What is a Category?

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1 Definition of Categories

Definition (Category): A category $\mathscr{C}$ has the following components:

- a collection of objects $ob(\mathscr{C})$
- $\forall\ A, B \in ob(\mathscr{C})$, a collection of maps (arrows, morphisms), denoted by, $\mathscr{C}(A, B)$
- $\forall\ A, B, C \in ob(\mathscr{C})$, a composition function

$$\mathscr{C}(B, C) \times \mathscr{C}(A, B) \to \mathscr{C}(A, C)$$

$$(g, f) \mapsto (g \circ f)$$

- $\forall A \in ob(\mathscr{C})$, an element $1_A \in \mathscr{C}(A, A)$, called the identity on A

The following axioms have to be satisfied:

- identity laws: $\forall f \in \mathscr{C}(A, B).\ f \circ 1_A = f = 1_B \circ f$
- associativity: $\forall f \in \mathscr{C}(A, B),\ g \in \mathscr{C}(B, C),\ h \in \mathscr{C}(C, D).\ (h \circ g) \circ f = h \circ (g \circ f)$.

2 Diagrams

It is often possible and useful to draw diagrams for a category or parts thereof. For example consider the following diagram.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"] & B \arrow[d, "h"] \\ C \arrow[r, "i"] & D \end{tikzcd}$$

This represents the category with the four objects $\{A, B, C, D\}$ and the morphisms $\{f, g, h, i, 1_A, 1_B, 1_C, 1_D\}$. Since every object is required to have an identity arrow by definition, we usually leave them implicit and do not include them in diagrams.

We say a diagram *commutes* if any two paths between two objects obtained by composing arrows are the same. The example diagram above commutes if $h \circ f = i \circ g$.

3 Epics and Monics

Monomorphisms and epimorphisms are two special kinds of morphisms.

Definition (Monomorphism): Let $\mathscr{C}$ be a category. A map $X \xrightarrow{f} Y$ is called a monomorphism if for all objects Z and maps $Z \underset{g'}{\overset{g}{\rightrightarrows}} X$

$$f \circ g = f \circ g' \Rightarrow g = g'$$

If f is a monomorphism, we say f is monic.

Definition (Epimorphism): Let $\mathscr{C}$ be a category. A map $X \xrightarrow{f} Y$ is called an epimorphism if for all objects Z and maps $Y \overset{g}{\underset{g'}{\rightrightarrows}} Z$

$$g' \circ f = g' \circ f \Rightarrow g = g'$$

If f is an epimorphism, we say f is epic.

In other words, a monomorphism is cancellable on the left, and an epimorphism is cancellable on the right of compositions. By the identity laws of categories, for any object A the identity map 1_A is always both monic and epic.

In the category **Set**, monic maps are exactly the injective functions, and epic maps are exactly the surjective functions. In ordinary set theory these properties of functions are formulated in terms of elements of a set. Epic and monic arrows enable us to talk about these properties without referring to elements at all. In the categories **Grp**, **Mon**, **Ring**, and **Vect**, monic maps are exactly the injective homomorphisms. However, in **Ring** there are maps that are epic, but not surjective.

4 Isomorphisms

Definition (Isomorphism): Let $\mathscr{C}$ be a category and $A, B \in ob(\mathscr{C})$.
A map $f\colon A \to B$ in $\mathscr{C}$ is an isomorphism if there is a $g\colon B \to A$ s.t. $g \circ f = 1_A$ and $f \circ g = 1_B$.
A and B are isomorphic in $\mathscr{C}$ ($A \cong B$) if there is an isomorphism between them.

In the category **Set** isomorphisms are exactly the invertible functions. It can be shown that invertible functions between two sets are exactly the bijective functions, i.e. functions that are both injective and surjective. Therefore, isomorphisms in the category **Set** are exactly those maps that are both monic and epic.

In the category **Grp** isomorphisms are exactly the bijective group homomorphisms. since those are the group homomorphisms that have an inverse homomorphism.

Similarly, in the category **Ring** isomorphisms are exactly the bijective ring homomorphisms.

We now show for a general category $\mathscr{C}$ that every isomorphism is both monic and epic.

Proof. Consider the diagram:

$$\begin{tikzcd}
A \arrow[r, shift left, "y"] \arrow[r, shift right, "x"'] & B \arrow[r, "m"] \arrow[dr, shift left, "1"] & C \arrow[r, shift left, "i"] \arrow[r, shift right, "j"'] \arrow[d, "e"] & D \\
& & B \arrow[ul, shift left, "1"] &
\end{tikzcd}$$

$m\colon B \to C$ is an isomorphism with inverse $e\colon C \to B$. Then:

m is monic:

$$\begin{aligned}
m \circ x &= m \circ y \\
e \circ (m \circ x) &= e \circ (m \circ y) \\
(e \circ m) \circ x &= (e \circ m) \circ y \\
1_B \circ x &= 1_B \circ y \\
x &= y
\end{aligned}$$

m is epic:

$$\begin{aligned}
i \circ m &= j \circ m \\
(i \circ m) \circ e &= (j \circ m) \circ e \\
i \circ (m \circ e) &= j \circ (m \circ e) \\
i \circ 1_C &= j \circ 1_C \\
i &= j
\end{aligned}$$

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Note that the converse of the statement holds for the category **Set**, but not in general. If a map is both monic and epic it is not necessarily an isomorphism. For a map f to be an isomorphism there needs to exist an inverse map f^{-1}. However monic and epic maps do not necessarily have inverses.

If a map f is an isomorphism, then its inverse is unique. Thus we speak of *the* inverse of f.

Proof. Let $f\colon A \to B$ be a map. Assume there exist two inverses $g\colon B \to A$, such that $gf = 1_A$, $fg = 1_B$ and $h\colon B \to A$, such that $hf = 1_A$, $fh = 1_B$. We know from the identity axiom: $1_A \circ g = g$ and $h \circ 1_B = h$. Therefore we get:

$$1_A \circ g = g \Longleftrightarrow (h \circ f) \circ g = g \Longleftrightarrow h \circ (f \circ g) = g \Longleftrightarrow h \circ 1_B = g \Longleftrightarrow h = g.$$

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5 Examples

- The category of sets, denoted as **Set**, is the category whose objects $ob(\textbf{Set})$ are sets.
  The arrows in **Set** are the functions between two $A, B \ \in ob(\textbf{Set})$.
  The identity- function is defined as $\forall A : set,\ id_A :\ A \to A,\ \forall x \in A\ f\ x = x$
  The composition $\circ$ is the composition of functions and this is associative. Proof:
  Assume sets A,B,C,D, functions $f :\ A \to B$, $g :\ B \to C$, $h :\ C \to D$ and x $\in$ A.
  $((h\ \circ\ g)\ \circ\ f)\ x = ((h\ \circ g\ )(f\ x) = h(g(f\ x))$
  $(h\ \circ\ (g\ \circ f))\ x = h((g\ \circ\ f)\ x) = h(g(f\ x))$
  Thus $(h\ \circ\ g)\ \circ\ f =\ h\ \circ\ (g\ \circ\ f)$ holds. $\Box$
- The category of relations, denoted as **Rel**, is the category whose objects are the sets
  The arrows are all binary relations between two $A, B\ \in ob(\textbf{Rel})$
  The identity arrow is the identity function $\forall A : set,\ id_A :\ A \to A,\ \forall x \in A\ f\ x = x$
  The composition $R \circ S,\ R \in \textbf{Rel}(A, B), S \in \textbf{Rel}(B, C), A, B, C \in ob(\textbf{Rel})$ is defined as $(x, y) \in R \circ S \leftrightarrow \exists z.(x, z) \in S \wedge (z, y) \in R$
- Categories similar to **Set** can be constructed for sets which have some additional structure and structure-preserving mappings between them. For example, there is a category **Poset** with partially ordered sets as objects. It is defined similarly to **Set**, with the difference that the functions are monotone.
- A single poset $(P, \leq)$ also gives rise to a category $\mathscr{P}$ if we take elements of P to be the objects. There is a unique arrow in $\mathscr{P}(A, B)$ iff $A \leq B$. The reflexivity requirement of $\leq$ ensures that identity arrows exist for all objects. Also, the category has composition because the order is transitive.
- $\textbf{Vect}_k$ is the category of vector spaces over a field k, with linear transformations as arrows.
- There is a category **Grp** whose objects are groups; maps are given by group homomorphisms between two groups $A, B \in ob(\textbf{Grp})$.
- A single group $(G, \cdot)$ can also be seen as a category $\mathcal{G}$. There is a unique object G which represents the group itself, and arrows in $\mathcal{G}(G, G)$ correspond to group elements.
  The identity arrow is the unit element 1 of the group.
  Composition operates on two group elements and has to be associative, so it is given by the group action.
  A group furthermore requires all elements to be invertible, thus we need each arrow $\mathcal{G}(G, G)$ to be an isomorphism.
  Dropping the last requirement, we get the category of a single monoid.
- Categories need not represent mathematical structures. They can be arbitrarily constructed by giving objects, arrows and arrow composition in a way that satisfies the axioms in the definition. For instance, the category **1** includes a single object and its identity map, without further specification what the object is.
  A category without non-trivial arrows is called discrete; removing all arrows except identities makes an arbitrary category discrete.

6 Dual categories

C^{op} denotes the dual category for any category C. One obtains C^{op} by reversing the arrows in C. For every sentence Σ in the language of category theory, the reversed sentence Σ^* exists, therefore any proof for any theorem yields for the dual theorem by the duality principle.

7 Terminal and Initial Objects

> **Definition (Initial and terminal Objects):** In any category C, an object 0 is called initial iff for any object $A \in C$, there is an unique morphism $0 \to A$. an object 1 is called terminal iff for any object $A \in C$, there is an unique morphism $A \to 1$. A terminal object in C is initial in C^{op}.

7.1 Proposition:

Initial and terminal objects are unique up to isomorphism.

7.1.1 Proof:

Assume 0,0' are both inital objects in some category C and show that $f : 0 \to 0', g : 0' \to 0$ form an unique isomorphism $f \circ g$ between 0,0'. One can draw the following diagram:

$$\begin{tikzcd}
 & 0' \arrow[rr, "id_{0'}"] \arrow[dr, "g"] & & 0' \\
0 \arrow[ur, "f"] \arrow[rr, "id_0"] & & 0 \arrow[ur, "f"] &
\end{tikzcd}$$

Since 0 is initial, we know that f is unique, from the same argument follows uniqueness of $g = f^{-1}$. Therefore $f \circ g$ and $g \circ f$ is unique.
The same holds for terminal objects by duality. □

Categories in which the terminal is identical to the initial object are called pointed category. Such objects zero objects.

7.1.2 Example:

How to show that $\emptyset$ is initial in Set and the one- element set $\{x\}$ terminal?

- There is only the binary union function from $\emptyset$ to any other set, since there are no arguments in the domain to use.
- Assume some f with
  $\forall A$: set with $A \neq \emptyset$, $f : A \to \{x\}$ f is obviously a constant function, since $\forall y \in A,\ f\ y = x$ holds, and therefore unique.