Differentiation and Partial Derivatives

Derivative f'(x)

$f'(x) = \lim_{h\to 0} (f(x+h) - f(x))/h$

Rate of change at a point

Partial Derivative ∂f/∂x

$\partial f/\partial x_i = \lim_{h\to 0} (f(x+he_i) - f(x))/h$

Derivative w.r.t. one variable, holding others constant

Chain Rule

$d/dx\ f(g(x)) = f'(g(x)) \cdot g'(x)$

Essential for backpropagation

Product Rule

$(fg)' = f'g + fg'$

Quotient Rule

$(f/g)' = (f'g - fg')/g^2$

Gradient ∇f(x)

$$\nabla f(x) = [\partial f/\partial x_1, \partial f/\partial x_2, ..., \partial f/\partial x_n]^T$$

Vector of all partial derivatives

Points in direction of steepest ascent

Magnitude = rate of increase

Hessian Matrix H(f)

$H_{ij} = \partial^2 f/\partial x_i \partial x_j$

Matrix of second derivatives

Captures curvature of function

Optimization Condition

At minimum/maximum:

$\nabla f(x^*) = 0$

ML Applications

- Gradient descent optimization
- Finding regression coefficients
- Minimizing loss functions

- $\partial L/\partial\beta = 0$ solves for optimal β

Example: Computing Derivatives and Gradient

Example 1: Chain Rule

Given: $f(x) = (3x^2 + 2x)^5$

Find: $f'(x)$

Solution: Let $u = 3x^2 + 2x$

$$f(x) = u^5$$

$$f'(x) = 5u^4 \cdot u'$$

$$u' = 6x + 2$$

$f'(x) = 5(3x^2 + 2x)^4(6x + 2)$

Example 2: Partial Derivatives

Given: $f(x,y) = x^2y + 3xy^2 + y^3$

Find: $\partial f/\partial x$ and $\partial f/\partial y$

$\partial f/\partial x$: (treat y as constant)

$$\partial f/\partial x = 2xy + 3y^2$$

$\partial f/\partial y$: (treat x as constant)

$$\partial f/\partial y = x^2 + 6xy + 3y^2$$

$\nabla f = [2xy + 3y^2, x^2 + 6xy + 3y^2]^T$

Example 3: Gradient Descent Step

Loss function: $L(w) = (w - 3)^2$

Current: $w = 0$, learning rate $\alpha = 0.1$

Step 1: Compute gradient

$$\nabla L(w) = 2(w - 3)$$

$$\nabla L(0) = 2(0 - 3) = -6$$

Step 2: Update parameter

$$w\_new = w - \alpha\nabla L(w)$$

$$w\_new = 0 - 0.1(-6) = 0.6$$

New w = 0.6 (closer to minimum at w = 3)

Example 4: Hessian Matrix

Given: $f(x,y) = x^2 + 2xy + 3y^2$

Find: Hessian H(f)

First derivatives:

$$\partial f/\partial x = 2x + 2y$$

$$\partial f/\partial y = 2x + 6y$$

Second derivatives:

$$\partial^2 f/\partial x^2 = 2, \ \partial^2 f/\partial x\partial y = 2$$

$$\partial^2 f/\partial y\partial x = 2, \ \partial^2 f/\partial y^2 = 6$$

H = [2 2]
    [2 6]