Instructions / Outline

Your final project submission should include a theory section, in which you explain the background of the method you use. For deterministic methods, this should include a mathematical description of how the algorithm guarantees local or global optimality. For stochastic methods, an in-depth analysis of the algorithm parameters as well as a mathematical explanation of the convergence properties of the algorithm is required.

Part 1 will be to define the problem, derive the equations required, and describe the assumptions you will make. This part may be hand-written, and should be 2-3 pages in length. Part 1 is due on Friday, November 7 at the beginning of class.

Part 2 of this project will be to solve the problem, either using the GAMS modeling system for an engineering problem-based project, or using a programming language for an algorithm-based project. In both cases, you will need to solve a specific instance of the problem, and give enough explanation such that a user could alter the constants within your program and solve a different instance. The written report will give an overview of the problem and its solution, along with a hard copy of the code written and will be due Wednesday, December 10.

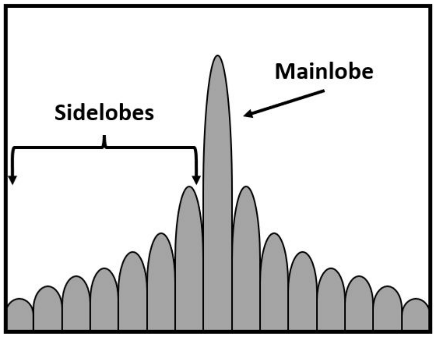
Part 1 Outline:

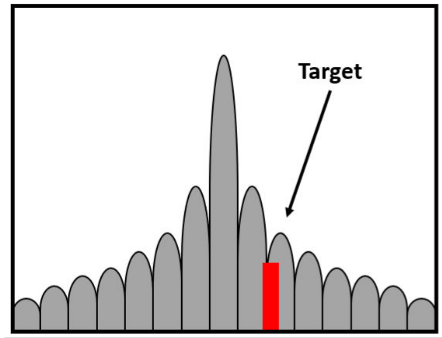
- Introduction: Introduce the idea of radar waveform design, what we want, and why we want it. This should include constant envelope, good autocorrelation, and spectral containtment. Additionally speak to cognitive radar and the importance of speed in some scenarios.
- Problem Formulation: From the derived problem, introduce the cost function, constraints, and supporting variables. Then build theory behind cost function by classifying its differentiability and convexity.
- Proposed Routine: From the theory built in the previous section, relate why the ADMM optimization routine is fit. Relate back to importance of speed in some scenarios and extend ADMM to NysADMM (incooperates randNLA ideaologies) for rapid convergence and problem dimensionality reduction.

Introduction

Radar system performance is fundamentally constrained by the optimality of the transmitted waveform. In particular, for modulated radar waveforms, the ability to distinguish targets in range depends jointly on the waveform's time–bandwidth product and its pulse compression response obtained through matched filtering. Matched filtering—defined as the process of filtering the received signal with a time-reversed, complex-conjugated version of the transmitted waveform—maximizes the output signal-to-noise ratio and produces a correlation-like response. These characteristics jointly determine the degree of range ambiguity that can be achieved and ultimately govern target resolution and separability.

The time–bandwidth product, a system-dependent parameter, primarily governs the main-lobe width of the correlation response and thus directly determines the achievable range resolution. In contrast, the modulation structure, which is a design-dependent property, dictates the shape and magnitude of the sidelobes in the correlation function. Excessive sidelobe energy can obscure weaker targets located near stronger reflections, leading to degraded detection and estimation performance. Therefore, the design of radar waveforms must carefully balance resolution and sidelobe suppression to achieve a desirable correlation structure.





Moreover, as modern radar systems increasingly adopt cognitive or adaptive architectures, the speed and flexibility of waveform synthesis have become critical design considerations. Real-time adaptability to environmental or mission constraints requires optimization frameworks capable of efficiently generating waveforms that meet operational requirements without sacrificing performance.

Problem Formulation:

The objective of the proposed waveform optimization framework is to minimize the autocorrelation sidelobes of a modulated radar waveform while maintaining a fixed system time–bandwidth product. The cost function is defined as

$$J = ||\mathbf{w}_{SL} \odot \mathbf{r}||_p^p$$

where

$$\mathbf{r} = \mathbf{A}^H((\mathbf{A}\mathbf{s}) \odot (\mathbf{A}\mathbf{s})^*)$$

represents the autocorrelation response of the waveform, formed as the inverse Fourier transform of its power spectral density. Here, $\mathbf{s} \in \mathbb{C}^{Nx1}$ denotes the discrete complex waveform, $\mathbf{A}$ is the discreate Fourier transform (DFT) operator, and $\mathbf{A}^H$ corresponds to its Hermitian (inverse) transform. The binary weighting mask $\mathbf{w}_{SL}$ suppresses the contribution of the mainlobe region, ensuring that the optimization process focuses exclusively on sidelobe minimization. The p-norm provides a tunable measure of the sidelobe energy, with $p = 2$ corresponding to the average sidelobe energy criterion and $p = \infty$ theoretically corresponding to the minimization of the maximum sidelobe level [**citation**].

In addition to sidelobe suppression, the waveform must satisfy two physical constraints to ensure both hardware compatibility and spectral compliance. To minimize distortion and maintain efficient operation of the transmit amplifier, the waveform is required to exhibit a constant-envelope property, such that the magnitude of each complex sample remains unity:

$$h = |\mathbf{s}| = \mathbf{s} \odot \mathbf{s}^* = \mathbf{1}_{N\times 1}$$

This constraint preserves phase-only modulation and prevents nonlinear distortion introduced by amplitude variations in the power amplifier. Furthermore, to satisfy spectral containment requirements imposed by the finite system bandwidth and FEC channelization, the waveform's frequency content must remain confined within a prescribed band. This is enforced through a spectral mask $\mathbf{w}_F$, which limits the out-of-band spectral energy relative to the total signal power:

$$g = ||w_F \odot (\mathbf{As})||_2^2 - \gamma||s||_2^2 \leq 0$$

The first term penalizes spectral energy in regions weighted by $\mathbf{w}_F$ while the second term scales the allowable leakage energy through the design parameter γ. Together, these constraints define a feasible design space of physically realizable, spectrally contained, constant-envelope waveforms. The optimization problem therefore seeks to achieve an optimal trade-off between autocorrelation performance and practical transmission constraints.

Summarizing the above, the waveform design optimization has been formulated to minimize the sidelobe energy of the resulting correlation structure while maintaining a constant envelope (equality constraint) and spectral containtment (inequality constraint).

$$\begin{aligned}
&\min_{\mathbf{s}} ||\mathbf{w}_{SL} \odot \mathbf{r}||_p^p \\
&\text{s.t } \ \mathbf{s} \odot \mathbf{s}^* = \mathbf{1}, \\
&||\mathbf{w}_F \odot (\mathbf{As})||_2^2 - \gamma||s||_2^2 \leq 0
\end{aligned}$$

Convexity Analysis

To assess the convexity of the defined problem and motivate an appropriate solver, each term in the formulation must be analyzed individually before combining them into a cumulative objective. Starting with the definition of

$$\mathbf{r} = \mathbf{A}^H((\mathbf{As}) \odot (\mathbf{As})^*),$$

note that both $\mathbf{As}$ and $(\mathbf{As})^*$ represent linear and conjugate-linear transformations of $\mathbf{s}$ individually preserving convexity. The hadamard product between the two, however, is bilinear in $\mathbf{s}$ (linear independently but not jointly) and therefore non-convex in general. For this special case where the same linear operator $\mathbf{A}$ appears in both terms, the resulting element wise multiplication simplifies to

$$|\mathbf{a}_k^H\mathbf{s}|^2 = \mathbf{s}^H(\mathbf{a}_k\mathbf{a}_k^H)\mathbf{s},$$

which is convex quadratic since $\mathbf{a}_k\mathbf{a}_k^H \succcurlyeq 0$. Applying the inverse Fourier transform matrix $\mathbf{A}^H$ corresponds to a linear combination of these quadtratic terms

$$r_i = \sum_k A_{ki}^* \mathbf{a}_k \mathbf{s}^2,$$

where convexity of each r_i is preserved only if all coefficients of $\mathbf{A}^H$ are real and nonnegative. Being that $\mathbf{A}^H$ does not satisfy this convexity-preserving requirement, $\mathbf{r}$ is non-convex. Finally, application of the binary mainlobe mask $\mathbf{w}_{SL} \in \{0, 1\}$ can be seem as a diagonal linear map, $\mathrm{Diag}(\mathbf{w}_{SL})\mathbf{r}$, and does preserve convexity albeit it was already lost through the previous operation.

The equality constraint $\mathbf{s} \odot \mathbf{s}^* = \mathbf{1}$ forces each entry to lie on the unit circle in $\mathbb{C}$ and therefore is non-convex. Rewriting the inequality constraint to in terms using the $\mathrm{Diag}(\mathbf{w}_F) = \mathbf{D}$ notation in place of the hadamard product yeilds

$$(\mathbf{DAs})^H(\mathbf{DAs}) - \gamma \mathbf{s}^H\mathbf{Is} = \mathbf{s}^H(\mathbf{A}^H\mathbf{D}^H\mathbf{DA} - \gamma\mathbf{I})\mathbf{s} \leq 0.$$

Being that all terms within the paretheses are constant, the function collapses to a concave quadtratic since γ is set to make $\mathbf{A}^H\mathbf{D}^H\mathbf{DA} - \gamma\mathbf{I}$ negative semidefinite. Building upon this analysis, concave function $f(\mathbf{x})$ in the inequality $f(\mathbf{x}) \leq 0$ results in the nonconvex region outside of the dome region the solution space to $f(\mathbf{x})$ occupies.

Propsed Routine

Using this derived undertanding of the convexity of the indivdual constraints, the overall optimization problem is highly non-convex due to non-convexities in both the objective and the two constraints. This precludes the use of standard convex optimization methods and motivating the selection of a more robust nonlinear programming (NLP) solver. In this context, the Alternating Direction Method of Multipliers (ADMM) offers an attractive framework because of its ability to decompose complex, coupled optimization problems into smaller subproblems that can be solved efficiently and in parallel. Although it is traditionally applied to convex problems, its augmented-Lagrangian structure and iterative splitting approach demonstrate favorable convergence behavoir even in non-convex settings.

To further improve convergence speed and computational scalability, a Nystrom-accelerated ADMM (NysADMM) solver is also considered. This variant leverages randomized numerical linear algebra (RandNLA) techniques—specifically low-rank Nystrom approximations—to reduce the computational cost of large matrix inversions that occur within the ADMM update steps. By approximating key subspace projections using a smaller randomized basis, the NysADMM algorithm achieves faster per-iteration updates while maintaining comparable solution accuracy.