What is a Category?

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Category

TODO

Epis and Monis

TODO

Isomorphisms

Examples

- The category of sets, denoted as **Set** is the category whose objects ob(**Set**) are sets. The arrows in **Set** are the functions between two $A, B \in ob(\mathbf{Set})$. The identity- function is defined as $\forall A: set, id_A: A \to A, \ \forall x \in A \ f \ x = x$ The composition \circ is the composition of functions and this is associative. Proof: Assume sets A,B,C,D, functions $f: A \to B, g: B \to C, h: C \to D$ and $x \in A$. $((h \circ g) \circ f) \ x = ((h \circ g)(f \ x) = h(g(f \ x))$ $(h \circ (g \circ f)) \ x = h((g \circ f) \ x) = h(g(f \ x))$ Thus $(h \circ g) \circ f = h \circ (g \circ f)$ holds. \square
- The category of relations, denoted as **Rel** is the category whose objects are the sets The arrows are all binary relations between two $A, B \in ob(\mathbf{Rel})$ The identity arrow is the identity function $\forall A : set, id_A : A \to A, \ \forall x \in A \ f \ x = x$ The composition $R \circ S, \ R \in \mathbf{Rel}(A, B), S \in \mathbf{Rel}(B, C), \ A, B, C \in ob(\mathbf{Rel})$ is defined as $(x, y) \in R \circ S \leftrightarrow \exists z. (x, z) \in S \land (z, y) \in R$
- TODO Grp
- TODO Vec
- TODO Pos
- TODO Discrete, 1, 2

Dual categories

 C^{op} denotes the dual category for any category C. One obtains C^{op} by reversing the arrows in C. For every sentence Σ in the language of category theory, the reversed sentence Σ^* exists, therefore any proof for any theorem yields for the dual theorem by the duality principle.

Terminal and Initial Objects

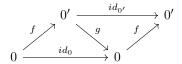
Definition (Initial and terminal Objects): In any category C, an object 0 is called initial iff for any object $A \in C$, there is an unique morphism $0 \to A$. an object 1 is called terminal iff for any object $A \in C$, there is an unique morphism $A \to 1$. A terminal object in C is initial in C^{op}

Proposition:

Initial and terminal objects are unique up to isomorphism.

Proof:

Assume 0,0' are both inital objects in some category C and show that $f: 0 \to 0', g: 0' \to 0$ form an unique isomorphism $f \circ g$ between 0,0'. One can draw the following diagram:



Since 0 is initial, we know that f is unique, from the same argument follows uniqueness of $g = f^{-1}$. Therefore $f \circ g$ and $g \circ f$ is unique.

The same holds for terminal objects by duality. \Box

Categories, in which the terminal is identical to the initial object, are called pointed category. Such objects zero objects.

Example:

How to show that \emptyset is initial in Set and the one- element set $\{x\}$ terminal?

- There is only the binary union function from \emptyset to any other set, since there are no arguments in the domain to use.
- Assume some f with $\forall A : \text{set with } A \neq \emptyset, f : \rightarrow \{x\}$ f is obviously a constant function, since $\forall y \in A, f y = x$ holds, and therefore unique.