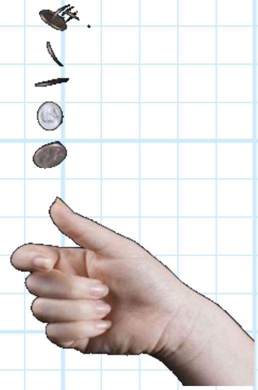
Problem 2



Under hypothesis $\mathcal{H}_0$, a coin is **"fair"**, such that the probability of "heads" when flipped is:

$$Pr(\text{"heads"}|\mathcal{H}_0) = 0.5$$

Under hypothesis $\mathcal{H}_1$, a coin is **"biased"**, such that the probability of "heads" when flipped is:

$$Pr(\text{"heads"}|\mathcal{H}_1) = 0.75$$

We flip an **unknown** coin **5 times**, and count (i.e., measure) the number of times it comes up **heads**.

This number is denoted as x:

$$x = \{\text{number of "heads"}\}$$

From this value x, we wish to **detect biased coins** (i.e., determine when hypothesis $\mathcal{H}_1$ is true)!

Note this number x will be an integer, such that:

$$x \in \{0,1,2,3,4,5\}$$

So for this **discrete** measurement case, the likelihood "functions" are expressed as a list of **probabilities** (not densities!), one for each integer value x:

$$Pr(x=0|\mathcal{H}_0) = p_{00} \qquad Pr(x=0|\mathcal{H}_1) = p_{01}$$

$$Pr(x=1|\mathcal{H}_0) = p_{10} \qquad Pr(x=1|\mathcal{H}_1) = p_{11}$$

$$Pr(x=2|\mathcal{H}_0) = p_{20} \qquad Pr(x=2|\mathcal{H}_1) = p_{21}$$

$$Pr(x=3|\mathcal{H}_0) = p_{30} \qquad Pr(x=3|\mathcal{H}_1) = p_{31}$$

$$Pr(x=4|\mathcal{H}_0) = p_{40} \qquad Pr(x=4|\mathcal{H}_1) = p_{41}$$

$$Pr(x=5|\mathcal{H}_0) = p_{50} \qquad Pr(x=5|\mathcal{H}_1) = p_{51}$$

Note then that, for **example**:

$$Pr(x \le 3|\mathcal{H}_0) = \sum_{x=0}^{3} Pr(x|\mathcal{H}_0)$$

or

$$Pr(x > 2|\mathcal{H}_1) = \sum_{x=3}^{5} Pr(x|\mathcal{H}_1) = 1 - \sum_{x=0}^{2} Pr(x|\mathcal{H}_1)$$

1. Determine the likelihood "functions" $Pr(x|\mathcal{H}_0)$ and $Pr(x|\mathcal{H}_1)$. In other words, determine the **numeric values** of the **12 probabilities** ($p_{00}, p_{10}, \ldots$ and $p_{01}, p_{11}, \ldots$) in the lists above.

2. Say the **decision rule** is:

$$\text{Choose } \mathcal{H}_0 \quad \text{if} \quad x < y'$$

$$\text{Choose } \mathcal{H}_1 \quad \text{if} \quad x \geq y'$$

where:

$$y' \in \{0,1,2,3,4,5,6\}$$

Sketch the **ROC** for this detection problem. This ROC should consist of **7 points** on the (P_{FA}, P_D) plane. Provide the **specific coordinates** of each of these seven points.

3. Say the *a priori* probability of a biased coin is $Pr(\mathcal{H}_1) = 1/3$. Use the **MAP** detection criterion to form a **decision rule** of the form:

$$\text{Choose } \mathcal{H}_0 \quad \text{if} \quad x < y'$$

$$\text{Choose } \mathcal{H}_1 \quad \text{if} \quad x \geq y'$$

where y' is an **integer (find its value**!).

4. Determine the **probability of error** $Pr\{\varepsilon\}$ of this MAP decision rule.