***1. Suppose that*** $x_1 = 2, x_2 = 0, x_3 = 4$ **is the optimal solution to the linear programming problem**

$$\max\{z = 4x_1 + 2x_2 + 3x_3\}$$

s.t.

$$2x_1 + 3x_2 + x_3 \leq 12$$

$$x_1 + 4x_2 + 2x_3 \leq 10$$

$$3x_1 + x_2 + x_3 \leq 10$$

$$x_1, x_2, x_3 \geq 0$$

***Formulate the dual of this problem, and using the principle of complementary slackness and the duality theorm, find an optimal solution to the dual problem. What value will the objective function of the dual problem have at this optimal solution?***

**Constructing the Dual**

To form the dual to the above linear programming problem, we transform the standard form

$$\max_{\mathbf{x}\in\mathbb{R}^n}\{\mathbf{c}^\mathbf{T}\mathbf{x}\} \text{ s.t. } \mathbf{A}\mathbf{x} \leq \mathbf{b}, \ \mathbf{x} \geq 0$$

to

$$\min_{\mathbf{w}\in\mathbb{R}^m}\{\mathbf{b}^\mathbf{T}\mathbf{w}\} \text{ s.t. } \mathbf{A}^\mathbf{T}\mathbf{w} \geq \mathbf{c}, \ \mathbf{w} \geq 0$$

thus the dual is

$$\min\{z' = 12w_1 + 10w_2 + 10w_3\}$$

s.t.

$$2w_1 + w_2 + 3w_3 \geq 4$$

$$3w_1 + 4w_2 + w_3 \geq 2$$

$$w_1 + 2w_2 + w_3 \geq 3$$

$$w_1, w_2, w_3 \geq 0$$

**Dual Optimal Solution**

**Theorem 1.1** (Complementary slackness). *Let* $\mathbf{x}$ *be a primal optimal solution and let* $\mathbf{u}$ *be a dual optimal solution. Then:*

- *For* $i = 1, 2, \dots, m$, *either* $\mathbf{x}$ *satisfies the* $i^{th}$ *constraint of* $(\mathbf{P})$ *with equality, or* $u_i = 0$.
- *For* $i = 1, 2, \dots, n$, *either* $x_i = 0$, *or* $\mathbf{u}$ *satisfies the* $i^{th}$ *constraint of* $(\mathbf{D})$ *with equality.*

For the optimal solutions $\mathbf{x}_{opt}$ and $\mathbf{w}_{opt}$, from the complementary slackness theorm we know that if $x_i > 04 then the corresponding dual constraint is tight and if $w_i > 0$ then the corresponding primal constraint is tight. Given the optimal solution to the primal: $x_1 = 2 > 0$ therefore the dual constraint is tight, $x_2 = 0 \not> 0$ therefore the dual can be slack, and $x_3 = 4 > 0$ so this dual constraint is tight as well. Examining the primal constraints, constraint (1) is not tight therefore by complementary slackness $w_1 = 0$. Primal constraints (2) and (3) are both tight therefore $w_2 \geq 0$ and $w_3 \geq 0$. From these conjections, we can directly sovle for dual variables $w_2$ and $w_3$ by setting up the system of linear equations

$$w_1 = 0 \quad w_2 + 3w_3 = 4 \quad 2w_2 + w_3 = 3$$

$$w_1 = 0, \quad w_2 = 1, \quad w_3 = 1$$

This solution is dual feasible and objective value is equivalent to that of the primal proving optimality.

**Primal:** $4(2) + 2(0) + 3(4) = 20$

**Dual:** $12(0) + 10(1) + 10(1) = 20$