**Problem 15**

Under hypothesis $\mathcal{H}_0$, scalar measurement $x$ is equal to **random variable** $w_1$:

$$x = w_1$$

Under hypothesis $\mathcal{H}_1$, scalar measurement $x$ is equal to the **sum** of random variable $w_1$ and $w_2$:

$$x = w_1 + w_2$$

Random variables $w_1$ and $w_2$ are independent, and each is described with an **exponential** *pdf*:

$$p(w_1) = \begin{cases} \lambda_1 e^{-\lambda_1 w_1} & \text{for } w_1 \geq 0 \\ 0 & \text{for } w_1 < 0 \end{cases}$$

$$p(w_2) = \begin{cases} \lambda_2 e^{-\lambda_2 w_2} & \text{for } w_2 \geq 0 \\ 0 & \text{for } w_2 < 0 \end{cases}$$

Note that $\lambda_1 \neq \lambda_2$!

Say we wish to apply the **MAP detection criterion**, and the two hypotheses are **equally probable**.

Determine the correct **threshold** $y'$ for a decision rule of the form:

$$\text{Choose } \mathcal{H}_1 \text{ if } T_d = x > y'$$

**Hint:** Implement your knowledge of EECS 861, being carefully to consider the **limits of integration**.

***1. Determine the correct threshold*** $y'$ ***for a decision rule of the form:*** **Choose** $H_1$ **if** $T_d = x > y'$

Given hypothesis $H_0$ describes the single random variable $w_1$ which is exponentially distributed, the likelihood of this hypothesis producing measurement $x$ can be written directly as

$$p(x|H_0) = \lambda_1 e^{-\lambda_1 x}$$

Hypothesis $H_1$ describes the measurement being the sum of two independent exponentially distributed random variables $w_1$ and $w_2$. The resulting likelihood distribution of $H_1$ producing this measurement is the convolution of the individual exponential distributions which, using [1], is

$$p(x|H_1) = \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1}\left(e^{-\lambda_i x} - e^{-\lambda_2 x}\right) \text{ with } \lambda_1 \neq \lambda_2$$

Given the two hypotheses $H_0 = w_1$ and $H_1 = w_1 + w_2$ are equally probable, there a posteriori probabilities are desribed as

$$p(H_0|x) = \frac{p(x|H_0)p(H_0)}{p(x|H_0)p(H_0) + p(x|H_1)p(H_1)}$$

$$p(H_1|x) = \frac{p(x|H_1)p(H_1)}{p(x|H_0)p(H_0) + p(x|H_1)p(H_1)}$$

From this a posteriori expression, the decision rule for $H_1$ is thus written as

$$\text{Choose } H_1 \text{ if } p(H_1|x) > p(H_0|x)$$

Dividing the entire rule by $p(H_0|x)$, noting that each a posteriori probability is equal to its liklihood probability of that hypothesis, frames the liklihood ratio test

$$\text{Choose } H_1 \text{ if } \frac{p(H_1|x)}{p(H_0|x)} = L(x) > 1.0$$

To determine this liklihood ratio in terms of the detection statistic $T_d(x)$, simplifications of $L(x)$ are required. First, recongnizing that the denominators of both a posteriori distributions are the equal, these denominators are cancelled. This leaves the liklihood ratio

$$L(x) = \frac{p(x|H_1)}{p(x|H_0)|} = \frac{\frac{\lambda_1\lambda_2}{\lambda_2 - \lambda_1}\left(e^{-\lambda_i x} - e^{-\lambda_2 x}\right)}{\lambda_1 e^{-\lambda_1 x}} = \frac{\lambda_2}{\lambda_2 - \lambda_1}\left(1 - e^{-(\lambda_2 - \lambda_1)x}\right)$$

Moving the scalar to the left handside of the inequality and taking the natural log of both sides leaves

$$-(\lambda_2 - \lambda_1)x > \ln\left(\frac{\lambda_1}{\lambda_2}\right)$$

Solving for $x$ leaves the detection statistic $T_d(x) = x$ on the left side of the equation and an equivalently scaled decision boundary on the right.

$$\text{if} \quad (\lambda_2 - \lambda_1) > 0 \quad \rightarrow \quad \text{Choose } H_1 \text{ if } T_d(x) = x > \frac{\ln(\lambda_2/\lambda_1)}{\lambda_2 - \lambda_1}$$

$$\text{if} \quad (\lambda_2 - \lambda_1) < 0 \quad \rightarrow \quad \text{Choose } H_1 \text{ if } T_d(x) = x < \frac{\ln(\lambda_2/\lambda_1)}{\lambda_2 - \lambda_1}$$

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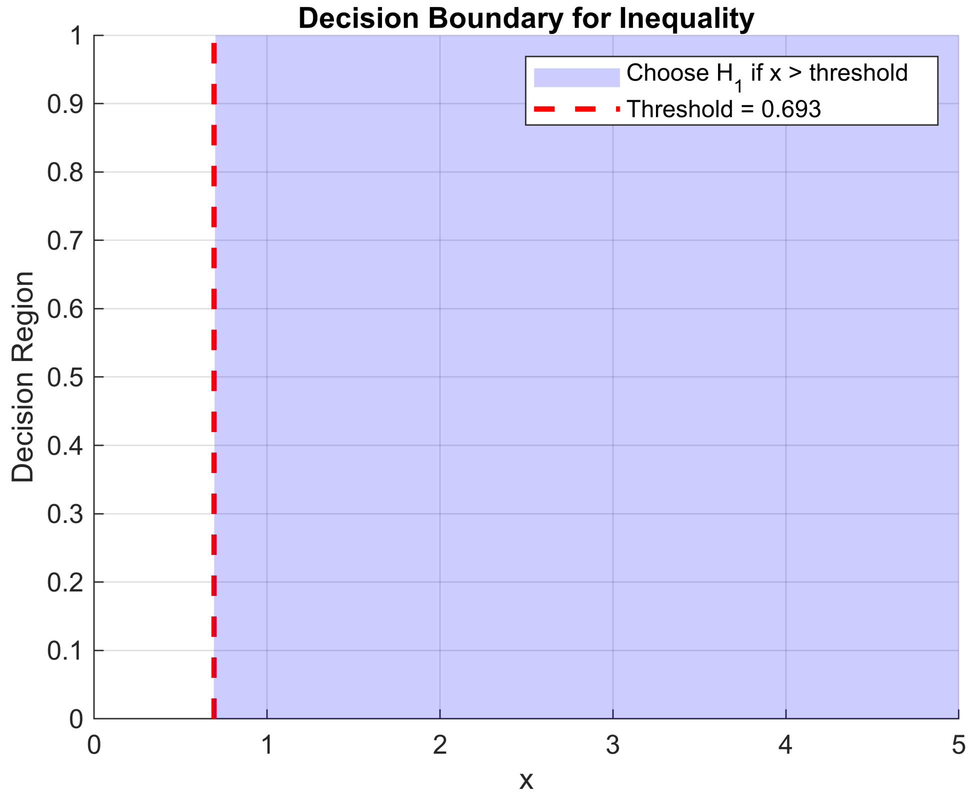
% exponential lambda values
lambda1 = 1;
lambda2 = 2;

% compute detection statistic threshold
threshold = log(lambda2/lambda1) / (lambda2 - lambda1);

% determine decision region depending on sign of (lambda2 - lambda1)
x = linspace(0, 5, 500);
if lambda2 > lambda1
    decisionRegion = (x > threshold);
else
    decisionRegion = (x < threshold);
end

% plot results
figure; hold on; grid on;
plot([threshold threshold], [0 1], 'r--', 'LineWidth', 2);
area(x, decisionRegion, 'FaceAlpha', 0.2, 'FaceColor', 'b', 'EdgeColor', 'none');
xlabel('x'); ylabel('Decision Region');
title('Decision Boundary for Inequality');
if lambda2 > lambda1
    legend(sprintf('Threshold = %.3f', threshold), 'Choose H_1 if x > threshold');
else
    legend(sprintf('Threshold = %.3f', threshold), 'Choose H_1 if x < threshold');
end
```




[1] https://en.wikipedia.org/wiki/Exponential_distribution