

**1. To decorrelate the elements of the noise vector, apply the Karhunen-Loeve transform. Specifically, determine (a) the transformed signal vectors  $s_A'$  and  $s_B'$  as well as the covariance matrix of the transformed noise vector  $w'$ .**

Given that the measurement vector  $\mathbf{x}$  is modeled as  $\mathbf{x} = \mathbf{s} + \mathbf{w}$  where the noise vector  $\mathbf{w}$  is normally distributed as  $\mathcal{N}(0, \mathbf{C}_w)$ , the Karhunen-Loeve transform can be applied to whiten the resulting measurement making the samples of  $\mathbf{x}$  independent and identically distributed (i.i.d) which greatly simplifies detection processing. Specifically this involves determining a linear transformation of the data such that resulting covariance matrix  $\mathbf{C}_x$  is diagonalized.

Being that each hypothesis consists of a deterministic signal,  $s_A$  or  $s_B$ , and additive noise  $\mathbf{w}$ , the measurements covariance structure is solely a result of the noise covariance being that  $\mathbf{w}$  is the only random component. This intuition is represented below in which the mean of the measurement under each hypothesis is

$$\mu_{x|H_A} = s_A \quad \text{and} \quad \mu_{x|H_B} = s_B,$$

with covariance

$$\mathbf{C}_{x|H_A} = E\{(\mathbf{x} - \mu_{x|H_A})(\mathbf{x} - \mu_{x|H_A})^T\} = E\{\mathbf{w}\mathbf{w}^T\} = \mathbf{C}_w,$$

$$\mathbf{C}_{x|H_B} = E\{(\mathbf{x} - \mu_{x|H_B})(\mathbf{x} - \mu_{x|H_B})^T\} = E\{\mathbf{w}\mathbf{w}^T\} = \mathbf{C}_w$$

resulting in each having normal distribution described by

$$\mathbf{x}|H_A \sim \mathcal{N}(s_A, \mathbf{C}_w) \quad \text{and} \quad \mathbf{x}|H_B \sim \mathcal{N}(s_B, \mathbf{C}_w).$$

An eigen-decomposition is performed on this measurement covariance matrix resulting in the the orthonormal matrix  $\mathbf{V}$  containing eigen vectors in each column, and matrix  $\mathbf{\Sigma}$  containing the corresponding eigen-values along its diagonal.

$$\text{eig}(\mathbf{C}_w) = \mathbf{V}\mathbf{\Sigma}\mathbf{V}^T \quad \mathbf{V} = \begin{bmatrix} +0.5 & +0.5 & +0.5 & +0.5 \\ -0.5 & -0.5 & +0.5 & +0.5 \\ +0.5 & -0.5 & -0.5 & +0.5 \\ -0.5 & +0.5 & -0.5 & +0.5 \end{bmatrix} \quad \mathbf{\Sigma} = \begin{bmatrix} 6.0 & 0 & 0 & 0 \\ 0 & 4.0 & 0 & 0 \\ 0 & 0 & 2.0 & 0 \\ 0 & 0 & 0 & 1.0 \end{bmatrix}$$

Defining a new signal basis consisting of the orthonormal eigen vectors in  $\mathbf{V}$ , the measurement vector can be rewritten as  $\mathbf{x} = \mathbf{V}\mathbf{x}' = \sum_{n=1}^4 x_n' v_n$  where  $\mathbf{x}'$  represents the discrete set of coefficients applied to each component of the new basis defined by  $\mathbf{x}' = \mathbf{V}^T \mathbf{x}$ . Writing each of the hypothesized measurements in this orthogonal basis yeilds

$$\mathbf{x}'|H_A = \mathbf{V}^T \mathbf{s}_A + \mathbf{V}^T \mathbf{w} \quad \text{and} \quad \mathbf{x}'|H_B = \mathbf{V}^T \mathbf{s}_B + \mathbf{V}^T \mathbf{w}.$$

Substituting  $\mathbf{V}^T$ ,  $\mathbf{s}_A$ , and  $\mathbf{s}_B$  into the above equations, the transformed signal vectors  $\mathbf{s}_A'$  and  $\mathbf{s}_B'$  are found as

$$\mathbf{s}_A' = \begin{bmatrix} +0.5 & -0.5 & +0.5 & -0.5 \\ +0.5 & -0.5 & -0.5 & +0.5 \\ +0.5 & +0.5 & -0.5 & -0.5 \\ +0.5 & +0.5 & +0.5 & +0.5 \end{bmatrix} \begin{bmatrix} 3.0 \\ -1.0 \\ 1.0 \\ -3.0 \end{bmatrix} = \begin{bmatrix} 4.0 \\ 0.0 \\ 2.0 \\ 0.0 \end{bmatrix} \quad \text{and} \quad \mathbf{s}_B' = \begin{bmatrix} +0.5 & -0.5 & +0.5 & -0.5 \\ +0.5 & -0.5 & -0.5 & +0.5 \\ +0.5 & +0.5 & -0.5 & -0.5 \\ +0.5 & +0.5 & +0.5 & +0.5 \end{bmatrix} \begin{bmatrix} -3.0 \\ 3.0 \\ -5.0 \\ 5.0 \end{bmatrix} = \begin{bmatrix} -8.0 \\ 2.0 \\ 0.0 \\ 0.0 \end{bmatrix}.$$

The transformed measurement covariance matrix is equal to  $\mathbf{\Sigma}$  shown through the relation of the projection matrix  $\mathbf{V}^T$  and the eigen-decomposition of the original covariance structure.

$$\mathbf{C}_w' = \mathbf{V}^T \mathbf{C}_w = \mathbf{V}^T \mathbf{V} \mathbf{\Sigma} \mathbf{V}^T \mathbf{V} = \mathbf{I} \mathbf{\Sigma} \mathbf{I} = \mathbf{\Sigma} = \begin{bmatrix} 6.0 & 0 & 0 & 0 \\ 0 & 4.0 & 0 & 0 \\ 0 & 0 & 2.0 & 0 \\ 0 & 0 & 0 & 1.0 \end{bmatrix}$$

**2. Now, using a LRT threshold of  $\gamma = e^2$ , determine a LRT for this detection problem, expressed in terms of the transformed measurement  $\mathbf{x}' = [x_1', x_2', x_3', x_4']^T$ .**

Using the i.i.d. transformed measurement vector  $\mathbf{x}'$  where each sample normally distributed with a diagonalized covariance matrix, the likelihood functions for each hypothesis can be rewritten as the sum of a set of 4 independent gaussian random variables.

$$p(\mathbf{x}'|H_A) = \frac{1}{(2\pi)^{4/2}(\det(\mathbf{\Sigma})^{1/2})} \exp\left(-\frac{1}{2}(\mathbf{x}' - \mathbf{s}_A')^T \mathbf{\Sigma}(\mathbf{x}' - \mathbf{s}_A')\right)$$

$$p(\mathbf{x}'|H_B) = \frac{1}{(2\pi)^{4/2}(\det(\mathbf{\Sigma})^{1/2})} \exp\left(-\frac{1}{2}(\mathbf{x}' - \mathbf{s}_B')^T \mathbf{\Sigma}(\mathbf{x}' - \mathbf{s}_B')\right)$$

Using these likelihood function definitions, and likelihood ratio  $L(x)$  is defined as

$$L(x) = \frac{\frac{1}{(2\pi)^{4/2}(\det(\mathbf{\Sigma})^{1/2})} \exp\left(-\frac{1}{2}(\mathbf{x}' - \mathbf{s}_B')^T \mathbf{\Sigma}(\mathbf{x}' - \mathbf{s}_B')\right)}{\frac{1}{(2\pi)^{4/2}(\det(\mathbf{\Sigma})^{1/2})} \exp\left(-\frac{1}{2}(\mathbf{x}' - \mathbf{s}_A')^T \mathbf{\Sigma}(\mathbf{x}' - \mathbf{s}_A')\right)}$$

which can be simplified to

$$L(x) == \frac{\exp\left(-\frac{1}{2}(\mathbf{x}' - \mathbf{s}_{B'})^T \mathbf{\Sigma}(\mathbf{x}' - \mathbf{s}_{B'})\right)}{\exp\left(-\frac{1}{2}(\mathbf{x}' - \mathbf{s}_{A'})^T \mathbf{\Sigma}(\mathbf{x}' - \mathbf{s}_{A'})\right)} = \exp(-72x_1 + 8x_2 - 4x_3 - 148)$$

and used in a likelihood ratio test (LRT) with the given threshold  $\gamma = e^2$ :

$$\text{Choose } H_B \text{ if } \exp(-72x_1 + 8x_2 - 4x_3 - 148) \geq e^2 \text{ else } H_A.$$

**3. Simplify this LRT into a decision rule of the form:  $T_d(x') > \gamma'$ .**

Further simplification of the LRT can be made to ultimately express the decision rule in terms of a decision statistic that consists solely of the transformed measurement vector (i.e.  $T_d(\mathbf{x}') = \mathbf{x}'$ ). Taking the natural logarithm of both sides of the LRT inequality derived in part 2 then moving all constants to the right hand side of the equation yields

$$\ln(\exp(-72x_1 + 8x_2 - 4x_3 - 148)) \geq \ln(e^2) \longrightarrow -72x_1 + 8x_2 - 4x_3 \geq 150.$$

Plugging this back into the LRT where  $T_d(\mathbf{x}') = -72x_1 + 8x_2 - 4x_3$  and  $\gamma' = 150$  yields the simplest form of the decision rule:

$$\text{Choose } H_B \text{ if } -72x_1 + 8x_2 - 4x_3 \geq 150 \text{ else } H_A.$$