EECS 965 Exam I

Fall 2025 - 75 Points

Name________________________________

Instructions

1) Your solutions must be detailed, unambiguous, and organized. I require that you show **all** your mathematics (i.e., no Matlab unless explicitly allowed).

2) Mathematically reduce your solutions to their **simplest** and most fundamental form.

2) Please **underline or circle** all numeric answers (e.g., $|\mathbf{A}| = 11.2$), or in some way make your final answer **clear**.

4) If you feel that a problem is unclear, contradictory, incomplete, or ambiguous, **ask for clarification**.

5) This is an exam; it must reflect **your** knowledge and effort—and yours only!

Please **sign** this statement: "*As an honorable scholar and human being, I pledge that this exam is a reflection of my knowledge only. I hereby pledge that I committed no act that a reasonable person could construe as academic misconduct.*"

Signed:________________________________

Problem 1 – 25 points

Consider a **real-valued** random variable x .

Under hypothesis $\mathcal{H}_0$, x is described by the *pdf*:

$$p(x|\mathcal{H}_0) = \begin{cases} (3/8)(2-x)^2 \;\; for\ 0 < x < 2 \\ 0 \;\; for\ x < 0,\ x > 2 \end{cases}$$

While under hypothesis $\mathcal{H}_1$, x is described by the *pdf*:

$$p(x|\mathcal{H}_1) = \begin{cases} (3/8)x^2 \;\; for\ 0 < x < 2 \\ 0 \;\; for\ x < 0,\ x > 2 \end{cases}$$

The *a priori* probabilities of these two hypotheses are:

$$P(\mathcal{H}_0) = \frac{3}{5} \qquad \text{and} \qquad Pr(\mathcal{H}_1) = \frac{2}{5}$$

From a **single** observation x, we must (attempt to) **choose** the correct hypothesis ($\mathcal{H}_0$ or $\mathcal{H}_1$).

1. Determine the Likelihood Ratio Test (i.e., **determine** $L(x)$ **and** γ) using the **MAP** detection criteria.

2. Simplify the LRT, such that the decision rule can be expressed in terms of this **decision statistic**:

$$T_d(x) = x$$

For **this** decision statistic, determine the value of threshold γ' for MAP criteria.

3. Say you now instead desire a threshold γ' that would result in a **probability of detection** of:

$$P_D = 0.90$$

Determine:

a) the value of this **threshold** γ', and

b) the resulting **probability of false alarm**.

Problem 2 - 25 points

Under hypothesis $\mathcal{H}_a$, an observation $\mathbf{x} = [x_1, x_2, x_3, x_4]^T$ is equal to a known signal $\mathbf{s}_a$ with additive Gaussian noise $\mathbf{w}$:

$$\mathbf{x} = \mathbf{s}_a + \mathbf{w}$$

Under hypothesis $\mathcal{H}_b$, an observation $\mathbf{x}$ is equal to a known signal $\mathbf{s}_b$ with additive Gaussian noise $\mathbf{w}$:

$$\mathbf{x} = \mathbf{s}_b + \mathbf{w}$$

The two **known signals** are:

$$\mathbf{s}_a = [3, -1, 1, -3]^T \qquad \text{and} \qquad \mathbf{s}_b = [-3, 3, -5, 5]^T$$

while the **Gaussian** noise vector $\mathbf{w} = [w_1, w_2, w_3, w_4]^T$ is distributed as:

$$\mathbf{w} \sim \mathcal{N}(\mathbf{0}, \mathbf{C}_w)$$

where:

$$\mathbf{C}_w = E\{\mathbf{w}\mathbf{w}^T\} = \sum_{n=1}^{4} \lambda_n \, \hat{\mathbf{v}}_n \hat{\mathbf{v}}_n^T$$

And also where:

$$\lambda_1 = 6.0 \qquad \hat{\mathbf{v}}_1 = [+0.5, -0.5, +0.5, -0.5]^T$$

$$\lambda_2 = 4.0 \qquad \hat{\mathbf{v}}_2 = [+0.5, -0.5, -0.5, +0.5]^T$$

$$\lambda_3 = 2.0 \qquad \hat{\mathbf{v}}_3 = [+0.5, +0.5, -0.5, -0.5]^T$$

$$\lambda_4 = 1.0 \qquad \hat{\mathbf{v}}_4 = [+0.5, +0.5, +0.5, +0.5]^T$$

1. To decorrelate the elements of the noise vector, apply the **Karhunen-Loeve transform**.

Specifically, determine

a) the **transformed** signal vectors $\mathbf{s}'_a$ and $\mathbf{s}'_b$, and

b) the **covariance matrix** of the transformed noise vector $\mathbf{w}'$.

2. Now, using a LRT threshold of $\gamma = e^2$, determine a LRT for this detection problem, expressed in terms of **transformed measurement** $\mathbf{x}' = [x'_1, x'_2, x'_3, x'_4]^T$ (i.e., in terms of variables x'_1, x'_2, x'_3, x'_4).

3. Simplify this LRT into a **decision rule** of the form:

$$T_d(\mathbf{x}') > \gamma'$$

I.E., provide **explicitly**—and in their **simplest** possible form—the statistic $T_d(\mathbf{x}')$ and threshold γ'.

Problem 3 - 25 points

Under hypothesis $\mathcal{H}_a$, a scalar measurement x is equal to random **variable** w_a:

$$x = w_a$$

Random variable w_a is described by the **Gaussian distribution**:

$$w_a \sim \mathcal{N}\left(\mu = 0, \sigma^2 = 1\right)$$

i.e.,:

$$p(w_a) = \frac{1}{\sqrt{2\pi}} \exp\left[\frac{-w_a^2}{2}\right] \qquad for \ -\infty < w_a < \infty$$

Under hypothesis $\mathcal{H}_b$, the scalar measurement x is equal to random **variable** w_b:

$$x = w_b$$

Random variable w_b is described by a **uniform distribution:**

$$p(w_b) = \begin{cases} 0.2 & for \ -2.5 \leq w_b \leq 2.5 \\ 0 & for \ w_b \leq -2.5,\ w_b \geq 2.5 \end{cases}$$

The two hypotheses are **equally probable**, *a priori* .

1. Use the **MAP** criterion to **determine a decision rule** using the detection statistic:

$$T_d(x) = x$$

Describe carefully and **completely** the **decision regions** $\mathcal{R}_a$ and $\mathcal{R}_b$ for this decision rule.

2. Determine the **probability of error** for this decision rule (you **may** use **MatLab** or other cumulative probability solver for this calculation).