Problem 24

Consider a **real-valued** random variable x .

Under hypothesis $\mathcal{H}_0$, x is described by the *pdf*:

$$p(x|\mathcal{H}_0) = \begin{cases} 3(1-x)^2 & \text{for } 0 < x < 1 \\ 0 & \text{for } x < 0,\ x > 1 \end{cases}$$

While under hypothesis $\mathcal{H}_1$, x is described by the *pdf*:

$$p(x|\mathcal{H}_1) = \begin{cases} 3x^2 & \text{for } 0 < x < 1 \\ 0 & \text{for } x < 0,\ x > 1 \end{cases}$$

The *a priori* probabilities of these two hypotheses are:

$$P(\mathcal{H}_0) = 0.8 \qquad \text{and} \qquad P(\mathcal{H}_1) = 0.2$$

From a **single** observation x, we must (attempt to) **choose** the correct hypothesis ($\mathcal{H}_0$ or $\mathcal{H}_1$).

1. Determine the Likelihood Ratio Test (i.e., **determine** $L(x)$ **and** γ) for both the **Maximum Likelihood** (ML) and **MAP** detection criteria.

2. Simplify the LRT, such that the decision rule can be expressed in terms of this **decision statistic**:

$$T_d(x) = x$$

For **this** decision statistic, determine the values of threshold γ' for **both** the ML and MAP criteria.

3. Say that the **Neyman-Pearson** criterion is instead used, so that the probability of false alarm is

$$P_{FA} = 0.2$$

Determine the resulting **probability of detection** P_D

4. Plot the **Receiver Operating Curve** (ROC) for this detection problem (you may use Matlab to **plot** your expression).

1. Determine the Likelihood Ratio Test (i.e. $L(x)$ and y) for both the ML and MAP detection criteria.

MAP Detection Criteria

Given the likelihood functions $p(x|H_0)$ and $p(x|H_1)$ as well as the a priori probabilities $p(H_0)$ and $p(H_1)$ for hypotheses H_0 and H_1, the maximum a posteriori probability densities can be computed using

$$p(H_0|x) = \frac{p(x|H_0)p(H_0)}{p(x|H_0)p(H_0) + p(x|H_1)p(H_1)} \qquad p(H_1|x) = \frac{p(x|H_1)p(H_1)}{p(x|H_0)p(H_0) + p(x|H_1)p(H_1)}$$

where

$$p(x|H_0) = \begin{cases} 3(1-x)^2 & \text{for } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}, \quad p(x|H_1) = \begin{cases} 3x^2 & \text{for } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$p(H_0) = 0.8\,, \quad p(H_1) = 0.2$$

Plugging these values into the a posteriori expression gives

$$p(H_0|x) = \begin{cases} \dfrac{3(1-x)^2(0.8)}{3(1-x)^2(0.8) + 3x^2(0.2)} & \text{for } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases} \quad p(H_0|x) = \begin{cases} \dfrac{3x^2(0.2)}{3(1-x)^2(0.8) + 3x^2(0.2)} & \text{for } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Simplifying the above expressions yeilds

$$p(H_0|x) = \begin{cases} \dfrac{4(1-x)^2}{4(1-x)^2 + x^2} & \text{for } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases} \quad p(H_1|x) = \begin{cases} \dfrac{x^2}{x^2 + 4(1-x)^2} & \text{for } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

From these posteriori expressions, the MAP binary decision rule is written and rearranged into a likelihood ratio.

$$\text{Choose } H_1 \text{ if } p(H_1|x) > p(H_0|x) \quad \rightarrow \quad \text{Choose } H_1 \text{ if } \frac{p(H_1|x)}{p(H_0|x)} = L(x) > 1.0$$

Plugging in the simplified versions of $p(H_0|x)$ and $p(H_1|x)$ and further simplifiying the likelihood ratio gives

$$L(x) = \frac{\dfrac{x^2}{x^2 + 4(1-x)^2}}{\dfrac{4(1-x)^2}{4(1-x)^2 + x^2}} = \frac{x^2}{4(1-x)^2}$$

Including this simplified form into the decision rule leaves

$$\text{Choose } H_1 \text{ if } \frac{x^2}{(1-x)^2} > 4.0 \text{ and } 0 < x < 1$$

ML Decision Criteria

The ML decision criteria is derived very similarly to the MAP except that no a priori understanding of either hypothesis is included and therefore the decision rule is expressed solely by the likelihood functions of H_0 and H_1 producing the measurement.

$$\text{Choose } H_1 \text{ if } p(x|H_1) > p(x|H_0) \quad \rightarrow \quad \text{Choose } H_1 \text{ if } \frac{p(x|H_1)}{p(x|H_0)} = L(x) > 1.0$$

Plugging in the know values of $p(x|H_0)$ and $p(x|H_1)$ to the likelihood ratio gives

$$L(x) = \frac{3x^2}{3(1-x)^2} = \frac{x^2}{(1-x)^2}$$

with the resulting decision rule ***determine the values of threshold y' for both the ML and MAP criteria.***

To convert the MAP decision rule into one where the detectio

$$\text{Choose } H_1 \text{ if } \frac{x^2}{(1-x)^2} > 1.0 \text{ and } 0 < x < 1$$

2. Simplify the LRT such that the decision rule can be expressed in terms of the decision statistic $T_d(x) = x$. For this decision statistic,

n statistic $T_d(x)$ is equal to the measurement itself, the denomonator of the LRT is moved to the right hand side of the equation and then expanded

$$x^2 > 4(1-x)^2 \rightarrow x^2 > 4 - 8x + 4x^2 \rightarrow -3x^2 + 8x - 4 > 0 \rightarrow (x-2)(3x-2) > 0$$

From the factorization, we see that there are two solutions to x but only one of which lies within the feasability region $0 < x < 1$ stated earlier in the problem therefore the resulting simplified MAP decision rule is

$$\text{Choose } H_1 \text{ if } T_d(x) = x > y' \text{ where } y' = \frac{2}{3} \text{ and } 0 < x < 1$$

Converting the ML decision rule to one using the simplified decisions statistic $T_d(x) = x$ follows a similar process where the denomonator of the LRT is brought over to the right hand side of the inequality, expanded, and like-terms grouped.

$$x^2 > 1(1-x)^2 \rightarrow x^2 > 1 - 2x + x^2 \rightarrow 2x > 1$$

From the grouping of terms and cancellation of the quadratic term, the resulting ML decision rule is thus

Choose H_1 if $T_d(x) = x > y'$ where $y' = \frac{1}{2}$ and $0 < x < 1$

3. Say that the Neyman-Pearson criterion is instead used such that the probability of false alarm is $P_{FA} = 0.2$. Determine the resulting probability of detection, P_D using this false alarm rate.

Building upon the fact that the detection statistic for both the ML and MAP detectors is the measurement itself, i.e. $T_d(x) = x$, we now with determine a new region such that the probability of error (specifically false alarm) is fixed. To do this the definitions of P_{FA} and P_D are used

$$P_{FA} = P(T_d(x) = x \in \mathscr{R}_1 | H_0) = \int_{x \in \mathscr{R}_1} p(x|H_0)dx$$

$$P_D = P(T_d(x) = x \in \mathscr{R}_1 | H_1) = \int_{x \in \mathscr{R}_1} p(x|H_1)dx$$

Fixing P_{FA} as 0.2, a new decision region $\mathscr{R}_1$ must be found that satisfies the equation

$$0.2 = \int_{x \in \mathscr{R}_1} p(x|H_0)dx \quad \rightarrow \quad 0.2 = \int_0^{y'} 3(1-x)^2 dx$$

Using the definition of a intergal, the above can be rearraned to directly solve for y' via

$$0.2 = (3x - 3x^2 + x^3)|_0^{y'} \rightarrow 0 = y'(3 - 3y' + y'^2) - 0.2 \rightarrow 0 = (y' - 1)^3 + 0.8$$

Moving the constant to the left hand side and taking the cubed root yeilds the detection threshold $y' = 0.0716$. This threshold is now used to determine the probability of detection across the interval $0 < x < y'$.

$$P_D = \int_0^{y'} 3x^2 dx = 0.000369$$

4. Plot the receiver operating curve (ROC) for this detection problem.

Using the above intergals for the probablity of false alarm and probability of detection, the ROC is constructed below in matlab.

```
% probability of false alarm span
P_FA = linspace(0.001, 0.999, 1000);
```

```
% loop through each probability of false alarm
P_D = zeros(1, length(P_FA));
for i = 1:length(P_FA)
    % compute region threshold y`
    threshold = 1 - (1-P_FA(i))^(1/3);

    % evaluate probability of detection using region threshold
    P_D(i) = threshold^3;
end

% plot P_FA vrs P_D
```