

- 1. Use the MAP criterion to determine a decision rule using the detection statistic: $T_d(x) = x$. Describe carefully and completely the decision regions \mathcal{R}_A and \mathcal{R}_B for this decision rule.**

Following the intuition built in *Problem 1*, the maximum a posteriori probability (MAP) density function is found via its relation to the likelihood and a priori densities through Bayes' theorem.

$$p(H_i|x) = \frac{p(x|H_i)p(H_i)}{p(x)} = \frac{p(x|H_i)p(H_i)}{\sum_i p(x|H_i)p(H_i)}$$

Substituting in the given likelihood functions and a priori densities into the above equation, the MAP density for hypotheses H_A and H_B expressed as

$$p(H_A|x) = \begin{cases} \frac{\frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right)(0.5)}{\frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right)(0.5) + (0.2)(0.5)} & \text{for } -2.5 \leq x \leq 2.5 \\ \frac{\frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right)(0.5)}{\frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right)(0.5) + 0(0.5)} & \text{otherwise} \end{cases},$$

$$p(H_B|x) = \begin{cases} \frac{(0.2)(0.5)}{\frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right)(0.5) + (0.2)(0.5)} & \text{for } -2.5 \leq x \leq 2.5 \\ 0 & \text{otherwise} \end{cases},$$

which can be simplified to

$$p(H_A|x) = \begin{cases} \frac{\frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right)}{\frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right) + (0.2)} & \text{for } -2.5 \leq x \leq 2.5 \\ 1 & \text{otherwise} \end{cases} \rightarrow \begin{cases} \frac{1}{1 + 0.2 \sqrt{2\pi} \exp\left(\frac{x^2}{2}\right)} & \text{for } -2.5 \leq x \leq 2.5 \\ 1 & \text{otherwise} \end{cases},$$

$$p(H_B|x) = \begin{cases} \frac{(0.2)}{\frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right) + (0.2)} & \text{for } -2.5 \leq x \leq 2.5 \\ 0 & \text{otherwise} \end{cases} \rightarrow \begin{cases} \frac{0.2 \sqrt{2\pi} \exp\left(\frac{x^2}{2}\right)}{1 + 0.2 \sqrt{2\pi} \exp\left(\frac{x^2}{2}\right)} & \text{for } -2.5 \leq x \leq 2.5 \\ 0 & \text{otherwise} \end{cases}.$$

From these a posteriori expressions, the MAP decision rule can be constructed:

Choose H_B if $p(H_B|x) \geq p(H_A|x)$ and $-2.5 \leq x \leq 2.5$ else choose H_A ,

and rearranged into a likelihood ratio using measurement dependent likelihood ratio $L(x)$:

Choose H_B if $\frac{p(H_B|x)}{p(H_A|x)} = L(x) \geq 1.0$ and $-2.5 \leq x \leq 2.5$ else choose H_A .

Substituting the derived equations for $p(H_A|x)$ and $p(H_B|x)$ into the likelihood ratio $L(x)$ yeilds

$$L(x) = \left(\frac{0.2 \sqrt{2\pi} \exp\left(\frac{x^2}{2}\right)}{1 + 0.2 \sqrt{2\pi} \exp\left(\frac{x^2}{2}\right)} \right) \left(\frac{1}{1 + 0.2 \sqrt{2\pi} \exp\left(\frac{x^2}{2}\right)} \right)^{-1},$$

which, noting the same terms in the numerator and denomonator after the inverse operator, can be simplified to

$$L(x) = 0.2 \sqrt{2\pi} \exp\left(\frac{x^2}{2}\right).$$

Plugging this representation of $L(x)$ back into the LRT yeilds,

Choose H_B if $0.2 \sqrt{2\pi} \exp\left(\frac{x^2}{2}\right) \geq 1.0$ and $-2.5 \leq x \leq 2.5$ else choose H_A

which can be further simplified down to a detection statistic equal to the measurement itself by taking the natural logarithm of both sides of the inequality, grouping constants on the right hand side, and solving for x .

$$= \ln\left(0.2 \sqrt{2\pi} \exp\left(\frac{x^2}{2}\right)\right) \geq \ln(1.0) \longrightarrow \ln(0.2 \sqrt{2\pi}) + \ln\left(\exp\left(\frac{x^2}{2}\right)\right) \geq 0 \longrightarrow \frac{x^2}{2} \geq -\ln(0.2 \sqrt{2\pi}) \longrightarrow x < -1.176 \text{ and } x > 1.176$$

This result means that the original LRT inequality $L(x) \geq \ln(1.0)$ holds only for values of x outside of the interval $[-1.176, 1.176]$. Being that this derived boundary relates to the detection statistic $T_d(x) = x$, is can be directly related to the original boundaries of the measurement x . Mathmatically, the culmination of these boundaries results in the LRT:

Choose H_B if $T_d(x) \in \mathcal{R}_B$ else choose H_A

where \mathcal{R}_A and \mathcal{R}_B are

$$\mathcal{R}_A = \{T_d(x) : T_d(x) < -2.5\} \cup \{T_d(x) : -1.176 < T_d(x) < 1.176\} \cup \{T_d(x) : 2.5 < T_d(x)\}$$

$$\mathcal{R}_B = \{T_d(x) : -2.5 \geq T_d(x) \leq -1.176\} \cup \{T_d(x) : 1.176 \leq T_d(x) \leq 2.5\}$$

which is equivalent to $\mathcal{R}_A = (-\infty, 2.5) \cup (-1.176, 1.176) \cup (2.5, \infty)$ and $\mathcal{R}_B = (-2.5, -1.176) \cup (1.176, 2.5)$.

Similar to that of *Problem 1*, the resulting simplified detection statistic exhibits bounds that supersede those found in the original MAP derivation. Using this region, the simplified likelihood ratio test can be written as

$$\text{Choose } H_B \text{ if } -\sqrt{0.456} \leq x \leq \sqrt{0.456} \text{ else choose } H_A.$$

Decision regions \mathcal{R}_A and \mathcal{R}_B describe, in a MAP sense, the region in which hypothesis H_A and H_B are most probable respectively. Using the simplified detection statistic from the LRT derived above, the decision regions are described by the boundaries in which H_B is to be chosen over H_A and vice versa.

$$\mathcal{R}_A = \{T_d x \in [-\infty, -\sqrt{0.456}] \cup [\sqrt{0.456}, \infty]\}$$

$$\mathcal{R}_B = \{T_d x \in [-\sqrt{0.456}, \sqrt{0.456}]\}$$

2. Determine the probability of error for this decision rule.

Defining the a posteriori probabilities for hypotheses H_A and H_B as a function of the measurement x , the probability of error given this measurement, for a binary detection case, is the a posteriori probability of the least probable hypothesis. This can be stated mathematically as choosing H_B when the measurement occurs within region \mathcal{R}_A and vice versa.

$$Pr(e|x) = \begin{cases} Pr(x|H_A) & \text{for } x \in \mathcal{R}_B \\ Pr(x|H_B) & \text{for } x \in \mathcal{R}_A \end{cases} \longrightarrow \begin{cases} 1 - Pr(x|H_A) & \text{for } x \in \mathcal{R}_A \\ 1 - Pr(x|H_B) & \text{for } x \in \mathcal{R}_B \end{cases}$$

From this conditional definition of error, the cumulative probability of error, $Pr(e)$, is found by integrating over the entire measurement density function $p(x)$. This is equivalently expressed as the expected value of $Pr(e|x)$ defined below.

$$Pr(e) = \mathbb{E}\{Pr(e|x)\} = \int Pr(e|x)p(x)$$

Using the definitions for \mathcal{R}_A and \mathcal{R}_B defined in part 1 for the detection statistic $T_d(x) = x$, the conditional error is evaluated as

$$Pr(e|x) = \begin{cases} 1 - Pr(x|H_A) & \text{for } x \in \mathcal{R}_A \\ 1 - Pr(x|H_B) & \text{for } x \in \mathcal{R}_B \end{cases}$$

