Problem 25

Under hypothesis $\mathcal{H}_a$, an observation $\mathbf{x} = [x_1, x_2]^T$ is equal to a known signal $\mathbf{s}_a$ with additive Gaussian noise $\mathbf{w}$:

$$\mathbf{x} = \mathbf{s}_a + \mathbf{w}$$

Under hypothesis $\mathcal{H}_b$, an observation $\mathbf{x}$ is equal to a known signal $\mathbf{s}_b$ with additive Gaussian noise $\mathbf{w}$:

$$\mathbf{x} = \mathbf{s}_b + \mathbf{w}$$

The two **known signals** are:

$$\mathbf{s}_a = \left[\frac{5}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]^T \qquad \text{and} \qquad \mathbf{s}_b = \left[\frac{-3}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]^T$$

while the **Gaussian** noise vector $\mathbf{w} = [w_1, w_2]^T$ is distributed as:

$$\mathbf{w} \sim \mathcal{N}(0, C_w)$$

where:

$$C_w = E\{\mathbf{w}\mathbf{w}^T\} = \sum_{n=1}^{2} \lambda_n \, \mathbf{v}_n \mathbf{v}_n^T$$

and:

$$\lambda_1 = 2.0 \qquad \mathbf{v}_1 = \left[\frac{+1}{\sqrt{2}}, \frac{+1}{\sqrt{2}}\right]^T$$

$$\lambda_2 = 1.0 \qquad \mathbf{v}_2 = \left[\frac{+1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right]^T$$

1. To decorrelate the elements of the noise vector, apply the **Karhunen-Loeve transform**.

Specifically, determine

a) the **transformed** signal vectors $\mathbf{s}'_a$ and $\mathbf{s}'_b$, and

b) the **covariance matrix** of the transformed noise vector $\mathbf{w}'$.

2. Now, using a LRT threshold of $\gamma = e^4$, determine a LRT for this detection problem, expressed in terms of **transformed measurement** $\mathbf{x}' = [x'_1, x'_2]^T$.

3. Simplify this LRT into a **decision rule** of the form:

$$T_d(\mathbf{x}') \;>\; \gamma'$$

I.E., provide **explicitly**—and in their **simplest** possible form—the statistic $T_d(\mathbf{x}')$ and threshold γ'.

1. To decorrelate the elements of the noise vector, apply the Karhunen-Loeve transform. Specifically, determine:

a) the transformed signal vectors $\mathbf{s}_a{}'$ and $\mathbf{s}_b{}'$.

b) the covariance matrix of the trasformed noise vector $\mathbf{w}'$.

Given the measurement $\mathbf{x}$ is a two dimensional vector modeled as $\mathbf{x} = \mathbf{s_{a,b}} + \mathbf{w}$ where $\mathbf{w}$ is correlated Guassian noise normally distributed as $\mathcal{N}(0, \mathbf{C}_w)$, the Karhunen-Loeve transform is applied to the data in order to whiten the measurement. Specifically, we wish to apply a linear transform to the measurement such that its resulting covariance matrix $\mathbf{C}_x$ is diagonalized and can be represented as $\sigma_w \mathbf{I}$ where $\mathbf{I}$ is the identity matrix.

First, the measurement covariance is shown to be equivalent to the noise covariance matrix

$$\mu_{x|H_a} = E\{\mathbf{x}|H_a\} = \mathbf{s}_a \quad \mu_{x|H_b} = E\{\mathbf{x}|H_b\} = \mathbf{s}_b$$

$$\mathbf{C}_{X|H_a} = E\{(\mathbf{x} - \mu_{x|H_a})(\mathbf{x} - \mu_{x|H_a})^T\} = E\{\mathbf{w}\mathbf{w}^\mathbf{T}\} = \mathbf{C}_w$$

$$\mathbf{C}_{X|H_b} = E\{(\mathbf{x} - \mu_{x|H_b})(\mathbf{x} - \mu_{x|H_b})^T\} = E\{\mathbf{w}\mathbf{w}^\mathbf{T}\} = \mathbf{C}_w$$

Next, an eigen decomposition is performed on the measurement covariance matrix

$$\mathbf{C}_x = \mathbf{V}\mathbf{\Lambda}_x\mathbf{V}^\mathbf{T}$$

Defining a new signal basis consisting of the orthonormal eigenvectors of matrix $\mathbf{V}$, the measurement vector can be expressed as

$$\mathbf{x} = \sum_{n=1}^{2} x_n{}'\mathbf{v}_n \quad \rightarrow \quad \mathbf{x} = \mathbf{V}\mathbf{x}' \quad \rightarrow \quad \mathbf{x}' = \mathbf{V}^\mathbf{T}\mathbf{x}$$

Expanding the measurent into its signal and noise components for both hypotheses H_a and H_b gives

$$\mathbf{x}'|H_a = \mathbf{V}^\mathbf{T}\mathbf{s_a} + \mathbf{V}^\mathbf{T}\mathbf{w} \quad \mathbf{x}'|H_b = \mathbf{V}^\mathbf{T}\mathbf{s_b} + \mathbf{V}^\mathbf{T}\mathbf{w}$$

Plugging in the values given for $\mathbf{V}$ and $\mathbf{s}_{a,b}$ given in the equation yeilds

$$\mathbf{x}'_{H_a} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}^T \begin{bmatrix} \frac{5}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} + \mathbf{V}^\mathbf{T}\mathbf{w} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} + \mathbf{V}^\mathbf{T}\mathbf{w}$$

$$\mathbf{x}'_{H_b} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}^T \begin{bmatrix} \frac{-3}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} + \mathbf{V^T w} = \begin{bmatrix} -1 \\ -2 \end{bmatrix} + \mathbf{V^T w}$$

therefore

$$\mathbf{s}_a = [3 \;\; 2]^T \qquad \mathbf{s}_b = [-1 \;\; -2]^T$$

As stated previously, through this process, the original noise covariance matrix has been diagonalized since

$$\mathbf{V^T C}_w = \mathbf{V^T V \Lambda}_w \mathbf{V^T V} = \mathbf{\Lambda}_w$$

therefore

$$\mathbf{C}_{w'} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

2. Now, using a LRT threshold of $\gamma = e^4$, determine a LRT for this detection problem, expressed in terms of the transformed measurement $\mathbf{x}' = [x_1', x_2']^T$

Using the transformed measurement vector $\mathbf{x}'$ made up of independent samples due to its covariance matrix being diagonal, the likelihood functions for each hypothesis can be expressed as the sum of independent guassian random variables.

$$p(\mathbf{x}|H_a) = \frac{1}{2\pi\sqrt{2}} \exp\left[\frac{-1}{2}\left(\frac{|x_1' - 3|^2}{2} + \frac{|x_2' - 2|^2}{1}\right)\right] \qquad p(\mathbf{x}|H_b) = \frac{1}{2\pi\sqrt{2}} \exp\left[\frac{-1}{2}\left(\frac{|x_1' + 1|^2}{2} + \frac{|x_2' + 2|^2}{1}\right)\right]$$

The likelihood ratio, $L(\mathbf{x}) = p(\mathbf{x}|H_b)/p(\mathbf{x}|H_a)$, can now be expressed and simplified using a threshold of $\gamma = e^4$

$$L(\mathbf{x}) = \frac{\exp\left[\frac{-1}{2}\left(\frac{|x_1' + 1|^2}{2} + \frac{|x_2' + 2|^2}{1}\right)\right]}{\exp\left[\frac{-1}{2}\left(\frac{|x_1' - 3|^2}{2} + \frac{|x_2' - 2|^2}{1}\right)\right]} = \exp(2 - 2x_1' - 4x_2') \quad \rightarrow \quad \text{Choose } H_b \text{ if } \exp(2 - 2x_1' - 4x_2') \geq e^4$$

3. Simplify this LRT into a decision rule of the form $T_d(\mathbf{x}') > \gamma'$.

Further simplifications can be made in order to express the decision rule in terms of decision statistic $T_d(\mathbf{x}) = (x_1' - x_2')$.

$$\text{Choose } H_b \text{ if } x_1' + 2x_2' \leq -1$$

In the above, $T(\mathbf{x}') = x_1' + 2x_2'$ and $\gamma' = -1$.