

1. Determine the likelihood functions $Pr(x|H_0)$ and $Pr(x|H_1)$. In other words, determine the numerical values of the 12 probabilities (p_{00}, p_{10}, \dots and p_{01}, p_{11}, \dots) in the lists above.

The discrete likelihood functions for the biased and unbiased coin are computed using a binomial distribution

$$Pr(x|H_i) = \binom{5}{x} P(\text{'heads'}|H_i)^x (1 - P(\text{'heads'}|H_i))^{5-x}$$

therefore the exhaustive list of probabilities for each hypothesis is

$$Pr(x = 0|H_0) = 0.0312 \quad Pr(x = 0|H_1) = 0.00097$$

$$Pr(x = 1|H_0) = 0.1562 \quad Pr(x = 1|H_1) = 0.0146$$

$$Pr(x = 2|H_0) = 0.3125 \quad Pr(x = 2|H_1) = 0.0878$$

$$Pr(x = 3|H_0) = 0.3125 \quad Pr(x = 3|H_1) = 0.2636$$

$$Pr(x = 4|H_0) = 0.1562 \quad Pr(x = 4|H_1) = 0.3955$$

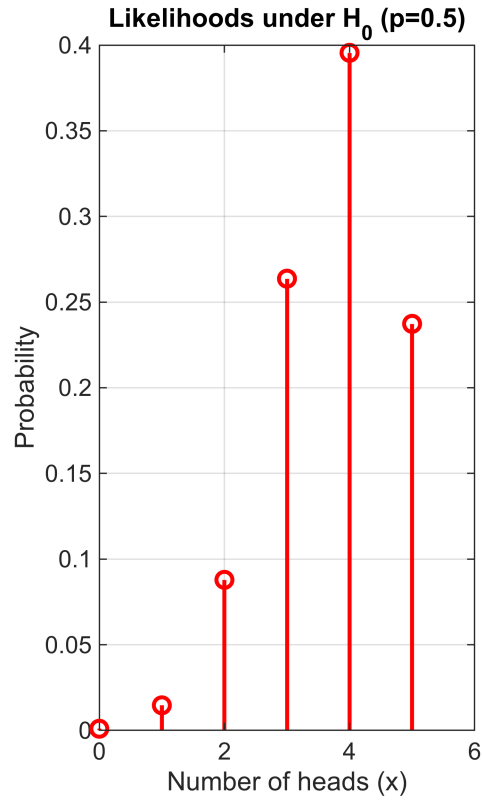
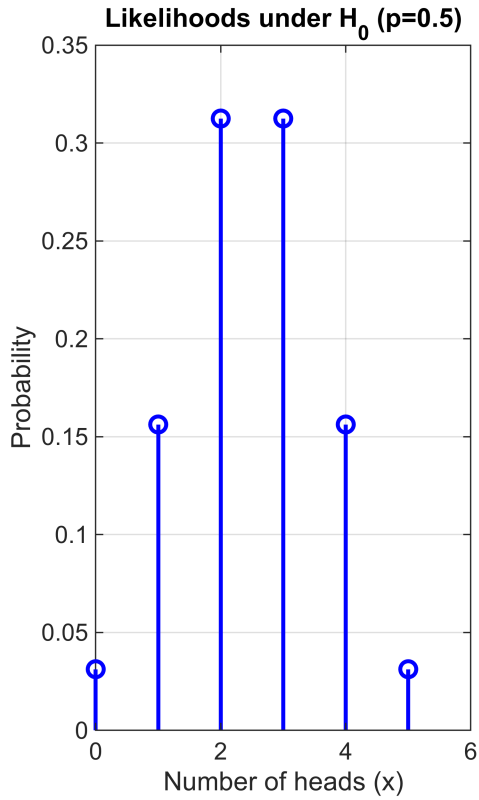
$$Pr(x = 5|H_0) = 0.0312 \quad Pr(x = 5|H_1) = 0.2373$$

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% number of flips
n = 5;
x = 0:n;

% hypothesis 0 distribution
p0 = binopdf(x, n, 0.5);

% hypothesis 1 distribution
p1 = binopdf(x, n, 0.75);

figure("Name", "Discrete Likelihood Functions");
subplot(1, 2, 1);
stem(x, p0, 'b', 'LineWidth', 1.5);
xlabel('Number of heads (x)');
ylabel('Probability');
title('Likelihoods under H_0 (p=0.5)');
grid on;
subplot(1, 2, 2);
stem(x, p1, 'r', 'LineWidth', 1.5);
xlabel('Number of heads (x)');
ylabel('Probability');
title('Likelihoods under H_1 (p=0.75)');
grid on;
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2. Say the decision rule is

Choose H_0 if $x < y'$

Choose H_1 if $x \geq y'$

where

$$y' \in \{0, 1, 2, 3, 4, 5, 6\}.$$

Sketch the ROC for this detection problem. This ROC should consist of 7 points on the (P_{FA}, P_D) plane. Provide the specific coordinates of each of these seven points.

To construct the receiver operator curve, the integral definitions for P_D and P_{FA} must first be defined over the interval $\mathcal{R}_1 = \{y', \infty\}$

$$P_{FA} = P(T_d(x) = x \in \mathcal{R}_1 | H_0) = \int_{x \in \mathcal{R}_1} p(x|H_0) dx$$

$$P_D = P(T_d(x) = x \in \mathcal{R}_1 | H_1) = \int_{x \in \mathcal{R}_1} p(x|H_1) dx$$

Because the likelihood functions are discrete, these integrals are rewritten as summations where $y' \in \{0, 1, 2, 3, 4, 5, 6\}$

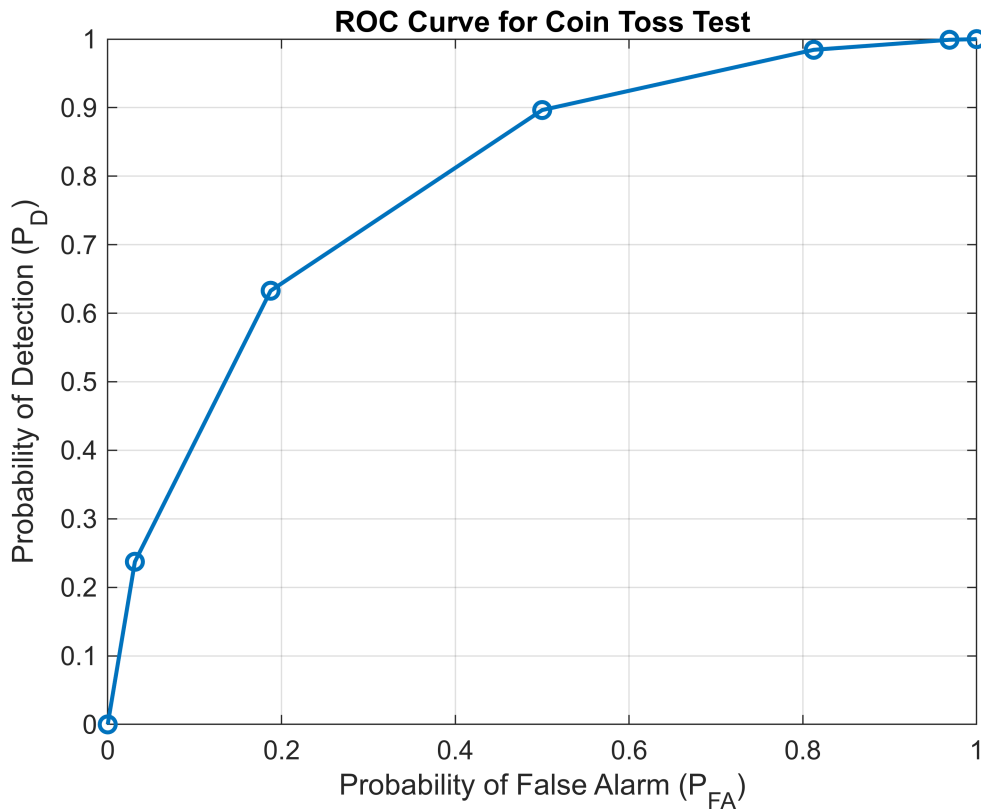
$$P_{FA} = \sum_{x=y'}^5 Pr(x|H_0) dx$$

$$P_D = \sum_{x=y'}^5 Pr(x|H_1) dx$$

and then plotted using the probabilities found in (a).

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n = 5;
x = 0:n;
P_FA = zeros(n+2,1);
P_D = zeros(n+2,1);

for i = 0:n
    P_FA(i+1) = sum(binopdf(i:n, n, 0.5));
    P_D(i+1) = sum(binopdf(i:n, n, 0.75));
end
% Plot ROC curve
figure("Name","ROC");
plot(P_FA, P_D, '-o', 'LineWidth', 1.5, 'MarkerSize', 6);
xlabel('Probability of False Alarm (P_{FA})');
ylabel('Probability of Detection (P_D)');
title('ROC Curve for Coin Toss Test');
grid on;
axis([0 1 0 1]);
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3. Say the a priori probability of a biased coin is $Pr(H_1) = 1/3$. Use the MAP detection criterion to form a decision rule of the form

Choose H_0 if $x < y'$

Choose H_1 if $x \geq y'$

where y' is an integer.

Given the a priori probabilities of seeing a biased and unbiased coin as $Pr(H_1) = 1/3$ and $Pr(H_0) = 2/3$ respectively, the maximum a posteriori probabilities can be determined.

$$Pr(H_0|x) = \frac{Pr(x|H_0)Pr(H_0)}{Pr(x|H_0)Pr(H_0) + Pr(x|H_1)Pr(H_1)} \quad Pr(H_1|x) = \frac{Pr(x|H_1)Pr(H_1)}{Pr(x|H_0)Pr(H_0) + Pr(x|H_1)Pr(H_1)}$$

Plugging in the discrete likelihood functions and a priori probabilities, the above is expressed and simplified as

$$Pr(H_0|x) = \frac{\left(\binom{5}{x}(0.5)^x(1-0.5)^{5-x}\right)(2/3)}{\left(\binom{5}{x}(0.5)^x(1-0.5)^{5-x}\right)(2/3) + \left(\binom{5}{x}(0.75)^x(1-0.75)^{5-x}\right)(1/3)} = \frac{64}{64 + 3^x}$$

$$Pr(H_1|x) = \frac{\left(\binom{6}{x}(0.75)^x(1-0.75)^{6-x}\right)(1/3)}{\left(\binom{6}{x}(0.5)^x(1-0.5)^{6-x}\right)(2/3) + \left(\binom{6}{x}(0.75)^x(1-0.75)^{6-x}\right)(1/3)} = \frac{3^x}{64 + 3^x}$$

Using these two a posteriori probabilities, a likelihood ratio test (LRT) is formed

$$\text{Choose } H_0 \text{ if } \frac{Pr(H_1|x)}{Pr(H_0|x)} = L(x) < 1.0 \quad \text{Choose } H_1 \text{ if } \frac{Pr(H_1|x)}{Pr(H_0|x)} = L(x) \geq 1.0$$

where

$$L(x) = \frac{\frac{3^x}{64 + 3^x}}{\frac{64}{64 + 3^x}} = \frac{3^x}{64}$$

Plugging this into the LRT and further simplifying gives

$$\text{Choose } H_0 \text{ if } x < \frac{\ln(64)}{\ln(3)} \approx 3.78 \quad \text{Choose } H_1 \text{ if } x \geq \frac{\ln(64)}{\ln(3)} \approx 3.78$$

4. Determine the probability of error $Pr(e)$ of this MAP decision rule.

The probability of error is found is the probabilistic view of making the wrong decision. Mathmatically, this is found as the expected value of the conditional distribution between the error e and measurement x .

$$Pr(e) = \sum Pr(e|x)Pr(x)$$

For the MAP decision rule derived in (3),

$$Pr(e|x) = \begin{cases} 1 - Pr(H_0|x) & \text{for } x \in \{0, \dots, 3\} \\ 1 - Pr(H_1|x) & \text{for } x \in \{4, \dots, 5\} \end{cases} \rightarrow Pr(e|x) = \begin{cases} 1 - \frac{64}{64 + 3^x} & \text{for } x \in \{0, \dots, 3\} \\ 1 - \frac{3^x}{64 + 3^x} & \text{for } x \in \{4, \dots, 5\} \end{cases}$$

$$Pr(x) = \left(\binom{5}{x}(0.5)^x(1-0.5)^{5-x}\right)(2/3) + \left(\binom{5}{x}(0.75)^x(1-0.75)^{5-x}\right)(1/3)$$

Plugging this into the equation for $Pr(e)$ yeilds

$$Pr(e) = \sum_{x=0}^3 \left(1 - \frac{64}{64 + 3^x}\right) Pr(x) + \sum_{x=4}^5 \left(1 - \frac{3^x}{64 + 3^x}\right) Pr(x)$$

$$Pr(e) = 0.247$$