Problem 26

Consider a random variable w_n, described with an **exponential** ***pdf***:

$$p(w_n) = \begin{cases} \lambda_n e^{-\lambda_n w_n} & for \quad w_n \geq 0 \\ 0 & for \quad w_n < 0 \end{cases}$$

Under hypothesis $\mathcal{H}_0$, scalar measurement x is equal to **random variable** w_0:

$$x = w_0$$

where $\lambda_0 = 4$.

Under hypothesis $\mathcal{H}_1$, scalar measurement x is equal to **random variable** w_1:

$$x = w_1$$

where $\lambda_1 = 2$.

Under hypothesis $\mathcal{H}_2$, scalar measurement x is equal to **random variable** w_2:

$$x = w_2$$

where $\lambda_2 = 1$.

Each hypothesis is **equally probable**, *a priori*.

1. Use the MAP criterion to **determine a decision rule** using the detection statistic:

$$T_d(x) = x$$

2. Calculate the resulting **confusion matrix** for this decision rule.

1. Use the MAP criterion to determine a decision rule using the detection statistic $T_d(x) = x$.

Before finding a sufficient statistic and forming the MAP decision criterion, the likelihood functions for each hypothesis producing the measurement x must be expressed.

$$p(x|H_0) = \lambda_0 e^{-\lambda_0 x} \text{ for } \lambda_0 = 4 \text{ and } x \geq 0$$

$$p(x|H_1) = \lambda_1 e^{-\lambda_1 x} \text{ for } \lambda_1 = 2 \text{ and } x \geq 0$$

$$p(x|H_2) = \lambda_2 e^{-\lambda_2 x} \text{ for } \lambda_0 = 4 \text{ and } x \geq 0$$

Given that each hypothesis is equally probable, the a posteriori probabilites can be written as

$$p(H_i|x) = \frac{p(x|H_u i)p(H_i)}{\sum_k p(x|H_k)p(k)} = \frac{\left(\lambda_i e^{-\lambda_i x}\right)(1/3)}{(1/3)\left(\sum_k \lambda_k e^{-\lambda_k x}\right)} \text{ for } k = 0, 1, 2$$

Cancelling the $(1/3)$ that occurs in both the numerator and denomonator, then moving the remaining denomonator to the left hand side of the equation, the sufficient statistic for hypothesis H_i is formed.

$$T_s(H_i|x) = \lambda_i e^{-\lambda_i x}$$

Applying the natural log to both sides allows for further simplications through the logarithmic power rules.

$$T_s(H_i|x) = \ln(\lambda_i) - \lambda_i x$$

Using this sufficicient statistic $T_s(H_i|x)$, the maximum a posteriori decision rules are formed where hypothesis H_i is choosen if its a posteriori probability $p(H_i|x)$ is the largest. Because all of the operations performed in simplifying this probability where linear and/or monotonically increasing, the sufficient statistic can be substituted in place of the original probability expression. This theory results in the following rules

$$\text{Choose } H_0 \text{ if } T_s(H_0|x) > T_s(H_1|x) \text{ and } T_s(H_0|x) > T_s(H_2|x)$$

$$\text{Choose } H_1 \text{ if } T_s(H_1|x) > T_s(H_0|x) \text{ and } T_s(H_1|x) > T_s(H_2|x)$$

$$\text{Choose } H_2 \text{ if } T_s(H_2|x) > T_s(H_0|x) \text{ and } T_s(H_2|x) > T_s(H_1|x)$$

For each of the decision rules above, the value of each sufficient statistic $T_s(H_i|x)$ can be subsituted in and, moving all measurement dependent variables to the left hand side of the inequality and all hypothesis dependent variables to the right, a detection statistic is formed as well as decision boundaries defined.

$$\text{Choose } H_0 \text{ if } x < 0.3466 \text{ and } x < 0.4621$$

$$\text{Choose } H_1 \text{ if } x > 0.3466 \text{ and } x < 0.6931$$

$$\text{Choose } H_2 \text{ if } x > 0.4621 \text{ and } x > 0.6931$$

Denoting x as the detection statistic $T_d(x)$ and taking the stricter boundaries from the expression above, decision regions are formed.

$$\mathscr{R}_0 = \{T_d(x) : T_d(x) < 0.3466\}$$

$$\mathscr{R}_1 = \{T_d(x) : 0.3466 < T_d(x) < 0.6931\}$$

$$\mathscr{R}_2 = \{T_d(x) : T_d(x) > 0.6931\}$$

The decision rules can now be rewritten in terms of the detection statistic $T_d(x)$ being within each of the decision regions $\mathscr{R}_i$ for $i = 0, 1, 2$.

$$\text{Choose } H_0 \text{ if } T_d(x) \in \mathscr{R}_0$$

$$\text{Choose } H_1 \text{ if } T_d(x) \in \mathscr{R}_1$$

$$\text{Choose } H_2 \text{ if } T_d(x) \in \mathscr{R}_2$$

2. Calculate the resulting confusion matrix for this decision rule.

Given the above decision rules are derived from taking the maximum a posteriori hypothesis given an observed measurement, we now need to quantify the probability of error that we make an incorrect decision. Constructing a confusion matrix provides information not only on the probability of error, but the different errors associated with all three hypothesis H_0, H_1, and H_2 over all decisions we could make.

Denoting P_{ij} as

$$P_{ij} = Pr(\text{choose } H_j, \text{but } H_i \text{ is true}) \text{ for } i = 0, 1, 2 \text{ and } j = 0, 1, 2$$

the confusion matrix for the problem above is

$$\mathbf{P} = \begin{bmatrix} P_{00} & P_{01} & P_{02} \\ P_{10} & P_{11} & P_{12} \\ P_{20} & P_{21} & P_{22} \end{bmatrix}$$

Mathmatically, each diagonal element in the matrix can be computed as

$$P_{ii} = Pr(\text{choose } H_i, \text{and } H_i \text{ is true}) = \int_{T_d(x) \in \mathscr{R}_i} Pr(H_i|x)p(x)dx$$

and each off-diagonal element computed as

$$P_{ij} = Pr(\text{choose } H_i, \text{but } H_j \text{ is true}) = \int_{T_d(x) \in \mathscr{R}_j} Pr(H_{i \neq j}|x)p(x)dx$$

From part 1, $Pr(H_i|x)$ was defined explicitly where its denominator is the expanded form of $p(x)$. Using this definition, the intergals above can be rewritten as

$$P_{ii} = \int_{T_d(x)\in\mathscr{R}_i} Pr(H_i|x)p(x)dx = \int_{T_d(x)\in\mathscr{R}_i} \frac{p(x|H_i)p(H_i)}{p(x)}(p(x))dx = p(H_i)\int_{T_d(x)\in\mathscr{R}_i} p(x|H_i)dx$$

$$P_{ij} = \int_{T_d(x)\in\mathscr{R}_j} Pr(H_{i\neq j}|x)p(x)dx = \int_{T_d(x)\in\mathscr{R}_j} \frac{p(x|H_{i\neq j})p(H_{i\neq j})}{p(x)}(p(x))dx = p(H_{i\neq j})\int_{T_d(x)\in\mathscr{R}_j} p(x|H_{i\neq j})dx$$

Although the intergals are now cleanly expressed in terms of known (and easily computed) probability distribution functions, the limits of integration remain in terms of the detection statistic $T_d(x)$. To perform this intergration, we must transform the liklihood functions from dependence on measurement x into a form dependent on the detection statistic $T_d(x)$.

Given our measurement varaible is exponentially distributed and all operations performed within $T_d(x)$ are linear (for this example there are no operations), $T_d(x)$ will also be exponentially distributed. Looking at the mapping

$$T_d(x) = x$$

We see that becuase no operations are performed on the measurement, the mean and variance of random variable $T_d(x)$ are equivalent to that of random variable x thus

$$T_d(x) \sim \lambda e^{\lambda x} \text{ for } x \geq 0$$

and

$$p(T_d(x)|H_i) = \lambda_i e^{\lambda_i x} \text{ for } i = 0, 1, 2.$$

Given this liklihood is now dependent on $T_d(x)$ matching the bounds of the intergration P_{ii} and P_{ij}, the confusion matix can be computed as

$$P_{00} = (1/3)\int_0^{0.35} p(T_d(x)|H_0)dT_d(x) \quad P_{11} = (1/3)\int_{0.34}^{0.70} p(T_d(x)|H_1)dT_d(x) \quad P_{22} = (1/3)\int_{0.70}^{\infty} p(T_d(x)|H_2)dT_d(x)$$

$$P_{01} = (2/3)\int_{0.35}^{0.70} (1 - p(T_d(x)|H_0))dT_d(x) \quad P_{02} = (2/3)\int_{0.70}^{\infty} (1 - p(T_d(x)|H_0))dT_d(x)$$

$$P_{10} = (2/3)\int_{0}^{0.35} (1 - p(T_d(x)|H_1))dT_d(x) \quad P_{12} = (2/3)\int_{0.70}^{\infty} (1 - p(T_d(x)|H_1))dT_d(x)$$

$$P_{20} = (2/3)\int_{0}^{0.35} (1 - p(T_d(x)|H_2))dT_d(x) \quad P_{21} = (2/3)\int_{0.35}^{0.70} (1 - p(T_d(x)|H_2))dT_d(x)$$

These intergals are evaluated in the code block below

```
% define detection statistic liklihood functions (x >= 0)
p0 = @(x) expcdf(x, 1/4);
```

```
p1 = @(x) expcdf(x, 1/2);
p2 = @(x) expcdf(x, 1/1);

% integration bounds
b0 = 0.35;
b1 = 0.70;

% compute each elements intergal
P00 = (1/3) * (p0(b0) - p0(0));
P11 = (1/3) * (p1(b1) - p1(b0));
P22 = (1/3) * (p2(Inf) - p2(b1));

P01 = (2/3) * (1 - (p0(b1) - p0(b0)));
P02 = (2/3) * (1 - (p0(Inf) - p0(b1)));

P10 = (2/3) * (1 - (p1(b0) - p1(0)));
P12 = (2/3) * (1 - (p1(Inf) - p1(b1)));

P20 = (2/3) * (1 - (p2(b0) - p2(0)));
P21 = (2/3) * (1 - (p2(b1) - p2(b0)));

% construct confusion matrix
P = [P00 P01 P02; P10 P11 P12; P20 P21 P22];
disp(P);
```

```
    0.2511    0.5428    0.6261
    0.3311    0.0833    0.5023
    0.4698    0.5279    0.1655
```

From the above, we see that the numerically computed confusion matrix is

$$\mathbf{P} = \begin{bmatrix} 0.2511 & 0.5428 & 0.6261 \\ 0.3311 & 0.0833 & 0.5023 \\ 0.4698 & 0.5279 & 0.1655 \end{bmatrix}$$

The matrix is validated knowing that the sum of the diagonal elements and sum of the off diagonal elemnts seperately are equal to the probability of error $p(e)$.

```
% copmute probability of error checks
peDiagonals = sum(diag(P));
peOffDiagonals = sum(P) - peDiagonals;
disp(peDiagonals == peDiagonals);
```

```
   1
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