Problem 1

Consider a **real-valued** random variable x .

Under hypothesis $\mathcal{H}_0$, x is described by the *pdf*:

$$p(x|\mathcal{H}_0) = \begin{cases} \cos x \ \text{ for } 0 < x < \pi/2 \\ 0 \ \text{ for } x < 0,\ x > \pi/2 \end{cases}$$

While under hypothesis $\mathcal{H}_1$, x is described by the *pdf*:

$$p(x|\mathcal{H}_1) = \begin{cases} \sin x \ \text{ for } 0 < x < \pi/2 \\ 0 \ \text{ for } x < 0,\ x > \pi/2 \end{cases}$$

The *a priori* probabilities of these two hypotheses are:

$$Pr(\mathcal{H}_0) = 0.75 \qquad \text{and} \qquad Pr(\mathcal{H}_1) = 0.25$$

From a **single** observation x, we must attempt to **choose** the correct hypothesis ($\mathcal{H}_0$ or $\mathcal{H}_1$).

1. Determine the Likelihood Ratio Test (i.e., **determine** $L(x)$ **and** γ) for both the **Maximum Likelihood** (ML) and **MAP** detection criteria.

2. Simplify the LRT, such that the decision rule can be expressed in terms of this **detection statistic**:

$$T_d(x) = x$$

For **this** detection statistic, determine the values of threshold y' for **both** the ML and MAP criteria.

3. Determine the **probability of error** $Pr(\varepsilon)$ for **both** the MAP **and** the ML decision rule.

Which decision rule exhibits the larger $Pr(\varepsilon)$? **Why** is this true?

4. Say that the **Neyman-Pearson** criterion is used, so that the probability of false alarm is

$$P_{FA} = 0.293$$

Determine the resulting **probability of detection** P_D

1. Determine the Likelihood Ratio Test (i.e. $L(x)$ and y) for both the ML and MAP detection criteria.

MAP Detection Criteria

Given the likelihood functions $p(x|H_0)$ and $p(x|H_1)$ as well as the a priori probabilities $p(H_0)$ and $p(H_1)$ for hypotheses H_0 and H_1, the maximum a posteriori probability densities can be computed using

$$p(H_0|x) = \frac{p(x|H_0)p(H_0)}{p(x|H_0)p(H_0) + p(x|H_1)p(H_1)} \qquad p(H_1|x) = \frac{p(x|H_1)p(H_1)}{p(x|H_0)p(H_0) + p(x|H_1)p(H_1)}$$

where

$$p(x|H_0) = \begin{cases} \cos(x) & \text{for } 0 < x < \frac{\pi}{2} \\ 0 & \text{for } x < 0,\ x > \frac{\pi}{2} \end{cases}, \quad p(x|H_1) = \begin{cases} \sin(x) & \text{for } 0 < x < \frac{\pi}{2} \\ 0 & \text{for } x < 0,\ x > \frac{\pi}{2} \end{cases}, \quad p(H_0) = 0.75, \quad p(H_1) = 0.25$$

Plugging these values into the a posteriori expression yeilds

$$p(H_0|x) = \begin{cases} \dfrac{\cos(x)(0.75)}{\cos(x)(0.75) + \sin(x)(0.25)} & \text{for } 0 < x < \frac{\pi}{2} \\ 0 & \text{for } x < 0,\ x > \frac{\pi}{2} \end{cases}$$

$$p(H_1|x) = \begin{cases} \dfrac{\sin(x)(0.25)}{\cos(x)(0.75) + \sin(x)(0.25)} & \text{for } 0 < x < \frac{\pi}{2} \\ 0 & \text{for } x < 0,\ x > \frac{\pi}{2} \end{cases}$$

Simplifying the above by factoring out constants and utilizing trigonometric identities leaves

$$p(H_0|x) = \begin{cases} \dfrac{1}{1 + (1/3)\tan(x)} & \text{for } 0 < x < \frac{\pi}{2} \\ 0 & \text{for } x < 0,\ x > \frac{\pi}{2} \end{cases} \quad p(H_1|x) = \begin{cases} \dfrac{\tan(x)}{3 + \tan(x)} & \text{for } 0 < x < \frac{\pi}{2} \\ 0 & \text{for } x < 0,\ x > \frac{\pi}{2} \end{cases}$$

From these posteriori expressions, the MAP binary decision rule is written and rearranged into a likelihood ratio.

$$\text{Choose } H_1 \text{ if } p(H_1|x) > p(H_0|x) \quad \rightarrow \quad \text{Choose } H_1 \text{ if } \frac{p(H_1|x)}{p(H_0|x)} = L(x) > 1.0$$

Plugging in the simplified versions of $p(H_0|x)$ and $p(H_1|x)$ and further simplifiying the likelihood ratio gives

$$L(x) = \frac{\frac{\tan(x)}{3 + \tan(x)}}{\frac{1}{1 + (1/3)\tan(x)}} = \frac{\frac{\tan(x)}{3 + \tan(x)}}{\frac{3}{3 + \tan(x)}} = \frac{\tan(x)}{3}$$

Including this simplified form into the decision rule as well as the bounds on x leaves

$$\text{Choose } H_1 \text{ if } \tan(x) > 3.0 \text{ and } 0 < x < \frac{\pi}{2}$$

ML Decision Criteria

The ML decision criteria is derived very similarly to the MAP except that no a priori understanding of either hypothesis is included and therefore the decision rule is expressed solely by the likelihood functions of H_0 and H_1 producing the measurement.

$$\text{Choose } H_1 \text{ if } p(x|H_1) > p(x|H_0) \quad \rightarrow \quad \text{Choose } H_1 \text{ if } \frac{p(x|H_1)}{p(x|H_0)} = L(x) > 1.0$$

Plugging in the know values of $p(x|H_0)$ and $p(x|H_1)$ then simplifying the likelihood ratio leaves

$$\text{Choose } H_1 \text{ if } \tan(x) > 1.0 \text{ and } 0 < x < \frac{\pi}{2}$$

2. Simplify the LRT such that the decision rule can be expressed in terms of the detection statistic: $T_d(x) = x$**. For this detection statistic, determine the values of threshold** y' **for both the ML and MAP criteria.**

To convert the ML and MAP decision rules into one where the detection statistic $T_d(x)$ is equal to the measurement itself, the inverse tangent is taken on both sides of each inequality.

$$\textbf{ML: } \text{Choose } H_1 \text{ if } T_d(x) = x > \tan^{-1}(1.0) \text{ and } 0 < x < \frac{\pi}{2}$$

$$\textbf{MAP: } \text{Choose } H_1 \text{ if } T_d(x) = x > \tan^{-1}(3.0) \text{ and } 0 < x < \frac{\pi}{2}$$

3. Determine the probability of error $Pr(e)$ **for both the MAP and ML decision rule.**

The probability of error is the probabilistic view of making the wrong decision--meaning we choose hypothesis H_i when the measurement instead lies in the region corresponding to hypothesis H_j. Mathmatically, this is found as the expected value of the conditional distribution between the error e and measurement x:

$$Pr(e) = E_x\{p(e|x)\} = \int Pr(e|x)p(x)dx$$

For the **ML** detector

$$p(e|x) = \begin{cases} 1 - p(x|H_0) & \text{for } 0 < x < y' \\ 1 - p(x|H_1) & \text{for } y' < x < \frac{\pi}{2} \end{cases} \quad \rightarrow \quad Pr(e|x) = \begin{cases} 1 - \cos(x) & \text{for } 0 < x < \tan^{-1}(1) \\ 1 - \sin(x) & \text{for } \tan^{-1}(1) < x < \frac{\pi}{2} \end{cases}$$

$$p(x) = \frac{\cos(x) + \sin(x)}{2}$$

Subbing these equations into the intergal given above for $Pr(e)$ yeilds

$$Pr(e) = \int_0^{\tan^{-1}(1)} (1 - \cos(x))\frac{\cos(x) + \sin(x)}{2}dx + \int_{\tan^{-1}(1)}^{\frac{\pi}{2}} (1 - \sin(x))\frac{\cos(x) + \sin(x)}{2}dx$$

$$Pr(e) = \frac{1}{2} - \frac{\pi}{8} = 0.1073$$

For the **MAP** detector

$$p(e|x) = \begin{cases} 1 - p(x|H_0) & \text{for } 0 < x < y' \\ 1 - p(x|H_1) & \text{for } y' < x < \frac{\pi}{2} \end{cases} \quad \rightarrow \quad Pr(e|x) = \begin{cases} 1 - \cos(x) & \text{for } 0 < x < \tan^{-1}(3) \\ 1 - \sin(x) & \text{for } \tan^{-1}(3) < x < \frac{\pi}{2} \end{cases}$$

$$p(x) = (0.75)\cos(x) + (0.25)\sin(x)$$

Subbing these equations into the intergal given above for $Pr(e)$ yeilds

$$Pr(e) = \int_0^{\tan^{-1}(3)} (1 - \cos(x))((0.75)\cos(x) + (0.25)\sin(x))dx + \int_{\tan^{-1}(3)}^{\frac{\pi}{2}} (1 - \sin(x))((0.75)\cos(x) + (0.25)\sin(x))dx$$

$$Pr(e) = \frac{7}{10} - \frac{\tan^{-1}(3)}{4} - \frac{\pi}{16} = 0.1914$$

4. Using a Neyman-Pearson criterion with a probability of false alarm rate of 0.293, determine the resulting probability of detection P_D.

Building upon the fact that the detection statistic for both the ML and MAP detectors is the measurement itself, i.e. $T_d(x) = x$, we now with determine a new region such that the probability of error (specifically false alarm) is fixed. To do this we recall that

$$P_{FA} = P(T_d(x) = x \in \mathscr{R}_1 | H_0) = \int_{x \in \mathscr{R}_1} p(x|H_0)dx$$

$$P_D = P(T_d(x) = x \in \mathscr{R}_1 | H_1) = \int_{x \in \mathscr{R}_1} p(x|H_1)dx$$

Fixing P_{FA} as 0.293, we must find the new decision region $\mathscr{R}_1$ that satisfies the equation

$$0.293 = \int_{x \in \mathscr{R}_1} p(x|H_0)dx \quad \rightarrow \quad 0.293 = \int_0^{y'} \cos(x)dx$$

Being that the cosine operator is bijective, meaning its direct inverse exists, we can rearrange to directly solve for the upper bound y'.

$$0.293 = \sin(y') \quad \rightarrow \quad y' = 0.297$$

Using the derived new region that keeps P_{FA} fixed to 0.293, the probability of detection is computed

$$P_D = \int_0^{y'} \sin(x)dx \quad \rightarrow \quad P_D = \int_0^{0.297} \sin(x)dx \quad \rightarrow \quad P_D = 0.044$$