1. Use the Lagrange Multiplier method to solve the following NLP and explain in words what the meaning of Lagrange multipliers are in this case. Additionally, write GAMS code to check your solution. Consider multiple starting points to help determine if there are multiple local optima.

$$\min \{z = 3x_1^2 + 4x_2^2 + x_1x_3 + x_2x_3\}$$

s.t.

$$x_1 + 3x_2 = 5$$

$$2x_1 + x_3 = 7$$

Analytical Solution: Using the method of Lagrange multipliers, the constraints in the above optimization problem can be included into the objective function as

$$\min \{L(\mathbf{x}, \boldsymbol{\lambda})\} \quad \rightarrow \quad L(\mathbf{x}, \boldsymbol{\lambda}) = 3x_1^2 + 4x_2^2 + x_1x_3 + x_2x_3 + \lambda_1(x_1 + 3x_2 - 5) + \lambda_2(2x_1 + x_3 - 7)$$

Solving the above includes finding where the Lagrangian $L(\mathbf{x}, \boldsymbol{\lambda})$ is equal to zero for all partial derivatives. The gradient of $L(\mathbf{x}, \boldsymbol{\lambda})$ is computed with respect to $\mathbf{x}$ and $\boldsymbol{\lambda}$ below.

$$\nabla L(\mathbf{x}, \boldsymbol{\lambda}) = \begin{cases} \dfrac{\partial L}{\partial x_1} = 6x_1 + x_3 + \lambda_1 + 2\lambda_2 \\ \dfrac{\partial L}{\partial x_2} = 8x_2 + x_3 + 3\lambda_1 \\ \dfrac{\partial L}{\partial x_3} = x_1 + x_2 + \lambda_2 \\ \dfrac{\partial L}{\partial \lambda_1} = x_1 + 3x_2 - 5 \\ \dfrac{\partial L}{\partial \lambda_2} = 2x_1 + x_3 - 7 \end{cases}$$

By setting this gradient equal to zero, a system of five linear equations is formed contianing five unknowns which can easily be computer numerically producing.

$$\mathbf{x}_{opt} = [0.7368 \;\; 1.421 \;\; 5.5263] \qquad \boldsymbol{\lambda}_{opt} = [-5.6433 \;\; -2.1579]$$

Plugging these values back into the objective function to find the optimal cost gives

$$\min \{z\} = 21.63$$

with each constraint satisfied proving the solutions feasability.

$$0.7368 + (3)(1.421) = 5 \qquad (2)(0.7368) + 5.5263 = 7$$

The lagrange multipliers in this equality constrained problem tell us a sensitivity of the objective funciton to each constraint and specifically the distance from optimality the objective function is if it were unconstrained.

Numerical Solution: The following outlines matlab code used to solve the same nonlinear optimization validating the above analytical solution.

```
% objective function
obj = @(x) 3*x(1)^2 + 4*x(2)^2 + x(1)*x(3) + x(2)*x(3);

% equality constraints
Aeq = [1 3 0; 2 0 1];
beq = [5; 7];

% inequality constraints
A = []; b = [];

% initial guess 01
x0 = [1; 1; 1];

% fmincon convex nonlinear solver
opts = optimoptions('fmincon','Display','iter','Algorithm','sqp');
[x_opt, fval] = fmincon(obj, x0, A, b, Aeq, beq, [], [], [], opts);
```

```
 Iter  Func-count            Fval   Feasibility   Step Length       Norm of   First-order
                                                                       step    optimality
    0           4    9.000000e+00     4.000e+00     1.000e+00     0.000e+00     9.000e+00
    1           9    2.104030e+01     1.200e+00     7.000e-01     1.260e+00     8.233e+00
    2          13    2.348537e+01     0.000e+00     1.000e+00     2.134e+00     3.659e+00
    3          17    2.164461e+01     0.000e+00     1.000e+00     1.941e+00     2.201e+00
    4          21    2.163158e+01     0.000e+00     1.000e+00     1.776e-01     1.223e-07

Local minimum found that satisfies the constraints.

Optimization completed because the objective function is non-decreasing in
feasible directions, to within the value of the optimality tolerance,
and constraints are satisfied to within the value of the constraint tolerance.

<stopping criteria details>
```

```
fprintf('Optimal x = [%.6f, %.6f, %.6f]\n', x_opt(1), x_opt(2), x_opt(3));
```

```
Optimal x = [0.736842, 1.421053, 5.526316]
```

```
fprintf('Minimum objective value = %.6f\n', fval);
```

```
Minimum objective value = 21.631579
```

```

% initial guess 02
x0 = [2; 0; 2];

% fmincon convex nonlinear solver
opts = optimoptions('fmincon','Display','iter','Algorithm','sqp');
```

```
[x_opt, fval] = fmincon(obj, x0, A, b, Aeq, beq, [], [], [], opts);
```

```
 Iter  Func-count            Fval   Feasibility   Step Length       Norm of   First-order
                                                                       step    optimality
    0           4    1.600000e+01     3.000e+00     1.000e+00     0.000e+00     1.400e+01
    1           9    1.692684e+01     9.000e-01     7.000e-01     2.968e+00     1.035e+01
    2          13    2.163235e+01     0.000e+00     1.000e+00     9.979e-01     2.753e-01
    3          17    2.163163e+01     0.000e+00     1.000e+00     5.426e-02     6.220e-02
    4          21    2.163158e+01     0.000e+00     1.000e+00     1.110e-02     8.655e-08

Local minimum found that satisfies the constraints.

Optimization completed because the objective function is non-decreasing in
feasible directions, to within the value of the optimality tolerance,
and constraints are satisfied to within the value of the constraint tolerance.

<stopping criteria details>
```

```
fprintf('Optimal x = [%.6f, %.6f, %.6f]\n', x_opt(1), x_opt(2), x_opt(3));
```

```
Optimal x = [0.736842, 1.421053, 5.526316]
```

```
fprintf('Minimum objective value = %.6f\n', fval);
```

```
Minimum objective value = 21.631579
```

```
% initial guess 03
x0 = [10; -1; -1];

% fmincon convex nonlinear solver
opts = optimoptions('fmincon','Display','iter','Algorithm','sqp');
[x_opt, fval] = fmincon(obj, x0, A, b, Aeq, beq, [], [], [], opts);
```

```
 Iter  Func-count            Fval   Feasibility   Step Length       Norm of   First-order
                                                                       step    optimality
    0           4    2.950000e+02     1.200e+01     1.000e+00     0.000e+00     5.900e+01
    1           8    5.685444e+01     1.776e-15     1.000e+00     2.021e+01     5.204e+01
    2          12    2.179247e+01     1.776e-15     1.000e+00     8.610e+00     6.434e+00
    3          16    2.163158e+01     8.882e-16     1.000e+00     6.241e-01     9.558e-08

Local minimum found that satisfies the constraints.

Optimization completed because the objective function is non-decreasing in
feasible directions, to within the value of the optimality tolerance,
and constraints are satisfied to within the value of the constraint tolerance.

<stopping criteria details>
```

```
fprintf('Optimal x = [%.6f, %.6f, %.6f]\n', x_opt(1), x_opt(2), x_opt(3));
```

```
Optimal x = [0.736842, 1.421053, 5.526316]
```

```
fprintf('Minimum objective value = %.6f\n', fval);
```

```
Minimum objective value = 21.631579
```

2. Consider the following minimization problem:

$$\min \{ f(\mathbf{x}) = -x_1^3 + x_2^3 - 2x_1x_3^2\}$$

s.t.

$$2x_1 + x_2^2 + x_3 - 5 = 0$$

$$5x_1^2 - x_2^2 - x_3 \geq 2$$

$$x_1, x_2, x_3 \geq 0$$

(a) Write the KKT conditions for the solution points of the problem.

The Lagrangian of the above problem is written as

$$L(\mathbf{x}, \lambda, \boldsymbol{\mu}) = -x_1^3 + x_2^3 - 2x_1x_3 + \lambda(2x_1 + x_2^2 - +x_3 - 5) + \mu_1(5x_1^2 - x_2^2 - x_3 - 2) + \mu_2(x_1) + \mu_3(x_2) + \mu_4(x_3)$$

From the definition of Lagrange multipliers at an optimum point, the gradient of the above is written as

$$\nabla L(\mathbf{x}, \lambda, \boldsymbol{\mu})\big|_{\mathbf{x}^*} = \nabla f(\mathbf{x}) + \lambda \nabla h(\mathbf{x}) + \sum_{i=1}^{4} \mu_i \nabla g_i(\mathbf{x}) = \mathbf{0}$$

$$\begin{bmatrix} -3x_1^2 - 2x_3^2 \\ 3x_2^2 \\ -2x_1 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ 2x_2 \\ 1 \end{bmatrix} + \mu_1 \begin{bmatrix} 10x_1 \\ -2x_2 \\ -1 \end{bmatrix} + \mu_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \mu_3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \mu_4 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Pairing this with the definition of complementary slackness of constrained problems at an optimum, the KKT conditions are written as

$$\begin{bmatrix} -3x_1^2 - 2x_3^2 \\ 3x_2^2 \\ -2x_1 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ 2x_2 \\ 1 \end{bmatrix} + \mu_1 \begin{bmatrix} 10x_1 \\ -2x_2 \\ -1 \end{bmatrix} + \mu_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \mu_3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \mu_4 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\mu_1(5x_1^2 - x_2^2 - x_3 - 2) = 0,\ \ \mu_2(x_1) = 0,\ \mu_3(x_2) = 0,\ \ \mu_4(x_3) = 0,\ \ \mu_1, \mu_2, \mu_3, \mu_4 \geq 0$$

$$5x_1^2 - x_2^2 - x_3 - 2 \geq 0,\ \ 2x_1 + x_2^2 + x_3 - 5 = 0$$

(b) Vector $\mathbf{x}^* = [1, 0, 3]^T$ is known to be a local minimizer. At $\mathbf{x}^*$, find λ_1^* and μ_i^* for $1 \leq i \leq 4$, and verify that $\mu_i^* \geq 0$ for $1 \leq i \leq 4$.

Plugging local optimal vector $\mathbf{x}_{opt}$ into the above creates a system of thirteen equations with 5 unkowns ($\lambda, \mu_1, \mu_2, \mu_3, \mu_4$) and can be solved using back substitution

$$\mathbf{x}^* = [1, 0, 3], \quad \lambda = \frac{41}{12}, \quad \boldsymbol{\mu} = [\frac{17}{12}, 0, 0, 0]$$

(c) Examine the second-order conditions for set $(\mathbf{x}^*, \lambda^*, \boldsymbol{\mu}^*)$.

Given the full Lagrangian $L(\mathbf{x}, \lambda, \boldsymbol{\mu})$ and its gradient $\nabla L(\mathbf{x}, \lambda, \boldsymbol{\mu})$ derived in (a), the Hessian with respect to $\mathbf{x}$ is shown below

$$\nabla_{\mathbf{x}}^2 L(\mathbf{x}, \lambda, \boldsymbol{\mu}) = \begin{bmatrix} -6x_1 + 10\mu_1 & 0 & -4x^3 \\ 0 & 6x_2 + 2\lambda - 2\mu_1 & 0 \\ -2 & 0 & 0 \end{bmatrix}$$

The definiteness of this hessian at the optimal point is used to determine the optimality where, becuase the problem is a minimization, if the resulting eigenvalues are less than or equal to zero, the solution is a local optimum.

$$\nabla_{\mathbf{x}}^2 L(\mathbf{x}, \lambda, \boldsymbol{\mu}) = \begin{bmatrix} -6x_1 + 10\mu_1 & 0 & -4x^3 \\ 0 & 6x_2 + 2\lambda - 2\mu_1 & 0 \\ -2 & 0 & 0 \end{bmatrix} @ \ \mathbf{x}^* = [1, 0, 3], \quad \lambda = \frac{41}{12}, \quad \boldsymbol{\mu} = [\frac{17}{12}, 0, 0, 0] \rightarrow$$

$$\begin{bmatrix} 8.16 & 0 & -108 \\ 0 & 4 & 0 \\ -2 & 0 & 0 \end{bmatrix}$$

$$\text{eig}\left(\begin{bmatrix} 8.16 & 0 & -108 \\ 0 & 4 & 0 \\ -2 & 0 & 0 \end{bmatrix}\right) \rightarrow \boldsymbol{\lambda}_{\text{eigenvalues}} = [4, 19, 33, -11.17]$$

Becuase the eigenvalues are mixed signed, we can validate that the proposed optimum $\mathbf{x}^* = [1, 0, 3]$ is not a optimum and is instead a saddle point.