**Problem 15**

Under hypothesis $\mathcal{H}_0$, scalar measurement $x$ is equal to **random variable** $w_1$:

$$x = w_1$$

Under hypothesis $\mathcal{H}_1$, scalar measurement $x$ is equal to the **sum** of random variable $w_1$ and $w_2$:

$$x = w_1 + w_2$$

Random variables $w_1$ and $w_2$ are independent, and each is described with an **exponential** *pdf*:

$$p(w_1) = \begin{cases} \lambda_1 e^{-\lambda_1 w_1} & \text{for } w_1 \geq 0 \\ 0 & \text{for } w_1 < 0 \end{cases}$$

$$p(w_2) = \begin{cases} \lambda_2 e^{-\lambda_2 w_2} & \text{for } w_2 \geq 0 \\ 0 & \text{for } w_2 < 0 \end{cases}$$

Note that $\lambda_1 \neq \lambda_2$!

Say we wish to apply the **MAP detection criterion**, and the two hypotheses are **equally probable**.

Determine the correct **threshold** $y'$ for a decision rule of the form:

$$\text{Choose } \mathcal{H}_1 \text{ if } T_d = x > y'$$

**Hint:** Implement your knowledge of EECS 861, being carefully to consider the **limits of integration.**

***1. Determine the correct threshold $y'$ for a decision rule of the form:*** **Choose $H_1$ if $T_d = x > y'$**

Given the two hypotheses $H_0 = w_1$ and $H_1 = w_1 + w_2$ are equally probable a priori and that $w_1$ and $w_2$ are both exponentially distributed random variables, there a posteriori probabilities are desribed by

$$p(H_0|x) = \frac{p(x|H_0)p(H_0)}{\sum_{i=0,1} p(x|H_i)p(H_i)} = \frac{\left(\lambda_1 e^{-\lambda_1 x}\right)(1/2)}{\left(\lambda_1 e^{-\lambda_1 x}\right)(1/2) + \left(\lambda_2 e^{-\lambda_2 x}\right)(1/2)} = \frac{\left(\lambda_1 e^{-\lambda_1 x}\right)}{\left(\lambda_1 e^{-\lambda_1 x}\right) + \left(\lambda_2 e^{-\lambda_2 x}\right)}$$

$$p(H_1|x) = \frac{p(x|H_1)p(H_1)}{\sum_{i=0,1} p(x|H_i)p(H_i)} = \frac{\left(\lambda_1 e^{-\lambda_1 x} + \lambda_2 e^{-\lambda_2 x}\right)(1/2)}{\left(\lambda_1 e^{-\lambda_1 x}\right)(1/2) + \left(\lambda_1 e^{-\lambda_1 x} + \lambda_2 e^{-\lambda_2 x}\right)(1/2)} = \frac{\left(\lambda_1 e^{-\lambda_1 x} + \lambda_2 e^{-\lambda_2 x}\right)}{\left(2\lambda_1 e^{-\lambda_1 x} + \lambda_2 e^{-\lambda_2 x}\right)}$$

From this a posteriori expression, the decision rule for $H_1$ is thus written as

Choose $H_1$ if $p(H_1|x) > p(H_0|x)$

Dividing the entire rule by $p(H_0|x)$, noting that each a posteriori probability is equal to its liklihood probability of that hypothesis, frames the liklihood ratio test

Choose $H_1$ if $\dfrac{p(H_1|x)}{p(H_0|x)} = L(x) > 1.0$

To determine this liklihood ratio in terms of the detection statistic $T_d(x)$, further simplifications of $L(x)$ are required. Taking the natural log of both sides of the inequality gives

$$\ln(p(H_1|x)) - \ln(p(H_0|x)) > \ln(1.0)$$

where

$$\ln(p(H_1|x) = \ln\left(\lambda_1 + \lambda_2 e^{-(\lambda_2-\lambda_1)x}\right) - \ln\left(2\lambda_1 + \lambda_2 e^{-(\lambda_2-\lambda_1)x}\right)$$

$$\ln(p(H_0|x)) = \ln\left(\frac{\lambda_1}{\lambda_1 + \lambda_2 e^{-(\lambda_2-\lambda_1)x}}\right)$$

thus the expression becomes

$$\ln\left(\lambda_1 + \lambda_2 e^{-(\lambda_2-\lambda_1)x}\right) - \ln\left(2\lambda_1 + \lambda_2 e^{-(\lambda_2-\lambda_1)x}\right) - \ln\left(\frac{\lambda_1}{\lambda_1 + \lambda_2 e^{-(\lambda_2-\lambda_1)x}}\right) > 0$$

Solving for and expression only containing $x$ and no other hypothesis dependent variables in the above inequality forms the detection statistic $T_d(x)$ in which all operations perserve the decision boundary.

$$T_d(x) = x < -\frac{1}{\lambda_2 - \lambda_1} \ln\left(\frac{\lambda_1}{2\lambda_2}(\sqrt{5} - 1)\right)$$

```
% exponential lambda values
lambda1 = 1;
lambda2 = 2;

% compute detection statistic threshold
threshold = -(1/(lambda2 - lambda1)) * log((lambda1/(2*lambda2)) * (sqrt(5) - 1));

% X-axis range
x = linspace(0, 5, 500);

% evaluate decision boudary inequality
x = linspace(0, 5, 500);
decisionRegion = x < threshold;

% plot results
figure; hold on; grid on;
plot([threshold threshold], [0 1], 'r--', 'LineWidth', 2);
area(x, decisionRegion, 'FaceAlpha', 0.2, 'FaceColor', 'b', 'EdgeColor', 'none');
xlabel('x'); ylabel('Decision Region');
title('Decision Boundary for Inequality');
legend(sprintf('Threshold = %.3f', threshold), 'Choose H_1 if x in this region');
```