varphi}\right)' e^{-im\varphi} d\varphi.
\end{aligned} \tag{5.37}$$

Apply the integration by parts,

$$\begin{aligned}
J_{m-1}(x) + J_{m+1}(x) =& \frac{2}{ix}\frac{1}{2\pi i^m}\left(\underbrace{\left[\left(e^{ix\cos\varphi} e^{-im\varphi}\right)\right]_{-\pi}^{\pi}}_{=0 \text{ by periodicity}} - \int_{-\pi}^{\pi} e^{ix\cos\varphi}\left(e^{-im\varphi}\right)' d\varphi\right)\\
=& \frac{2m}{x}\frac{1}{2\pi i^m}\int_{-\pi}^{\pi} e^{ix\cos\varphi} e^{-im\varphi} d\varphi = \frac{2m}{x} J_m(x).
\end{aligned} \tag{5.38}$$

Here is the proof of (5.34)

$$\begin{aligned}
J_m'(x) =& \frac{1}{2\pi i^m}\int_{-\pi}^{\pi} \frac{d}{dx}\left(e^{ix\cos\varphi}\right) e^{-im\varphi} d\varphi\\
=& \frac{1}{2\pi i^m}\int_{-\pi}^{\pi} e^{ix\cos\varphi} i\cos\varphi\, e^{-im\varphi} d\varphi\\
=& \frac{1}{2\pi i^m}\int_{-\pi}^{\pi} e^{ix\cos\varphi} i\frac{e^{i\varphi} + e^{-i\varphi}}{2} e^{-im\varphi} d\varphi\\
=& \frac{1}{2}\left(\frac{1}{2\pi i^{m-1}}\int_{-\pi}^{\pi} e^{ix\cos\varphi} e^{-i(m-1)\varphi} d\varphi - \frac{1}{2\pi i^{m+1}}\int_{-\pi}^{\pi} e^{ix\cos\varphi} e^{-i(m+1)\varphi} d\varphi\right)\\
=& \frac{1}{2}(J_{m-1}(x) - J_{m+1}(x)).
\end{aligned} \tag{5.39}$$

Here is the proof of (5.35). Consider (5.33) $\times\, mx^{m-1}$ + (5.34) $\times\, x^m$,

$$\begin{aligned}
(x^m J_m)' =& mx^{m-1} J_m + x^m J_m' \\
=& mx^{m-1}\frac{x}{2m}(J_{m-1} + J_{m+1}) + \frac{x^m}{2}(J_{m-1} - J_{m+1}) \\
=& \frac{x^m}{2}(J_{m-1} + J_{m+1} + J_{m-1} - J_{m+1}) \\
=& x^m J_{m-1}.
\end{aligned} \tag{5.40}$$

Here is the proof of (5.36). Consider $-$(5.33) $\times\, mx^{-m-1}$ + (5.34) $\times\, x^{-m}$,

$$\begin{aligned}
(x^{-m} J_m)' =& -mx^{-m-1} J_m + x^{-m} J_m' \\
=& -mx^{-m-1}\frac{x}{2m}(J_{m-1} + J_{m+1}) + \frac{x^{-m}}{2}(J_{m-1} - J_{m+1}) \\
=& \frac{x^{-m}}{2}(-J_{m-1} - J_{m+1} + J_{m-1} - J_{m+1}) \\
=& -x^{-m} J_{m+1}.
\end{aligned} \tag{5.41}$$

Therefore, the proof has been finished. □

5.3.4 The orthogonality relation

Consider the following Sturm-Liouville eigenvalues problem,

$$\begin{cases} x(xR')' + (\lambda x^2 - m^2)R = 0, \\ R(0) < \infty,\ R(1)\cos\beta + R'(1)\sin\beta = 0. \end{cases} \tag{5.42}$$

We know that $R(x) = J_m(\sqrt{\lambda}x)$ is a solution to the above equation. Using the boundary condition, we know that $R(1)\cos\beta + R'(1)\sin\beta = 0 \Rightarrow J_m(\sqrt{\lambda})\cos\beta + \sqrt{\lambda}J_m'(\sqrt{\lambda})\sin\beta = 0$. Therefore, if $\left\{x_n^{(m)}\right\}$ are the nonnegative solutions of the equation

$$J_m\left(x_n^{(m)}\right)\cos\beta + x_n^{(m)} J_m'\left(x_n^{(m)}\right)\sin\beta = 0. \tag{5.43}$$

Then $\lambda_n = \left(x_n^{(m)}\right)^2$. Applying the Sturm-Liouville theorem Theorem 3.15, we get the following proposition.

Proposition 5.9. *Let* $\left\{x_n^{(m)}\right\}$ *be the nonnegative solutions of the equation*

$$J_m\left(x_n^{(m)}\right)\cos\beta + x_n^{(m)} J_m'\left(x_n^{(m)}\right)\sin\beta = 0, \tag{5.44}$$

where $m \geq 0$ *and* $0 \leq \beta \leq \pi/2$.

Then we have the following orthogonality relations.

$$\int_0^1 J_m\left(xx_{n_1}^{(m)}\right) J_m\left(xx_{n_2}^{(m)}\right) xdx = 0,\ n_1 \neq n_2 \tag{5.45}$$

$$\begin{cases} \displaystyle\int_0^1 J_m\left(xx_n^{(m)}\right)^2 xdx = \frac{1}{2} J_{m+1}\left(x_n^{(m)}\right)^2, & \text{if } \beta = 0, \\ \displaystyle\int_0^1 J_m\left(xx_n^{(m)}\right)^2 xdx = \frac{x_n^2 - m^2 + \cot^2\beta}{2x_n^2} J_m\left(x_n^{(m)}\right)^2, & \text{if } 0 < \beta \leq \frac{\pi}{2}. \end{cases} \tag{5.46}$$

Proof. For the ease of notation, let us drop the superscript in $x_n^{(m)}$. We know that $J_m(xx_n)$ are solutions of the following equation.

$$x(xR')' + (x_n^2x^2 - m^2)R = 0. \tag{5.47}$$

If we view m as fixed, then the above equation is a Sturm-Liouville eigenvalue problem with eigenvalues $\lambda = x_n^2$, $s(x) = \rho(x) = x$ and $q(x) = m^2$. From the Sturm-Liouville theory (Theorem 3.15), the first equation (5.45) (orthogonality) holds. For the second and third equations (5.46), we multiply (5.47) by $2R'$.

$$2xR'(xR')' + \left(x_n^2x^2 - m^2\right) 2RR' = 0$$

We can rewrite this as

$$\left[(xR')^2\right]' + \left(x_n^2x^2 - m^2\right)\left(R^2\right)' = 0$$

By integrating both sides and using integration by parts, we get

$$(xR')^2\Big|_{x=1} - (xR')^2\Big|_{x=0} + \left(x_n^2x^2 - m^2\right) R^2\Big|_0^1 - \int_0^1 2x_n^2 xR^2 dx = 0.$$

Note that $R(x) = J_m(xx_n)$ and $R'(x) = x_m J_m'(xx_n)$. Hence $R(0) = J_m(0) = 0$ $(m = 1, 2, \cdots)$ by (5.30). We obtain

$$\left[x_n J_m'(x_n)\right]^2 + \left(x_n^2 - m^2\right) J_m(x_n)^2 - 2x_n^2 \int_0^1 J_m(xx_n)^2 xdx = 0.$$

Therefore, when $\beta = 0$ (in other words, $J_m(x_n) = 0$), we have

$$\int_0^1 J_m(xx_n)^2 xdx = \frac{x_n^2\left[J_m'(x_n)\right]^2}{2x_n^2} = \frac{\left[\frac{m}{x_n}J_m(x_n) - J_{m+1}(x_n)\right]^2}{2} = \frac{J_{m+1}(x_n)^2}{2}$$

where we applied the recurrence formula (5.36). (Notice that $\frac{d}{dx}\left[x^{-m}J_m(x)\right] = -x^{-m}J_{m+1}(x) \Leftrightarrow J_m'(x) = \frac{m}{x}J_m(x) - J_{m+1}(x)$).

When $0 < \beta \leq \pi/2$, or in other words, $J_m(x_n)\cos\beta + x_nJ_m'(x_n)\sin\beta = 0$ (which implies that $J_m'(x_n) = -\frac{1}{x_n}J_m(x_n)\cot\beta$), we have

$$
\begin{aligned}
\int_0^1 J_m\left(x x_n\right)^2 x d x &= \frac{x_n^2\left[J_m^{\prime}(x_n)\right]^2+\left(x_n^2-m^2\right) J_m(x_n)^2}{2 x_n^2} \\
&= \frac{x_n^2\left[\frac{-1}{x_n} J_m(x_n) \cot \beta\right]^2+\left(x_n^2-m^2\right) J_m(x_n)^2}{2 x_n^2} \\
&= \frac{\left(x_n^2-m^2+\cot^2 \beta\right) J_m(x_n)^2}{2 x_n^2}
\end{aligned}
$$

This completes the proof of (5.46). □

Now we introduce the notion of Fourier–Bessel series

Definition 5.10 (Fourier-Bessel series)**.** *Let us consider the expansion of a piecewise smooth function $f(x)$, $0 < x < 1$, in a series of the form*

$$
f(x)=\sum_{n=1}^{\infty} A_n J_m\left(x x_n^{(m)}\right), \quad 0<x<1,
$$

where $\{x_n\}$ are the nonnegative solutions of $J_m(x)\cos\beta + xJ_m'(x)\sin\beta = 0$. This is called a Fourier–Bessel series.

We have the following theorem.

Theorem 5.11. *Let $m \geq 0$, $0 \leq \beta \leq \pi/2$, and let $\{x_n : n \geq 1\}$ be the nonnegative solutions of (5.44). If $f(x)$, defined on $0 < x < 1$, is a piecewise smooth function, then $f(x)$ can be expanded in terms of Fourier–Bessel series*

$$
f(x)=\sum_{n=1}^{\infty} A_n J_m\left(x x_n^{(m)}\right), \quad 0<x<1,
$$

where the coefficients A_n can be computed by

$$
A_n=\frac{\int_0^1 f(x) J_m\left(x x_n^{(m)}\right) x d x}{\int_0^1 J_m\left(x x_n^{(m)}\right)^2 x d x}, \quad n=1,2, \cdots
$$

Moreover, the series $\sum_{n=1}^{\infty} A_n J_m\left(x x_n^{(m)}\right)$ converges for each $x \in [0,1]$ to $\frac{1}{2}[f(x+0)+f(x-0)]$ for $0 < x < 1$.

Proof. **TODO:** By multiplying (3.10) by $J_m\left(x x_n^{(m)}\right)$ and integrating both sides, we obtain □

Example 5.1. Let us compute the Fourier–Bessel series of the function $f(x) = 1$, $0 < x < 1$, where $m = 0$ and $\beta = 0$. We have $1 = \sum_{n=1}^{\infty} A_n J_0\left(xx_n^{(0)}\right)$, where $J_0(x_n) = 0$ and

$$A_n = \frac{\int_0^1 J_m\left(xx_n^{(0)}\right) xdx}{\int_0^1 J_m\left(xx_n^{(0)}\right)^2 xdx}, \quad n = 1, 2, \cdots$$

First applying a change of variable $t = xx_n$, then applying (5.35) and (5.46), we obtain

$$A_n = \frac{\frac{1}{x_n^2}\int_0^{x_n} tJ_0(t)dt}{\int_0^1 J_0\left(xx_n\right)^2 xdx} \overset{(5.35)}{=} \frac{\frac{1}{x_1^2} tJ_1(t)\Big|_0^{x_n}}{\int_0^1 J_0\left(xx_n\right)^2 xdx} \overset{(5.46)}{=} \frac{\frac{1}{x_n}J_1(x_n)}{\frac{1}{2}J_1(x_n)^2} = \frac{2}{x_n J_1(x_n)}.$$

where we drop the superscript in $x_n^{(0)}$.

Therefore,

$$1 = 2\sum_{n=1}^{\infty} \frac{J_0\left(xx_n^{(0)}\right)}{x_n^{(0)} J_1\left(x_n^{(0)}\right)}.$$

5.4 The Taylor series

Finally, we find the Taylor series solution of the Bessel's equation $x(xJ_m'(x))' + (x^2 - m^2)J_m(x) = 0$ (5.25). Let $J_m(x) = \sum_{n=0}^{\infty} a_n x^{n+\gamma}$ ($a_0 \neq 0$, $\gamma \geq 0$) be a solution to (5.25). Inserting this expansion into (5.25), we obtain

$$\left(\gamma^2 - m^2\right) a_0 x^{\gamma} + \left((1+\gamma)^2 - m^2\right) a_1 x^{\gamma+1} + \sum_{n=2}^{\infty} \left[\left((n+\gamma)^2 - m^2\right) a_n + a_{n-2}\right] x^{n+\gamma} = 0.$$

Hence,

$$\gamma = m, \quad a_1 = 0, \quad a_n = \frac{-a_{n-2}}{n(n+2m)}(n \geq 2)$$

From this, we obtain

$$J_m(x) = a_0 x^m \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2(2+2m)4(4+2m)\cdots 2n(2n+2m)}\right]$$

Let us choose $a_0 = 2^m/m!$. Then, we have

$$J_m(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+m}}{2^{m+2n}(m+n)!n!}$$

5.5 The vibrating drumhead

Let us consider small transverse vibrations of a circular membrane.

$$\begin{cases} u_{tt} = c^2 \Delta u = c^2 \left(u_{\rho\rho} + \dfrac{1}{\rho} u_\rho + \dfrac{1}{\rho^2} u_{\varphi\varphi} \right), & 0 \le \rho < a, \quad t > 0, \\ u(a, \varphi, t) = 0, & t > 0, \\ u(\rho, \varphi, 0) = 1, \quad u_t(\rho, \varphi, 0) = 0, & 0 \le \rho < a. \end{cases} \tag{5.48}$$

We solve the equation in terms of Bessel's function.

We first look for separated solutions in the form

$$u(\rho, \varphi, t) = R(\rho)\Phi(\varphi)T(t).$$

Inserting into (5.48), we get

$$\frac{1}{c^2}\frac{T''}{T} = \frac{R'' + \frac{1}{\rho}R'}{R} + \frac{1}{\rho^2}\frac{\Phi_{\varphi\varphi}}{\Phi}.$$

The left (resp. right) hand side is a function of t (resp. ρ, φ). Therefore, they must be a constant independent of all of three variables.

$$\frac{1}{c^2}\frac{T''}{T} = -\lambda, \quad \frac{R'' + \frac{1}{\rho}R'}{R} + \frac{1}{\rho^2}\frac{\Phi_{\varphi\varphi}}{\Phi} = \lambda.$$

The second equation can be transformed to

$$\frac{\rho^2 R'' + \rho R'}{R} - \lambda\rho^2 = -\frac{\Phi_{\varphi\varphi}}{\Phi}$$

Again, the left (resp. right) hand side is a function of ρ (resp. φ). By introducing the separation constant μ, we have

$$\frac{1}{c^2}\frac{T''}{T} = -\lambda, \quad \frac{\rho^2 R'' + \rho R'}{R} - \lambda\rho^2 = \mu, \quad -\frac{\Phi_{\varphi\varphi}}{\Phi} = \mu.$$

From $u(a, \varphi, t) = 0$ and $u_t(\rho, \varphi, 0) = 0$, we know that $R(a) = 0$ respectively. Then we obtain

$$\begin{aligned} &\Phi''(\varphi) + \mu\Phi(\varphi) = 0, \quad \Phi(-\pi) = \Phi(\pi), \quad \Phi'(-\pi) = \Phi'(\pi) \\ &R''(\rho) + \tfrac{1}{\rho}R'(\rho) + \left(\lambda - \tfrac{\mu}{\rho^2}\right) R(\rho) = 0, \quad R(a) = 0 \\ &T''(t) + \lambda c^2 T(t) = 0. \end{aligned} \tag{5.49}$$

In the first equation of (5.49), nontrivial solutions are obtained when $\sqrt{\mu} = m = 1, 2, \cdots$,

$$\Phi(\varphi) = A\cos m\varphi + B\sin m\varphi, \quad m = 0, 1, 2, \cdots$$

With $\mu = m^2$ $(m = 0, 1, 2, \cdots)$ in the second equation of (5.49), we obtain $R(\rho) = J_m(\rho\sqrt{\lambda})$. For $R(a) = 0$, we obtain $\sqrt{\lambda} = x_n^{(m)}/a$ where $x_n^{(m)}$ are the nonnegative roots of $J_m(x) = 0$. With $\lambda = \left(x_n^{(m)}/a\right)^2$ in the third equation of (5.49), we get the solution $T(t) = \left(\bar{A}\cos\frac{ctx_n^{(m)}}{a} + \bar{B}\sin\frac{ctx_n^{(m)}}{a}\right)$. The separated solutions are obtained as

$$u(\rho,\varphi,t) = J_m\left(\frac{\rho x_n^{(m)}}{a}\right)(A\cos m\varphi + B\sin m\varphi)\left(\bar{A}\cos\frac{ctx_n^{(m)}}{a} + \bar{B}\sin\frac{ctx_n^{(m)}}{a}\right). \tag{5.50}$$

We will now take the initial conditions into account. The general solution is given as a linear combination (superposition) of (5.50). To satisfy $u_t(\rho,\varphi,0) = 0$, we set $\bar{B} = 0$. Now the general solution is written as

$$u(\rho,\varphi,t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left\{\left[A_{mn}J_m\left(\frac{\rho x_n^{(m)}}{a}\right)\right]\cos m\varphi + \left[B_{mn}J_m\left(\frac{\rho x_n^{(m)}}{a}\right)\right]\sin m\varphi\right\}\cos\frac{ctx_n^{(m)}}{a} \tag{5.51}$$

After matching the general solution with the last boundary condition $u(\rho,\varphi,0) = 1$, we obtain that

$$1 = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left(A_{mn}J_m\left(\frac{\rho x_n^{(m)}}{a}\right)\cos m\varphi + B_{mn}J_m\left(\frac{\rho x_n^{(m)}}{a}\right)\sin m\varphi\right) \tag{5.52}$$

To compute the coefficients A_{mn} (resp. B_{mn}), we multiply both sides by $J_m\left(\frac{\rho x_n^{(m)}}{a}\right)\cos m\varphi$ (resp. $J_m\left(\frac{\rho x_n^{(m)}}{a}\right)\sin m\varphi$) and then take integral. By $\frac{1}{\pi}\int_{-\pi}^{\pi}\cos(mx)dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\sin(mx)dx = 0$ (if $m \neq 0$), we know that $A_{mn} = B_{mn} = 0$ if $m \neq 0$. Therefore, we take $m = 0$ in (5.51) and (5.52),

$$u(\rho,\varphi,t) = \sum_{n=1}^{\infty}A_{0n}J_0\left(\frac{\rho x_n^{(0)}}{a}\right)\cos\frac{ctx_n^{(0)}}{a}.$$

$$1 = \sum_{n=1}^{\infty}A_{0n}J_0\left(\frac{\rho x_n^{(0)}}{a}\right)$$

In Example 5.1, we calculated the Fourier-Bessel expansion $1 = 2\sum_{n=1}^{\infty}\frac{J_0\left(xx_n^{(0)}\right)}{x_n^{(0)}J_1\left(x_n^{(0)}\right)}$. By comparison, we obtain $A_{0n} = \frac{2}{x_n^{(0)}J_1\left(x_n^{(0)}\right)}$, which implies that

$$u(\rho,\varphi,t) = \sum_{n=1}^{\infty}\frac{2}{x_n^{(0)}J_1\left(x_n^{(0)}\right)}J_0\left(\frac{\rho x_n^{(0)}}{a}\right)\cos\frac{ctx_n^{(0)}}{a}.$$

5.6 Heat flow in the infinite cylinder

Let us consider the heat transfer in the infinite cylinder $0 \leq \rho < a$. We will solve the heat equation in polar coordinates

$$
\begin{cases}
u_t = K\Delta u, & t>0, \quad 0\le\rho<a, \quad -\pi\le\varphi\le\pi,\\
u\left(a,\varphi,t\right) = T_1, & t>0, \quad -\pi\le\varphi\le\pi,\\
u(\rho,\varphi,0) = T_2, & 0\le\rho<a, \quad -\pi\le\varphi\le\pi,
\end{cases}
$$

where T_1 and T_2 are positive constants.

We first the steady-state solution. Let us try $U(\rho)$ because the boundary conditions is independent of φ.

$$
K\Delta U = K\left(\frac{\partial^2 U}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial U}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2 U}{\partial\varphi^2}\right) = K\left(U'' + \frac{1}{\rho}U'\right) = 0.
$$

The general solution is obtained as

$$
U(\rho) = A + B\ln\rho.
$$

Let us exclude the second term and set $B=0$ (otherwise $U(0)$ diverges). To satisfy $U\left(a\right)=T_1$, we choose $A=T_1$. We thus obtain

$$
U(\rho) = T_1.
$$

Define $v(\rho,\varphi,t) = u(\rho,\varphi,t) - U(\rho)$. We have

$$
\begin{cases}
v_t = K\Delta v, & t>0, \quad 0\le\rho<a, \quad -\pi\le\varphi\le\pi,\\
v\left(a,\varphi,t\right) = 0, & t>0, \quad -\pi\le\varphi\le\pi,\\
v(\rho,\varphi,0) = T_2 - T_1, & 0\le\rho<a, \quad -\pi\le\varphi\le\pi,
\end{cases}
$$

Using separation of variables with $u(\rho,\varphi,t) = R(\rho)\Phi(\varphi)T(t)$, we obtain Inserting into (5.48), we get

$$
\frac{1}{K}\frac{T'}{T} = \frac{R'' + \frac{1}{\rho}R'}{R} + \frac{1}{\rho^2}\frac{\Phi_{\varphi\varphi}}{\Phi}.
$$

The left (resp. right) hand side is a function of t (resp. ρ, φ). Therefore, they must be a constant independent of all of three variables.

$$
\frac{1}{K}\frac{T'}{T} = -\lambda, \quad \frac{R'' + \frac{1}{\rho}R'}{R} + \frac{1}{\rho^2}\frac{\Phi_{\varphi\varphi}}{\Phi} = \lambda.
$$

The second equation can be transformed to

$$
\frac{\rho^2 R'' + \rho R'}{R} - \lambda\rho^2 = -\frac{\Phi_{\varphi\varphi}}{\Phi}
$$

Again, the left (resp. right) hand side is a function of ρ (resp. φ). By introducing the separation constant μ, we have

$$
\frac{1}{K}\frac{T'}{T} = -\lambda, \quad \frac{\rho^2 R'' + \rho R'}{R} - \lambda\rho^2 = \mu, \quad -\frac{\Phi_{\varphi\varphi}}{\Phi} = \mu.
$$

From $u(a, \varphi, t) = 0$ and $u_t(\rho, \varphi, 0) = 0$, we know that $R(a) = 0$ respectively. Then we obtain

$$\begin{aligned}
&\Phi''(\varphi) + \mu\Phi(\varphi) = 0, \quad \Phi(-\pi) = \Phi(\pi), \quad \Phi'(-\pi) = \Phi'(\pi) \\
&R''(\rho) + \tfrac{1}{\rho}R'(\rho) + \left(\lambda - \tfrac{\mu}{\rho^2}\right)R(\rho) = 0, \quad R(a) = 0 \\
&T'(t) + \lambda K T(t) = 0.
\end{aligned} \tag{5.53}$$

In the first equation of (5.49), nontrivial solutions are obtained when $\sqrt{\mu} = m = 1, 2, \cdots$,

$$\Phi(\varphi) = A\cos m\varphi + B\sin m\varphi, \quad m = 0, 1, 2, \cdots$$

With $\mu = m^2$ $(m = 0, 1, 2, \cdots)$ in the second equation of (5.49), we obtain $R(\rho) = J_m(\rho\sqrt{\lambda})$. For $R(a) = 0$, we obtain $\sqrt{\lambda} = x_n^{(m)}/a$ where $x_n^{(m)}$ are the nonnegative roots of $J_m(x) = 0$. With $\lambda = \left(x_n^{(m)}/a\right)^2$ in the third equation of (5.49), we get the solution $T(t) = \exp\left[-\frac{\left(x_n^{(m)}\right)^2 Kt}{a^2}\right]$. The separated solutions are obtained as

$$v(\rho, \varphi, t) = J_m\left(\frac{\rho x_n^{(m)}}{a}\right)(A\cos m\varphi + B\sin m\varphi)\exp\left(-\frac{\left(x_n^{(m)}\right)^2 Kt}{a^2}\right). \tag{5.54}$$

The general solutions are obtained as

$$v(\rho, \varphi, t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m\left(\frac{\rho x_n^{(m)}}{a}\right)(A_{mn}\cos m\varphi + B_{mn}\sin m\varphi)\exp\left(-\frac{\left(x_n^{(m)}\right)^2 Kt}{a^2}\right). \tag{5.55}$$

After matching the general solution with the last boundary condition $u(\rho, \varphi, 0) = T_2 - T_1$, we obtain that

$$T_2 - T_1 = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left(A_{mn}J_m\left(\frac{\rho x_n^{(m)}}{a}\right)\cos m\varphi + B_{mn}J_m\left(\frac{\rho x_n^{(m)}}{a}\right)\sin m\varphi\right) \tag{5.56}$$

To compute the coefficients A_{mn} (resp. B_{mn}), we multiply both sides by $J_m\left(\frac{\rho x_n^{(m)}}{a}\right)\cos m\varphi$ (resp. $J_m\left(\frac{\rho x_n^{(m)}}{a}\right)\sin m\varphi$) and then take integral. By $\frac{1}{\pi}\int_{-\pi}^{\pi}\cos(mx)dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\sin(mx)dx = 0$ (if $m \neq 0$), we know that $A_{mn} = B_{mn} = 0$ if $m \neq 0$. Therefore, we take $m = 0$ in (5.55) and (5.56),

$$u(\rho, \varphi, t) = \sum_{n=1}^{\infty} A_{0n}J_0\left(\frac{\rho x_n^{(0)}}{a}\right)\exp\left(-\frac{\left(x_n^{(0)}\right)^2 Kt}{a^2}\right).$$

$$T_2 - T_1 = \sum_{n=1}^{\infty} A_{0n}J_0\left(\frac{\rho x_n^{(0)}}{a}\right)$$

Noting that $1 = 2\sum_{n=1}^{\infty}\frac{J_0\left(x x_n^{(m)}\right)}{x_n J_1(x_n)}$ $\left(0 < x < 1, J_0\left(x_n^{(0)}\right) = 0\right)$, we get $A_{0n} = \frac{2(T_2 - T_1)}{x_n^{(0)}J_1\left(x_n^{(0)}\right)}$, which implies that

$$u(\rho, \varphi, t) = T_1 + \sum_{n=1}^{\infty}\frac{2(T_2 - T_1)}{x_n^{(0)}J_1\left(x_n^{(0)}\right)}J_0\left(\frac{\rho x_n^{(0)}}{a}\right)\exp\left(-\frac{\left(x_n^{(0)}\right)^2 Kt}{a^2}\right).$$

6 PDEs in spherical coordinates

6.1 Laplace equation in spherical coordinates

In this section, we will discuss how to solve the Laplace equation $\Delta u = 0$ in spherical coordinates. The Laplacian operator Δ is defined by the following

$$\Delta = \partial_{xx} + \partial_{yy} + \partial_{zz} \quad \text{(in 3D)} \tag{6.1}$$

We consider the spherical domain $\{(x, y, z) : x^2 + y^2 + z^2 \leq R^2\}$, for which spherical coordinates (r, θ, φ) are particularly useful. The transformation from cartesian coordinates (x, y, z) to spherical coordinates can simplify the domain to $\{(x, y, z) : r \leq R\}$.

The relationship between Cartesian coordinates (x, y, z) and spherical coordinates (r, θ, φ) is given by

$$x = r \sin\theta \cos\varphi, \quad y = r \sin\theta \sin\varphi, \quad z = r\cos\theta. \tag{6.2}$$

Additionally, we have the inverse relationships,

$$r = \sqrt{x^2 + y^2 + z^2}, \quad \theta = \arctan\left(\frac{z}{\sqrt{x^2 + y^2}}\right), \quad \varphi = \arctan\left(\frac{y}{x}\right). \tag{6.3}$$

To convert PDEs from Cartesian to spherical coordinates, we need to express the partial derivatives with respect to x, y and z in terms of r, θ and φ.

Here is a picture of spherical coordinates. **TODO:**

Proposition 6.1. *The Laplacian transforms as in the spherical coordinates*

$$\Delta f = f_{xx} + f_{yy} + f_{zz} = \frac{1}{r^2}\left(r^2 u_r\right)_r + \frac{1}{r^2 \sin\theta}\left(\sin\theta\, u_\theta\right)_\theta + \frac{1}{r^2 \sin^2\theta} u_{\varphi\varphi}. \tag{6.4}$$

Proof. Let $\rho = \sqrt{x^2 + y^2}$, then we know that we have

$$z = r\cos\theta, \quad \rho = r\sin\theta,$$

which is the same as the transformation rule of the polar coordinates (6.2).

By (5.8) to $u_{zz} + u_{\rho\rho}$, we know that

$$u_{zz} + u_{\rho\rho} = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta}. \tag{6.5}$$

For x, y, we also have $x = \rho\cos\varphi$, $y = \rho\sin\varphi$. Apply (5.8) again to $u_{xx} + u_{yy}$, we get

$$\begin{aligned} u_{xx} + u_{yy} + u_{zz} =& u_{\rho\rho} + \frac{1}{\rho} u_\rho + \frac{1}{\rho^2} u_{\theta\theta} + u_{zz} \\ =& u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} + \frac{1}{\rho} u_\rho + \frac{1}{\rho^2} u_{\varphi\varphi}. \end{aligned} \tag{6.6}$$

We note that

$$r=\sqrt{\rho^2+z^2} \quad \Rightarrow \quad r_\rho=\frac{\rho}{r}=\sin\theta,$$

$$\tan\theta=\frac{\rho}{z} \quad \Rightarrow \quad \frac{d\tan\theta}{d\theta}\frac{\partial\theta}{\partial\rho}=\frac{1}{z} \quad \Rightarrow \quad \frac{1}{\cos^2\theta}\frac{\partial\theta}{\partial\rho}=\frac{1}{r\cos\theta} \quad \Rightarrow \quad \theta_\rho=\frac{\cos\theta}{r}.$$

Therefore, we obtain that

$$u_\rho=r_\rho u_r+\theta_\rho u_\theta=\sin\theta\, u_r+\frac{\cos\theta}{r}u_\theta$$

Inserting into (6.6), we obtain

$$\begin{aligned} u_{xx}+u_{yy}+u_{zz}=&u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}+\frac{1}{r\sin\theta}\left(\sin\theta\, u_r+\frac{\cos\theta}{r}u_\theta\right)+\frac{1}{r^2\sin^2\theta}u_{\varphi\varphi} \\ =&\frac{1}{r^2}\left(r^2u_r\right)_r+\frac{1}{r^2\sin\theta}\left(\sin\theta\, u_\theta\right)_\theta+\frac{1}{r^2\sin^2\theta}u_{\varphi\varphi}. \end{aligned} \tag{6.7}$$

which completes the proof of the proposition. □

This expression is crucial for solving the Laplace equation in spherical coordinates.

6.2 Separation of Variables for the 3D Laplace equation

Let's consider solving the equation on a spherical domain

$$\Delta u=0, \quad \sqrt{x^2+y^2+z^2}\le R \tag{6.8}$$

where Δ is the Laplacian operator.

We assume that the solution is rotational symmetric with respect to the z-axis, i.e. it is independent of the φ variable, $u=u(r,\theta)$. Under this assumption, the Laplace equation becomes

$$\Delta u=\frac{1}{r^2}\left(r^2u_r\right)_r+\frac{1}{r^2\sin\theta}\left(\sin\theta\, u_\theta\right)_\theta=0, \tag{6.9}$$

Let us find separated solutions of Laplace's equation $\Delta u=0$ on the domain $0\le\sqrt{x^2+y^2+z^2}\le R$. In terms of spherical coordinates, the domain becomes $0\le r\le R$, $0\le\theta\le\pi$. The boundary value problem should be written as

$$\begin{cases} \Delta u=\frac{1}{r^2}\left(r^2u_r\right)_r+\frac{1}{r^2\sin\theta}\left(\sin\theta\, u_\theta\right)_\theta=0, & 0\le r\le R, \quad 0\le\theta\le\pi \\ u(R,\theta)=G(\theta), & -\pi\le\theta\le\pi. \end{cases} \tag{6.10}$$

By plugging $u(r,\theta)=R(r)\Theta(\theta)$ into $\Delta u=0$, we obtain

$$0=\frac{1}{r^2}(r^2u_r)_r+\frac{1}{r^2\sin\theta}\left(\sin\theta\, u_\theta\right)_\theta=\frac{1}{r^2}(r^2R')'\Theta+\frac{1}{r^2\sin\theta}(\sin\theta\,\Theta')'R$$

Dividing by $R\Theta$ and multiplying by r^2, we have

$$\frac{(r^2R')'}{R} = -\frac{(\sin\theta\,\Theta')'}{\sin\theta\,\Theta}.$$

By introducing the separation constant μ, we have

$$\begin{cases} (\sin\theta\,\Theta')' + \mu\sin\theta\,\Theta = 0, \quad \Theta(0),\, \Theta(\pi) < \infty, \\ (r^2R')' - \mu R = 0. \end{cases} \tag{6.11}$$

We can solve $R(r)$ as follows. Inserting the change of variable $r = e^s$, the equation of R becomes $R''(s) + R'(s) - \mu R(s) = 0$. Denote the solutions of the characteristic equation $x^2 + x - \mu = 0$ by $x_\pm$. We get $x_\pm = \frac{-1\pm\sqrt{1+4\mu}}{2}$ and the general solution is given by $R = Ae^{x_+s} + Be^{x_-s} = Ar^{x_+} + Br^{x_-}$. To get a smooth solution, it is required that $x_\pm$ are integers. We can check that if $\mu = k(k+1)$, then the solutions are integers: $x_+ = k$ and $x_- = -k-1$. The finiteness of $R(0)$ implies that $B = 0$. In summary, we have

$$\mu = k(k+1), \quad R(r) = Ar^k. \tag{6.12}$$

Inserting $\mu = k(k+1)$ into the equation of θ, we get

$$(\sin\theta\,\Theta')' + k(k+1)\sin\theta\,\Theta = 0, \quad \Theta(0),\, \Theta(\pi) < \infty. \tag{6.13}$$

This is the Legendre equation. If we denote the solution of it by $P_k(\cos\theta)$, referred to as Legendre polynomial. Then we have

$$\Theta(\theta) = P_k(\cos\theta). \tag{6.14}$$

Therefore, the separated solutions are obtained as

$$u(r,\theta) = A_k r^k P_k(\cos\theta) \tag{6.15}$$

The general solution is

$$u(r,\theta) = \sum_{k=0}^{\infty} A_k r^k P_k(\cos\theta)$$

Finally, to decide the coefficients A_k, we need to discuss the properties of the Legendre polynomial.

6.3 The Legendre polynomial

When applying separation of variables to the 3D Laplace equation, a new class of functions, known as Legendre polynomial, will emerge.

6.3.1 The definition of Legendre polynomial

Definition 6.2 (The Legendre polynomial)**.** *The Legendre equation is given by the follows*

$$(\sin\theta\,\Theta')' + k(k+1)\sin\theta\,\Theta = 0. \tag{6.16}$$

The solution $P_k(\cos\theta)$ to the above equation satisfying $\Theta(-\pi) = \Theta(\pi)$, $\Theta'(-\pi) = \Theta'(\pi)$ is refered to as the Legendre polynomial.

Remark 6.3. The Bessel function defined in this manner has an ambiguity in constants, as any solution of the form $CJ_m(x)$ also satisfies the above equation. This ambiguity is resolved by defining $J_m(x)$ using (5.26) below.

Lemma 6.4. *The Legendre is equivalent to the following equations.*

$$\begin{aligned}
(\sin\theta\,\Theta')' + k(k+1)\sin\theta\,\Theta &= 0,\\
\Theta'' + \cot\theta\,\Theta' + k(k+1)\Theta &= 0,\\
((1-s^2)y')' + k(k+1)y &= 0\\
(1-s^2)y'' - 2sy' + k(k+1)y &= 0
\end{aligned} \tag{6.17}$$

Proof. The first two follows from a simple computation. The second two follows from a change of variable $s = \cos\theta$. (We have $y(s) = \Theta(\arccos s)$ and $y(\cos\theta) = \Theta(\theta)$.) □

6.3.2 Formulas of the Legendre polynomial

We can solve the Legendre polynomial using Taylor series as in the proof of the following proposition.

Proposition 6.5. *The Legendre polynomial is a polynomial in s of degree k.*

Proof. Assume that the Taylor series of the solution $y(s)$ is given by $y(s) = \sum_{n=0}^{\infty} a_n s^n$. The first derivatives can be computed as

$$y'(s) = \sum_{n=0}^{\infty} n a_n s^{n-1} = \sum_{n=0}^{\infty} n a_n s^{n-1}$$

$$(1-s^2)y'(s) = \sum_{n=0}^{\infty} n a_n s^{n-1} - \sum_{n=0}^{\infty} n a_n s^{n+1} = \sum_{n=-2}^{-1} (n+2)a_{n+2}s^{n+1} + \sum_{n=0}^{\infty}((n+2)a_{n+2} - na_n)s^{n+1}$$

where we used the fact that $\sum_{n=0}^{\infty} n a_n s^{n-1} = \sum_{n=-2}^{-1}(n+2)a_{n+2}s^{n+1} + \sum_{n=0}^{\infty}(n+2)a_{n+2}s^{n+1}$.

The second order derivative can be computed as

$$((1-s^2)y'(s))' = \sum_{n=-2}^{-1} (n+2)(n+1)a_{n+2}s^n + \sum_{n=0}^{\infty}((n+2)a_{n+2} - na_n)(n+1)s^n$$

Inserting into the Legendre equation $((1-s^2)y')' + k(k+1)y = 0$, we get

$$\sum_{n=0}^{\infty}[(n+1)(n+2)a_{n+2} + (k(k+1) - n(n+1))a_n]s^n = 0$$

which implies that

$$a_{n+2} = \frac{n(n+1) - k(k+1)}{(n+1)(n+2)} a_n = \frac{(n-k)(n+k+1)}{(n+1)(n+2)} a_n \tag{6.18}$$

From the above recursion formula, we get the following observations.

1. Given $a_k \neq 0$, we can calculate $a_{k+2}, a_{k+4}, \dots$ and so on, providing one set of solutions. Similarly, given a_{k+1}, we can calculate $a_{k+3}, a_{k+5}, \dots$, providing a second set of solutions.

2. By (6.18), $a_{k+2} = 0$, which implies that $a_{k+2} = a_{k+4} = \dots = 0$. If we take $a_{k+1} = 0$, we also have $a_{k+3} = a_{k+5} = \cdots = 0$. In this case, the series terminates at k, making $y(s)$ a polynomial.

3. By (6.18), if $a_{k+1} \neq 0$, then we have $a_{k+3} \neq 0$, $a_{k+5} \neq 0, \cdots$. A complicated analysis implies that $a_k \not\to 0$ and the series is not convergent. Therefore, a_k must be zero.

Combining the above analysis, the Legendre polynomial is a polynomial in s. Because $a_n = 0$ and $a_k \neq 0$ for $n > k$, the degree of the polynomial is equal to k. □

Proposition 6.6 (Rodrigue formula)**.** *We have (up to a constant)*

$$P_k(s) = \frac{1}{2^k k!}\left(\frac{d}{ds}\right)^k \left(s^2-1\right)^k, \quad k = 0, 1, 2, \dots. \tag{6.19}$$

Proof. Let us expand the polynomial $\left(\frac{d}{ds}\right)^k \left(s^2-1\right)^k$ with Legendre polynomials.

$$\left(\frac{d}{ds}\right)^k \left(s^2-1\right)^k = \sum_{j=0}^{k} c_j P_j(s).$$

As $\left(\frac{d}{ds}\right)^k \left(s^2-1\right)$ is a degree k polynomial, the upper limit of the above sum is k.

As $P_k(s)$ is a solution to a Sturm-Liouville problem, they are orthogonal. From this, we obtain

$$c_j \int_{-1}^{1} (P_j(s))^2 \, \mathrm{d}s = \int_{-1}^{1} \left(\frac{d}{ds}\right)^k \left(s^2-1\right)^k \; P_j(s) \, \mathrm{d}s$$

Using integration by parts, for $j < k$, we get

$$\begin{aligned} c_j \int_{-1}^{1} (P_j(s))^2 \, \mathrm{d}s &= \left[\left(\frac{d}{ds}\right)^{k-1} \left(s^2-1\right)^k \; P_j(s)\right]_{-1}^{1} - \int_{-1}^{1} \left(\frac{d}{ds}\right)^{k-1} \left(s^2-1\right)^k \; P_j'(s) \, \mathrm{d}s \\ &= \int_{-1}^{1} \left(\frac{d}{ds}\right)^{k-1} \left(s^2-1\right)^k \; P_j'(s) \, \mathrm{d}s = -\int_{-1}^{1} \left(\frac{d}{ds}\right)^{k-2} \left(s^2-1\right)^k \; P_j''(s) \, \mathrm{d}s = \cdots \\ &= (-1)^k \int_{-1}^{1} \left(s^2-1\right)^k \; P_j^{(k)}(s) \, \mathrm{d}s = 0 \end{aligned}$$

where we used the fact that $\left(\frac{d}{ds}\right)^j \left(s^2-1\right)^k |_{s=\pm 1} = 0$ for any $j < k$ (see Lemma 6.7) and $P_j^{(k)}(s) = 0$ (the k-th derivative of a degree k polynomial is 0).

Therefore, we know that $c_j = 0$ for $j < k$, which implies that $\left(\frac{d}{ds}\right)^k \left(s^2-1\right)^k = c_k\, P_k(s)$. The choice of $c_k = 2^k k!$ implies (6.19). $\square$

The following lemma is important in the proof of the Rodrigue formula.

Lemma 6.7. $\left(\frac{d}{ds}\right)^j \left(s^2-1\right)^k |_{s=\pm 1} = 0$ *for* $j < k$.

Proof. Let us only prove the case of $s = 1$. We know that $\left(\frac{d}{ds}\right)^j \left(s^2-1\right)^k |_{s=1}$ is proportional to the j-th Taylor coefficient of $\left(s^2-1\right)^k$ at $s = 1$. By $\left(s^2-1\right)^k = (s-1)^k(s+1)^k = 2^k(s-1)^k(1+\frac{s-1}{2})^k$, the j-th Taylor coefficient is 0 when $j < k$. Therefore, $\left(\frac{d}{ds}\right)^j \left(s^2-1\right)^k |_{s=1} = 0$. $\square$

The following lemma is important in the computation of Legendre polynomial

Lemma 6.8. $\left(\frac{d}{ds}\right)^k \Big|_{s=0} s^l = l! \cdot \delta_{kl}$

Proof. For $k > l$, we have$\left(\frac{d}{ds}\right)^k s^l = l! \cdot \delta_{kl} = 0$. For $k \le l$, $\left(\frac{d}{ds}\right)^k s^l = \frac{l!}{k!} s^{l-k}$ when $k \le l$. When $k < l$, $\left(\frac{d}{ds}\right)^k \Big|_{s=0} s^l = \frac{l!}{l-k!} s^{l-k}|_{s=0} = 0$. When $k = l$, $\left(\frac{d}{ds}\right)^l s^l = l!$. $\square$

Using the Rodrigue formula, we can show that

Proposition 6.9 (Recursion formula of Legendre polynomial)**.** *We have*

$$\begin{gathered} nP_n(s) = (2n-1)sP_{n-1}(s) - (n-1)P_{n-2}(s) \quad n = 2, 3, \cdots, \\ P_0(s) = 1, \quad P_1(s) = s. \end{gathered} \tag{6.20}$$

Proof. **TODO:** $\square$

Example 6.1. By Rodrigue formula, we can compute that $P_0(s) = 1$, $P_1(s) = s$ and $P_2(s) = \frac{1}{2^2 2!}\frac{d^2}{ds^2}(s^2-1)^2 = \frac{1}{2}(3s^2-1)$.

6.3.3 The orthogonality relation

Consider the following Sturm-Liouville eigenvalues problem,

$$\begin{cases} (\sin\theta\, \Theta')' + k(k+1)\sin\theta\, \Theta = 0, \\ \Theta(0),\ \Theta(\pi) < \infty. \end{cases} \tag{6.21}$$

or equivalently

$$\begin{cases} ((1-s^2)y'(s))' + k(k+1)y(s) = 0, \\ y(-1),\, y(1) < \infty. \end{cases} \tag{6.22}$$

We know that $y(s) = P_k(s)$ is a solution to the above equation. Applying the Sturm-Liouville theorem Theorem 3.15, we get the following proposition.

Proposition 6.10. *We have the following orthogonality relations of the Legendre polynomial.*

$$\int_{-1}^{1} P_k(s)P_{k'}(s)ds = \int_0^{\pi} P_k(\cos\theta)P_{k'}(\cos\theta)\sin\theta d\theta = \frac{2}{2k+1}\delta_{kk'}. \tag{6.23}$$

Proof. Using the Sturm-Liouville theorem, it is straight forward to show that the integral is 0 when $k \neq k'$. Therefore, it suffices to show that $\int_{-1}^{1}(P_k(s))^2 ds = \frac{2}{2k+1}$.

By Rodrigue formula,

$$\int_{-1}^{1}(P_k(s))^2 ds = \frac{1}{2^{2k}(k!)^2}\int_{-1}^{1}\left(\frac{d}{ds}\right)^k \left(s^2-1\right)^k \left(\frac{d}{ds}\right)^k \left(s^2-1\right)^k ds \tag{6.24}$$

Apply integration by parts for k times and the boundary terms vanishes by Lemma 6.7, which implies

$$\int_{-1}^{1}(P_k(s))^2 ds = \frac{1}{2^{2k}(k!)^2}\int_{-1}^{1}\left(s^2-1\right)^k \cdot \left(\frac{d}{ds}\right)^{2k}\left(s^2-1\right)^k ds = \frac{(2k)!}{2^{2k}(k!)^2}\int_{-1}^{1}\left(s^2-1\right)^k ds \tag{6.25}$$

where we applied the fact that $\left(\frac{d}{ds}\right)^{2k}\left(s^2-1\right)^k = (2k)!$, which follows from the fact that

$$\left(\frac{d}{ds}\right)^{2k}\left(s^2-1\right)^k = \left(\frac{d}{ds}\right)^{2k}\left(s^{2k} + \text{lower order terms}\right) = \left(\frac{d}{ds}\right)^{2k} s^{2k} \tag{6.26}$$

Therefore, we get

$$\int_{-1}^{1}(P_k(s))^2 ds = \frac{(2k)!}{2^{2k}(k!)^2}\int_{-1}^{1}\left(s^2-1\right)^k ds = \frac{(2k)!}{2^{2k}(k!)^2}\int_{-1}^{1}(s-1)^k(s+1)^k ds \tag{6.27}$$

$$\begin{aligned}
&\int_{-1}^{1}(s-1)^k(s+1)^k ds = \frac{1}{k+1}\int_{-1}^{1}(s-1)^k \frac{d}{ds}(s+1)^{k+1} ds \\
=&\left[\frac{1}{k+1}(s-1)^k \frac{d}{ds}(s+1)^{k+1}\right]_{-1}^{1} - \frac{1}{k+1}\int_{-1}^{1}\frac{d}{ds}(s-1)^k (s+1)^{k+1} ds \\
=&\frac{k}{k+1}\int_{-1}^{1}(s-1)^{k-1}(s+1)^{k+1} ds = \cdots = \frac{k\cdots 1}{k+1\cdots(k+k)}\int_{-1}^{1}(s-1)^{k-k}(s+1)^{k+k} ds \\
=&\frac{(k!)^2}{(2k)!}\int_{-1}^{1}(s+1)^{2k} ds = \frac{(k!)^2}{(2k)!}\frac{1}{2k+1}\left[(s+1)^{2k}\right]_{-1}^{1} = \frac{(k!)^2}{(2k)!}\frac{2^{2k+1}}{2k+1}
\end{aligned} \tag{6.28}$$

Therefore,

$$\int_{-1}^{1}(P_k(s))^2 ds = \frac{(2k)!}{2^{2k}(k!)^2}\int_{-1}^{1}(s-1)^k(s+1)^k\,ds = \frac{(2k)!}{2^{2k}(k!)^2}\frac{(k!)^2}{(2k)!}\frac{2^{2k+1}}{2k+1} = \frac{2}{2k+1}, \tag{6.29}$$

which completes the proof. □

Now we introduce the notion of Fourier–Legendre series

Definition 6.11 (Fourier-Legendre series)**.** *Let us consider the expansion of a piecewise smooth function $f(x)$, $0 < x < 1$, in a series of the form*

$$f(\theta) = \sum_{n=0}^{\infty} A_k P_k(\cos\theta), \qquad f(s) = \sum_{n=0}^{\infty} A_k P_k(s).$$

This is called a Fourier–Legendre series.

We have the following theorem.

Theorem 6.12. *If $f(\theta)$, defined on $0 < \theta < \pi$ (or $f(s)$, defined on $-1 < s < 1$), is a piecewise smooth function, then $f(\theta)$ can be expanded in terms of Fourier–Bessel series*

$$f(\theta) = \sum_{n=0}^{\infty} A_k P_k(\cos\theta), \quad 0 < \theta < \pi, \qquad f(s) = \sum_{n=0}^{\infty} A_k P_k(s), \quad -1 < s < 1.$$

where the coefficients A_n can be computed by

$$A_k = \frac{2k+1}{2}\int_0^{\pi} f(\theta)P_k(\cos\theta)\sin\theta\,\mathrm{d}\theta, \quad \textit{or} \quad A_k = \frac{2k+1}{2}\int_0^{\pi} f(s)P_k(s)\,\mathrm{d}s, \quad k = 0, 2, \cdots \tag{6.30}$$

Moreover, the series converges for each θ or s to $\frac{1}{2}[f(\theta+0)+f(\theta-0)]$ or $\frac{1}{2}[f(s+0)+f(s-0)]$ respectively.

Proof. **TODO:** By multiplying (3.10) by $P_k(\cos\theta)$ □

Example 6.2. Let us compute the Fourier–Legendre series of the function

$$f(s) = \begin{cases} 1, & s \in (0,1), \\ -1, & s \in (-1,0). \end{cases}$$

Solution. The Fourier-Legendre series is given by:

$$f(s) = \sum_{k=0}^{\infty} a_k P_k(s),$$

where the coefficients a_k are determined by (6.30)

$$a_k = \frac{2k+1}{2}\int_{-1}^{1} f(s)P_k(s)\,ds.$$

As $f(s)$ is an odd function, we get $a_k = 0$ for $k = 2m$. Therefore, the expansion simplifies to:

$$f(s) = \sum_{m=0}^{\infty} a_{2m+1}P_{2m+1}(s),$$

To compute these coefficients, we evaluate the integral:

$$a_{2m+1} = \frac{2(2m+1)+1}{2}\int_{-1}^{1} P_{2m+1}(s)\,ds = (4m+3)\int_{0}^{1} P_{2m+1}(s)\,ds.$$

Using the Rodrigue formula, we get

$$\begin{aligned}
\int_0^1 P_{2m+1}(s)\,ds =& \frac{1}{2^{2m+1}(2m+1)!}\int_0^1 \left(\frac{d}{ds}\right)^{2m+1}\left(s^2-1\right)^{2m+1} ds \\
=& \frac{1}{2^{2m+1}(2m+1)!}\left[\left(\frac{d}{ds}\right)^{2m}\left(s^2-1\right)^{2m+1}\right]_0^1 \\
=& \frac{1}{2^{2m+1}(2m+1)!}\left(\frac{d}{ds}\right)^{2m}\Big|_{s=0}\left(s^2-1\right)^{2m+1}
\end{aligned}$$

Using the Binomial formula $(a+b)^k = \sum_{l=0}^{k} C_k^l a^l b^{k-l}$ and the fact that $\left(\frac{d}{ds}\right)^k\Big|_{s=0} s^l = l!\cdot\delta_{kl}$, we get

$$\begin{aligned}
\int_0^1 P_{2m+1}(s)\,ds =& \frac{1}{2^{2m+1}(2m+1)!}\left(\frac{d}{ds}\right)^{2m}\Big|_{s=0}\left(s^2-1\right)^{2m+1} \\
=& \frac{1}{2^{2m+1}(2m+1)!}\left(\frac{d}{ds}\right)^{2m}\Big|_{s=0}\left(\sum_{l=0}^{2m+1} C_{2m+1}^l(-1)^{2m+1-l}s^{2l}\right) \\
=& \frac{1}{2^{2m+1}(2m+1)!}\left(\frac{d}{ds}\right)^{2m}\Big|_{s=0}\left(C_{2m+1}^m(-1)^{m+1}s^{2m}\right) \\
=& \frac{1}{2^{2m+1}(2m+1)!}C_{2m+1}^m(-1)^{m+1}(2m)! \\
=& \frac{1}{2^{2m+1}(2m+1)}C_{2m+1}^m(-1)^{m+1}
\end{aligned}$$

Inserting into the equation of a_{2m+1}, we get

$$a_{2m+1} = (4m+3)\int_0^1 P_{2m+1}(s)\,ds = \frac{4m+3}{2^{2m+1}(2m+1)}C_{2m+1}^m(-1)^{m+1}.$$

Thus, the Fourier-Legendre series of $f(s)$ is:

$$f(s) = \sum_{m=0}^{\infty}\frac{(-1)^{m+1}(4m+3)}{2^{2m+1}(2m+1)}C_{2m+1}^m\,P_{2m+1}(s).$$

□

6.4 The electric potential

Let us consider electric potential satisfying a boundary value problem of the Laplace equation.

$$\begin{cases} \Delta u = \dfrac{1}{r^2}\left(r^2 u_r\right)_r + \dfrac{1}{r^2 \sin\theta}\left(\sin\theta\, u_\theta\right)_\theta = 0, & 0 \le r < a, \quad 0 \le \theta \le \pi \\ u(a,\theta) = G(\theta), & 0 \le \theta \le \pi, \end{cases} \tag{6.31}$$

where

$$G(\theta) = \begin{cases} 1, & \theta \in (0, \pi/2), \\ 0, & s \in (\pi/2, \pi). \end{cases} \tag{6.32}$$

We solve the equation in terms of Legendre polynomial.

We first look for separated solutions in the form

$$u(r,\theta) = R(r)\Theta(\theta).$$

Inserting into (5.48), we get

$$\frac{(r^2 R')'}{R} = -\frac{(\sin\theta\,\Theta')'}{\sin\theta\,\Theta}.$$

By introducing the separation constant μ, we have

$$\begin{cases} (\sin\theta\,\Theta')' + \mu \sin\theta\,\Theta = 0, \quad \Theta(0),\, \Theta(\pi) < \infty, \\ (r^2 R')' - \mu R = 0. \end{cases} \tag{6.33}$$

We can solve $R(r)$ as follows. Inserting the change of variable $r = e^s$, the equation of R becomes $R''(s) + R'(s) - \mu R(s) = 0$. Denote the solutions of the characteristic equation $x^2 + x - \mu = 0$ by $x_\pm$. We get $x_\pm = \frac{-1 \pm \sqrt{1+4\mu}}{2}$ and the general solution is given by $R = Ae^{x_+ s} + Be^{x_- s} = Ar^{x_+} + Br^{x_-}$. To get a smooth solution, it is required that $x_\pm$ are integers. We can check that if $\mu = k(k+1)$, then the solutions are integers: $x_+ = k$ and $x_- = -k-1$. The finiteness of $R(0)$ implies that $B = 0$. In summary, we have

$$\mu = k(k+1), \quad R(r) = Ar^k. \tag{6.34}$$

Inserting $\mu = k(k+1)$ into the equation of θ, we get

$$(\sin\theta\,\Theta')' + k(k+1)\sin\theta\,\Theta = 0, \quad \Theta(0),\, \Theta(\pi) < \infty. \tag{6.35}$$

This is the Legendre equation and the solution is given by

$$\Theta(\theta) = P_k(\cos\theta). \tag{6.36}$$

Therefore, the separated solutions are obtained as

$$u(r,\theta) = A_k r^k P_k(\cos\theta) \tag{6.37}$$

The general solution is

$$u(r,\theta) = \sum_{k=0}^{\infty} A_k r^k P_k(\cos\theta)$$

Finally, to decide the coefficients A_k, we apply the boundary condition $u(a,\theta) = G(\theta)$, which implies that

$$G(\theta) = \sum_{k=0}^{\infty} A_k a^k P_k(\cos\theta)$$

By the formula of coefficients (6.30), we get

$$\begin{aligned} a_k =& \frac{2k+1}{2a^k} \int_0^{\pi} G(\theta) P_k(\cos\theta)\, d\theta = \frac{2k+1}{2a^k} \int_{-1}^{1} G(\arccos s) P_k(s)\, ds \\ =& \frac{2k+1}{2a^k} \int_0^1 P_k(s)\, ds. \end{aligned}$$

To compute these coefficients, we evaluate the integral $\int_0^1 P_k(s)\, ds$. Using the Rodrigue formula, we get

$$\begin{aligned} \int_0^1 P_k(s)\, ds =& \frac{1}{2^k k!} \int_0^1 \left(\frac{d}{ds}\right)^k \left(s^2-1\right)^k \, ds \\ =& \frac{1}{2^k k!} \left[\left(\frac{d}{ds}\right)^{k-1} \left(s^2-1\right)^k\right]_0^1 \\ =& \frac{1}{2^k k!} \left(\frac{d}{ds}\right)^{k-1} \Bigg|_{s=0} \left(s^2-1\right)^k \end{aligned}$$

Using the Binomial formula $(a+b)^k = \sum_{l=0}^{k} C_k^l a^l b^{k-l}$ and the fact that $\left(\frac{d}{ds}\right)^k \Big|_{s=0} s^l = l! \cdot \delta_{kl}$, we get

$$\begin{aligned} \int_0^1 P_k(s)\, ds =& \frac{1}{2^k k!} \left(\frac{d}{ds}\right)^{k-1} \Bigg|_{s=0} \left(s^2-1\right)^k \\ =& \frac{1}{2^k k!} \left(\frac{d}{ds}\right)^{k-1} \Bigg|_{s=0} \left(\sum_{l=0}^{k} C_k^l (-1)^{k-l} s^{2l}\right) \end{aligned}$$

If $k = 2m+1$, then $\frac{1}{2^k k!} \left(\frac{d}{ds}\right)^{k-1} \Big|_{s=0} \left(C_k^m (-1)^{m+1} s^{2m}\right) \neq 0$ is the only non-vanishing term. If

$k = 2m$, all term vanishes. Therefore, $a_{2m} = 0$. Inserting $k = 2m + 1$ in the above formula, we get

$$\begin{aligned}\int_0^1 P_{2m+1}(s)\,ds =& \frac{1}{2^{2m+1}(2m+1)!}\left(\frac{d}{ds}\right)^{2m}\Big|_{s=0}\left(C_{2m+1}^m(-1)^{m+1}s^{2m}\right)\\ =& \frac{1}{2^{2m+1}(2m+1)!}C_{2m+1}^m(-1)^{m+1}(2m)!\\ =& \frac{1}{2^{2m+1}(2m+1)}C_{2m+1}^m(-1)^{m+1}\end{aligned}$$

Inserting into the equation of a_{2m+1}, we get

$$a_{2m+1} = \frac{(4m+3)}{2a^{2m+1}}\int_0^1 P_{2m+1}(s)\,ds = \frac{4m+3}{2m+1}\frac{(-1)^{m+1}}{2^{2m+2}a^{2m+1}}C_{2m+1}^m.$$

Thus, the solution $u(r,\theta)$ is given by

$$u(r,\theta) = \sum_{m=0}^{\infty}\frac{(-1)^{m+1}(4m+3)}{2^{2m+2}(2m+1)}C_{2m+1}^m\left(\frac{r}{a}\right)^{2m+1}P_{2m+1}(s).$$

7 Fourier transform and PDEs on unbounded domains

7.1 The Fourier transform

7.1.1 Motivation: from Fourier series to Fourier transform

Recall the complex form of the Fourier series on $[-L, L]$:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n\, e^{in\pi x/L}, \qquad c_n = \frac{1}{2L}\int_{-L}^{L} f(x')\, e^{-in\pi x'/L}\, dx'.$$

Substituting c_n back into the expansion gives

$$f(x) = \sum_{n=-\infty}^{\infty}\frac{1}{2L}\int_{-L}^{L} f(x')\, e^{-in\pi x'/L}\, dx' \;\cdot\; e^{in\pi x/L}.$$

Define the discrete frequencies

$$p_n = \frac{n\pi}{L}, \qquad \Delta p = p_{n+1} - p_n = \frac{\pi}{L}.$$

Then we can rewrite the Fourier series as a Riemann sum:

$$f(x) = \sum_{n=-\infty}^{\infty}\frac{1}{2\pi}\underbrace{\int_{-L}^{L} f(x')\, e^{-ip_n x'}\, dx'}_{\to\ \hat{f}(p_n)} \cdot e^{ip_n x}\,\Delta p.$$

As $L \to \infty$, $\Delta p \to 0$ and the discrete variable p_n becomes a continuous variable p. The Riemann sum converges to an integral:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(x')\, e^{-ipx'}\, dx' \right) e^{ipx}\, dp. \tag{7.1}$$

This motivates the following definition.

7.1.2 Definition and properties

Definition 7.1 (Fourier transform)**.** *Let $f(x)$ be a sufficiently fast-decaying function defined on $\mathbb{R}$. The Fourier transform of f is defined by*

$$\hat{f}(p) = \int_{-\infty}^{\infty} f(x)\, e^{-ipx}\, dx. \tag{7.2}$$

The inverse Fourier transform is given by

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(p)\, e^{ipx}\, dp. \tag{7.3}$$

Theorem 7.2 (Properties of the Fourier transform)**.** *Let $f(x)$ be a function with Fourier transform $\hat{f}(p)$. The following properties hold.*

1. ***Derivative property:*** $\widehat{f'}(p) = ip\,\hat{f}(p)$*, provided* $f(x) \to 0$ *as* $|x| \to \infty$.
2. ***Multiplication by** x**:*** $\widehat{xf}(p) = i\,\hat{f}'(p)$.
3. ***Parseval's identity:*** $\displaystyle\int_{-\infty}^{\infty} |f(x)|^2\, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{f}(p)|^2\, dp.$
4. ***Generalized Parseval's identity:*** $\displaystyle\int_{-\infty}^{\infty} f(x)\overline{g(x)}\, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(p)\overline{\hat{g}(p)}\, dp.$

Proof of (1). We compute the Fourier transform of f' by integration by parts:

$$\widehat{f'}(p) = \int_{-\infty}^{\infty} f'(x)\, e^{-ipx}\, dx = \left[f(x)\, e^{-ipx}\right]_{-\infty}^{\infty} + ip \int_{-\infty}^{\infty} f(x)\, e^{-ipx}\, dx = ip\,\hat{f}(p), \tag{7.4}$$

where the boundary term vanishes since $f(x) \to 0$ as $|x| \to \infty$. □

Proof of (2). Differentiating $\hat{f}(p) = \int_{-\infty}^{\infty} f(x)\, e^{-ipx}\, dx$ with respect to p under the integral sign gives

$$\hat{f}'(p) = \frac{d}{dp} \int_{-\infty}^{\infty} f(x)\, e^{-ipx}\, dx = \int_{-\infty}^{\infty} f(x)\,(-ix)\, e^{-ipx}\, dx = -i\,\widehat{xf}(p). \tag{7.5}$$

Therefore $\widehat{xf}(p) = i\,\hat{f}'(p)$. □

An important consequence of property (1) is the following.

Corollary 7.3. *If $f(x) \to 0$ and $f'(x) \to 0$ as $|x| \to \infty$, then* $\widehat{f''}(p) = (ip)^2 \hat{f}(p) = -p^2 \hat{f}(p)$.

We also record the following property of real-valued functions.

Proposition 7.4. *If $f(x)$ is real-valued, then* $\hat{f}(p) = \overline{\hat{f}(-p)}$.

7.2 Heat equation on the full line

Consider the heat equation on $\mathbb{R}$:

$$u_t = k\, u_{xx}, \quad x \in \mathbb{R}, \quad t > 0, \qquad u(x,0) = f(x). \tag{7.6}$$

7.2.1 Solution by Fourier transform

We apply the Fourier transform in x to both sides of (7.6). By Corollary 7.3, the right-hand side transforms as $k\, \widehat{u_{xx}} = -kp^2\, \hat{u}$. Since the Fourier transform does not act on t, we obtain the ODE

$$\hat{u}_t = -kp^2\, \hat{u}. \tag{7.7}$$

This is a separable first-order ODE in t (with p as a parameter). Solving by separation of variables,

$$\frac{d\hat{u}}{dt} = -kp^2\, \hat{u} \quad \Longrightarrow \quad \ln \hat{u} = -kp^2\, t + C \quad \Longrightarrow \quad \hat{u}(p,t) = \hat{u}(p,0)\, e^{-kp^2 t}. \tag{7.8}$$

The initial condition gives $\hat{u}(p,0) = \hat{f}(p)$, so

$$\hat{u}(p,t) = \hat{f}(p)\, e^{-kp^2 t}. \tag{7.9}$$

Applying the inverse Fourier transform (7.3),

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(p)\, e^{-kp^2 t}\, e^{ipx}\, dp. \tag{7.10}$$

7.2.2 The heat kernel

We can simplify (7.10) into a convolution form. Substituting the definition $\hat{f}(p) = \int_{-\infty}^{\infty} f(x')\, e^{-ipx'}\, dx'$ and interchanging the order of integration,

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x') \int_{-\infty}^{\infty} e^{-kp^2 t + ip(x-x')}\, dp\, dx'. \tag{7.11}$$

To evaluate the inner integral, we use the following Gaussian integral formula.

Theorem 7.5 (Gaussian integral)**.**

$$\int_{-\infty}^{\infty} e^{-\alpha p^2+\beta p}\,dp = \sqrt{\frac{\pi}{\alpha}}\,e^{\beta^2/(4\alpha)}, \qquad \mathrm{Re}(\alpha) > 0. \tag{7.12}$$

We apply Theorem 7.5 with $\alpha = kt$ and $\beta = i(x - x')$, so that $\beta^2 = -(x - x')^2$:

$$\int_{-\infty}^{\infty} e^{-ktp^2+i(x-x')p}\,dp = \sqrt{\frac{\pi}{kt}}\,e^{-(x-x')^2/(4kt)}. \tag{7.13}$$

Substituting back into (7.11),

$$\begin{aligned} u(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x')\,\sqrt{\frac{\pi}{kt}}\,e^{-(x-x')^2/(4kt)}\,dx' \\ &= \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} f(x')\,e^{-(x-x')^2/(4kt)}\,dx'. \end{aligned} \tag{7.14}$$

We define the heat kernel (also called the Green's function for the heat equation on $\mathbb{R}$):

$$\boxed{G(x,x';t) = \frac{1}{\sqrt{4\pi kt}}\,e^{-(x-x')^2/(4kt)}.} \tag{7.15}$$

The solution to the heat equation (7.6) is then

$$u(x,t) = \int_{-\infty}^{\infty} G(x,x';t)\,f(x')\,dx'. \tag{7.16}$$

7.2.3 Connection to eigenfunction expansion

Consider the heat equation $u_t = k\,u_{xx}$ on $x \in \mathbb{R}$. Define the operator $A = -\partial_{xx}$ acting on bounded functions $\phi(x)$ on $\mathbb{R}$. The eigenvalue problem $A\phi = \lambda\phi$, i.e. $-\phi'' = \lambda\phi$, has solutions

$$\phi(x) = e^{ipx}, \qquad \lambda = p^2, \qquad p \in \mathbb{R}. \tag{7.17}$$

The eigenfunctions e^{ipx} are labeled by a continuous parameter p, and the eigenvalues satisfy $\lambda = p^2 \geq 0$. This is called the continuous spectrum, in contrast to the discrete spectrum that arises on bounded domains. The "eigenfunction expansion" becomes

$$u(x,t) = \int_{-\infty}^{\infty} \hat{u}(p,t)\,e^{ipx}\,\frac{dp}{2\pi}, \tag{7.18}$$

which is exactly the inverse Fourier transform.

7.3 Fourier sine and cosine transforms

The Fourier transform requires that $f(x)$ is defined on all of $\mathbb{R}$ and decays sufficiently fast as $|x| \to \infty$. In many applications, we need to solve PDEs on the half-line $x > 0$. In this case, we extend f to the whole line and use the Fourier sine or cosine transform.

7.3.1 Heat equation on the half-line

Consider the heat equation on the half-line:

$$u_t = k\, u_{xx}, \quad x > 0, \quad t > 0, \qquad u(x,0) = f(x), \tag{7.19}$$

together with one of the following boundary conditions:

- **Dirichlet boundary condition:** $u(0,t) = 0$.
- **Neumann boundary condition:** $u_x(0,t) = 0$.

Since the Fourier transform requires $u(x,t)$ to be defined on $x \in (-\infty, \infty)$, we need to extend f from $x > 0$ to all of $\mathbb{R}$.

7.3.2 Odd and even extensions

The odd extension of $f(x)$ (for Dirichlet boundary conditions) is defined by

$$f_o(x) = \begin{cases} f(x), & x > 0, \\ -f(-x), & x < 0. \end{cases} \tag{7.20}$$

The even extension of $f(x)$ (for Neumann boundary conditions) is defined by

$$f_e(x) = \begin{cases} f(x), & x > 0, \\ f(-x), & x < 0. \end{cases} \tag{7.21}$$

The key observation is:

- If f is odd, then $\hat{f}(p)$ is purely imaginary and odd.
- If f is even, then $\hat{f}(p)$ is real and even.

Therefore, for Dirichlet boundary conditions (f_o is odd), we use the Fourier sine transform, and for Neumann boundary conditions (f_e is even), we use the Fourier cosine transform.

Definition 7.6 (Fourier sine transform)**.** *The Fourier sine transform and its inverse are defined by*

$$\hat{f}_s(p) = \int_0^\infty f(x)\sin(px)\,dx, \qquad f(x) = \frac{2}{\pi}\int_0^\infty \hat{f}_s(p)\sin(px)\,dp. \tag{7.22}$$

Definition 7.7 (Fourier cosine transform)**.** *The Fourier cosine transform and its inverse are defined by*

$$\hat{f}_c(p) = \int_0^\infty f(x)\cos(px)\,dx, \qquad f(x) = \frac{2}{\pi}\int_0^\infty \hat{f}_c(p)\cos(px)\,dp. \tag{7.23}$$

7.3.3 Dirichlet case: solution via sine transform

We use the odd extension f_o to extend f to $\mathbb{R}$, then apply the full-line solution.

Taking the Fourier sine transform of $u_t = k\, u_{xx}$ gives the ODE

$$(\hat{u}_s)_t = -kp^2\, \hat{u}_s. \tag{7.24}$$

With the initial condition $\hat{u}_s(p,0) = \hat{f}_s(p)$, the solution is

$$\hat{u}_s(p,t) = \hat{f}_s(p)\, e^{-kp^2 t}. \tag{7.25}$$

Applying the inverse sine transform,

$$u(x,t) = \frac{2}{\pi} \int_0^\infty \hat{f}_s(p)\, \sin(px)\, e^{-kp^2 t}\, dp. \tag{7.26}$$

Substituting $\hat{f}_s(p) = \int_0^\infty f(x') \sin(px')\, dx'$ and using the full-line result with the odd extension, we obtain

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_0^\infty f(x') \left[e^{-(x-x')^2/(4kt)} - e^{-(x+x')^2/(4kt)} \right] dx'. \tag{7.27}$$

We can write this as $u(x,t) = \int_0^\infty G^D(x,x';t)\, f(x')\, dx'$, where the Dirichlet Green's function on the half-line is

$$G^D(x,x';t) = \frac{1}{\sqrt{4\pi kt}} \left[e^{-(x-x')^2/(4kt)} - e^{-(x+x')^2/(4kt)} \right]. \tag{7.28}$$

7.3.4 Neumann case: solution via cosine transform

Using the even extension f_e and the cosine transform, we get the same type of ODE:

$$(\hat{u}_c)_t = -kp^2\, \hat{u}_c, \qquad \hat{u}_c(p,0) = \hat{f}_c(p). \tag{7.29}$$

The solution is $\hat{u}_c(p,t) = \hat{f}_c(p)\, e^{-kp^2 t}$. Applying the inverse cosine transform,

$$u(x,t) = \frac{2}{\pi} \int_0^\infty \hat{f}_c(p)\, \cos(px)\, e^{-kp^2 t}\, dp. \tag{7.30}$$

Substituting $\hat{f}_c$ and computing as before, we obtain

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_0^\infty f(x') \left[e^{-(x-x')^2/(4kt)} + e^{-(x+x')^2/(4kt)} \right] dx'. \tag{7.31}$$

The Neumann Green's function on the half-line is

$$\boxed{G^N(x,x';t) = \frac{1}{\sqrt{4\pi kt}} \left[e^{-(x-x')^2/(4kt)} + e^{-(x+x')^2/(4kt)} \right].} \tag{7.32}$$

Note that the Dirichlet Green's function uses a minus sign (odd extension), while the Neumann Green's function uses a plus sign (even extension).

7.4 Wave equation on $\mathbb{R}$

Consider the wave equation on the real line:

$$u_{tt} = c^2\, u_{xx}, \quad x \in \mathbb{R}, \quad t > 0, \tag{7.33}$$

with initial conditions $u(x, 0) = f(x)$ and $u_t(x, 0) = g(x)$.

7.4.1 Solution by Fourier transform

Taking the Fourier transform in x and using Corollary 7.3, we obtain the ODE

$$\hat{u}_{tt} + c^2 p^2\, \hat{u} = 0. \tag{7.34}$$

This is a constant-coefficient second-order ODE in t. The characteristic equation is $r^2 + c^2p^2 = 0$, giving $r = \pm icp$. By Theorem 1.15, the general solution is

$$\hat{u}(p, t) = A(p)\, e^{icpt} + B(p)\, e^{-icpt}. \tag{7.35}$$

Applying the initial conditions $\hat{u}(p, 0) = \hat{f}(p)$ and $\hat{u}_t(p, 0) = \hat{g}(p)$, we get the system

$$A(p) + B(p) = \hat{f}(p), \qquad icp\big[A(p) - B(p)\big] = \hat{g}(p). \tag{7.36}$$

Solving for $A(p)$ and $B(p)$,

$$A(p) = \frac{1}{2}\hat{f}(p) + \frac{\hat{g}(p)}{2icp}, \qquad B(p) = \frac{1}{2}\hat{f}(p) - \frac{\hat{g}(p)}{2icp}. \tag{7.37}$$

Substituting back into (7.35) and using $\frac{e^{icpt}+e^{-icpt}}{2} = \cos(cpt)$ and $\frac{e^{icpt}-e^{-icpt}}{2} = i\sin(cpt)$, we get

$$\hat{u}(p, t) = \hat{f}(p)\cos(cpt) + \frac{\hat{g}(p)}{icp}\sin(cpt). \tag{7.38}$$

7.4.2 D'Alembert's formula

Applying the inverse Fourier transform to (7.38), the solution splits into two terms $u(x, t) = I_1 + I_2$.

For the first term, using $\cos(cpt) = \frac{1}{2}(e^{icpt} + e^{-icpt})$,

$$\begin{aligned} I_1 &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(p)\, e^{ipx}\cos(cpt)\, dp \\ &= \frac{1}{2}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(p)\, e^{ip(x+ct)}\, dp + \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(p)\, e^{ip(x-ct)}\, dp\right] \\ &= \frac{1}{2}\big[f(x+ct) + f(x-ct)\big], \end{aligned} \tag{7.39}$$

where the last step follows from the inverse Fourier transform formula (7.3).

For the second term,

$$I_2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\hat{g}(p)}{icp} \, e^{ipx} \, \sin(cpt) \, dp. \tag{7.40}$$

We use the identity $\frac{\sin(cpt)}{cp} = \int_0^t \cos(cps) \, ds$ to write

$$I_2 = \frac{1}{2\pi} \int_0^t \int_{-\infty}^{\infty} \frac{\hat{g}(p)}{i} \, e^{ipx} \, \cos(cps) \, dp \, ds = \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) \, ds. \tag{7.41}$$

Combining I_1 and I_2, we obtain the following classical result.

Theorem 7.8 (D'Alembert's formula)**.** *The solution to the wave equation* (7.33) *with initial conditions* $u(x,0) = f(x)$ *and* $u_t(x,0) = g(x)$ *is*

$$\boxed{u(x,t) = \frac{1}{2}\big[f(x+ct) + f(x-ct)\big] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) \, ds.} \tag{7.42}$$

Example 7.1. Consider $u_{tt} = c^2 \, u_{xx}$ with $u(x,0) = e^{-x^2}$ and $u_t(x,0) = 0$. Since $g = 0$, D'Alembert's formula gives

$$u(x,t) = \frac{1}{2}\big[e^{-(x+ct)^2} + e^{-(x-ct)^2}\big]. \tag{7.43}$$

The initial profile e^{-x^2} splits into two bumps traveling in opposite directions, each with speed c and half the original amplitude.

7.5 Nonhomogeneous heat equation

Consider the nonhomogeneous heat equation

$$u_t - k \, u_{xx} = h(x,t), \qquad u(x,0) = f(x). \tag{7.44}$$

We decompose $u = v + w$, where v solves the homogeneous problem with initial data, and w solves the nonhomogeneous problem with zero initial data:

$$\begin{aligned} &v_t = k \, v_{xx}, \qquad v(x,0) = f(x), \\ &w_t - k \, w_{xx} = h(x,t), \qquad w(x,0) = 0. \end{aligned} \tag{7.45}$$

The solution for v is already known from section 7.2.2: $\hat{v}(p,t) = \hat{f}(p) \, e^{-kp^2 t}$.

For w, we take the Fourier transform in x to obtain the first-order linear ODE

$$\hat{w}_t + kp^2 \, \hat{w} = \hat{h}(p,t), \qquad \hat{w}(p,0) = 0. \tag{7.46}$$

We solve this by the integrating factor method (Theorem 1.12). Multiplying by the integrating factor e^{kp^2t},

$$\frac{d}{dt}\big[e^{kp^2t}\,\hat{w}\big] = e^{kp^2t}\,\hat{h}(p,t). \tag{7.47}$$

Integrating from 0 to t and using $\hat{w}(p,0) = 0$,

$$e^{kp^2t}\,\hat{w}(p,t) = \int_0^t e^{kp^2s}\,\hat{h}(p,s)\,ds \quad\Longrightarrow\quad \hat{w}(p,t) = \int_0^t \hat{h}(p,s)\,e^{-kp^2(t-s)}\,ds. \tag{7.48}$$

Applying the inverse Fourier transform and substituting back $\hat{f}$ and $\hat{h}$, and using the Gaussian integral (Theorem 7.5), we obtain the full solution:

$$\begin{aligned} u(x,t) = &\int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi kt}}\,e^{-(x-x')^2/(4kt)}\,f(x')\,dx' \\ &+ \int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi k(t-s)}}\,e^{-(x-x')^2/[4k(t-s)]}\,h(x',s)\,dx'\,ds. \end{aligned} \tag{7.49}$$

In terms of the Green's function (7.15),

$$\boxed{u(x,t) = \int_{-\infty}^{\infty} G(x,x';t)\,f(x')\,dx' + \int_0^t \int_{-\infty}^{\infty} G(x,x';t-s)\,h(x',s)\,dx'\,ds.} \tag{7.50}$$

7.6 The Dirac delta function

Definition 7.9 (Dirac delta function)**.** *The Dirac delta "function" $\delta(x)$ is defined by the two properties*

$$\delta(x) = \begin{cases} \infty, & x = 0, \\ 0, & x \neq 0, \end{cases} \qquad \text{and} \qquad \int_{-\infty}^{\infty} \delta(x)\,dx = 1. \tag{7.51}$$

It represents a "point charge" (or point source).

For any domain $\Omega \subset \mathbb{R}$,

$$\int_\Omega \delta(x)\,dx = \begin{cases} 1, & 0 \in \Omega, \\ 0, & 0 \notin \Omega. \end{cases} \tag{7.52}$$

Note that the heat kernel (7.15) approaches a delta function as $t \to 0^+$:

$$\frac{1}{\sqrt{4\pi kt}}\,e^{-(x-x')^2/(4kt)} \to \delta(x-x') \qquad \text{as } t \to 0^+. \tag{7.53}$$

7.6.1 Connection to Green's function

The Green's function G for the heat equation satisfies

$$\begin{cases} G_t = k\,G_{xx}, \\ G\big|_{t=0} = \delta(x - x'). \end{cases} \tag{7.54}$$

That is, G is the response to a point-source initial condition at $x = x'$.

7.6.2 Properties of $\delta(x)$

Theorem 7.10 (Properties of $\delta(x)$)**.** *The Dirac delta function has the following properties.*

1. ***Sifting property:*** $\displaystyle\int_{-\infty}^{\infty} \delta(x-a)\,g(x)\,dx = g(a)$.
2. *$\delta(x)$ is even:* $\delta(-x) = \delta(x)$.
3. $\delta(cx) = \dfrac{1}{|c|}\,\delta(x)$, *for* $c \neq 0$.
4. $x\,\delta(x) = 0$.

7.6.3 δ as a limit of sequences

The delta function can be realized as a limit of ordinary functions:

$$\delta(x) = \lim_{\varepsilon \to 0} \delta_\varepsilon(x), \tag{7.55}$$

where $\delta_\varepsilon(x)$ is any family of functions satisfying $\int_{-\infty}^{\infty} \delta_\varepsilon(x)\,dx = 1$ and $\delta_\varepsilon(x) \to 0$ for $x \neq 0$ as $\varepsilon \to 0$. A standard example is the Gaussian approximation

$$\delta_\varepsilon(x) = \frac{1}{\varepsilon\sqrt{\pi}}\,e^{-x^2/\varepsilon^2}. \tag{7.56}$$

7.6.4 Fourier transform of δ

By the sifting property with $g(x) = e^{-ipx}$ and $a = 0$,

$$\hat{\delta}(p) = \int_{-\infty}^{\infty} \delta(x)\,e^{-ipx}\,dx = e^0 = 1. \tag{7.57}$$

Applying the inverse Fourier transform to $\hat{\delta}(p) = 1$, we obtain the Fourier representation of δ:

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ipx}\,dp. \tag{7.58}$$

7.7 Green's function: general theory

7.7.1 Verification that G is the Green's function

We verify that the heat kernel $G(x,t) = \frac{1}{\sqrt{4\pi kt}}\, e^{-x^2/(4kt)}$ satisfies $G_t = k\, G_{xx}$ and $G\big|_{t\to 0^+} = \delta(x)$.

Proof. We work in Fourier space. The Fourier transform of G is

$$\hat{G}(p,t) = \int_{-\infty}^{\infty} G(x,t)\, e^{-ipx}\, dx = e^{-kp^2 t}. \tag{7.59}$$

We check both conditions:

1. **PDE:** $\hat{G}_t = -kp^2\, e^{-kp^2 t} = -kp^2\, \hat{G} = k\, \widehat{G_{xx}}$, which is the Fourier transform of $G_t = k\, G_{xx}$.
2. **Initial condition:** $\hat{G}(p,0) = 1 = \hat{\delta}(p)$, so $G\big|_{t=0} = \delta(x)$. □

7.7.2 Convolution and the convolution theorem

Definition 7.11 (Convolution)**.** *The* <u>*convolution*</u> *of two functions f and g is defined by*

$$(f \star g)(x) = \int_{-\infty}^{\infty} f(y)\, g(x-y)\, dy. \tag{7.60}$$

Theorem 7.12 (Convolution theorem)**.** *The Fourier transform of a convolution is the product of the Fourier transforms:*

$$\widehat{f \star g}(p) = \hat{f}(p) \cdot \hat{g}(p). \tag{7.61}$$

Proof. We compute directly from the definitions:

$$\begin{aligned} \widehat{f \star g}(p) &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(y)\, g(x-y)\, dy \right) e^{-ipx}\, dx \\ &= \int_{-\infty}^{\infty} f(y)\, e^{-ipy} \left(\int_{-\infty}^{\infty} g(x-y)\, e^{-ip(x-y)}\, dx \right) dy \\ &= \hat{f}(p) \cdot \hat{g}(p), \end{aligned} \tag{7.62}$$

where in the second line we substituted $\mu = x - y$ in the inner integral. □

As a consistency check, the convolution theorem works for δ: convolution with δ is the identity ($\delta \star g = g$), and in Fourier space $\hat{\delta} \cdot \hat{g} = 1 \cdot \hat{g} = \hat{g}$.

7.7.3 Superposition principle for Green's functions

The sifting property of δ implies that any function $f(x)$ can be decomposed as

$$f(x) = \int_{-\infty}^{\infty} f(x')\,\delta(x - x')\,dx'. \tag{7.63}$$

This means that f is a "continuous linear combination" of delta functions $\delta(x-x')$, with coefficients $f(x')$.

Since $G(x, x'; t)$ solves the heat equation with initial data $\delta(x - x')$, by linearity (superposition), the solution for general initial data f is

$$u(x,t) = \int_{-\infty}^{\infty} G(x, x'; t)\, f(x')\,dx'. \tag{7.64}$$

Lemma 7.13. *If $G(x, x'; t)$ satisfies (7.54), then $u(x,t) = \int_{-\infty}^{\infty} G(x,x';t)\,f(x')\,dx'$ is a solution to $u_t = k\,u_{xx}$, $u|_{t=0} = f(x)$.*

Proof. We verify both conditions.

PDE: Differentiating under the integral sign,

$$u_t = \int_{-\infty}^{\infty} G_t(x, x'; t)\, f(x')\,dx' = \int_{-\infty}^{\infty} k\,G_{xx}(x, x'; t)\, f(x')\,dx' = k\,u_{xx}. \tag{7.65}$$

Initial condition:

$$u\big|_{t=0} = \int_{-\infty}^{\infty} G(x, x'; 0)\, f(x')\,dx' = \int_{-\infty}^{\infty} \delta(x - x')\, f(x')\,dx' = f(x). \tag{7.66}$$

$\square$

7.7.4 Computing G by Fourier transform

We can also derive the heat kernel by directly solving the Green's function problem (7.54). Taking the Fourier transform in x,

$$\hat{G}_t = -kp^2\,\hat{G}. \tag{7.67}$$

The initial condition $G|_{t=0} = \delta(x - x')$ gives

$$\hat{G}(p, 0) = \int_{-\infty}^{\infty} \delta(x - x')\,e^{-ipx}\,dx = e^{-ipx'}. \tag{7.68}$$

Solving the ODE gives $\hat{G}(p,t) = e^{-ipx'}\,e^{-kp^2 t}$. Applying the inverse Fourier transform and using the Gaussian integral (Theorem 7.5),

$$\begin{aligned} G(x, x'; t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ipx}\,e^{-ipx'}\,e^{-kp^2 t}\,dp = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-kp^2 t + ip(x-x')}\,dp \\ &= \frac{1}{\sqrt{4\pi k t}}\,e^{-(x-x')^2/(4kt)}. \end{aligned} \tag{7.69}$$

This confirms the heat kernel (7.15).

7.8 Green's function for the Laplace equation

7.8.1 Poisson's equation in 2D

Consider Poisson's equation in 2D:

$$u_{xx} + u_{yy} = f(x, y). \tag{7.70}$$

The Green's function satisfies

$$G_{xx} + G_{yy} = \delta(x - x')\,\delta(y - y'). \tag{7.71}$$

For $(x, y) \neq (x', y')$, the right-hand side vanishes, so $G_{xx} + G_{yy} = 0$.

7.8.2 Free-space Green's function

By symmetry, G depends only on the distance $r = \sqrt{(x - x')^2 + (y - y')^2}$ from the source point (x', y'). In polar coordinates centered at (x', y'), the Laplacian of a radially symmetric function gives

$$0 = \frac{1}{r}\big(r\,G_r\big)_r \qquad \text{for } r > 0. \tag{7.72}$$

Integrating once gives $r\,G_r = B$, and integrating again gives

$$G(r) = A + B\ln r. \tag{7.73}$$

To determine B, we integrate the equation $G_{xx} + G_{yy} = \delta(x - x')\delta(y - y')$ over a disk D_ε of radius ε centered at (x', y') and apply the divergence theorem:

$$1 = \iint_{D_\varepsilon} (G_{xx} + G_{yy})\,dA = \oint_{\partial D_\varepsilon} \frac{\partial G}{\partial r}\,ds = \frac{B}{r}\cdot 2\pi r\Big|_{r=\varepsilon} = 2\pi B. \tag{7.74}$$

Therefore $B = \frac{1}{2\pi}$, and we may take $A = 0$. The <u>free-space Green's function</u> for Poisson's equation in 2D is

$$\boxed{G(x, y; x', y') = \frac{1}{2\pi}\ln r = \frac{1}{2\pi}\ln\sqrt{(x - x')^2 + (y - y')^2}.} \tag{7.75}$$

The solution to Poisson's equation is then

$$u(x, y) = \iint G(x, y; x', y')\,f(x', y')\,dx'\,dy'. \tag{7.76}$$

7.8.3 Method of mirror images

The method of mirror images allows us to construct Green's functions on half-spaces by placing "mirror charges" to enforce boundary conditions.

Example 7.2 (Dirichlet on the half-line). Consider the heat equation on $x > 0$ with Dirichlet boundary condition:

$$u_t = k\,u_{xx}, \quad u\big|_{x=0} = 0, \quad u\big|_{t=0} = f(x). \tag{7.77}$$

The Green's function must satisfy

$$\begin{cases} G_t = k\,G_{xx}, \\ G\big|_{t=0} = \delta(x - x'), \\ G\big|_{x=0} = 0. \end{cases} \tag{7.78}$$

We place a mirror charge of opposite sign $-\delta(x + x')$ at $-x'$. The free-space Green's function is $G^{\text{free}} = \frac{1}{\sqrt{4\pi kt}}\, e^{-(x-x')^2/(4kt)}$ and the image contribution is $G^{\text{image}} = -\frac{1}{\sqrt{4\pi kt}}\, e^{-(x+x')^2/(4kt)}$. Therefore

$$G^D(x, x'; t) = \frac{1}{\sqrt{4\pi kt}} \left[e^{-(x-x')^2/(4kt)} - e^{-(x+x')^2/(4kt)} \right]. \tag{7.79}$$

One can verify that the boundary condition is satisfied: $G^D(0, x'; t) = \frac{1}{\sqrt{4\pi kt}} \left[e^{-x'^2/(4kt)} - e^{-x'^2/(4kt)} \right] = 0$.

Example 7.3 (Neumann on the half-line). For the Neumann boundary condition $u_x(0, t) = 0$, the image has the *same* sign (even reflection):

$$G^N(x, x'; t) = \frac{1}{\sqrt{4\pi kt}} \left[e^{-(x-x')^2/(4kt)} + e^{-(x+x')^2/(4kt)} \right]. \tag{7.80}$$

The solution is then $u(x, t) = \int_0^\infty G^N(x, x'; t)\, f(x')\, dx'$.

Example 7.4 (2D Laplace equation on the half-plane). For the 2D Laplace equation on the half-plane $x > 0$ with Dirichlet boundary condition $G|_{x=0} = 0$, we place a mirror charge at $(-x', y')$:

$$G^D(x, y; x', y') = \frac{1}{4\pi} \ln\left[\frac{(x - x')^2 + (y - y')^2}{(x + x')^2 + (y - y')^2} \right]. \tag{7.81}$$