a. Use this measurement to determine the matched filter estimate of $\widehat{\theta}_{MF}$.

In *Lecture 53* the matched filter estimate was shown to be

$$\widehat{\theta}_{MF} = \mathbf{N}_H^{-1}\mathbf{H}^T\mathbf{x}$$

where $\mathbf{N}_H^{-1} = \mathrm{diag}\{\mathbf{H}^T\mathbf{H}\}^{-1}$. From this definition, it is clear by the diagonal operator that the non-diagonal elements are unused and assumed to be zero. This is ultimately an assumption about the properties of the cross-correlation matrix of system $\mathbf{H}$ that simplifies the solution to a tractable and simple matrix inverse at the lose of measurement information contained in these unsed components of the system. The matched filter estimate is solved in the matlab excerpt below.

```
H = [10 20; 20 0; 10 -10];
x = [-28; 23; 29];
theta_MF = inv(diag(diag(H'*H)))*H'*x
```

```
theta_MF = 2×1
    0.7833
   -1.7000
```

$$\widehat{\theta}_{MF} = [0.7833 \quad -1.700]^T$$

b. Use this measurement to determine the maximum likelihood estimate $\widehat{\theta}_{MLE}$.

The maximum likelihood estiamte (MLE) derived in *Lecture 48*, was defined as the estimate of source θ that maximized the log-likelihood function resulting in

$$\widehat{\theta}_{MLE} = (\mathbf{H^TH})^{-1}\mathbf{H^Tx}.$$

The MLE estimate here can be viewed as equivalent to the matched filter when the system cross correlation matrix, $\mathbf{H}^T\mathbf{H}$, is diagonal, however will provide a slightly different estimate otherwise. The MLE estimate is solved in the matlab excerpt below.

```
H = [10 20; 20 0; 10 -10];
x = [-28; 23; 29];
theta_MLE = inv(H'*H)*H'*x
```

```
theta_MLE = 2×1
    1.1034
   -1.9207
```

$$\widehat{\theta}_{MF} = [1.1034 \quad -1.9207]^T$$

c. Use this measurement to determine the MAP/MMSE estimate $\widehat{\theta}_{MAP}$.

Similarly to the maximum likelihood estimate, the maximum a posteriori estimate is the estimate of source θ that maximizes the log a posteriori function resulting in

$$\widehat{\theta}_{MAP} = \left(\mathbf{H}^{\mathrm{T}}\mathbf{C}_{\mathrm{w}}^{-1}\mathbf{H} + \mathbf{C}_{\theta}^{-1}\right)^{-1}\left(\mathbf{C}_{\theta}^{-1}\mu_{\theta} + \mathbf{H}^{\mathrm{T}}\mathbf{C}_{\mathrm{w}}^{-1}\mathbf{x}\right),$$

in which when μ_{θ} is equal to zero, as in this case, the estimate simplifies to

$$\widehat{\theta}_{MAP} = \left(\mathbf{H}^{\mathrm{T}}\mathbf{C}_{\mathrm{w}}^{-1}\mathbf{H} + \mathbf{C}_{\theta}^{-1}\right)^{-1}\left(\mathbf{H}^{\mathrm{T}}\mathbf{C}_{\mathrm{w}}^{-1}\mathbf{x}\right).$$

This estimator includes both the diagonal elements of the system cross correlation matrix as well as both the noise and a priori source correlation matrices. This estimate, opposed to the previous two discussed, maximizes that amount of system and a priori information used and uses this information to decorrelate and scale all off diagonal elements of the resulting system correlation matrix. The MAP estimate is solved in the matlab excerpt below.

```
H = [10 20; 20 0; 10 -10];
x = [-28; 23; 29];
C_theta = 10*eye(size(H, 2));
C_noise = 9*eye(size(H, 1));
theta_MAP = inv(H'*inv(C_noise)*H + inv(C_theta))*(H'*inv(C_noise)*x)
```

```
theta_MAP = 2×1
    1.1011
   -1.9168
```

$$\widehat{\theta}_{MAP} = [1.1011 \quad -1.9168]^{T}$$

d. Compare and contrast these results. Provide profound thoughts.

The estimators shown above build upon eachother through including more system and a priori information.

In the matched filter (MF) case, all off-diagonal elements of the system cross correlation matrix $\mathbf{H}^{T}\mathbf{H}$ were assumned zero. This allows for a simple tractable matrix inverse yet does not properly normalize elements of measurement $\mathbf{x}$ that may occupy this space, potentially corrupting and biasing the estimate if the assumption is false (as in this example).

The MLE estimator removes this diagonal assumption and includes the off diagonal elements into the normalization providing a efficient estimate that maximizes the amount of information obtained from the measurement $\mathbf{x}$. From *Lecture 48* this estimate was proven to be both unbiased and minimum variance.

The MAP estimate builds upon the MLE estimate in utilizing all measurement dependent information with no assumptions about mutual orthogonality of the system cross correlation matrix in addition to including a priori

information on of both the source and noise correlation matrices. From the MAP estimator definition, we see that the same normalization through matrix inverse is applied however now the cross correlation $\mathbf{H}^T\mathbf{H}$ is altered by both the noise and source correlations. The initial projection onto the space spanned by the system $\mathbf{H}$ is additionally altered via the matrix inverse of the source noise correlation matrix which effectively diagonalizes the measurements cross correlation in this new space.