Gradient Derivations William Powers 2025

This accompanying document derives the gradients of the sidelobe objective, the constant envelope equality constraint, and the spectral-containment inequality constraint using Wirtinger calculus. Throughout, the complex decision vector $\mathbf{s} \in \mathbb{C}^N$, and real-valued objective and constraint functions are treated as functions of $(\mathbf{s}, \mathbf{s}^*)$.

Preliminaries

For a real-valued function $f(\mathbf{s}, \mathbf{s}^*)$, the Wirtinger gradient is $\nabla_{\mathbf{s}} f = 2\dfrac{\partial f}{\partial \mathbf{s}^*}$. Other useful definitions utilized in the

derivation include $\dfrac{\partial |s_i|^2}{\partial s_i^*} = s_i$, and $\nabla_{\mathbf{z}} \mathbf{z}^p = p\mathbf{z}^{p-2}\mathbf{z}$.

Gradient of the Objective

The sidelobe objective function is given as

$$J = ||\mathbf{w}_{SL} \odot \mathbf{r}||_p^p, \quad \mathbf{r} = \mathbf{A}^H((\mathbf{A}\mathbf{s}) \odot (\mathbf{A}\mathbf{s})^*).$$

Denoting $\mathbf{y} = \mathbf{A}\mathbf{s}$ and $\mathbf{q} = |\mathbf{y}|^2$, the gradient is computed via the backpropogation through $J \to \mathbf{r} \to \mathbf{q} \to \mathbf{y} \to \mathbf{s}$. The gradient w.r.t. $\mathbf{r}$ is first computed ad represented in vector form as

$$\nabla_{\mathbf{r}} J = p|\mathbf{w}_{SL}|^p \odot |\mathbf{r}|^{p-2} \odot \mathbf{r}.$$

Backpropogating through $\mathbf{q}$ yeilds

$$\nabla_{\mathbf{q}} J = \mathbf{A}\nabla_{\mathbf{r}} J$$

since $\mathbf{r} = \mathbf{A}^H \mathbf{q}$. Continuing the chain, backpropogating through $\mathbf{y}$, gives

$$\nabla_{\mathbf{y}} J = 2\mathbf{y} \odot \nabla_{\mathbf{q}} J = 2\mathbf{y} \odot (\mathbf{A}\nabla_{\mathbf{r}} J).$$

Finally backpropogation through $\mathbf{s}$ yeilds the full objective gradient of

$$\nabla_{\mathbf{s}} J = \mathbf{A}^H \nabla_{\mathbf{y}} J = 2\mathbf{A}^H[\mathbf{y} \odot (\mathbf{A}\nabla_{\mathbf{r}} J)] = 2\mathbf{A}^H\left[(\mathbf{A}\mathbf{s}) \odot \mathbf{A}(p|\mathbf{w}_{SL}|^p \odot |\mathbf{r}|^{p-2} \odot \mathbf{r})\right].$$

Gradient of the Equality Constraint

The constant envelope constraint was defined as

$$h(\mathbf{s}) = |\mathbf{s}|^2 - 1.$$

The linear multiplier term $\lambda^T h(\mathbf{s})$ of the full ALM satisfies

$$\nabla_\mathbf{s}[\lambda^T h(\mathbf{s})] = 2\lambda \odot \mathbf{s}$$

whereas the quadtratic penalty term $\frac{\rho}{2}||h(\mathbf{s})||_2^2$ satisfies

$$\nabla_\mathbf{s}\left[\frac{\rho}{2}||h(\mathbf{s})||_2^2\right] = 2\rho(|\mathbf{s}|^2 - 1) \odot \mathbf{s}$$

which when combined results in the full equality constraint gradient

$$\nabla_\mathbf{s} h_{ALM}(\mathbf{s}) = 2\lambda \odot \mathbf{s} + 2\rho(|\mathbf{s}|^2 - 1) \odot \mathbf{s}.$$

Gradient of the Inequality Constraint

The spectral containment constraint was stated as

$$g(\mathbf{s}) = ||\mathbf{s}_F \odot (\mathbf{A}\mathbf{s})||_2^2 - \gamma||\mathbf{s}||_2^2.$$

Denoting $\mathbf{y} = \mathbf{A}\mathbf{s}$ and $\widehat{\mathbf{y}} = \mathbf{w}_F \odot \mathbf{y}$, the gradient of the first term is thus

$$\nabla_\mathbf{y}||\widehat{\mathbf{y}}||_2^2 = 2|\mathbf{w}_F|^2 \odot \mathbf{y}$$

and therefore

$$\nabla_\mathbf{s}||\widehat{\mathbf{y}}||_2^2 = 2\mathbf{A}^H\left(|\mathbf{w}_F|^2 \odot (\mathbf{A}\mathbf{s})\right).$$

The gradient of the second term is simply

$$\nabla_\mathbf{s}\left[-\gamma||\mathbf{s}||_2^2\right] = -2\gamma\mathbf{s}$$

bringing the combined full inequality constraint gradient

$$\nabla_\mathbf{s} g(\mathbf{s}) = 2\mathbf{A}^H\left(|\mathbf{w}_F|^2 \odot (\mathbf{A}\mathbf{s})\right) - 2\gamma\mathbf{s}.$$

In the ALM formulation, the inequality constraint additionally contributes $\mu g(\mathbf{s}) + \frac{\eta}{2}[\max(0, g(\mathbf{s}))]^2$ resulting in the modified gradient

$$\nabla_\mathbf{s} g_{ALM}(\mathbf{s}) = \begin{cases} \mu \nabla_\mathbf{s} g(\mathbf{s}) & \text{for } g(\mathbf{s}) \leq 0, \\ (\mu + \eta g(\mathbf{s}))\nabla_\mathbf{s} g(\mathbf{s}) & \text{for } g(\mathbf{s}) > 0. \end{cases}$$

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