

Consider the likelihood function where both θ and x are real scalar values, but θ is "unknown but not random". Determine the Maximum Likelihood Estimate of θ and compute the Cramer-Rao Lower Bound of the error variance of this estimate.

$$p(x|\theta) = \begin{cases} \frac{2}{\pi} \cos^2(x - \theta) & \text{for } -\frac{\pi}{2} < (x - \theta) < \frac{\pi}{2} \\ 0 & \text{otherwise} \end{cases}$$

a. MLE Estimate

To determine the maximum likelihood estimate $\hat{\theta}_{MLE}$, we must determine the value of θ that maximizes the given likelihood function. Mathematically, this is represented as

$$\hat{\theta}_{MLE} = \arg \left[\max_{\theta} \sqrt{(p(x|\theta))} \right]$$

where the square root, beign that the likelihood function is $p(x|\theta) \in (0, 2/\pi]$ over the given domain, allows for simplification to:

$$\hat{\theta}_{MLE} = \arg \left[\max_{\theta} \left(\sqrt{\frac{2}{\pi}} \cos(x - \theta) \right) \right].$$

Finding the maximum of the simplified likelihood function involves finding the local extreme points, and subsequently applying the second derivative test to prove concavity. The derivative of the above is found to be

$$\frac{\partial \sqrt{p(x|\theta)}}{\partial \theta} = \frac{d}{d\theta} \left[\sqrt{\frac{2}{\pi}} \cos(x - \theta) \right] = \sqrt{\frac{2}{\pi}} \sin(x - \theta),$$

where setting this result equal to 0 and solving for θ yeilds the extreme point

$$\sqrt{\frac{2}{\pi}} \sin(x - \theta) = 0 \longrightarrow \theta = x \text{ for } -\frac{\pi}{2} < (x - \theta) < \frac{\pi}{2}.$$

The second derivative of $\sqrt{p(x|\theta)}$ is

$$\frac{d}{d\theta} \left[\sqrt{\frac{2}{\pi}} \sin(x - \theta) \right] = -\sqrt{\frac{2}{\pi}} \cos(x - \theta)$$

which when evaluated at $\theta = x$ yeilds $-\sqrt{2/\pi}$ proving the extreme point is a strict local maximum (and therefore a global maximum over the given domain being that it is the only extreme point). From this conclusion, our maximum likelihood estimate is thus

$$\hat{\theta}_{MLE} = x.$$

b. Cramer-Rao Lower Bound

The Cramer-Rao Lower Bound (CRLB) sets a lower limit on the estimator variance that can be achieved. For classical estimate, it is expressed utilizing expectations over the likelihood function $p(x|\theta)$ as

$$\text{var}_{x|\theta}(\epsilon) \geq I_{x|\theta}(\theta)^{-1}$$

where $\text{var}_{x|\theta}(\epsilon)$ is the error variance of the maximum likelihood estimator (MLE) and $I_{x|\theta}(\theta)^{-1}$ is the inverse of Fisher's information for source θ . From the above, to find this lower bound on the likelihood definition of error variance, we must solve for the source's Fisher information value defined as

$$I_{x|\theta}(\theta) = \mathbb{E}_{x|\theta} \left[\left(\frac{\partial \sqrt{p(x|\theta)}}{\partial \theta} \right)^2 \right].$$

Substituting in the first derivative of $\sqrt{p(x|\theta)}$ found in (a), the expectation over the likelihood function is expressed as

$$I_{x|\theta}(\theta) = \mathbb{E}_{x|\theta} \left[\left(\sqrt{\frac{2}{\pi}} \sin(x - \theta) \right)^2 \right] = \int_{-\frac{\pi}{2} + \theta}^{\frac{\pi}{2} + \theta} \left(\sqrt{\frac{2}{\pi}} \sin(x - \theta) \right)^2 p(x|\theta) dx = \int_{-\frac{\pi}{2} + \theta}^{\frac{\pi}{2} + \theta} \left(\sqrt{\frac{2}{\pi}} \sin(x - \theta) \right)^2 \left(\sqrt{\frac{1}{\pi}} \cos(x - \theta) \right) dx$$

with solution (evaluated numerically in matlab below)

$$I_{x|\theta}(\theta) = \int_{-\frac{\pi}{2} + \theta}^{\frac{\pi}{2} + \theta} \left(\sqrt{\frac{2}{\pi}} \sin(x - \theta) \right)^2 \left(\sqrt{\frac{1}{\pi}} \cos(x - \theta) \right) dx = 0.2394.$$

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% evaluate the integral numerically in matlab
syms x theta real
integrand = (sqrt(2/pi) * sin(x - theta))^2 * (sqrt(1/pi) * cos(x - theta));

lower = -pi/2 + theta;
upper = pi/2 + theta;
I_sym = simplify(int(integrand, x, lower, upper));
I_num = double(subs(I_sym, theta, 0))

I_num = 0.2394
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The inverse of this result therefore bounds the error variance of the MLE estimator as

$$\text{var}_{x|\theta}(\epsilon) \geq \frac{1}{0.2394} \longrightarrow \text{var}_{x|\theta} \geq 4.1771.$$