What is a Category?

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Category

TODO

Epis and Monis

TODO

Isomorphisms

Examples

- The category of sets, denoted as **Set** is the category whose objects ob(**Set**) are sets.
  The arrows in **Set** are the functions between two $A, B \ \in ob(\textbf{Set})$.
  The identity- function is defined as $\forall A : set,\ id_A :\ A \to A,\ \forall x \in A\ f\ x = x$
  The composition $\circ$ is the composition of functions and this is associative. Proof:
  Assume sets A,B,C,D, functions $f :\ A \to B,\ g :\ B \to C,\ h :\ C \to D$ and x $\in$ A.
  $((h\ \circ\ g)\ \circ\ f)\ x = ((h\ \circ g\ )(f\ x) = h(g(f\ x))$
  $(h\ \circ\ (g\ \circ f))\ x = h((g\ \circ\ f)\ x) = h(g(f\ x))$
  Thus $(h\ \circ\ g)\ \circ\ f =\ h\ \circ\ (g\ \circ\ f)$ holds. $\square$

- The category of relations, denoted as **Rel** is the category whose objects are the sets
  The arrows are all binary relations between two $A, B\ \in ob(\textbf{Rel})$
  The identity arrow is the identity function $\forall A : set,\ id_A :\ A \to A,\ \forall x \in A\ f\ x = x$
  The composition $R \circ S,\ R \in \textbf{Rel}(A, B), S\ \in \textbf{Rel}(B, C),\ A, B, C \in ob(\textbf{Rel})$ is defined as $(x, y) \in R \circ S \leftrightarrow \exists z.(x, z) \in S \wedge (z, y) \in R$
- TODO Grp
- TODO Vec
- TODO Pos
- TODO Discrete, 1, 2

Dual categories

C^{op} denotes the dual category for any category C. One obtains C^{op} by reversing the arrows in C. For every sentence Σ in the language of category theory, the reversed sentence Σ^* exists, therefore any proof for any theorem yields for the dual theorem by the duality principle.

Terminal and Initial Objects

> **Definition (Initial and terminal Objects):** In any category C, an object 0 is called initial iff for any object $A \in C$, there is an unique morphism $0 \to A$. an object 1 is called terminal iff for any object $A \in C$, there is an unique morphism $A \to 1$. A terminal object in C is initial in C^{op}

Proposition:

Initial and terminal objects are unique up to isomorphism.

Proof:

Assume 0,0' are both inital objects in some category C and show that $f : 0 \to 0', g : 0' \to 0$ form an unique isomorphism $f \circ g$ between 0,0'. One can draw the following diagram:

$$\begin{tikzcd}
& 0' \arrow[rr, "id_{0'}"] \arrow[dr, "g"] & & 0' \\
0 \arrow[ur, "f"] \arrow[rr, "id_0"] & & 0 \arrow[ur, "f"] &
\end{tikzcd}$$

Since 0 is initial, we know that f is unique, from the same argument follows uniqueness of $g = f^{-1}$. Therefore $f \circ g$ and $g \circ f$ is unique.
The same holds for terminal objects by duality. $\square$

Categories, in which the terminal is identical to the initial object, are called pointed category. Such objects zero objects.

Example:

How to show that $\emptyset$ is initial in Set and the one- element set $\{x\}$ terminal?

- There is only the binary union function from $\emptyset$ to any other set, since there are no arguments in the domain to use.
- Assume some f with
  $\forall A$: set with $A \neq \emptyset$, $f :\to \{x\}$ f is obviously a constant function, since $\forall y \in A$, $f\ y = x$ holds, and therefore unique.