1. Determine the a posteriori probabilities $Pr(H_0|x)$ and $Pr(H_1|x)$.

Denoting vendors 0 and 1 as hypotheses H_0 and H_1 with the number of defects being the measured variable x, the a posteriori probabilities will be of the form

$$Pr(H_0|x) = \frac{Pr(x|H_0)p(H_0)}{\sum_i Pr(x|H_i)p(H_i)} \qquad Pr(H_1|x) = \frac{Pr(x|H_1)p(H_1)}{\sum_i Pr(x|H_i)p(H_i)}$$

Naturally, given that we are choosing between two hypotheses, the a priori liklihood function will be of the form of a Binomial distribution.

$$Pr(x|H_i) = \frac{N!}{x!(N-x)!} p(x|H_i)^x (1 - p(x|H_i))^{N-x}$$

Putting the liklihood function and given a prioris, the combined a posteriori distributions are

$$Pr(H_0|x) = \frac{Pr(x|H_0)(0.8)}{Pr(x|H_0)(0.8) + Pr(x|H_1)(0.2)} \quad Pr(H_1|x) = \frac{Pr(x|H_1)(0.2)}{Pr(x|H_0)(0.8) + Pr(x|H_1)(0.2)}$$

2. Manipulate these a posteriori probabilities to determine the simplest possible sufficient statistics $T_s(H_0|x)$ and $T_s(H_1|x)$.

Becuase the denominators for both a posteriori distributions $Pr(H_0|x)$ and $Pr(H_1|x)$ are the same, they can be multiplied to the left hand side and grouped into a significant statistic term $T_s(H_i|x)$.

$$T_s(H_i|x) = Pr(x|H_i)p(H_i)$$

The factorial constant from the binomial liklihood distribution dependeds only on the measurement and therefore can be grouped into the sufficient statistic as well leaving

$$T_s(H_i|x) = \left(p(x|H_i)^x (1 - p(x|H_i))^{N-x}\right) p(H_i)$$

Applying the monotonically increasing natural logarithm operator to both sides allows for the 'removal' of the exponential terms and to be simplified into a first order polynomial.

$$T_s(H_i|x) = x\ln(p(x|H_i)) + (N-x)\ln(1 - p(x|H_i)) + \ln(p(H_i))$$

$$T_s(H_i|x) = x\ln\left(\frac{p(x|H_i)}{1 - p(x|H_i)}\right) + N\ln(1 - p(x|H_i)) + \ln(p(H_i))$$

Plugging in the known liklihoods of defects and a prioris for each vendor, the two simplified sufficient statistics are

$$T_s(H_0|x) = x\ln(0.526) - 0.7361$$

$$T_s(H_1|x) = x\ln(0.5385) - 5.9172$$

3. Use these sufficient statistics to express a MAP decision rule.

Using the two sufficient statistics derived above, the maximum a posteriori decision rule for each hypothesis is

$$\text{Choose } H_0 \text{ if } T_s(H_0|x) > T_s(H_1|x)$$

$$\text{Choose } H_1 \text{ if } T_s(H_1|x) > T_s(H_0|x)$$

4. Manipulate this MAP decision rule to determine a decision statistic $T_d(x)$, as well as the decision regions $\mathscr{R}_0$ and $\mathscr{R}_1$.

For each of the decision rules above, a singular scalar detection statistic $T_d(x)$ is expressed by moving all terms dependent on the measurement to the left hand side of the equation, and all constants to the right.

$$T_s(H_0|x) > T_s(H_1|x) \quad \rightarrow \quad x > \frac{-5.1812}{\ln(0.526) - \ln(0.539)} \quad \rightarrow \quad x < 220.6$$

$$T_s(H_1|x) > T_s(H_0|x) \quad \rightarrow \quad x > \frac{-12.4766}{\ln(1/4) - \ln(4)} \quad \rightarrow \quad x > 220.6$$

Therefore our detection statistic is just the measurement x itself and the regions are

$$\mathscr{R}_0 = \{T_d(x) : T_d(x) < 220.6\}$$

$$\mathscr{R}_1 = \{T_d(x) : T_d(x) > 220.6\}$$

Putting it all together, the MAP decision rule can be rewritten using $T_d(x)$, $\mathscr{R}_0$, and $\mathscr{R}_1$.

$$\text{Choose } H_0 \text{ if } T_d(x) \in \mathscr{R}_0$$

$$\text{Choose } H_1 \text{ if } T_d(x) \in \mathscr{R}_1$$

5. Determine the probability of error $Pr(e)$ **for this decision rule.**

The probability of error is the probabilistic view of after we make our decision using the rule above, that decision turns out to be false and another hypothesis is true instead. Mathmatically, this is found as the expected value of the conditional error given the measurement $p(e|x)$.

$$p(e) = \mathrm{E}_x\{p(e|x)\} = \int Pr(e|x)p(x)dx$$

where

$$Pr(e|x) = \begin{cases} 1 - Pr(H_0|x) & \text{for} \quad x < 220.6 \\ 1 - Pr(H_1|x) & \text{for} \quad x > 220.6 \end{cases} \quad \text{and} \quad p(x) = \sum_i Pr(x|H_i)p(H_i)$$

Plugging in with known terms we obtain the intergal

$$p(e) = \int Pr(e|x)(Pr(x|H_0)p(H_0) + Pr(x|H_1)p(H_1))dx$$

The intergal can then be broken apart into each of these regions

$$p(e) = \int_{-\infty}^{4.5} (1 - Pr(H_1|x))(0.11)dx + \int_{4.5}^{\infty} (1 - Pr(H_0|x))(0.11)dx$$

The intergal above is evaluated and results plotted below

The probability of error is the probabilistic view of making the wrong decision, despite out MAP detection rule telling us it was the most probable. Mathmatically, this is found as the expected value of the conditional distribution between the error e and measurement x.

$$p(e) = \mathrm{E}_x\{p(e|x)\} = \int Pr(e|x)p(x)dx$$

For our binomial case, $Pr(e|x)$ is

$$Pr(e|x) = \begin{cases} 1 - Pr(H_0|x) & \text{for} \quad x < 220.6 \\ 1 - Pr(H_1|x) & \text{for} \quad x > 220.6 \end{cases}$$

and the a priori probabilities $p(H_0)$ and $p(H_1)$ are given as 0.8 and 0.2 respectively. Plugging these values into the probability of error equation for each of the decision regions gives

$$p(e) = \int_{-\infty}^{220.6} (1 - Pr(H_0|x))(Pr(x|H_0)(0.8) + Pr(x|H_1)(0.2))dx + \int_{220.6}^{\infty} (1 - Pr(H_1|x))(Pr(x|H_0)(0.8) + Pr(x|H_1)(0.2))dx$$

This intergal is evaluated numerically in the code block below.

```
% problem parameters
N   = 10;
pi0 = 0.8;
pi1 = 0.2;
p0  = 0.05;
p1  = 0.35;
x = 0:N;

% likelihood functions
f0 = binopdf(x, N, p0);
```

```
f1 = binopdf(x, N, p1);

% a posterior functions
P    = pi0*f0 + pi1*f1;
postH0 = (pi0*f0) ./ P;
postH1 = (pi1*f1) ./ P;

% MAP decision rule boundary
boundary = 220.6;

% intergal calculation
maskH0 = (x <= boundary);
maskH1 = (x >= boundary+1);
pError = sum( (1 - postH0(maskH0)) .* P(maskH0) ) + sum( (1 - postH1(maskH1)) .*
P(maskH1) );
fprintf('pError = %.6f\n', pError);
```

```
pError = 0.200000
```

From the above excerpt we have a probability of error of 0.2.