1. Determine the Likelihood Ratio Test (i.e. determine $L(x)$ and γ) using the MAP detection criterion.

Given the likelihood functions $p(x|H_0)$ and $p(x|H_1)$ as well as the a priori probabilities $p(H_0)$ and $p(H_1)$ for hypotheses H_0 and H_1, the maximum a posteriori probability (MAP) density for each hypothesis is computed using the relation through Bayes' theorm.

$$p(H_i|x) = \frac{p(x|H_i)p(H_i)}{p(x)} = \frac{p(x|H_i)p(H_i)}{\sum_i p(x|H_i)p(x)dx}$$

Substituing in the given likelihood and a priori densities into the above equation, the MAP densities for each hypothesis are given as

$$p(H_0|x) = \begin{cases} \dfrac{(3/8)(2-x)^2(3/5)}{(3/8)(2-x)^2(3/5)+(3/8)(x)^2(2/5)} & \text{for } 0 < x < 2 \\ 0 & \text{for } x < 0,\ x > 2 \end{cases},$$

$$p(H_1|x) = \begin{cases} \dfrac{(3/8)(x)^2(2/5)}{(3/8)(2-x)^2(3/5)+(3/8)(x)^2(2/5)} & \text{for } 0 < x < 2 \\ 0 & \text{for } x < 0,\ x > 2 \end{cases}.$$

Simpilifying the above expressions yeilds

$$p(H_0|x) = \begin{cases} \dfrac{3(2-x)^2}{3(2-x)^2+2x^2} & \text{for } 0 < x < 2 \\ 0 & \text{for } x < 0,\ x > 2 \end{cases} \quad \text{and} \quad p(H_1|x) = \begin{cases} \dfrac{2x^2}{3(2-x)^2+2x^2} & \text{for } 0 < x < 2 \\ 0 & \text{for } x < 0,\ x > 2 \end{cases}.$$

From these a posteriori expressions, the MAP decision rule can be constructed as

Choose H_1 if $p(H_1|x) > p(H_0|x)$ and $0 < x < 2$ else choose H_0,

and rearranged into a likelihood ratio test (LRT) using measurement dependent likelihood ratio $L(x)$:

Choose H_1 if $\dfrac{p(H_1|x)}{p(H_0|x)} = L(x) > 1.0$ and $0 < x < 2$ else choose H_0.

Because both a posteriori definitions are bounded along the interval $[0, 2]$ and are equivalently evaluated to 0 outside of this region, the two hypothesis are treated as equally likely when the measurement lies outside of this region and H_0 is selected as default. The simplified representations for the a posteriori probability densities are substituted into the likelihood ratio definition and further simplified below.

$$L(x) = \frac{\dfrac{2x^2}{3(2-x)^2+2x^2}}{\dfrac{3(2-x)^2}{3(2-x)^2+2x^2}} = \left(\frac{2x^2}{3(2-x)^2+2x^2}\right)\left(\frac{3(2-x)^2+2x^2}{3(2-x)^2}\right) = \frac{2x^2}{3(2-x)^2}$$

The full MAP decision rule is therefore

Choose H_1 if $\dfrac{2x^2}{3(2-x)^2} > 1$ and $0 < x < 2$ else choose H_0.

2. Simplify the LRT, such that the decision rule can be expressed in terms of the detection statistic: $T_d(x) = x$. For this decision statistic, determine the value of threshold γ' for the MAP criterion.

To convert the MAP decision rule into one where the detection statistic $T_d(x)$ is equal to the measurment itself, the denominator of the likelihood function is moved to the left hand side of the inequality and further manipulated to isolate x on the right hand side.

$$\frac{2x^2}{3(2-x)^2} > 1 \longrightarrow 2x^2 > 3(2-x)^2 \longrightarrow 2x^2 > 12 - 12x + 3x^2 \longrightarrow x^2 - 12x + 12 < 0$$

Solving the above quadratic for x via the quadratic formula yeilds $6 - 2\sqrt{6} < x < 6 + 2\sqrt{6}$. Relating this result back to the original decision rule, the lower bound of $6 - 2\sqrt{6}$ supersedes the original decision boudary of 0 while the original upper bound of 2 remains tighter than $6 + 2\sqrt{6}$. The updated decision rule using detection statistic $T_d(x) = x$ and decision boundary $\gamma' = \left[6 - 2\sqrt{6}, 2\right]$ is therefore

Choose H_1 if $6 - 2\sqrt{6} < x < 2$ else choose H_0.

3. Say you now instead desire a threshold γ' that would result in a probability of detection of $P_D = 0.90$. Determine (a) the value of this threshold γ', and (b) the resulting probability of false alarm.

Denoting the decision boundary derived in part 2 as the region $\mathscr{R}_1 \in \left[6 - 2\sqrt{6}, 2\right]$, the probability of detection is defined as the intergal of the a posteriori probability density of H_1 over this region. This intuitively means that the measured detection statistic lies within the region of H_1 and therefore we choose H_1.

$$P_D = P(T_d(x) \in \mathscr{R}_1 | H_1) = \int_{T_d(x) \in \mathscr{R}_1} p(T_d(x)|H_1) dT_d(x)$$

Becuase the detection statistic $T_d(x)$ in the above intergal has been simplified down to $T_d(x) = x$, the detection staistic likelihood function $p(T_d(x)|H_1)$ is equivalent to the measurement likelihood function $p(x|H_1)$, and therefore the intergal can be rewritten as

$$P_D = P(x \in \mathscr{R}_1 | H_1) = \int_{x \in \mathscr{R}_1} p(x|H_1) dx$$

From this we could evaluate the probability of detection for the derived MAP decision rule in part 2. For this problem however, the resulting probabiltiy of detection is set and instead we are tasked to find the region in which this integration results in $P_D = 0.90$. Substituting in the given likelihood definition of H_1, the probability of detection intergal becomes

$$0.90 = \int_{\gamma'}^{2} \left(\frac{3}{8}x^2\right) dx$$

and is solved by finding the lower boundary γ' such that the equality is true.

$$\int_{\gamma'}^{2} \left(\frac{3}{8}x^2\right) dx \longrightarrow \left[\frac{x^3}{8}\right]_{\gamma'}^{2} \longrightarrow 0.90 = \frac{2^3 - \alpha^3}{8} \longrightarrow (0.8)^{1/3} = \gamma' \longrightarrow 0.928 = \gamma'$$

Using this new decision region $\mathscr{R}_1 = [0.928, 2]$, the resulting probability of false alarm can be found by integrating the likelihood function of H_0 over this region.

$$P_{FA} = \int_{x \in \mathscr{R}_1} p(x|H_0) dx = \int_{0.928}^{2} \left(\frac{3}{8}(2-x)^2\right) dx = \left[\frac{1}{8}(x^3 - 6x^2 + 12x)\right]_{0.928}^{2} = 0.1539$$