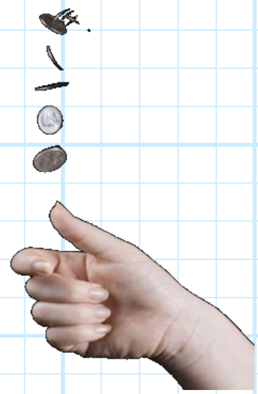
Problem 2



Under hypothesis $\mathcal{H}_0$, a coin is **"fair"**, such that the probability of "heads" when flipped is:

$$Pr("heads"|\mathcal{H}_0) = 0.5$$

Under hypothesis $\mathcal{H}_1$, a coin is **"biased"**, such that the probability of "heads" when flipped is:

$$Pr("heads"|\mathcal{H}_1) = 0.75$$

We flip an **unknown** coin **5 times**, and count (i.e., measure) the number of times it comes up **heads**.

This number is denoted as x:

$$x = \{\text{number of "heads"}\}$$

From this value x, we wish to **detect biased coins** (i.e., determine when hypothesis $\mathcal{H}_1$ is true)!

Note this number x will be an integer, such that:

$$x \in \{0,1,2,3,4,5\}$$

So for this **discrete** measurement case, the likelihood "functions" are expressed as a list of **probabilities** (not densities!), one for each integer value x:

$$
\begin{aligned}
Pr(x=0|\mathcal{H}_0) &= p_{00} \qquad & Pr(x=0|\mathcal{H}_1) &= p_{01}\\
Pr(x=1|\mathcal{H}_0) &= p_{10} \qquad & Pr(x=1|\mathcal{H}_1) &= p_{11}\\
Pr(x=2|\mathcal{H}_0) &= p_{20} \qquad & Pr(x=2|\mathcal{H}_1) &= p_{21}\\
Pr(x=3|\mathcal{H}_0) &= p_{30} \qquad & Pr(x=3|\mathcal{H}_1) &= p_{31}\\
Pr(x=4|\mathcal{H}_0) &= p_{40} \qquad & Pr(x=4|\mathcal{H}_1) &= p_{41}\\
Pr(x=5|\mathcal{H}_0) &= p_{50} \qquad & Pr(x=5|\mathcal{H}_1) &= p_{51}
\end{aligned}
$$

Note then that, for **example**:

$$Pr(x \le 3|\mathcal{H}_0) = \sum_{x=0}^{3} Pr(x|\mathcal{H}_0)$$

or

$$Pr(x > 2|\mathcal{H}_1) = \sum_{x=3}^{5} Pr(x|\mathcal{H}_1) = 1 - \sum_{x=0}^{2} Pr(x|\mathcal{H}_1)$$

1. Determine the likelihood "functions" $Pr(x|\mathcal{H}_0)$ and $Pr(x|\mathcal{H}_1)$. In other words, determine the **numeric values** of the **12 probabilities** ($p_{00}, p_{10}, \dots$ and $p_{01}, p_{11}, \dots$) in the lists above.

2. Say the **decision rule** is:

$$\text{Choose } \mathcal{H}_0 \quad \text{if} \quad x < y'$$

$$\text{Choose } \mathcal{H}_1 \quad \text{if} \quad x \geq y'$$

where:

$$y' \in \{0,1,2,3,4,5,6\}$$

Sketch the **ROC** for this detection problem. This ROC should consist of **7 points** on the (P_{FA}, P_D) plane. Provide the **specific coordinates** of each of these seven points.

3. Say the *a priori* probability of a biased coin is $Pr(\mathcal{H}_1) = \frac{1}{3}$. Use the **MAP** detection criterion to form a **decision rule** of the form:

$$\text{Choose } \mathcal{H}_0 \quad \text{if} \quad x < y'$$

$$\text{Choose } \mathcal{H}_1 \quad \text{if} \quad x \geq y'$$

where y' is an **integer** (**find its value**!).

4. Determine the **probability of error** $Pr\{\varepsilon\}$ of this MAP decision rule.

1. Determine the likelihood functions $Pr(x|H_0)$ and $Pr(x|H_1)$. In other words, determine the numerical values of the 12 probabilities ($p_{00}, p_{10}, \ldots$ and $p_{01}, p_{11}, \ldots$) in the lists above.

The discrete likelihood functions for the biased and unbiased coin are computed using a binomial distribution

$$Pr(x|H_i) = \binom{5}{x} P(\text{'heads'}|H_i)^x (1 - P(\text{'heads'}|H_i))^{5-x}$$

therefore the exhaustive list of probabilities for each hypothesis is

$$Pr(x=0|H_0) = 0.0312 \quad Pr(x=0|H_1) = 0.00097$$

$$Pr(x=1|H_0) = 0.1562 \quad Pr(x=1|H_1) = 0.0146$$

$$Pr(x=2|H_0) = 0.3125 \quad Pr(x=2|H_1) = 0.0878$$

$$Pr(x=3|H_0) = 0.3125 \quad Pr(x=3|H_1) = 0.2636$$

$$Pr(x=4|H_0) = 0.1562 \quad Pr(x=4|H_1) = 0.3955$$

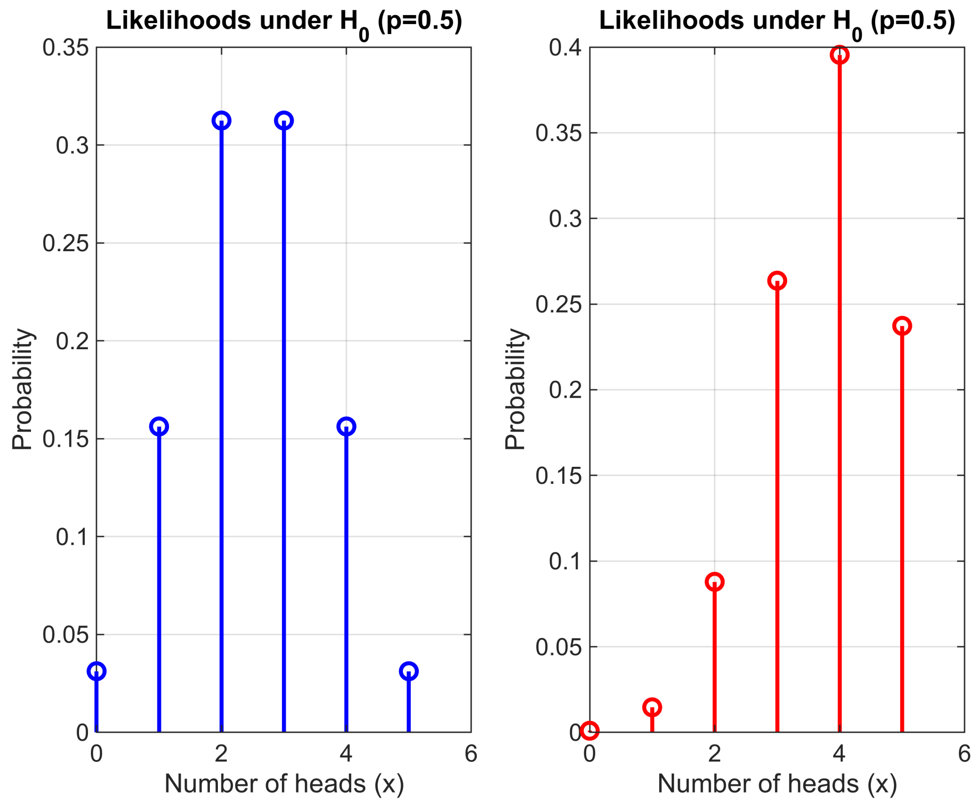
$$Pr(x=5|H_0) = 0.0312 \quad Pr(x=5|H_1) = 0.2373$$

```
% number of flips
n = 5;
x = 0:n;

% hypothesis 0 distribution
p0 = binopdf(x, n, 0.5);

% hypothesis 1 distribution
p1 = binopdf(x, n, 0.75);

figure("Name", "Discrete Likelihood Functions");
subplot(1, 2, 1);
stem(x, p0, 'b', 'LineWidth', 1.5);
xlabel('Number of heads (x)');
ylabel('Probability');
title('Likelihoods under H_0 (p=0.5)');
grid on;
subplot(1, 2, 2);
stem(x, p1, 'r', 'LineWidth', 1.5);
xlabel('Number of heads (x)');
ylabel('Probability');
title('Likelihoods under H_0 (p=0.5)');
grid on;
```



2. Say the decision rule is

$$\textbf{Choose } H_0 \textbf{ if } x < y'$$

$$\textbf{Choose } H_1 \textbf{ if } x \geq y'$$

where

$$y' \in \{0, 1, 2, 3, 4, 5, 6\}.$$

Sketch the ROC for this detection problem. This ROC should consist of 7 points on the (P_{FA}, P_D) ***plane. Provide the specific coordinates of each of these seven points.***

To construct the receiver operator curve, the intergal definitions for P_D and P_{FA} must first be defined over the interval $\mathscr{R}_1 = \{y', \infty\}$

$$P_{FA} = P(T_d(x) = x \in \mathscr{R}_1 | H_0) = \int_{x \in \mathscr{R}_1} p(x|H_0)dx$$

$$P_D = P(T_d(x) = x \in \mathscr{R}_1 | H_1) = \int_{x \in \mathscr{R}_1} p(x|H_1)dx$$

Becuase the likelihood functions are discrete, these intergals are rewritten as summations where $y' \in \{0, 1, 2, 3, 4, 5, 6\}$

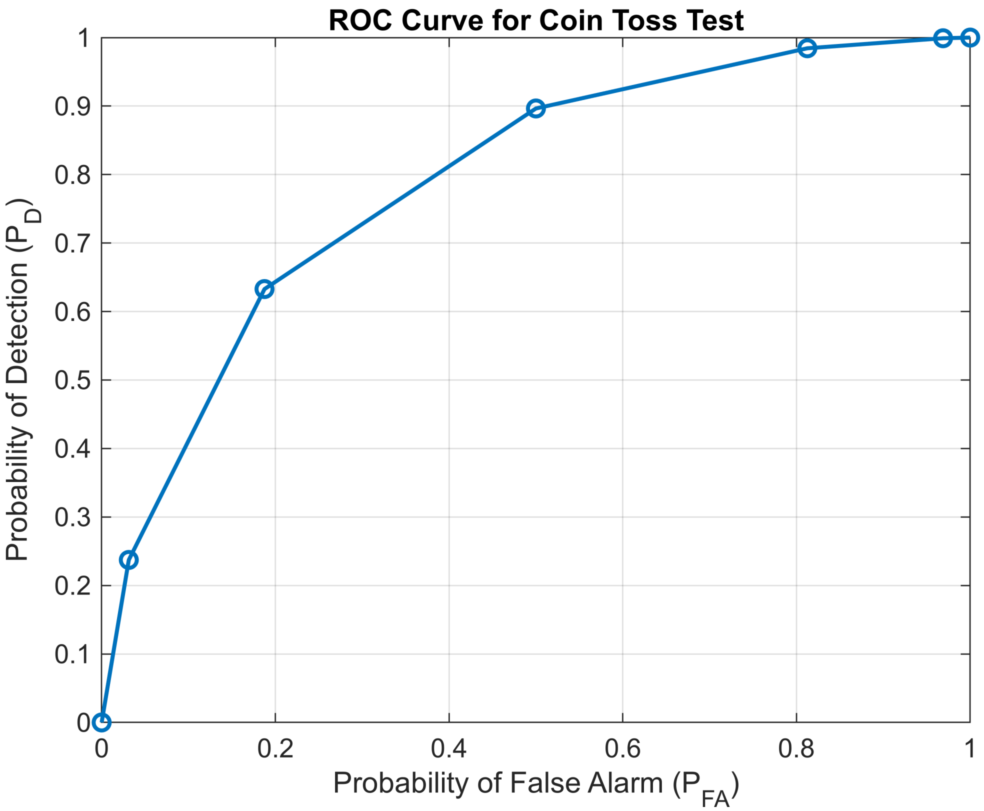
$$P_{FA} = \sum_{x=y'}^{5} Pr(x|H_0)dx$$

$$P_D = \sum_{x=y'}^{5} Pr(x|H_1)dx$$

and then plotted using the probabilities found in (a).

```
n = 5;
x = 0:n;
P_FA = zeros(n+2,1);
P_D  = zeros(n+2,1);

for i = 0:n
    P_FA(i+1) = sum(binopdf(i:n, n, 0.5));
    P_D(i+1) = sum(binopdf(i:n, n, 0.75));
end
% Plot ROC curve
figure("Name","ROC");
plot(P_FA, P_D, '-o', 'LineWidth', 1.5, 'MarkerSize', 6);
xlabel('Probability of False Alarm (P_{FA})');
ylabel('Probability of Detection (P_D)');
title('ROC Curve for Coin Toss Test');
grid on;
axis([0 1 0 1]);
```



3. Say the a priori probability of a biased coin is $Pr(H_1) = 1/3$. Use the MAP detection criterion to form a decision rule of the form

$$\textbf{Choose } H_0 \textbf{ if } x < y'$$

$$\textbf{Choose } H_1 \textbf{ if } x \geq y'$$

where y' is an interger.

Given the a prioris of seeing a biased and unbiased coin as $Pr(H_1) = 1/3$ and $Pr(H_0) = 2/3$ respectively, the maximum a posteriori probabilities can be determined.

$$Pr(H_0|x) = \frac{Pr(x|H_0)Pr(H_0)}{Pr(x|H_0)Pr(H_0) + Pr(x|H_1)Pr(H_1)} \qquad Pr(H_1|x) = \frac{Pr(x|H_1)Pr(H_1)}{Pr(x|H_0)Pr(H_0) + Pr(x|H_1)Pr(H_1)}$$

Plugging in the discrete likelihood functions and a prioris, the above is expressed and simplified as

$$Pr(H_0|x) = \frac{\left(\binom{5}{x}(0.5)^x(1-0.5)^{5-x}\right)(2/3)}{\left(\binom{5}{x}(0.5)^x(1-0.5)^{5-x}\right)(2/3) + \left(\binom{5}{x}(0.75)^x(1-0.75)^{5-x}\right)(1/3)} = \frac{64}{64+3^x}$$

$$Pr(H_1|x) = \frac{\left(\binom{6}{x}(0.75)^x(1-0.75)^{6-x}\right)(1/3)}{\left(\binom{6}{x}(0.5)^x(1-0.5)^{6-x}\right)(2/3) + \left(\binom{6}{x}(0.75)^x(1-0.75)^{6-x}\right)(1/3)} = \frac{3^x}{64+3^x}$$

Using these two a posteriori probabilities, a likelihood ratio test (LRT) is formed

$$\text{Choose } H_0 \text{ if } \frac{Pr(H_1|x)}{Pr(H_0|x)} = L(x) < 1.0 \qquad \text{Choose } H_1 \text{ if } \frac{Pr(H_1|x)}{Pr(H_0|x)} = L(x) \geq 1.0$$

where

$$L(x) = \frac{\frac{3^x}{64+3^x}}{\frac{64}{64+3^x}} = \frac{3^x}{64}$$

Plugging this into the LRT and further simplifying gives

$$\text{Choose } H_0 \text{ if } x < \frac{\ln(64)}{\ln(3)} \approx 3.78 \qquad \text{Choose } H_1 \text{ if } x \geq \frac{\ln(64)}{\ln(3)} \approx 3.78$$

4. Determine the probability of error $Pr(e)$ of this MAP decision rule.

The probability of error is found is the probabilistic view of making the wrong decision. Mathmatically, this is found as the expected value of the conditional distribution between the error e and measurement x.

$$Pr(e) = \sum Pr(e|x)Pr(x)$$

For the MAP decision rule derived in (3),

$$Pr(e|x) = \begin{cases} 1 - Pr(H_0|x) & \text{for } x \in \{0, ..., 3\}] \\ 1 - Pr(H_1|x) & \text{for } x \in \{4, ..., 5\}] \end{cases} \quad \rightarrow \quad Pr(e|x) = \begin{cases} 1 - \dfrac{64}{64+3^x} & \text{for } x \in \{0, ..., 3\}] \\ 1 - \dfrac{3^x}{64+3^x} & \text{for } x \in \{4, ..., 5\}] \end{cases}$$

$$Pr(x) = \left(\binom{5}{x}(0.5)^x(1-0.5)^{5-x}\right)(2/3) + \left(\binom{5}{x}(0.75)^x(1-0.75)^{5-x}\right)(1/3)$$

Plugging this into the equation for $Pr(e)$ yeilds

$$Pr(e) = \sum_{x=0}^{3} \left(1 - \frac{64}{64 + 3^x}\right) Pr(x) + \sum_{x=4}^{5} \left(1 - \frac{3^x}{64 + 3^x}\right) Pr(x)$$

$$Pr(e) = 0.247$$