Problem 23

Consider a two-dimensional, **real-valued** measurement:

$$\mathbf{x} = [x_1, x_2]^T$$

This measurement **vector** is related to a **deterministic** vector $\mathbf{s}$:

$$\mathbf{s} = [s_1, s_2]^T$$

and **random** vector $\mathbf{w}$ as:

$$\mathbf{w} = [w_1, w_2]^T$$

as:

$$\mathbf{x} = \mathbf{s} + \mathbf{w}$$

The elements of $\mathbf{w}$ are **independent** and identically distributed (*i.i.d.*); specifically zero-mean Gaussian:

$$w_1 \sim \mathcal{N}\left(0, \sigma_w^2\right) \qquad \text{and} \qquad w_2 \sim \mathcal{N}\left(0, \sigma_w^2\right)$$

Under hypothesis $\mathcal{H}_a$, the deterministic signal is:

$$\mathbf{s}_a = [0, 0]^T$$

Under hypothesis $\mathcal{H}_b$, the deterministic signal is:

$$\mathbf{s}_b = \left[\sqrt{2}, \sqrt{2}\right]^T$$

Under hypothesis $\mathcal{H}_c$, the deterministic signal is:

$$\mathbf{s}_c = 2\,\mathbf{s}_b = \left[2\sqrt{2}, 2\sqrt{2}\right]^T$$

The three hypotheses are **equally probable**, *a priori*.

1. Using the **MAP** decision criterion, determine a **simplified sufficient statistic** $T_s(\mathbf{x},\mathcal{H}_i)$ for **each** hypothesis.

2. **Using these sufficient statistics,** express a **specific MAP decision rule** for **each** of the three hypotheses.

3. From these decision rules, extract a single ***scalar* detection statistic** $T_d(\mathbf{x})$, and define specifically the **decision region** for each hypothesis (in **terms** of this detection statistic).

1. Using the MAP decision criterion, determine a simplified sufficient statistic $T_s(x, H_i)$ for each hypothesis.

Before finding a sufficient statistic and forming the MAP decision criterion, the likelihood functions for each hypotheses H_a, H_B, and H_c must be expressed. Given that sources s_a, s_b, and s_c are all deterministic two dimensional vectors contaminated with Normally distrubted white noise, the liklihood functions of each hypothesis producing the measurement will also be two dimensional Normally distributed.

$$p(\mathbf{x}|H_a) = \frac{1}{\sqrt{(2\pi)^k \det(\mathbf{\Sigma})}} \exp\left(-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu}_\mathbf{a})^H \mathbf{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu}_\mathbf{a})\right)$$

$$p(\mathbf{x}|H_b) = \frac{1}{\sqrt{(2\pi)^k \det(\mathbf{\Sigma})}} \exp\left(-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu}_\mathbf{b})^H \mathbf{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu}_\mathbf{b})\right)$$

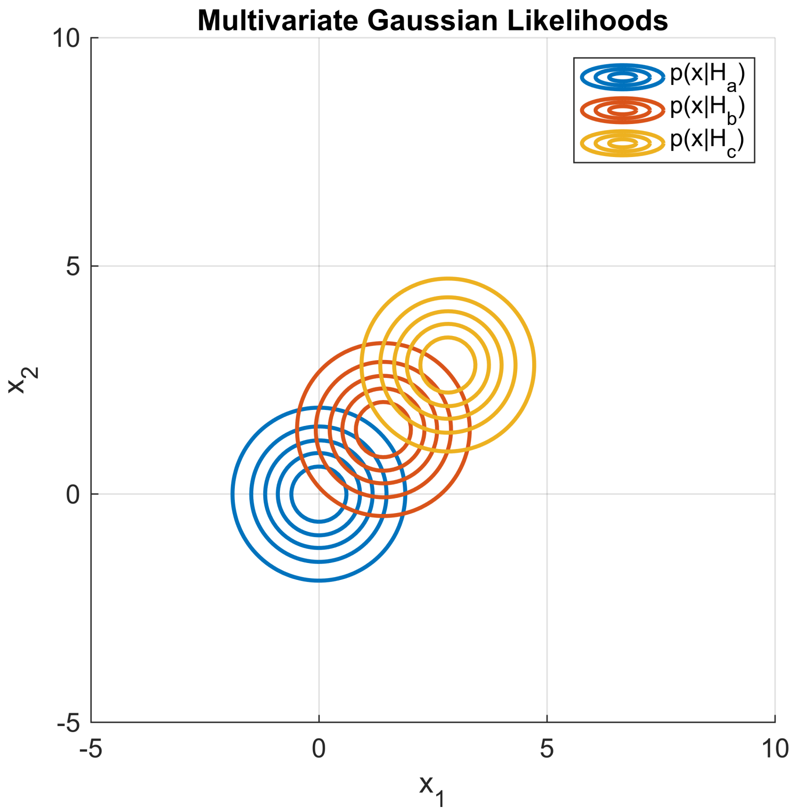
$$p(\mathbf{x}|H_c) = \frac{1}{\sqrt{(2\pi)^k \det(\mathbf{\Sigma})}} \exp\left(-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu}_\mathbf{c})^H \mathbf{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu}_\mathbf{c})\right)$$

$$\mathbf{\Sigma} = \begin{bmatrix} \sigma_w^2 & 0 \\ 0 & \sigma_w^2 \end{bmatrix} \quad \boldsymbol{\mu}_\mathbf{a} = [0, 0]^T \quad \boldsymbol{\mu}_\mathbf{b} = [\sqrt{2}, \sqrt{2}]^T \quad \boldsymbol{\mu}_\mathbf{c} = [2\sqrt{2}, 2\sqrt{2}]^T$$

These liklihood functions are plotted below for a arbitrary standard deviation.

```
% parameters
sigma_w = 1;
Sigma = sigma_w^2 * eye(2);

% define mean vectors for each hypothesis
mu = [0, 0; sqrt(2), sqrt(2); 2*sqrt(2), 2*sqrt(2)];

% x1, x2 support grid
x1 = linspace(-5, 10, 200);
x2 = linspace(-5, 10, 200);
[X1, X2] = meshgrid(x1, x2);
X = [X1(:), X2(:)];

% plot each PDF
figure('Name', 'Multivariate Liklihood Functions'); hold on; grid on;
colors = lines(size(mu,1));

for i = 1:size(mu,1)
    % multivariate Gaussian pdf values
    pdf_vals = mvnpdf(X, mu(i,:), Sigma);
    pdf_vals = reshape(pdf_vals, length(x2), length(x1));
```

```
    % contour plot
    contour(X1, X2, pdf_vals, 5, 'LineWidth', 1.5, 'Color', colors(i,:));
end

xlabel('x_1'); ylabel('x_2');
title('Multivariate Gaussian Likelihoods');
legend('p(x|H_a)', 'p(x|H_b)', 'p(x|H_c)');
axis equal;
```



The problem states that each of the three hypotheses are equally probable, using this a priori assumption, the a posteriori probabilities are calcualted as

$$Pr(H_a|\mathbf{x}) = \frac{p(\mathbf{x}|H_a)(1/3)}{\sum_{i=1,2,3} p(\mathbf{x}|H_i)(1/3)} = \frac{p(\mathbf{x}|H_a)}{p(\mathbf{x}|H_a) + p(\mathbf{x}|H_b) + p(\mathbf{x}|H_c)}$$

$$Pr(H_b|\mathbf{x}) = \frac{p(\mathbf{x}|H_b)(1/3)}{\sum_{i=1,2,3} p(\mathbf{x}|H_i)(1/3)} = \frac{p(\mathbf{x}|H_b)}{p(\mathbf{x}|H_a) + p(\mathbf{x}|H_b) + p(\mathbf{x}|H_c)}$$

$$Pr(H_c|\mathbf{x}) = \frac{p(\mathbf{x}|H_c)(1/3)}{\sum_{i=1,2,3} p(\mathbf{x}|H_i)(1/3)} = \frac{p(\mathbf{x}|H_c)}{p(\mathbf{x}|H_a) + p(\mathbf{x}|H_b) + p(\mathbf{x}|H_c)}$$

To form a sufficient statistic, we observe that it is unnecessary to compute the full posterior probability for each hypothesis; instead we just need to determine which posterior is greatest. Because the denominators in the expressions are identical, they are multiplied to the right hand side and grouped into sufficient statistic $T_s(\mathbf{x}, H_i)$.

$$T_s(\mathbf{x}, H_i) = (p(\mathbf{x}|H_a) + p(\mathbf{x}|H_b) + p(\mathbf{x}|H_c))Pr(H_i|\mathbf{x}) = p(\mathbf{x}|H_i)$$

Further algebraic simplifications are as follows

$$T_s(\mathbf{x}, H_i) = \frac{1}{2\pi\sigma_w^4}\exp\left(-\frac{1}{2}\begin{bmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{bmatrix}^H \begin{bmatrix} 1/\sigma_w^2 & 0 \\ 0 & 1/\sigma_w^2 \end{bmatrix} \begin{bmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{bmatrix}\right)$$

$$T_s(\mathbf{x}, H_i) = \frac{1}{2\pi\sigma_w^2}\exp\left(-\frac{(x_1-\mu_1)^2 + (x_2-\mu_2)^2}{2\sigma_w^2}\right)$$

$$T_s(\mathbf{x}, H_i) = (x_1-\mu_{i1})^2 + (x_2-\mu_{i2})^2$$

Rewriting this generalized form for each hypothesis gives three scalar sufficient statistics!

$$T_s(\mathbf{x}, H_a) = (x_1)^2 + (x_2)^2 \quad T_s(\mathbf{x}, H_b) = (x_1-\sqrt{2})^2 + (x_2-\sqrt{2})^2 \quad T_s(\mathbf{x}, H_c) = (x_1-2\sqrt{2})^2 + (x_2-2\sqrt{2})^2$$

2. Using these sufficient statistics, express a specific MAP decision rule for each of the three hypotheses.

The three sufficient statistics defined in part 1 can now be used to determine the maximum a posteriori decision rule. The three rules are,

$$\text{Choose } H_a \text{ if } T_s(\mathbf{x}, H_a) > T_s(\mathbf{x}, H_b) \quad \text{and} \quad T_s(\mathbf{x}, H_a) > T_s(\mathbf{x}, H_c)$$

$$\text{Choose } H_b \text{ if } T_s(\mathbf{x}, H_b) > T_s(\mathbf{x}, H_a) \quad \text{and} \quad T_s(\mathbf{x}, H_b) > T_s(\mathbf{x}, H_c)$$

$$\text{Choose } H_c \text{ if } T_s(\mathbf{x}, H_c) > T_s(\mathbf{x}, H_a) \quad \text{and} \quad T_s(\mathbf{x}, H_c) > T_s(\mathbf{x}, H_b)$$

3. From these decision rules, extract a single scalar detection statistic $T_d(\mathbf{x})$**, and define specifically the decision region for each hypothesis in terms of this detection statistic.**

For each of the decision rules above, a single scalar detection statistic $T_d(\mathbf{x})$ is expressed by moving all terms dependent on the measurement to the left hand side of the equation, and all terms dependent on the hypotheses to the left.

$$\text{Choose } H_a \quad \text{if} \quad x_1 + x_2 > \sqrt{2} \quad \text{and} \quad x_1 + x_2 > 2\sqrt{2}$$

$$\text{Choose} \quad H_b \quad \text{if} \quad x_1 + x_2 < \sqrt{2} \quad \text{and} \quad x_1 + x_2 > 3\sqrt{2}$$

$$\text{Choose} \quad H_c \quad \text{if} \quad x_1 + x_2 < 2\sqrt{2} \quad \text{and} \quad x_1 + x_2 < 3\sqrt{2}$$

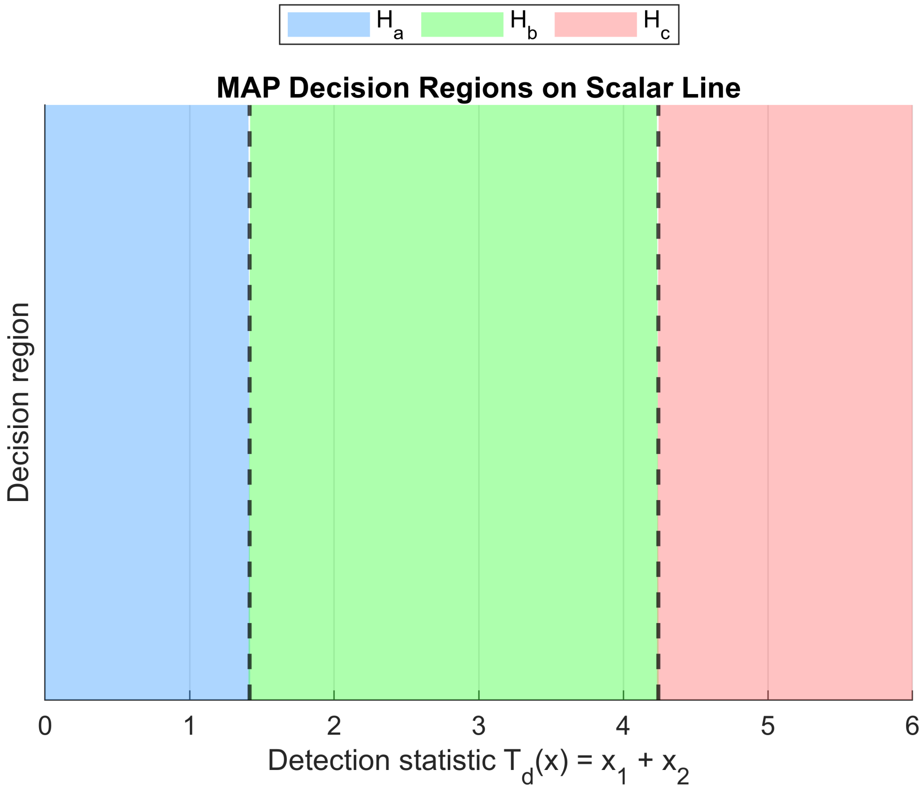
The redundant terms are removed and the stricter domains are kept producing the following decision regions using the decision statistic $T_d(\mathbf{x}) = x_1 + x_2$.

$$\text{Choose } H_a \text{ if } T_d(\mathbf{x}) < \sqrt{2}$$

$$\text{Choose } H_b \text{ if } \sqrt{2} < T_d(\mathbf{x}) < 3\sqrt{2}$$

$$\text{Choose } H_c \text{ if } T_d(\mathbf{x}) > 3\sqrt{2}$$

```
% thresholds
tau1 = sqrt(2);
tau2 = 3*sqrt(2);

% range for detection statistic
Td = linspace(0, 6, 500);

% decision regions: Ha (low), Hb (middle), Hc (high)
region = zeros(size(Td));
region(Td < tau1) = 1;
region(Td >= tau1 & Td <= tau2) = 2;
region(Td > tau2) = 3;

% plot decision regions
figure('Name', 'Decision Regions'); hold on;
area(Td, region==1, 'FaceColor',[0.2 0.6 1], 'EdgeColor','none', 'FaceAlpha',0.4);
area(Td, region==2, 'FaceColor',[0.2 1 0.2], 'EdgeColor','none', 'FaceAlpha',0.4);
area(Td, region==3, 'FaceColor',[1 0.4 0.4], 'EdgeColor','none', 'FaceAlpha',0.4);
xline(tau1,'k--','LineWidth',1.5);
xline(tau2,'k--','LineWidth',1.5);
xlabel('Detection statistic T_d(x) = x_1 + x_2');
ylabel('Decision region');
yticks([]);
legend({'H_a','H_b','H_c'}, 'Location','northoutside','Orientation','horizontal');
title('MAP Decision Regions on Scalar Line');
grid on;
```


H_a
H_b
H_c
MAP Decision Regions on Scalar Line
Decision region
0
1
2
3
4
5
6
Detection statistic T_d(x) = x_1 + x_2