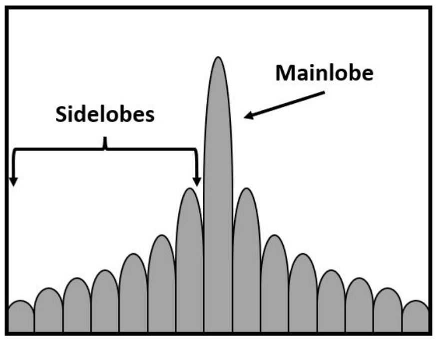
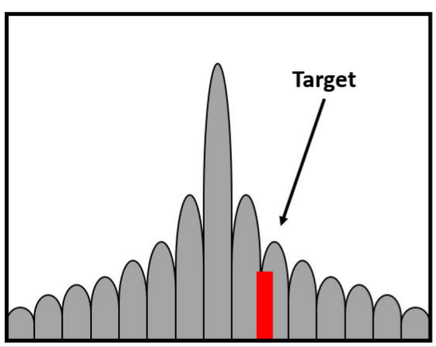
Introduction

Radar system performance is fundamentally constrained by the optimality of the transmitted waveform. For modulated radar signals, the ability to resolve and distinguish targets in range is jointly determined by the waveform's time–bandwidth product and its pulse compression response. The matched filter—defined as the conjugated, time-reversed version of the transmitted waveform—maximizes the output signal-to-noise ratio and thus represents the optimal pulse compression filter. Consequently, waveform design is often formulated as the task of shaping the transmitted signal to yield a desired matched-filter response, with particular emphasis on controlling the mainlobe width and sidelobe structure.



The time–bandwidth product, a system-dependent parameter, primarily governs the main-lobe width of the correlation response and thus directly determines the achievable range resolution. In contrast, the modulation structure, which is a design-dependent property, dictates the shape and magnitude of the sidelobes in the correlation function. Excessive sidelobe energy can obscure weaker targets located near stronger reflections, leading to degraded detection and estimation performance [1]. Therefore, the design of radar waveforms must carefully balance resolution and sidelobe suppression to achieve a desirable correlation structure.



Moreover, as modern radar systems increasingly adopt cognitive or adaptive architectures, the speed and flexibility of waveform synthesis have become critical design considerations. Real-time adaptability to environmental or mission constraints requires optimization frameworks capable of efficiently generating waveforms that meet operational requirements without sacrificing performance.

Problem Formulation:

The objective of the proposed waveform optimization framework is to minimize the matched-filter response sidelobes of a modulated radar waveform while maintaining a fixed system time–bandwidth product. The cost function is defined as

$$J = ||\mathbf{w}_{SL} \odot \mathbf{r}||_p^p$$

where

$$\mathbf{r} = \mathbf{A}^H((\mathbf{A}\mathbf{s}) \odot (\mathbf{A}\mathbf{s})^*)$$

represents the autocorrelation response of the waveform, formed as the inverse Fourier transform of its power spectral density. Here, $\mathbf{s} \in \mathbb{C}^{Nx1}$ denotes the discrete complex waveform samples, $\mathbf{A}$ denotes the discrete Fourier transform (DFT) operator, and $\mathbf{A}^H$ corresponds to its Hermitian (conjugate-inverse) transform. The binary weighting mask $\mathbf{w}_{SL}$ removes the contribution of the mainlobe region, ensuring that the optimization process focuses exclusively on sidelobe minimization. The p-norm provides a tunable measure of the sidelobe energy, with $p = 2$ corresponding to the average sidelobe energy criterion and $p = \infty$ theoretically corresponding to the minimization of the maximum sidelobe level [2].

In addition to sidelobe suppression, the waveform must satisfy two physical constraints to ensure both hardware compatibility and spectral compliance. To minimize distortion and maintain efficient operation of the transmit amplifier, the waveform is required to exhibit a constant-envelope property, such that the magnitude of each complex sample hsa unit magnitude:

$$h = |\mathbf{s}| = \mathbf{s} \odot \mathbf{s}^* = \mathbf{1}_{N\times 1}.$$

This constraint preserves phase-only modulation and prevents nonlinear distortion introduced by amplitude variations in the amplifier. Furthermore, to satisfy spectral containment requirements imposed by the finite system bandwidth and FCC regulation, the waveform's frequency content must remain confined within a prescribed band. This is enforced through spectral mask $\mathbf{w}_F$, which limits the out-of-band spectral energy relative to the total signal power:

$$g = ||w_F \odot (\mathbf{A}\mathbf{s})||_2^2 - \gamma ||s||_2^2 \leq 0.$$

The first term in the above inequality constraint penalizes spectral energy in regions weighted by $\mathbf{w}_F$ while the second term scales the allowable leakage energy through the design parameter γ. Together, these constraints define a feasible design space of physically realizable, spectrally contained, constant-envelope waveforms. The optimization problem is therefore formulated to achieve an optimal balance between autocorrelation performance and practical transmission constraints, as fully defined by the objective function and associated constraints below.

$$\begin{aligned}
&\min_{\mathbf{s}} ||\mathbf{w}_{SL} \odot \mathbf{r}||_p^p \\
&\text{s.t } \ \mathbf{s} \odot \mathbf{s}^* = \mathbf{1}, \\
&||\mathbf{w}_F \odot (\mathbf{A}\mathbf{s})||_2^2 - \gamma ||s||_2^2 \leq 0
\end{aligned}$$

Convexity Analysis

To assess the convexity of the defined problem and motivate an appropriate solver, each term in the formulation must be analyzed individually before combining them into a cumulative objective. Starting with the definition of

$$\mathbf{r} = \mathbf{A}^H((\mathbf{A}\mathbf{s}) \odot (\mathbf{A}\mathbf{s})^*),$$

note that both $\mathbf{A}\mathbf{s}$ and $(\mathbf{A}\mathbf{s})^*$ represent linear and conjugate-linear transformations of $\mathbf{s}$; both individually preserving convexity. The hadamard product between the two, however, is bilinear in $\mathbf{s}$ (linear independently but not jointly) and therefore non-convex in general. For this special case where the same linear operator $\mathbf{A}$ appears in both terms, each row of the resulting element wise multiplication simplifies to

$$|\mathbf{A}\mathbf{s}|_k^2 = |\mathbf{a}_k^H\mathbf{s}|^2 = \mathbf{s}^H(\mathbf{a}_k\mathbf{a}_k^H)\mathbf{s},$$

which is convex quadratic since $\mathbf{a}_k\mathbf{a}_k^H \succcurlyeq 0$. Applying the inverse Fourier transform matrix $\mathbf{A}^H$ corresponds to a linear combination of these quadtratic terms

$$r_i = \sum_k A_{ki}^*\mathbf{a}_k\mathbf{s}^2,$$

where convexity of each filter-response term r_i is preserved if and only if all coefficients of $\mathbf{A}^H$ are real and nonnegative. Being that $\mathbf{A}^H$ does not satisfy this convexity-preserving requirement, $\mathbf{r}$ is non-convex. The application of the binary mainlobe mask can be expressed as a linear mapping using the diagonal operator ($\mathrm{Diag}(\mathbf{w}_{SL})\mathbf{r}$) and therefore does not affect convexity; however, since convexity of $\mathbf{r}$ was already lost in the preceding operation, this transformation has no further effect on the problem's convexity.

The equality constraint $\mathbf{s} \odot \mathbf{s}^* = \mathbf{1}$ forces each entry to lie on the unit circle in $\mathbb{C}$ and therefore is non-convex. The inequality constraint, however, can be written in as a linear mapping using $\mathrm{Diag}(\mathbf{w}_F) = \mathbf{D}$ in place of the hadamard product and therefore can be alternately written as

$$|\mathbf{D}(\mathbf{A}\mathbf{s})|^2 - \gamma||s||_2 = (\mathbf{D}\mathbf{A}\mathbf{s})^H(\mathbf{D}\mathbf{A}\mathbf{s}) - \gamma\mathbf{s}^H\mathbf{I}\mathbf{s} \leq 0.$$

and further simplified into a generalized inner-product form

$$(\mathbf{D}\mathbf{A}\mathbf{s})^H(\mathbf{D}\mathbf{A}\mathbf{s}) - \gamma\mathbf{s}^H\mathbf{I}\mathbf{s} = \mathbf{s}^H(\mathbf{A}^H\mathbf{D}^H\mathbf{D}\mathbf{A} - \gamma\mathbf{I})\mathbf{s} \leq 0.$$

Being that all terms within the paretheses are constant, γ is set such that $\mathbf{A}^H\mathbf{D}^H\mathbf{D}\mathbf{A} - \gamma\mathbf{I}$ is indefinite and therefore solutions that satisfy the inequality constraint correspond to a nonconvex quadtraic solution space.

Propsed Routine

Using the analysis from the previous section, the overall optimization problem is highly non-convex due to non-convexities in both the objective and the two constraints. This precludes the use of standard convex optimization methods and motivates the selection of a more robust nonlinear programming (NLP) solver. In this context, the Alternating Direction Method of Multipliers (ADMM) offers an attractive framework because of its ability to decompose complex, coupled optimization problems into smaller subproblems that can be solved efficiently (and in parallel). Although it is traditionally applied to convex problems, its augmented-Lagrangian structure and iterative splitting approach demonstrate favorable convergence behavoir even in non-convex settings.

Furthermore, convergence speed and computational scalability can be improved by using the recent purposed Nystrom-accelerated ADMM (NysADMM) method [3]. This variant leverages randomized numerical linear algebra (RandNLA) techniques—specifically low-rank Nystrom approximations—to reduce the computational cost of large matrix inversions that occur within the ADMM update steps. By approximating subspace projections using a smaller randomized basis, the NysADMM algorithm achieves faster per-iteration updates while maintaining comparable solution accuracy.

Citations

[1] M. A. Richards, *Fundamentals of Radar Signal Processing*, 3rd ed. New York, NY, USA: McGraw-Hill, 2022.

[2] J. Owen, D. Felton, P. Asuzu, V. Amendolare, and S. Blunt, “Hybrid Random FM Waveforms for Enhanced Range Sidelobe Performance,” in *Proc. IEEE Int. Radar Conf.*, Rennes, France, 2024.

[3] S. Zhao, Z. Frangella, and M. Udell, “NysADMM: faster composite convex optimization via low-rank approximation,” in *Proc. 39th Int. Conf. on Machine Learning (ICML)*, Baltimore, MD, USA, 2022.