EECS 965 Exam I

Fall 2025 - 75 Points

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Instructions

1) Your solutions must be detailed, unambiguous, and organized. I require that you show **all** your mathematics (i.e., no Matlab unless explicitly allowed).

2) Mathematically reduce your solutions to their **simplest** and most fundamental form.

2) Please **underline or circle** all numeric answers (e.g., $|\mathbf{A}| = 11.2$), or in some way make your final answer **clear**.

4) If you feel that a problem is unclear, contradictory, incomplete, or ambiguous, **ask for clarification**.

5) This is an exam; it must reflect **your** knowledge and effort—and yours only!

Please **sign** this statement: "*As an honorable scholar and human being, I pledge that this exam is a reflection of my knowledge only. I hereby pledge that I committed no act that a reasonable person could construe as academic misconduct.*"

Signed: W. Powers 11/4/2025

Problem 1 – 25 points

Consider a **real-valued** random variable x .

Under hypothesis $\mathcal{H}_0$, x is described by the *pdf*:

$$p(x|\mathcal{H}_0) = \begin{cases} (3/8)(2-x)^2 \;\; for\ 0 < x < 2 \\ 0 \;\; for\ x < 0,\ x > 2 \end{cases}$$

While under hypothesis $\mathcal{H}_1$, x is described by the *pdf*:

$$p(x|\mathcal{H}_1) = \begin{cases} (3/8)x^2 \;\; for\ 0 < x < 2 \\ 0 \;\; for\ x < 0,\ x > 2 \end{cases}$$

The *a priori* probabilities of these two hypotheses are:

$$P(\mathcal{H}_0) = \frac{3}{5} \qquad \text{and} \qquad Pr(\mathcal{H}_1) = \frac{2}{5}$$

From a **single** observation x, we must (attempt to) **choose** the correct hypothesis ($\mathcal{H}_0$ or $\mathcal{H}_1$).

1. Determine the Likelihood Ratio Test (i.e., **determine** $L(x)$ **and** γ) using the **MAP** detection criteria.

2. Simplify the LRT, such that the decision rule can be expressed in terms of this **decision statistic**:

$$T_d(x) = x$$

For **this** decision statistic, determine the value of threshold γ' for MAP criteria.

3. Say you now instead desire a threshold γ' that would result in a **probability of detection** of:

$$P_D = 0.90$$

Determine:

a) the value of this **threshold** γ', and

b) the resulting **probability of false alarm**.

1. Determine the Likelihood Ratio Test (i.e. determine $L(x)$ and γ) using the MAP detection criterion.

Given the likelihood functions $p(x|H_0)$ and $p(x|H_1)$ as well as the a priori probabilities $p(H_0)$ and $p(H_1)$ for hypotheses H_0 and H_1, the maximum a posteriori probability (MAP) density for each hypothesis is computed using the relation through Bayes' theorm.

$$p(H_i|x) = \frac{p(x|H_i)p(H_i)}{p(x)} = \frac{p(x|H_i)p(H_i)}{\sum_i p(x|H_i)p(x)dx}$$

Substituing in the given likelihood and a priori densities into the above equation, the MAP densities for each hypothesis are given as

$$p(H_0|x) = \begin{cases} \dfrac{(3/8)(2-x)^2(3/5)}{(3/8)(2-x)^2(3/5)+(3/8)(x)^2(2/5)} & \text{for } 0 < x < 2 \\ 0 & \text{for } x < 0,\ x > 2 \end{cases},$$

$$p(H_1|x) = \begin{cases} \dfrac{(3/8)(x)^2(2/5)}{(3/8)(2-x)^2(3/5)+(3/8)(x)^2(2/5)} & \text{for } 0 < x < 2 \\ 0 & \text{for } x < 0,\ x > 2 \end{cases}.$$

Simpilifying the above expressions yeilds

$$p(H_0|x) = \begin{cases} \dfrac{3(2-x)^2}{3(2-x)^2+2x^2} & \text{for } 0 < x < 2 \\ 0 & \text{for } x < 0,\ x > 2 \end{cases} \quad \text{and} \quad p(H_1|x) = \begin{cases} \dfrac{2x^2}{3(2-x)^2+2x^2} & \text{for } 0 < x < 2 \\ 0 & \text{for } x < 0,\ x > 2 \end{cases}.$$

From these a posteriori expressions, the MAP decision rule can be constructed as

Choose H_1 if $p(H_1|x) > p(H_0|x)$ and $0 < x < 2$ else choose H_0,

and rearranged into a likelihood ratio test (LRT) using measurement dependent likelihood ratio $L(x)$:

Choose H_1 if $\dfrac{p(H_1|x)}{p(H_0|x)} = L(x) > 1.0$ and $0 < x < 2$ else choose H_0.

Because both a posteriori definitions are bounded along the interval $[0, 2]$ and are equivalently evaluated to 0 outside of this region, the two hypothesis are treated as equally likely when the measurement lies outside of this region and H_0 is selected as default. The simplified representations for the a posteriori probability densities are substituted into the likelihood ratio definition and further simplified below.

$$L(x) = \frac{\frac{2x^2}{3(2-x)^2 + 2x^2}}{\frac{3(2-x)^2}{3(2-x)^2 + 2x^2}} = \left(\frac{2x^2}{3(2-x)^2 + 2x^2}\right)\left(\frac{3(2-x)^2 + 2x^2}{3(2-x)^2}\right) = \frac{2x^2}{3(2-x)^2}$$

The full MAP decision rule is therefore

$$\text{Choose } H_1 \text{ if } \frac{2x^2}{3(2-x)^2} > 1 \text{ and } 0 < x < 2 \text{ else choose } H_0.$$

2. Simplify the LRT, such that the decision rule can be expressed in terms of the detection statistic: $T_d(x) = x$***. For this decision statistic, determine the value of threshold*** γ' ***for the MAP criterion.***

To convert the MAP decision rule into one where the detection statistic $T_d(x)$ is equal to the measurment itself, the denominator of the likelihood function is moved to the left hand side of the inequality and further manipulated to isolate x on the right hand side.

$$\frac{2x^2}{3(2-x)^2} > 1 \longrightarrow 2x^2 > 3(2-x)^2 \longrightarrow 2x^2 > 12 - 12x + 3x^2 \longrightarrow x^2 - 12x + 12 < 0$$

Solving the above quadratic for x via the quadratic formula yeilds $6 - 2\sqrt{6} < x < 6 + 2\sqrt{6}$. Relating this result back to the original decision rule, the lower bound of $6 - 2\sqrt{6}$ supersedes the original decision boudary of 0 while the original upper bound of 2 remains tighter than $6 + 2\sqrt{6}$. The updated decision rule using detection statistic $T_d(x) = x$ and decision boundary $\gamma' = \left[6 - 2\sqrt{6}, 2\right]$ is therefore

$$\text{Choose } H_1 \text{ if } 6 - 2\sqrt{6} < x < 2 \text{ else choose } H_0.$$

3. Say you now instead desire a threshold γ' ***that would result in a probability of detection of*** $P_D = 0.90$***. Determine (a) the value of this threshold*** γ'***, and (b) the resulting probability of false alarm.***

Denoting the decision boundary derived in part 2 as the region $\mathscr{R}_1 \in \left[6 - 2\sqrt{6}, 2\right]$, the probability of detection is defined as the intergal of the a posteriori probability density of H_1 over this region. This intuitively means that the measured detection statistic lies within the region of H_1 and therefore we choose H_1.

$$P_D = P(T_d(x) \in \mathscr{R}_1 | H_1) = \int_{T_d(x) \in \mathscr{R}_1} p(T_d(x)|H_1) dT_d(x)$$

Becuase the detection statistic $T_d(x)$ in the above intergal has been simplified down to $T_d(x) = x$, the detection staistic likelihood function $p(T_d(x)|H_1)$ is equivalent to the measurement likelihood function $p(x|H_1)$, and therefore the intergal can be rewritten as

$$P_D = P(x \in \mathscr{R}_1 | H_1) = \int_{x \in \mathscr{R}_1} p(x|H_1) dx$$

From this we could evaluate the probability of detection for the derived MAP decision rule in part 2. For this problem however, the resulting probabiltiy of detection is set and instead we are tasked to find the region in which this integration results in $P_D = 0.90$. Substituting in the given likelihood definition of H_1, the probability of detection intergal becomes

$$0.90 = \int_{\gamma'}^{2} \left(\frac{3}{8} x^2\right) dx$$

and is solved by finding the lower boundary γ' such that the equality is true.

$$\int_{\gamma'}^{2} \left(\frac{3}{8} x^2\right) dx \longrightarrow \left[\frac{x^3}{8}\right]_{\gamma'}^{2} \longrightarrow 0.90 = \frac{2^3 - \alpha^3}{8} \longrightarrow (0.8)^{1/3} = \gamma' \longrightarrow 0.928 = \gamma'$$

Using this new decision region $\mathscr{R}_1 = [0.928, 2]$, the resulting probability of false alarm can be found by integrating the likelihood function of H_0 over this region.

$$P_{FA} = \int_{x \in \mathscr{R}_1} p(x|H_0) dx = \int_{0.928}^{2} \left(\frac{3}{8}(2 - x)^2\right) dx = \left[\frac{1}{8}(x^3 - 6x^2 + 12x)\right]_{0.928}^{2} = 0.1539$$

Problem 2 - 25 points

Under hypothesis $\mathcal{H}_a$, an observation $\mathbf{x} = [x_1, x_2, x_3, x_4]^T$ is equal to a known signal $\mathbf{s}_a$ with additive Gaussian noise $\mathbf{w}$:

$$\mathbf{x} = \mathbf{s}_a + \mathbf{w}$$

Under hypothesis $\mathcal{H}_b$, an observation $\mathbf{x}$ is equal to a known signal $\mathbf{s}_b$ with additive Gaussian noise $\mathbf{w}$:

$$\mathbf{x} = \mathbf{s}_b + \mathbf{w}$$

The two **known signals** are:

$$\mathbf{s}_a = [3, -1, 1, -3]^T \qquad \text{and} \qquad \mathbf{s}_b = [-3, 3, -5, 5]^T$$

while the **Gaussian** noise vector $\mathbf{w} = [w_1, w_2, w_3, w_4]^T$ is distributed as:

$$\mathbf{w} \sim \mathcal{N}(\mathbf{0}, \mathbf{C}_w)$$

where:

$$\mathbf{C}_w = E\{\mathbf{w}\mathbf{w}^T\} = \sum_{n=1}^{4} \lambda_n \, \hat{\mathbf{v}}_n \hat{\mathbf{v}}_n^T$$

And also where:

$$\lambda_1 = 6.0 \qquad \hat{\mathbf{v}}_1 = [+0.5, -0.5, +0.5, -0.5]^T$$

$$\lambda_2 = 4.0 \qquad \hat{\mathbf{v}}_2 = [+0.5, -0.5, -0.5, +0.5]^T$$

$$\lambda_3 = 2.0 \qquad \hat{\mathbf{v}}_3 = [+0.5, +0.5, -0.5, -0.5]^T$$

$$\lambda_4 = 1.0 \qquad \hat{\mathbf{v}}_4 = \left[+0.5, +0.5, +0.5, +0.5\right]^T$$

1. To decorrelate the elements of the noise vector, apply the **Karhunen-Loeve transform**.

Specifically, determine

a) the **transformed** signal vectors $\mathbf{s}'_a$ and $\mathbf{s}'_b$, and

b) the **covariance matrix** of the transformed noise vector $\mathbf{w}'$.

2. Now, using a LRT threshold of $\gamma = e^2$, determine a LRT for this detection problem, expressed in terms of **transformed measurement** $\mathbf{x}' = \left[x'_1, x'_2, x'_3, x'_4\right]^T$ (i.e., in terms of variables x'_1, x'_2, x'_3, x'_4).

3. Simplify this LRT into a **decision rule** of the form:

$$T_d\left(\mathbf{x}'\right) > \gamma'$$

I.E., provide **explicitly**—and in their **simplest** possible form—the statistic $T_d\left(\mathbf{x}'\right)$ and threshold γ'.

1. To decorrelate the elements of the noise vector, apply the Karhunen-Loeve transform. Specfically, determine (a) the transformed signal vectors $\mathbf{s}_A{}'$ and $\mathbf{s}_B{}'$ as well as the covariance matrix of the transformed noise vector $\mathbf{w}'$.

Given that the measurement vector $\mathbf{x}$ is modeled as $\mathbf{x} = \mathbf{s} + \mathbf{w}$ where the noise vector $\mathbf{w}$ is normally distributed as $\mathcal{N}(0, \mathbf{C}_w)$, the Karhunen-Loeve transform can be applied to whiten the resulting measurement making the samples of $\mathbf{x}$ independent and identically distributed (i.i.d) which greatly simplifies detection processing. Specifically this involves determining a linear transformation of the data such that resulting covariance matrix $\mathbf{C}_\mathbf{x}$ is diagonalized.

Being that each hypothesis consists of a deterministic signal, $\mathbf{s}_A$ or $\mathbf{s}_B$, and additive noise $\mathbf{w}$, the measurements covariance structure is solely a result of the noise covariance being that $\mathbf{w}$ is the only random component. This intuition is represented below in which the mean of the measurement under each hypothesis is

$$\mu_{x|H_A} = \mathbf{s}_A \quad \text{and} \quad \mu_{x|H_B} = \mathbf{s}_B,$$

with covariance

$$\mathbf{C}_{X|H_A} = E\{(\mathbf{x} - \mu_{x|H_A})(\mathbf{x} - \mu_{x|H_A})^T\} = E\{\mathbf{w}\mathbf{w}^\mathbf{T}\} = \mathbf{C}_w,$$

$$\mathbf{C}_{X|H_B} = E\{(\mathbf{x} - \mu_{x|H_B})(\mathbf{x} - \mu_{x|H_B})^T\} = E\{\mathbf{w}\mathbf{w}^\mathbf{T}\} = \mathbf{C}_w$$

resulting in each having normal distribution described by

$$\mathbf{x}|H_A \sim \mathcal{N}(\mathbf{s}_A, \mathbf{C}_\mathbf{w}) \quad \text{and} \quad \mathbf{x}|H_B \sim \mathcal{N}(\mathbf{s}_B, \mathbf{C}_\mathbf{w}).$$

An eigen-decomposition is performed on this measurement covariance matrix resulting in the the orthonormal matrix $\mathbf{V}$ containing eigen vectors in each column, and matrix $\mathbf{\Sigma}$ containing the corresponding eigen-values along its diagonal.

$$\text{eig}(\mathbf{C}_\mathbf{w}) = \mathbf{V}\mathbf{\Sigma}\mathbf{V}^\mathbf{T} \qquad \mathbf{V} = \begin{bmatrix} +0.5 & +0.5 & +0.5 & +0.5 \\ -0.5 & -0.5 & +0.5 & +0.5 \\ +0.5 & -0.5 & -0.5 & +0.5 \\ -0.5 & +0.5 & -0.5 & +0.5 \end{bmatrix} \qquad \mathbf{\Sigma} = \begin{bmatrix} 6.0 & 0 & 0 & 0 \\ 0 & 4.0 & 0 & 0 \\ 0 & 0 & 2.0 & 0 \\ 0 & 0 & 0 & 1.0 \end{bmatrix}$$

Defining a new signal basis consisting of the orthonormal eigen vectors in $\mathbf{V}$, the measurement vector can be rewritten as $\mathbf{x} = \mathbf{V}\mathbf{x}' = \sum_{n=1}^{4} x_n{}' v_n$ where $\mathbf{x}'$ represents the discrete set of coefficients applied to each component of the new basis defined by $\mathbf{x}' = \mathbf{V}^T\mathbf{x}$. Writing each of the hypothesized measurements in this orthogonal basis yeilds

$$\mathbf{x}'|H_A = \mathbf{V^T s_A} + \mathbf{V^T w} \qquad \text{and} \qquad \mathbf{x}'|H_B = \mathbf{V^T s_B} + \mathbf{V^T w}.$$

Substituting $\mathbf{V}^T$, $\mathbf{s}_A$, and $\mathbf{s}_B$ into the above equations, the transformed signal vectors $\mathbf{s}_A{}'$ and $\mathbf{s}_B{}'$ are found as

$$\mathbf{s}_A{}' = \begin{bmatrix} +0.5 & -0.5 & +0.5 & -0.5 \\ +0.5 & -0.5 & -0.5 & +0.5 \\ +0.5 & +0.5 & -0.5 & -0.5 \\ +0.5 & +0.5 & +0.5 & +0.5 \end{bmatrix} \begin{bmatrix} 3.0 \\ -1.0 \\ 1.0 \\ -3.0 \end{bmatrix} = \begin{bmatrix} 4.0 \\ 0.0 \\ 2.0 \\ 0.0 \end{bmatrix} \quad \text{and} \quad \mathbf{s}_B{}' = \begin{bmatrix} +0.5 & -0.5 & +0.5 & -0.5 \\ +0.5 & -0.5 & -0.5 & +0.5 \\ +0.5 & +0.5 & -0.5 & -0.5 \\ +0.5 & +0.5 & +0.5 & +0.5 \end{bmatrix} \begin{bmatrix} -3.0 \\ 3.0 \\ -5.0 \\ 5.0 \end{bmatrix} = \begin{bmatrix} -8.0 \\ 2.0 \\ 0.0 \\ 0.0 \end{bmatrix}.$$

The transformed measurement covariance matrix is equal to $\mathbf{\Sigma}$ shown through the relation of the projection matrix $\mathbf{V}^T$ and the eigen-decomposition of the original covariance structure.

$$\mathbf{C_w}' = \mathbf{V^T C_w} = \mathbf{V^T V \Sigma V^T V} = \mathbf{I \Sigma I} = \mathbf{\Sigma} = \begin{bmatrix} 6.0 & 0 & 0 & 0 \\ 0 & 4.0 & 0 & 0 \\ 0 & 0 & 2.0 & 0 \\ 0 & 0 & 0 & 1.0 \end{bmatrix}$$

2. Now, using a LRT threshold of $\gamma = e^2$, determine a LRT for this detection problem, expressed in terms of the transformed measurement $\mathbf{x}' = [x_1{}', x_2{}', x_3{}', x_4{}']^T$.

Using the i.i.d. transformed measurement vector $\mathbf{x}'$ where each sample normally distributed with a diagonalized covariance matrix, the likelihood functions for each hypothesis can be rewritten as the sum of a set of 4 independent guassian random variables.

$$p(\mathbf{x}'|H_A) = \frac{1}{(2\pi)^{4/2}(\det(\mathbf{\Sigma})^{1/2})} \exp\Big(-\frac{1}{2}(\mathbf{x}' - \mathbf{s}_A{}')^T \mathbf{\Sigma} (\mathbf{x}' - \mathbf{s}_A{}')\Big)$$

$$p(\mathbf{x}'|H_B) = \frac{1}{(2\pi)^{4/2}(\det(\mathbf{\Sigma})^{1/2})} \exp\Big(-\frac{1}{2}(\mathbf{x}' - \mathbf{s}_B{}')^T \mathbf{\Sigma} (\mathbf{x}' - \mathbf{s}_B{}')\Big)$$

Using these likelihood function definitions, and likelihood ratio $L(x)$ is defined as

$$L(x) = \frac{\frac{1}{(2\pi)^{4/2}(\det(\mathbf{\Sigma})^{1/2})} \exp\Big(-\frac{1}{2}(\mathbf{x}' - \mathbf{s}_B{}')^T \mathbf{\Sigma} (\mathbf{x}' - \mathbf{s}_B{}')\Big)}{\frac{1}{(2\pi)^{4/2}(\det(\mathbf{\Sigma})^{1/2})} \exp\Big(-\frac{1}{2}(\mathbf{x}' - \mathbf{s}_A{}')^T \mathbf{\Sigma} (\mathbf{x}' - \mathbf{s}_A{}')\Big)}$$

which can be simplified to

$$L(x) == \frac{\exp\Big(-\frac{1}{2}(\mathbf{x}' - \mathbf{s}_B{}')^T \mathbf{\Sigma}(\mathbf{x}' - \mathbf{s}_B{}')\Big)}{\exp\Big(-\frac{1}{2}(\mathbf{x}' - \mathbf{s}_A{}')^T \mathbf{\Sigma}(\mathbf{x}' - \mathbf{s}_A{}')\Big)} = \exp(-72x_1 + 8x_2 - 4x_3 - 148)$$

and used in a likelihood ratio test (LRT) with the given threshold $\gamma = e^2$:

Choose H_B if $\exp(-72x_1 + 8x_2 - 4x_3 - 148) \geq e^2$ else H_A.

3. Simplify this LRT into a decision rule of the form: $T_d(x') > \gamma'$ ***.***

Further simplification of the LRT can be made to ultimately express the decision rule in terms of a decision statistic that consists solely of the transformed measurement vector (i.e. $T_d(\mathbf{x}') = \mathbf{x}'$). Taking the natural logarithm of both sides of the LRT inequality derived in part 2 then moving all constants to the right hand side of the equation yeilds

$$\ln\left(\exp(-72x_1 + 8x_2 - 4x_3 - 148)\right) \geq \ln(e^2) \quad \longrightarrow \quad -72x_1 + 8x_2 - 4x_3 \geq 150.$$

Plugging this back into the LRT where $T_d(\mathbf{x}') = -72x_1 + 8x_2 - 4x_3$ and $\gamma' = 150$ yeilds the simplest form of the decision rule:

Choose H_B if $-72x_1 + 8x_2 - 4x_3 \geq 150$ else H_A.

Problem 3 - 25 points

Under hypothesis $\mathcal{H}_a$, a scalar measurement x is equal to random **variable** w_a:

$$x = w_a$$

Random variable w_a is described by the **Gaussian distribution**:

$$w_a \sim \mathcal{N}\left(\mu = 0, \sigma^2 = 1\right)$$

i.e.,:

$$p(w_a) = \frac{1}{\sqrt{2\pi}} \exp\left[\frac{-w_a^2}{2}\right] \qquad for\ -\infty < w_a < \infty$$

Under hypothesis $\mathcal{H}_b$, the scalar measurement x is equal to random **variable** w_b:

$$x = w_b$$

Random variable w_b is described by a **uniform distribution:**

$$p(w_b) = \begin{cases} 0.2 & for\ -2.5 \le w_b \le 2.5 \\ 0 & for\ w_b \le -2.5,\ w_b \ge 2.5 \end{cases}$$

The two hypotheses are **equally probable**, *a priori* .

1. Use the **MAP** criterion to **determine a decision rule** using the detection statistic:

$$T_d(x) = x$$

Describe carefully and **completely** the **decision regions** $\mathcal{R}_a$ and $\mathcal{R}_b$ for this decision rule.

2. Determine the **probability of error** for this decision rule (you **may** use **MatLab** or other cumulative probability solver for this calculation).

1. Use the MAP criterion to determine a decision rule using the detction statistic: $T_d(x) = x$. Describe carefully and completely the decision regions $\mathscr{R}_A$ and $\mathscr{R}_B$ for this decision rule.

Following the intiution built in *Problem 1*, the maximum a posteriori probability (MAP) density function is found via its relation to the likelihood and a priori densities through Bayes' theorm.

$$p(H_i|x) = \frac{p(x|H_i)p(H_i)}{p(x)} = \frac{p(x|H_i)p(H_i)}{\sum_i p(x|H_i)p(x)dx}$$

Substituing in the given likelihood functions and a priori densities into the above equation, the MAP density for hypotheses H_A and H_B are expressed as

$$p(H_A|x) = \begin{cases} \dfrac{\frac{1}{\sqrt{2\pi}}\exp\left(\frac{-x^2}{2}\right)(0.5)}{\frac{1}{\sqrt{2\pi}}\exp\left(\frac{-x^2}{2}\right)(0.5) + (0.2)(0.5)} & \text{for } -2.5 \le x \le 2.5 \\ \dfrac{\frac{1}{\sqrt{2\pi}}\exp\left(\frac{-x^2}{2}\right)(0.5)}{\frac{1}{\sqrt{2\pi}}\exp\left(\frac{-x^2}{2}\right)(0.5) + 0(0.5)} & \text{otherwise} \end{cases},$$

$$p(H_B|x) = \begin{cases} \dfrac{(0.2)(0.5)}{\frac{1}{\sqrt{2\pi}}\exp\left(\frac{-x^2}{2}\right)(0.5) + (0.2)(0.5)} & \text{for } -2.5 \le x \le 2.5 \\ 0 & \text{otherwise} \end{cases},$$

which can be simplified to

$$p(H_A|x) = \begin{cases} \dfrac{\frac{1}{\sqrt{2\pi}}\exp\left(\frac{-x^2}{2}\right)}{\frac{1}{\sqrt{2\pi}}\exp\left(\frac{-x^2}{2}\right) + (0.2)} & \text{for } -2.5 \le x \le 2.5 \\ 1 & \text{otherwise} \end{cases} \longrightarrow \begin{cases} \dfrac{1}{1 + 0.2\sqrt{2\pi}\exp\left(\frac{x^2}{2}\right)} & \text{for } -2.5 \le x \le 2.5 \\ 1 & \text{otherwise} \end{cases},$$

$$p(H_B|x) = \begin{cases} \dfrac{(0.2)}{\frac{1}{\sqrt{2\pi}}\exp\left(\frac{-x^2}{2}\right) + (0.2)} & \text{for } -2.5 \le x \le 2.5 \\ 0 & \text{otherwise} \end{cases} \longrightarrow \begin{cases} \dfrac{0.2\sqrt{2\pi}\exp\left(\frac{x^2}{2}\right)}{1 + 0.2\sqrt{2\pi}\exp\left(\frac{x^2}{2}\right)} & \text{for } -2.5 \le x \le 2.5 \\ 0 & \text{otherwise} \end{cases}.$$

From these a posteriori expressions, the MAP decision rule can be constructed:

$$\text{Choose } \; H_B \; \text{ if } \; p(H_B|x) \geq p(H_A|x) \; \text{ and } \; -2.5 \leq x \leq 2.5 \; \text{ else choose } \; H_A,$$

and rearranged into a likelihood ratio using measurement dependent likelihood ratio $L(x)$:

$$\text{Choose } \; H_B \text{ if } \; \frac{p(H_B|x)}{p(H_A|x)} = L(x) \geq 1.0 \; \text{ and } \; -2.5 \leq x \leq 2.5 \; \text{ else choose } \; H_A.$$

Substituting the derived equations for $p(H_A|x)$ and $p(H_B|x)$ into the likelihood ratio $L(x)$ yeilds

$$L(x) = \left(\frac{0.2\sqrt{2\pi}\exp\left(\frac{x^2}{2}\right)}{1 + 0.2\sqrt{2\pi}\exp\left(\frac{x^2}{2}\right)} \right) \left(\frac{1}{1 + 0.2\sqrt{2\pi}\exp\left(\frac{x^2}{2}\right)} \right)^{-1},$$

which, noting the same terms in the numerator and denomonator after the inverse operator, can be simplified to

$$L(x) = 0.2\sqrt{2\pi}\exp\left(\frac{x^2}{2}\right).$$

Plugging this representation of $L(x)$ back into the LRT yeilds:

$$\text{Choose } \; H_B \text{ if } \; 0.2\sqrt{2\pi}\exp\left(\frac{x^2}{2}\right) \geq 1.0 \; \text{ and } \; -2.5 \leq x \leq 2.5 \; \text{ else choose } \; H_A$$

which can be further simplified down to a detection statistic equal to the measurement itself by taking the natural logarithm of both sides of the inequality, grouping constants on the right hand side, and solving for x.

$$= \ln\left(0.2\sqrt{2\pi}\exp\left(\frac{x^2}{2}\right)\right) \geq \ln(1.0) \longrightarrow \ln\left(0.2\sqrt{2\pi}\right) + \ln\left(\exp\left(\frac{x^2}{2}\right)\right) \geq 0 \longrightarrow \frac{x^2}{2} \geq -\ln\left(0.2\sqrt{2\pi}\right) \longrightarrow x < -1.176 \;\text{ and }\; x > 1$$

This result means that the original LRT inequality $L(x) \geq \ln(1.0)$ holds only for values of x outside of the interval $[-1.176, 1.176]$. Being that this derived boundary was found for the detection statistic $T_d(x) = x$, it can be directly related to the original boundaries of the measurement x. Mathmatically, the culmination of these boundaries results in the LRT:

$$\text{Choose } \; H_B \text{ if } \; T_d(x) \in \mathscr{R}_B \; \text{ else choose } \; H_A$$

where $\mathscr{R}_A$ and $\mathscr{R}_B$ are

$$\mathscr{R}_A = \{T_d(x) : T_d(x) < -2.5\} \cup \{T_d(x) : -1.176 < T_d(x) < 1.176\} \cup \{T_d(x) : 2.5 < T_d(x)\}$$

$$\mathscr{R}_B = \{T_d(x) : -2.5 \geq T_d(x) \leq -1.176\} \cup \{T_d(x) : 1.176 \leq T_d(x) \leq 1.5\}$$

more clearly written in set notation as

$$\mathscr{R}_A = (-\infty, 2.5) \cup (-1.176, 1.176) \cup (2.5, \infty)$$

$$\mathscr{R}_B = (-2.5, -1.176) \cup (1.176, 2.5).$$

__2. Determine the probability of error for this decision rule.__

Defining the a posteriori probabilities for hypotheses H_A and H_B as a function of the measurement x, the probability of error given this measurment, for a binary detection case, is the a posteriori probability of the least probable hypothesis. This can be stated intuitevly as choosing H_B when the measurement occurs within region $\mathscr{R}_A$ and vice versa.

$$Pr(e|x) = \begin{cases} Pr(x|H_A) & \text{for } x \in \mathscr{R}_B \\ Pr(x|H_B) & \text{for } x \in \mathscr{R}_A \end{cases} \longrightarrow \begin{cases} 1 - Pr(x|H_A) & \text{for } x \in \mathscr{R}_A \\ 1 - Pr(x|H_B) & \text{for } x \in \mathscr{R}_B \end{cases}$$

From this conditional definition of error, the cumulative probability of error, $Pr(e)$, is found by integrating over the entire measurement density function $p(x)$. This is equivalently expressed as the expected value of $Pr(e|x)$ defined below.

$$Pr(e) = \mathbb{E}\{Pr(e|x)\} = \int Pr(e|x)p(x)$$

Using the definitions for $\mathscr{R}_A$ and $\mathscr{R}_B$ derived in part 1 for the detection statistic $T_d(x) = x$, the conditional error is written as

$$Pr(e|x) = \begin{cases} 1 - \left(\frac{1}{\sqrt{2\pi}}\exp\left(\frac{-x^2}{2}\right)\right) & \text{for } -\infty < x < -2.5 \\ 1 - 0.2 & \text{for } -2.5 \le x \le -1.176 \\ 1 - \left(\frac{1}{\sqrt{2\pi}}\exp\left(\frac{-x^2}{2}\right)\right) & \text{for } -1.176 < x < 1.176 \\ 1 - 0.2 & \text{for } 1.176 \le x \le 2.5 \\ 1 - \left(\frac{1}{\sqrt{2\pi}}\exp\left(\frac{-x^2}{2}\right)\right) & \text{for } 2.5 < x < \infty \end{cases}.$$

Through Bayes' theorm in part 1, it was shown that the measurement probability density function was equal to the the sum of the products of each likelihood function and the corresponding a priori probability of the hypothesis. Substituting these values into this definition, $p(x)$ is obtained as

$$p(x) = p(x|H_A)p(H_A) + p(x|H_B)p(H_B) = \begin{cases} \left(\frac{1}{\sqrt{2\pi}}\exp\left(\frac{-x^2}{2}\right)\right)(0.5) + (0.2)(0.5) & \text{for } -2.5 \le x \le 2.5 \\ \left(\frac{1}{\sqrt{2\pi}}\exp\left(\frac{-x^2}{2}\right)\right)(0.5) + 0 & \text{otherwise} \end{cases}.$$

Combinding this representation with the derived formulation of $Pr(e|x)$, the intergal used to find the cumulative probability of error can be written as

$$Pr(e) = \int_{-\infty}^{-2.5} \left(1 - \frac{1}{\sqrt{2\pi}}\exp\left(\frac{-x^2}{2}\right)\right)\left(\frac{1}{\sqrt{2\pi}}\exp\left(\frac{-x^2}{2}\right)(0.5)\right)dx + \int_{-2.5}^{-1.176} (1-0.2)\left(\frac{1}{\sqrt{2\pi}}\exp\left(\frac{-x^2}{2}\right)(0.5) + (0.2)(0.5)\right)dx$$

$$+ \int_{-1.176}^{1.176} \left(1 - \frac{1}{\sqrt{2\pi}}\exp\left(\frac{-x^2}{2}\right)\right)\left(\frac{1}{\sqrt{2\pi}}\exp\left(\frac{-x^2}{2}\right)(0.5) + (0.2)(0.5)\right)dx + \int_{1.176}^{2.5} (1-0.2)\left(\frac{1}{\sqrt{2\pi}}\exp\left(\frac{-x^2}{2}\right)(0.5) + (0.2)(0.5)\right)dx$$

$$+ \int_{2.5}^{\infty} \left(1 - \frac{1}{\sqrt{2\pi}}\exp\left(\frac{-x^2}{2}\right)\right)\left(\frac{1}{\sqrt{2\pi}}\exp\left(\frac{-x^2}{2}\right)(0.5)\right)dx.$$

This intergal exhibits symmetery between the first two and last two terms which allows for the simplification to

$$Pr(e) = 2\int_{-\infty}^{-2.5} \left(1 - \frac{1}{\sqrt{2\pi}}\exp\left(\frac{-x^2}{2}\right)\right)\left(\frac{1}{\sqrt{2\pi}}\exp\left(\frac{-x^2}{2}\right)(0.5)\right)dx + 2\int_{-2.5}^{-1.176} (1-0.2)\left(\frac{1}{\sqrt{2\pi}}\exp\left(\frac{-x^2}{2}\right)(0.5) + (0.2)(0.5)\right)dx$$

$$+ \int_{-1.176}^{1.176} \left(1 - \frac{1}{\sqrt{2\pi}}\exp\left(\frac{-x^2}{2}\right)\right)\left(\frac{1}{\sqrt{2\pi}}\exp\left(\frac{-x^2}{2}\right)(0.5) + (0.2)(0.5)\right)dx.$$

Each component of the above intergal is computed in the matlab excerpt below resulting in $Pr(e) = 0.7208$.

```
% intergal from (-infty, -2.5)
f = @(x) 2*((1 - (1/sqrt(2*pi)) .* exp(-x.^2/2)) .* ((1/sqrt(2*pi)) .* exp(-x.^2/2)
* 0.5));
i1 = integral(f, -Inf, -2.5);

% intergal from (-2.5, -1.176)
f = @(x) 2*((1 - 0.2) .* ((1/sqrt(2*pi)) .* exp(-x.^2/2) * 0.5 + (0.2 * 0.5)));
i2 = integral(f, -2.5, -1.176);

% intergal from (-1.176, 1.176)
f = @(x) (1 - (1/sqrt(2*pi)) .* exp(-x.^2/2)) .* ((1/sqrt(2*pi)) .* exp(-x.^2/2) *
0.5 + (0.2 * 0.5));
i3 = integral(f, -1.176, 1.176);

% sum each component
i = i1 + i2 + i3;
fprintf('Pr(e) = %.6f\n', i);
```

```
Pr(e) = 0.720759
```