A scalar random variable x is Gaussian distributed as $x \sim \mathcal{N}(0, \sigma^2)$. (a) Say we wish to estimate the standard deviation from an observation x; does an efficient estimator exist? If so, what is this estimator and what is its error variance. (b) Say we wish to estimate the variance σ^2 from an observation x; an efficient estimator exist? If so, what is this estimator and what is its error variance?

a. Estimate of σ

From a classical estimation perspective, given an observation from random variable x, a efficient estimator can be proven to exist via the Cramer-Rao Lower Bound (CRLB) theorm. This states that if the derivative of the log-likelihood function can be written be written in the form

$$\frac{\partial \ln(p(x|\theta))}{\partial \theta} = f(\theta)(g(x) - \theta)$$

where $f(\theta)$ and $g(x)$ are functions of θ and x respectively, then an efficient estimator exists and that the efficient estimate of θ is equal to $g(x)$. Substituting $\theta \rightarrow \sigma$ as the parameter we are trying to estimate and $x \rightarrow x_i$ as a single measurement of random variable x, the CRLB theorm is rewritten as

$$\frac{\partial \ln(p(x_i|\sigma))}{\partial \sigma} = f(\sigma)(g(x_i) - \sigma)$$

where our maximum likelihood estimate of σ is equal to $g(x_i)$ (assuming the first derivative can be written in this form). The likelihood function of σ generating sample x_i is expressed using a normal distribution

$$p(x_i|\theta = \sigma) = \frac{1}{\sqrt{2\pi\theta^2}} \exp\left(-\frac{x_i}{2\theta^2}\right)$$

with the first derivative of the log of this equation given as

$$\frac{d}{d\theta}\left[\frac{1}{\sqrt{2\pi\theta^2}} \exp\left(-\frac{x_i}{2\theta^2}\right)\right] = \frac{d}{d\theta}\left[-\ln(\sqrt{2\pi\theta^2}) + \ln\left(-\frac{x_i}{2\theta^2}\right)\right] = \frac{d}{d\theta}\left[-\ln(\sqrt{2\pi\theta^2})\right] + \frac{d}{d\theta}\left[\ln\left(-\frac{x_i}{2\theta^2}\right)\right] = -\frac{3}{\theta}.$$

Relating this result back to the CRLB theorm, it is clear that we will not be able to write the log likelihood derivative in the form $f(\sigma)(g(x_i) - \sigma)$ therefore it has been proven that no efficient estimator exists.

b. Estimate of σ^2

Following the logic built in part a, proving that their exists an efficient estimator of σ^2 is done through showing that the derivative of the log-likelihood function can be written as

$$\frac{\partial \ln(p(x|\theta))}{\partial\theta} = f(\theta)(g(x) - \theta)$$

where $\theta = \sigma^2$. The likelihood function of σ^2 generatign sample x_i is again expressed using the normal distribution

$$p(x_i|\theta = \sigma^2) = \frac{1}{\sqrt{2\pi\theta}}\exp\left(-\frac{x_i}{2\theta}\right)$$

with the first derivative of the log of this equation given as

$$\frac{d}{d\theta}\left[\frac{1}{\sqrt{2\pi\theta}}\exp\left(-\frac{x_i}{2\theta}\right)\right] = \frac{d}{d\theta}\left[-\ln(\sqrt{2\pi\theta}) + \ln\left(-\frac{x_i}{2\theta}\right)\right] = \frac{d}{d\theta}\left[-\ln(\sqrt{2\pi\theta})\right] + \frac{d}{d\theta}\left[\ln\left(-\frac{x_i}{2\theta}\right)\right] = \frac{x_i}{2\theta^2} - \frac{1}{2\theta}.$$

Here, the resulting derivative is written in the form $f(\theta)(g(x) - \theta)$ therefore there exists a efficient estimator and that estimator is equal to $g(x)$. Factoring out $\frac{1}{2\theta}$ from both terms, we see that $g(x)$ is equivalent to

$$\frac{1}{2\theta}\left(\frac{x_i}{\theta} - 1\right) \longrightarrow g(x) = \frac{x_i}{\theta}.$$