1. Determine the Likelihood Ratio Test (i.e. determine $L(x)$ and γ) using the MAP detection criterion.

Given the likelihood functions $p(x|H_0)$ and $p(x|H_1)$ as well as the a priori probabilities $p(H_0)$ and $p(H_1)$ for hypotheses H_0 and H_1, the maximum a posteriori probability (MAP) densities for each hypothesis are computed using the Bayes' rule relation

$$p(H_i|x) = \frac{p(x|H_i)p(H_i)}{p(x)} = \frac{p(x|H_i)p(H_i)}{\sum_i p(x|H_i)p(x)dx}.$$

Substituing in the given likelihood and a priori densities into the above equation, the MAP densities for each hypothesis are given as

$$p(H_0|x) = \begin{cases} \dfrac{(3/8)(2-x)^2(3/5)}{(3/8)(2-x)^2(3/5) + (3/8)(x)^2(2/5)} & \text{for } 0 < x < 2 \\ 0 & \text{for } x < 0,\ x > 2 \end{cases},$$

$$p(H_1|x) = \begin{cases} \dfrac{(3/8)(x)^2(2/5)}{(3/8)(2-x)^2(3/5) + (3/8)(x)^2(2/5)} & \text{for } 0 < x < 2 \\ 0 & \text{for } x < 0,\ x > 2 \end{cases}.$$

Simpilifying the above expressions yeilds

$$p(H_0|x) = \begin{cases} \dfrac{3(2-x)^2}{3(2-x)^2 + 2x^2} & \text{for } 0 < x < 2 \\ 0 & \text{for } x < 0,\ x > 2 \end{cases} \quad \text{and} \quad p(H_1|x) = \begin{cases} \dfrac{2x^2}{3(2-x)^2 + 2x^2} & \text{for } 0 < x < 2 \\ 0 & \text{for } x < 0,\ x > 2 \end{cases}.$$

From these a posteriori expressions, the MAP decision rule can be constructed and rearranged into a likelihood ratio using measurement dependent likelihood function $L(x)$. Because both a posteriori definitions are bounded along the interval $[0, 2]$ and are equivalently 0 outside of this region, the two hypothesis are treated as equally likely.

$$\text{Choose } H_1 \text{ if } p(H_1|x) > p(H_0|x) \text{ and } 0 < x < 2 \text{ else choose } H_0$$

$$\downarrow$$

$$\text{Choose } H_1 \text{ if } \frac{p(H_1|x)}{p(H_0|x)} = L(x) > 1.0 \text{ else choose } H_0$$

2. Simplify the LRT, such that the decision rule can be expressed in terms of the detection statistic: $T_d(x) = x$***. For this decision statistic, determine the value of threshold*** γ' ***for the MAP criterion.***

3. Say you now instead desire a threshold γ' ***that would result in a probability of detection of*** $P_D = 0.90$***. Determine (a) the value of this threshold*** γ'***, and (b) the resulting probability of false alarm.***