What is a Category?

sarah, leonhard, Andreas Meyer

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1 Definition of Categories

Definition (Category): A category $\mathscr{C}$ has the following components:

- a collection of objects $ob(\mathscr{C})$
- $\forall\ A, B \in ob(\mathscr{C})$, a collection of maps (arrows, morphisms), denoted by, $\mathscr{C}(A, B)$
- $\forall\ A, B, C \in ob(\mathscr{C})$, a composition function

$$\mathscr{C}(B, C) \times \mathscr{C}(A, B) \to \mathscr{C}(A, C)$$

$$(g, f) \mapsto (g \circ f)$$

- $\forall A \in ob(\mathscr{C})$, an element $1_A \in \mathscr{C}(A, A)$, called the identity on A

The following axioms have to be satisfied:

- identity laws: $\forall f \in \mathscr{C}(A, B).\ f \circ 1_A = f = 1_B \circ f$
- associativity: $\forall f \in \mathscr{C}(A, B),\ g \in \mathscr{C}(B, C),\ h \in \mathscr{C}(C, D).\ (h \circ g) \circ f = h \circ (g \circ f)$.

2 Epis and Monis

TODO

3 Isomorphisms

Definition (Isomorphism): Let $\mathscr{C}$ be a category and $A, B \in ob(\mathscr{C})$.
A map $f\colon A \to B$ in $\mathscr{C}$ is an isomorphism if there is a $g\colon B \to A$ s.t. $g \circ f = 1_A$ and $f \circ g = 1_B$.
A and B are isomorphic in $\mathscr{C}$ ($A \cong B$) if there is an isomorphism between them.

4 Examples

- The category of sets, denoted as **Set**, is the category whose objects $ob(\textbf{Set})$ are sets.
  The arrows in **Set** are the functions between two $A, B\ \in ob(\textbf{Set})$.
  The identity- function is defined as $\forall A : set,\ id_A :\ A \to A,\ \forall x \in A\ f\ x = x$
  The composition $\circ$ is the composition of functions and this is associative. Proof:
  Assume sets A,B,C,D, functions $f :\ A \to B$, $g :\ B \to C$, $h :\ C \to D$ and x $\in$ A.
  $((h\ \circ\ g)\ \circ\ f)\ x = ((h\ \circ g\ )(f\ x) = h(g(f\ x))$
  $(h\ \circ\ (g\ \circ f))\ x = h((g\ \circ\ f)\ x) = h(g(f\ x))$
  Thus $(h\ \circ\ g)\ \circ\ f =\ h\ \circ\ (g\ \circ\ f)$ holds. $\square$
- The category of relations, denoted as **Rel**, is the category whose objects are the sets
  The arrows are all binary relations between two $A, B\ \in ob(\textbf{Rel})$
  The identity arrow is the identity function $\forall A : set,\ id_A :\ A \to A,\ \forall x \in A\ f\ x = x$
  The composition $R \circ S$, $R \in \textbf{Rel}(A, B), S \in \textbf{Rel}(B, C), A, B, C \in ob(\textbf{Rel})$ is defined as $(x, y) \in R \circ S \leftrightarrow \exists z.(x, z) \in S \wedge (z, y) \in R$
- Categories similar to **Set** can be constructed for sets which have some additional structure and structure-preserving mappings between them. For example, there is a category **Poset** with partially ordered sets as

objects. It is defined similarly to **Set**, with the difference that the functions are monotone.

- A single poset $(P, \leq)$ also gives rise to a category $\mathscr{P}$ if we take elements of P to be the objects. There is a unique arrow in $\mathscr{P}(A, B)$ iff $A \leq B$. The reflexivity requirement of $\leq$ ensures that identity arrows exist for all objects. Also, the category has composition because the order is transitive.
- $\mathbf{Vect}_k$ is the category of vector spaces over a field k, with linear transformations as arrows.
- There is a category **Grp** whose objects are groups; maps are given by group homomorphisms between two groups $A, B \in ob(\mathbf{Grp})$.
- A single group $(G, \cdot)$ can also be seen as a category $\mathcal{G}$. There is a unique object G which represents the group itself, and arrows in $\mathcal{G}(G, G)$ correspond to group elements.
  The identity arrow is the unit element 1 of the group.
  Composition operates on two group elements and has to be associative, so it is given by the group action.
  A group furthermore requires all elements to be invertible, thus we need each arrow $\mathcal{G}(G, G)$ to be an isomorphism.
  Dropping the last requirement, we get the category of a single monoid.
- Categories need not represent mathematical structures. They can be arbitrarily constructed by giving objects, arrows and arrow composition in a way that satisfies the axioms in the definition. For instance, the category **1** includes a single object and its identity map, without further specification what the object is. A category without non-trivial arrows is called discrete; removing all arrows except identities makes an arbitrary category discrete.

5 Dual categories

C^{op} denotes the dual category for any category C. One obtains C^{op} by reversing the arrows in C. For every sentence Σ in the language of category theory, the reversed sentence Σ^* exists, therefore any proof for any theorem yields for the dual theorem by the duality principle.

6 Terminal and Initial Objects

> **Definition (Initial and terminal Objects):** In any category C, an object 0 is called initial iff for any object $A \in C$, there is an unique morphism $0 \to A$. an object 1 is called terminal iff for any object $A \in C$, there is an unique morphism $A \to 1$. A terminal object in C is initial in C^{op}.

6.1 Proposition:

Initial and terminal objects are unique up to isomorphism.

6.1.1 Proof:

Assume 0,0' are both inital objects in some category C and show that $f : 0 \to 0', g : 0' \to 0$ form an unique isomorphism $f \circ g$ between 0,0'. One can draw the following diagram:

$$\begin{tikzcd}
 & 0' \arrow[rr, "id_{0'}"] \arrow[dr, "g"] & & 0' \\
0 \arrow[ur, "f"] \arrow[rr, "id_0"] & & 0 \arrow[ur, "f"] &
\end{tikzcd}$$

Since 0 is initial, we know that f is unique, from the same argument follows uniqueness of $g = f^{-1}$. Therefore $f \circ g$ and $g \circ f$ is unique.
The same holds for terminal objects by duality. $\square$

Categories in which the terminal is identical to the initial object are called pointed category. Such objects zero objects.

6.1.2 Example:

How to show that $\emptyset$ is initial in Set and the one- element set $\{x\}$ terminal?

- There is only the binary union function from $\emptyset$ to any other set, since there are no arguments in the domain to use.

- Assume some f with
  $\forall A$: set with $A \neq \emptyset,\ f : A \to \{x\}$ f is obviously a constant function, since $\forall y \in A,\ \ f\ y = x$ holds, and therefore unique.