What is a Category?

sarah, leonhard, Andreas Meyer

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1 Definition of Categories

Definition (Category): A category \mathscr{C} has the following components:

- a collection of objects $ob(\mathscr{C})$
- $\forall A, B \in ob(\mathscr{C})$, a collection of maps (arrows, morphisms), denoted by, $\mathscr{C}(A, B)$
- $\forall A, B, C \in ob(\mathscr{C})$, a composition function

$$\mathscr{C}(B,C) \times \mathscr{C}(A,B) \to \mathscr{C}(A,C)$$

 $(g,f) \mapsto (g \circ f)$

• $\forall A \in ob(\mathscr{C})$, an element $1_A \in \mathscr{C}(A,A)$, called the identity on A

The following axioms have to be satisfied:

- identity laws: $\forall f \in \mathcal{C}(A, B). \ f \circ 1_A = f = 1_B \circ f$
- associativity: $\forall f \in \mathscr{C}(A, B), \ g \in \mathscr{C}(B, C), \ h \in \mathscr{C}(C, D). \ (h \circ g) \circ f = h \circ (g \circ f).$

2 Epis and Monis

TODO

3 Isomorphisms

Definition (Isomorphism): Let $\mathscr C$ be a category and $A, B \in ob(\mathscr C)$. A map $f: A \to B$ in $\mathscr C$ is an isomorphism if there is a $g: B \to A$ s.t. $g \circ f = 1_A$ and $f \circ g = 1_B$. A and B are isomorphic in $\mathscr C$ ($A \cong B$) if there is an isomorphism between them.

4 Examples

- The category of sets, denoted as **Set**, is the category whose objects $ob(\mathbf{Set})$ are sets. The arrows in **Set** are the functions between two $A, B \in ob(\mathbf{Set})$. The identity- function is defined as $\forall A : set$, $id_A : A \to A$, $\forall x \in A \ f \ x = x$ The composition \circ is the composition of functions and this is associative. Proof: Assume sets A,B,C,D, functions $f: A \to B, g: B \to C, h: C \to D$ and $x \in A$. $((h \circ g) \circ f) \ x = ((h \circ g)(f \ x) = h(g(f \ x))$ $(h \circ (g \circ f)) \ x = h((g \circ f) \ x) = h(g(f \ x))$ Thus $(h \circ g) \circ f = h \circ (g \circ f)$ holds. \square
- The category of relations, denoted as \mathbf{Rel} , is the category whose objects are the sets The arrows are all binary relations between two $A, B \in ob(\mathbf{Rel})$ The identity arrow is the identity function $\forall A : set, id_A : A \to A, \ \forall x \in A \ f \ x = x$ The composition $R \circ S, R \in \mathbf{Rel}(A, B), S \in \mathbf{Rel}(B, C), A, B, C \in ob(\mathbf{Rel})$ is defined as $(x, y) \in R \circ S \leftrightarrow \exists z. (x, z) \in S \land (z, y) \in R$
- Categories similar to **Set** can be constructed for sets which have some additional structure and structurepreserving mappings between them. For example, there is a category **Poset** with partially ordered sets as

objects. It is defined similarly to **Set**, with the difference that the functions are monotone.

- A single poset (P, \leq) also gives rise to a category \mathscr{P} if we take elements of P to be the objects. There is a unique arrow in $\mathscr{P}(A, B)$ iff $A \leq B$. The reflexivity requirement of \leq ensures that identity arrows exist for all objects. Also, the category has composition because the order is transitive.
- Vect_k is the category of vector spaces over a field k, with linear transformations as arrows.
- There is a category **Grp** whose objects are groups; maps are given by group homomorphisms between two groups $A, B \in ob(\mathbf{Grp})$.
- A single group (G, \cdot) can also be seen as a category \mathcal{G} . There is a unique object G which represents the group itself, and arrows in $\mathcal{G}(G, G)$ correspond to group elements.

The identity arrow is the unit element 1 of the group.

Composition operates on two group elements and has to be associative, so it is given by the group action. A group furthermore requires all elements to be invertible, thus we need each arrow $\mathcal{G}(G,G)$ to be an isomorphism.

Dropping the last requirement, we get the category of a single monoid.

• Categories need not represent mathematical structures. They can be arbitrarily constructed by giving objects, arrows and arrow composition in a way that satisfies the axioms in the definition. For instance, the category 1 includes a single object and its identity map, without further specification what the object is. A category without non-trivial arrows is called discrete; removing all arrows except identities makes an arbitrary category discrete.

5 Dual categories

 C^{op} denotes the dual category for any category C. One obtains C^{op} by reversing the arrows in C. For every sentence Σ in the language of category theory, the reversed sentence Σ^* exists, therefore any proof for any theorem yields for the dual theorem by the duality principle.

6 Terminal and Initial Objects

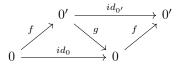
Definition (Initial and terminal Objects): In any category C, an object 0 is called initial iff for any object $A \in C$, there is an unique morphism $0 \to A$. an object 1 is called terminal iff for any object $A \in C$, there is an unique morphism $A \to 1$. A terminal object in C is initial in C^{op} .

6.1 Proposition:

Initial and terminal objects are unique up to isomorphism.

6.1.1 **Proof:**

Assume 0,0' are both inital objects in some category C and show that $f: 0 \to 0', g: 0' \to 0$ form an unique isomorphism $f \circ g$ between 0,0'. One can draw the following diagram:



Since 0 is initial, we know that f is unique, from the same argument follows uniqueness of $g = f^{-1}$. Therefore $f \circ g$ and $g \circ f$ is unique.

The same holds for terminal objects by duality. \Box

Categories in which the terminal is identical to the initial object are called pointed category. Such objects zero objects.

6.1.2 Example:

How to show that \emptyset is initial in Set and the one- element set $\{x\}$ terminal?

• There is only the binary union function from \emptyset to any other set, since there are no arguments in the domain to use.

