(a) Determine the ML estimate of λ **(i.e. determine** $\widehat{\lambda}_{MLE}(\mathbf{x})$**).**

Given the joint likelihood function of distribution parameter λ producing measurement vector $\mathbf{x}$ as

$$p(\mathbf{x}|\lambda) = \prod_{n=1}^{N} \lambda \exp[-\lambda x_n],$$

the task of determining the maximum likelihood (ML) estimate is that of finding the value of λ which maximizes this distribution. Utilizing the exponential power rule of $e^x e^y = e^{x+y}$, the above distribution can be alternately expressed as

$$p(\mathbf{x}|\lambda) = \lambda^N \exp\left[-\lambda \sum_{n=1}^{N} x_n\right].$$

The next step is to take the gradient of the multivariate distribution with respect to the estimation parameter λ which is first preceeded by applying the natural logarithm operator to algebraically simplify subsequent operations.

$$\ln(p(\mathbf{x}|\lambda) = N \ln(\lambda) - \lambda \sum_{n=1}^{N} x_n$$

$$\frac{\partial \ln(p(\mathbf{x}|\lambda))}{\partial \lambda} = \frac{N}{\lambda} - \sum_{n=1}^{N} x_n$$

Setting the result above equal to zero and solving for λ yeilds

$$\frac{N}{\lambda} - \sum_{n=1}^{N} x_n = 0 \quad \longrightarrow \quad \lambda = N\left(\sum_{n=1}^{N} x_n\right)^{-1}$$

To classify this extrema the second derivative test is used

$$\frac{\partial^2 \ln(p(\mathbf{x}|\lambda))}{\partial^2 \lambda} = -\frac{N}{\lambda^2} < 0 \ \ \forall \ \ \mathbf{x},$$

which, as shown above, is less than zero for all measurement vectors $\mathbf{x}$. This proves that the function is concave, the stationary point found is a global maximum, and being that this value of λ maximizes the likelihood function, it is the ML estimate.

$$\widehat{\lambda}_{MLE}(\mathbf{x}) = N\left(\sum_{n=1}^{N} x_n\right)^{-1}$$