1. Determine exactly the optimal signal vectors θ_{yes} and θ_{no}.

Assuming the signal is correlated with zero-mean Gaussian noise and that each of the two messages is equally probable, the likelihood functions are expressed as

$$p(\mathbf{x}|H_{no}) = \frac{1}{(2\pi^{N/2})\det(\mathbf{C}_w)}\exp\left[-\frac{1}{2}(\mathbf{x}-\boldsymbol{\theta}_{no})^T\mathbf{C}_w^{-1}(\mathbf{x}-\boldsymbol{\theta}_{no})\right]$$
$$p(\mathbf{x}|H_{yes}) = \frac{1}{(2\pi^{N/2})\det(\mathbf{C}_w)}\exp\left[-\frac{1}{2}(\mathbf{x}-\boldsymbol{\theta}_{yes})^T\mathbf{C}_w^{-1}(\mathbf{x}-\boldsymbol{\theta}_{yes})\right]$$

with resulting likelihood ratio

$$L(\mathbf{x}) = \exp\left[\frac{1}{2}(\mathbf{x}-\boldsymbol{\theta}_{no})^T\mathbf{C}_w^{-1}(\mathbf{x}-\boldsymbol{\theta}_{no}) - \frac{1}{2}(\mathbf{x}-\boldsymbol{\theta}_{yes})^T\mathbf{C}_w^{-1}(\mathbf{x}-\boldsymbol{\theta}_{yes})\right] = \exp\left[(\boldsymbol{\theta}_{yes}-\boldsymbol{\theta}_{no})^T\mathbf{C}_w^{-1}\mathbf{x} + \frac{1}{2}\boldsymbol{\theta}_{no}^T\mathbf{C}_w^{-1}\boldsymbol{\theta}_{no} - \frac{1}{2}\boldsymbol{\theta}_{yes}^T\mathbf{C}_w^{-1}\boldsymbol{\theta}_{yes}\right]$$

and resulting decision rule

$$L(\mathbf{x}) = \exp\left[(\boldsymbol{\theta}_{yes}-\boldsymbol{\theta}_{no})^T\mathbf{C}_w^{-1}\mathbf{x} + \frac{1}{2}\boldsymbol{\theta}_{no}^T\mathbf{C}_w^{-1}\boldsymbol{\theta}_{no} - \frac{1}{2}\boldsymbol{\theta}_{yes}^T\mathbf{C}_w^{-1}\boldsymbol{\theta}_{yes}\right] \lessgtr 1\,.$$

Simplifying the above decision rule into sufficient detection statistic $T_d(\mathbf{x})$ and threshold γ yeilds

$$T_d(\mathbf{x}) = (\boldsymbol{\theta}_{yes}-\boldsymbol{\theta}_{no})^T\mathbf{C}_w^{-1}\mathbf{x} \lessgtr \ln(1) + \frac{1}{2}\boldsymbol{\theta}_{yes}^T\mathbf{C}_w^{-1}\boldsymbol{\theta}_{yes} - \frac{1}{2}\boldsymbol{\theta}_{no}^T\mathbf{C}_w^{-1}\boldsymbol{\theta}_{no}.$$

This simplified detection is expressed as the difference between the generalized matched filter outputs of the expected values of H_{yes} and H_{no}. Given that the measurement is the cumulation of the transmitted signal $\boldsymbol{\theta}$ and noise vector $\mathbf{w}$, each of the matched filter outputs can be expanded to

$$T_{gmf,yes} = \boldsymbol{\theta}_{yes}^T\mathbf{C}_w^{-1}\mathbf{x} = \boldsymbol{\theta}_{yes}^T\mathbf{C}_w^{-1}\boldsymbol{\theta} + \boldsymbol{\theta}_{yes}^T\mathbf{C}_w^{-1}\mathbf{w}$$

$$T_{gmf,no} = \boldsymbol{\theta}_{no}^T\mathbf{C}_w^{-1}\mathbf{x} = \boldsymbol{\theta}_{no}^T\mathbf{C}_w^{-1}\boldsymbol{\theta} + \boldsymbol{\theta}_{no}^T\mathbf{C}_w^{-1}\mathbf{w}$$

with resulting signal-to-noise-ratio (SNR)

$$SNR_{out} = \frac{E\{|\boldsymbol{\theta}^T\mathbf{C}_w^{-1}\boldsymbol{\theta}|^2\}}{E\{|\boldsymbol{\theta}^T\mathbf{C}_w^{-1}\mathbf{w}|^2\}} = \frac{|\boldsymbol{\theta}^T\mathbf{C}_w^{-1}\boldsymbol{\theta}|^2}{|\boldsymbol{\theta}^T\mathbf{C}_w^{-1}\boldsymbol{\theta}|} = |\boldsymbol{\theta}^T\mathbf{C}_w^{-1}\boldsymbol{\theta}| \quad \text{where} \quad \boldsymbol{\theta} = \boldsymbol{\theta}_{yes} - \boldsymbol{\theta}_{no}$$

Keeping signal energy constant for both hypotheses, the resulting generalized matched filter output SNR can be maximized by examining the eigen expansion of the noise correlation matrix in the above expression.

$$SNR_{out} = |\boldsymbol{\theta}^T\mathbf{C}_w^{-1}\boldsymbol{\theta}| = |(\mathbf{V}\boldsymbol{\theta})^T\boldsymbol{\Lambda}_w^{-1}(\mathbf{V}\boldsymbol{\theta})| = \left|\sum_{n=1}^{N}\frac{|\mathbf{v}_n(\theta_{n,yes}-\theta_{n,no})|^2}{\lambda_n}\right|$$

Using the above eigen representation, we see that to maximize SNR_{out} the difference between the two signals must be distributed fully into the term associated with the smallest eigenvalue λ_n. Using this intuition, the optimal signals are thus

$$|\boldsymbol{\theta}_{yes} - \boldsymbol{\theta}_{no}|_{opt} = \mathbf{v}_{min}|(\boldsymbol{\theta}_{yes} - \boldsymbol{\theta}_{no})|$$

The above equation tells us what the difference between the two hypotheses signal vectors should be but not the individual signal vectors themselves. To further maximize the resulting SNR_{out} value, the difference between these two signal vectors must also be maximized which, when relating them to their corresponding vector space, results in each being colinear pointing in oppositite directions $\rightarrow \boldsymbol{\theta}_{yes} = -\boldsymbol{\theta}_{no}$. Using the logic derived above, the optimal signals for the given detection problem are

$$\boldsymbol{\theta}_{yes,opt} = \mathbf{v}_{min}|\theta_{yes}| = \mathbf{v}_{min}\sqrt{\boldsymbol{\theta}_{yes}^T\boldsymbol{\theta}_{yes}} = [+0.5, -0.5, -0.5, +0.5]$$

$$\boldsymbol{\theta}_{no,opt} = \mathbf{v}_{min}(-|\theta_{no}|) = \mathbf{v}_{min}\left(-\sqrt{\boldsymbol{\theta}_{no}^T\boldsymbol{\theta}_{no}}\right) = [-0.5, +0.5, +0.5, -0.5]$$

2. If these optimal signal vectors are indeed implemented, determine exactly the decision rule that minimizes the probability of error.

Utilizing the optimal signal vectors found in (1) in the maximum a posteroiri decision rule also derived in (1) yeilds (calculated using matlab)

$$T_d(\mathbf{x}) = (\boldsymbol{\theta}_{yes} - \boldsymbol{\theta}_{no})^T\mathbf{C}_w^{-1}\mathbf{x} \le \ln(1) + \frac{1}{2}\boldsymbol{\theta}_{yes}^T\mathbf{C}_w^{-1}\boldsymbol{\theta}_{yes} - \frac{1}{2}\boldsymbol{\theta}_{no}^T\mathbf{C}_w^{-1}\boldsymbol{\theta}_{no}$$

```
% Optimal signal vectors
theta_yes = [0.5; -0.5; -0.5; 0.5];
theta_no  = -theta_yes;

% Inverse noise correlation matrix (reconstructed)
Lambda = diag([7, 1, 2, 6]);
V = [
    0.5, -0.5,  0.5, -0.5;
    0.5, -0.5,  0.5,  0.5;
    0.5,  0.5, -0.5, -0.5;
    0.5,  0.5,  0.5,  0.5
];

% Compute inverse covariance matrix
invCw = V * inv(Lambda) * inv(V);  %#ok

% Compute decision threshold
```

```
decision_threshold = log(1) + ...
    0.5 * (theta_yes' * invCw * theta_yes) - ...
    0.5 * (theta_no'  * invCw * theta_no);

fprintf("Decision threshold = %.4f\n", decision_threshold);
```

```
Decision threshold = 0.0000
```

Thus the decision rule is

$$\text{Choose } \ H_{yes} \ \text{ if } \ [0.5 - 0.5, -0.5, 0.5]^T \mathbf{x} \leq 0$$

$$\text{Choose } \ H_{no} \ \text{ if } \ [0.5 - 0.5, -0.5, 0.5]^T \mathbf{x} \leq 0$$

3. Determine exactly the deflection coefficient associated with this decision rule.

The deflection coefficient is used in the determination of the ROC for two signal detection examples with white noise. Specifically, it describes the generalized matched filter output using the difference between the two signals as the measured and matched signal.

$$d^2 = (\mathbf{V}(\boldsymbol{\theta}_{yes} - \boldsymbol{\theta}_{no})_{opt})^T \mathbf{C}_w^{-1} (\mathbf{V}(\boldsymbol{\theta}_{yes} - \boldsymbol{\theta}_{no})_{opt})$$

```
deflection_coef = (V * (theta_yes - theta_no))' * invCw * (V * (theta_yes -
theta_no));
fprintf("Deflection coefficient = %.4f\n", sqrt(deflection_coef));
```

```
Deflection coefficient = 0.6362
```