Problem 25

Under hypothesis $\mathcal{H}_a$, an observation $\mathbf{x} = [x_1, x_2]^T$ is equal to a known signal $\mathbf{s}_a$ with additive Gaussian noise $\mathbf{w}$:

$$\mathbf{x} = \mathbf{s}_a + \mathbf{w}$$

Under hypothesis $\mathcal{H}_b$, an observation $\mathbf{x}$ is equal to a known signal $\mathbf{s}_b$ with additive Gaussian noise $\mathbf{w}$:

$$\mathbf{x} = \mathbf{s}_b + \mathbf{w}$$

The two **known signals** are:

$$\mathbf{s}_a = \left[\frac{5}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]^T \qquad \text{and} \qquad \mathbf{s}_b = \left[\frac{-3}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]^T$$

while the **Gaussian** noise vector $\mathbf{w} = [w_1, w_2]^T$ is distributed as:

$$\mathbf{w} \sim \mathcal{N}(\mathbf{0}, \mathbf{C}_w)$$

where:

$$\mathbf{C}_w = E\{\mathbf{w}\mathbf{w}^T\} = \sum_{n=1}^{2} \lambda_n \, \mathbf{v}_n \mathbf{v}_n^T$$

and:

$$\lambda_1 = 2.0 \qquad \mathbf{v}_1 = \left[\frac{+1}{\sqrt{2}}, \frac{+1}{\sqrt{2}}\right]^T$$

$$\lambda_2 = 1.0 \qquad \mathbf{v}_2 = \left[\frac{+1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right]^T$$

1. To decorrelate the elements of the noise vector, apply the **Karhunen-Loeve transform**.

Specifically, determine

a) the **transformed** signal vectors $\mathbf{s}'_a$ and $\mathbf{s}'_b$, and

b) the **covariance matrix** of the transformed noise vector $\mathbf{w}'$.

2. Now, using a LRT threshold of $\gamma = e^4$, determine a LRT for this detection problem, expressed in terms of **transformed measurement** $\mathbf{x}' = [x'_1, x'_2]^T$.

3. Simplify this LRT into a **decision rule** of the form:

$$T_d(\mathbf{x}') > \gamma'$$

I.E., provide **explicitly**—and in their **simplest** possible form—the statistic $T_d(\mathbf{x}')$ and threshold γ'.