1. To decorrelate the elements of the noise vector, apply the Karhunen-Loeve transform. Specfically, determine (a) the transformed signal vectors $\mathbf{s}_A{}'$ and $\mathbf{s}_B{}'$ as well as the covariance matrix of the transformed noise vector $\mathbf{w}'$.

Given that the measurement vector $\mathbf{x}$ is modeled as $\mathbf{x} = \mathbf{s} + \mathbf{w}$ where the noise vector $\mathbf{w}$ is normally distributed as $\mathcal{N}(0, \mathbf{C}_w)$, the Karhunen-Loeve transform can be applied to whiten the resulting measurement making the samples of $\mathbf{x}$ independent and identically distributed (i.i.d) which greatly simplifies detection processing. Specifically this involves determining a linear transformation of the data such that resulting covariance matrix $\mathbf{C_x}$ is diagonalized.

Being that each hypothesis consists of a deterministic signal, $\mathbf{s}_A$ or $\mathbf{s}_B$, and additive noise $\mathbf{w}$, the measurements covariance structure is solely a result of the noise covariance being that $\mathbf{w}$ is the only random component. This intuition is represented below in which the mean of the measurement under each hypothesis is

$$\mu_{x|H_A} = \mathbf{s}_A \qquad \text{and} \qquad \mu_{x|H_B} = \mathbf{s}_B,$$

with covariance

$$\mathbf{C}_{X|H_A} = E\{(\mathbf{x} - \mu_{x|H_A})(\mathbf{x} - \mu_{x|H_A})^T\} = E\{\mathbf{w}\mathbf{w}^\mathbf{T}\} = \mathbf{C}_w,$$

$$\mathbf{C}_{X|H_B} = E\{(\mathbf{x} - \mu_{x|H_B})(\mathbf{x} - \mu_{x|H_B})^T\} = E\{\mathbf{w}\mathbf{w}^\mathbf{T}\} = \mathbf{C}_w$$

resulting in each having normal distribution described by

$$\mathbf{x}|H_A \sim \mathcal{N}(\mathbf{s}_A, \mathbf{C_w}) \qquad \text{and} \qquad \mathbf{x}|H_B \sim \mathcal{N}(\mathbf{s}_B, \mathbf{C_w}).$$

An eigen-decomposition is performed on this measurement covariance matrix resulting in the the orthonormal matrix $\mathbf{V}$ containing eigen vectors in each column, and matrix $\mathbf{\Sigma}$ containing the corresponding eigen-values along its diagonal.

$$\text{eig}(\mathbf{C_w}) = \mathbf{V\Sigma V^T} \qquad \mathbf{V} = \begin{bmatrix} +0.5 & +0.5 & +0.5 & +0.5 \\ -0.5 & -0.5 & +0.5 & +0.5 \\ +0.5 & -0.5 & -0.5 & +0.5 \\ -0.5 & +0.5 & -0.5 & +0.5 \end{bmatrix} \qquad \mathbf{\Sigma} = \begin{bmatrix} 6.0 & 0 & 0 & 0 \\ 0 & 4.0 & 0 & 0 \\ 0 & 0 & 2.0 & 0 \\ 0 & 0 & 0 & 1.0 \end{bmatrix}$$

Defining a new signal basis consisting of the orthonormal eigen vectors in $\mathbf{V}$, the measurement vector can be rewritten as $\mathbf{x} = \mathbf{V}\mathbf{x}' = \sum_{n=1}^{4} x_n{}' v_n$ where $\mathbf{x}'$ represents the discrete set of coefficients applied to each component of the new basis defined by $\mathbf{x}' = \mathbf{V}^T\mathbf{x}$. Writing each of the hypothesized measurements in this orthogonal basis yeilds

$$\mathbf{x}'|H_A = \mathbf{V^T s_A} + \mathbf{V^T w} \quad \text{and} \quad \mathbf{x}'|H_B = \mathbf{V^T s_B} + \mathbf{V^T w}.$$

Substituting $\mathbf{V}^T$, $\mathbf{s}_A$, and $\mathbf{s}_B$ into the above equations, the transformed signal vectors $\mathbf{s}_A{}'$ and $\mathbf{s}_B{}'$ are found as

$$\mathbf{s}_A{}' = \begin{bmatrix} +0.5 & -0.5 & +0.5 & -0.5 \\ +0.5 & -0.5 & -0.5 & +0.5 \\ +0.5 & +0.5 & -0.5 & -0.5 \\ +0.5 & +0.5 & +0.5 & +0.5 \end{bmatrix} \begin{bmatrix} 3.0 \\ -1.0 \\ 1.0 \\ -3.0 \end{bmatrix} = \begin{bmatrix} 4.0 \\ 0.0 \\ 2.0 \\ 0.0 \end{bmatrix} \quad \text{and} \quad \mathbf{s}_B{}' = \begin{bmatrix} +0.5 & -0.5 & +0.5 & -0.5 \\ +0.5 & -0.5 & -0.5 & +0.5 \\ +0.5 & +0.5 & -0.5 & -0.5 \\ +0.5 & +0.5 & +0.5 & +0.5 \end{bmatrix} \begin{bmatrix} -3.0 \\ 3.0 \\ -5.0 \\ 5.0 \end{bmatrix} = \begin{bmatrix} -8.0 \\ 2.0 \\ 0.0 \\ 0.0 \end{bmatrix}.$$

The transformed measurement covariance matrix is equal to $\mathbf{\Sigma}$ shown through the relation of the projection matrix $\mathbf{V}^T$ and the eigen-decomposition of the original covariance structure.

$$\mathbf{C_w}' = \mathbf{V^T C_w} = \mathbf{V^T V \Sigma V^T V} = \mathbf{I \Sigma I} = \mathbf{\Sigma} = \begin{bmatrix} 6.0 & 0 & 0 & 0 \\ 0 & 4.0 & 0 & 0 \\ 0 & 0 & 2.0 & 0 \\ 0 & 0 & 0 & 1.0 \end{bmatrix}$$

2. Now, using a LRT threshold of $\gamma = e^2$, determine a LRT for this detection problem, expressed in terms of the transformed measurement $\mathbf{x}' = [x_1{}', x_2{}', x_3{}', x_4{}']^T$.

Using the i.i.d. transformed measurement vector $\mathbf{x}'$ where each sample normally distributed with a diagonalized covariance matrix, the likelihood functions for each hypothesis can be rewritten as the sum of a set of 4 independent guassian random variables.

$$p(\mathbf{x}'|H_A) = \frac{1}{(2\pi)^{4/2}(\det(\mathbf{\Sigma})^{1/2})} \exp\left(-\frac{1}{2}(\mathbf{x}' - \mathbf{s}_A{}')^T \mathbf{\Sigma}(\mathbf{x}' - \mathbf{s}_A{}')\right)$$

$$p(\mathbf{x}'|H_B) = \frac{1}{(2\pi)^{4/2}(\det(\mathbf{\Sigma})^{1/2})} \exp\left(-\frac{1}{2}(\mathbf{x}' - \mathbf{s}_B{}')^T \mathbf{\Sigma}(\mathbf{x}' - \mathbf{s}_B{}')\right)$$

Using these likelihood function definitions, and likelihood ratio $L(x)$ is defined as

$$L(x) = \frac{\frac{1}{(2\pi)^{4/2}(\det(\mathbf{\Sigma})^{1/2})} \exp\left(-\frac{1}{2}(\mathbf{x}' - \mathbf{s}_B{}')^T \mathbf{\Sigma}(\mathbf{x}' - \mathbf{s}_B{}')\right)}{\frac{1}{(2\pi)^{4/2}(\det(\mathbf{\Sigma})^{1/2})} \exp\left(-\frac{1}{2}(\mathbf{x}' - \mathbf{s}_A{}')^T \mathbf{\Sigma}(\mathbf{x}' - \mathbf{s}_A{}')\right)}$$

which can be simplified to

$$L(x) == \frac{\exp\left(-\frac{1}{2}(\mathbf{x}' - \mathbf{s}_B{}')^T \boldsymbol{\Sigma}(\mathbf{x}' - \mathbf{s}_B{}')\right)}{\exp\left(-\frac{1}{2}(\mathbf{x}' - \mathbf{s}_A{}')^T \boldsymbol{\Sigma}(\mathbf{x}' - \mathbf{s}_A{}')\right)} = \exp(-72x_1 + 8x_2 - 4x_3 - 148)$$

and used in a likelihood ratio test (LRT) with the given threshold $\gamma = e^2$:

Choose H_B if $\exp(-72x_1 + 8x_2 - 4x_3 - 148) \geq e^2$ else H_A.

3. Simplify this LRT into a decision rule of the form: $T_d(x') > \gamma'$***.***

Further simplification of the LRT can be made to ultimately express the decision rule in terms of a decision statistic that consists solely of the transformed measurement vector (i.e. $T_d(\mathbf{x}') = \mathbf{x}'$). Taking the natural logarithm of both sides of the LRT inequality derived in part 2 then moving all constants to the right hand side of the equation yeilds

$$\ln\left(\exp(-72x_1 + 8x_2 - 4x_3 - 148)\right) \geq \ln(e^2) \quad \longrightarrow \quad -72x_1 + 8x_2 - 4x_3 \geq 150.$$

Plugging this back into the LRT where $T_d(\mathbf{x}') = -72x_1 + 8x_2 - 4x_3$ and $\gamma' = 150$ yeilds the simplest form of the decision rule:

Choose H_B if $-72x_1 + 8x_2 - 4x_3 \geq 150$ else H_A.