***1. Use the MAP criterion to determine a decision rule using the detction statistic: $T_d(x) = x$. Describe carefully and completely the decision regions $\mathscr{R}_A$ and $\mathscr{R}_B$ for this decision rule.***

Following the intiution built in *Problem 1*, the maximum a posteriori probability (MAP) density function is found via its relation to the likelihood and a priori densities through Bayes' theorm.

$$p(H_i|x) = \frac{p(x|H_i)p(H_i)}{p(x)} = \frac{p(x|H_i)p(H_i)}{\sum_i p(x|H_i)p(x)dx}$$

Substituing in the given likelihood functions and a priori densities into the above equation, the MAP density for hypotheses $H_A$ and $H_B$ are expressed as

$$p(H_A|x) = \begin{cases} \dfrac{\frac{1}{\sqrt{2\pi}}\exp\left(\frac{-x^2}{2}\right)(0.5)}{\frac{1}{\sqrt{2\pi}}\exp\left(\frac{-x^2}{2}\right)(0.5) + (0.2)(0.5)} & \text{for } -2.5 \le x \le 2.5 \\ \dfrac{\frac{1}{\sqrt{2\pi}}\exp\left(\frac{-x^2}{2}\right)(0.5)}{\frac{1}{\sqrt{2\pi}}\exp\left(\frac{-x^2}{2}\right)(0.5) + 0(0.5)} & \text{otherwise} \end{cases},$$

$$p(H_B|x) = \begin{cases} \dfrac{(0.2)(0.5)}{\frac{1}{\sqrt{2\pi}}\exp\left(\frac{-x^2}{2}\right)(0.5) + (0.2)(0.5)} & \text{for } -2.5 \le x \le 2.5 \\ 0 & \text{otherwise} \end{cases},$$

which can be simplified to

$$p(H_A|x) = \begin{cases} \dfrac{\frac{1}{\sqrt{2\pi}}\exp\left(\frac{-x^2}{2}\right)}{\frac{1}{\sqrt{2\pi}}\exp\left(\frac{-x^2}{2}\right) + (0.2)} & \text{for } -2.5 \le x \le 2.5 \\ 1 & \text{otherwise} \end{cases} \longrightarrow \begin{cases} \dfrac{1}{1 + 0.2\sqrt{2\pi}\exp\left(\frac{x^2}{2}\right)} & \text{for } -2.5 \le x \le 2.5 \\ 1 & \text{otherwise} \end{cases},$$

$$p(H_B|x) = \begin{cases} \dfrac{(0.2)}{\frac{1}{\sqrt{2\pi}}\exp\left(\frac{-x^2}{2}\right) + (0.2)} & \text{for } -2.5 \le x \le 2.5 \\ 0 & \text{otherwise} \end{cases} \longrightarrow \begin{cases} \dfrac{0.2\sqrt{2\pi}\exp\left(\frac{x^2}{2}\right)}{1 + 0.2\sqrt{2\pi}\exp\left(\frac{x^2}{2}\right)} & \text{for } -2.5 \le x \le 2.5 \\ 0 & \text{otherwise} \end{cases}.$$

From these a posteriori expressions, the MAP decision rule can be constructed:

$$\text{Choose } H_B \text{ if } p(H_B|x) \geq p(H_A|x) \text{ and } -2.5 \leq x \leq 2.5 \text{ else choose } H_A,$$

and rearranged into a likelihood ratio using measurement dependent likelihood ratio $L(x)$:

$$\text{Choose } H_B \text{ if } \frac{p(H_B|x)}{p(H_A|x)} = L(x) \geq 1.0 \text{ and } -2.5 \leq x \leq 2.5 \text{ else choose } H_A.$$

Substituting the derived equations for $p(H_A|x)$ and $p(H_B|x)$ into the likelihood ratio $L(x)$ yeilds

$$L(x) = \left(\frac{0.2\sqrt{2\pi}\exp\left(\frac{x^2}{2}\right)}{1 + 0.2\sqrt{2\pi}\exp\left(\frac{x^2}{2}\right)}\right)\left(\frac{1}{1 + 0.2\sqrt{2\pi}\exp\left(\frac{x^2}{2}\right)}\right)^{-1},$$

which, noting the same terms in the numerator and denomonator after the inverse operator, can be simplified to

$$L(x) = 0.2\sqrt{2\pi}\exp\left(\frac{x^2}{2}\right).$$

Plugging this representation of $L(x)$ back into the LRT yeilds:

$$\text{Choose } H_B \text{ if } 0.2\sqrt{2\pi}\exp\left(\frac{x^2}{2}\right) \geq 1.0 \text{ and } -2.5 \leq x \leq 2.5 \text{ else choose } H_A$$

which can be further simplified down to a detection statistic equal to the measurement itself by taking the natural logarithm of both sides of the inequality, grouping constants on the right hand side, and solving for $x$.

$$= \ln\left(0.2\sqrt{2\pi}\exp\left(\frac{x^2}{2}\right)\right) \geq \ln(1.0) \longrightarrow \ln\left(0.2\sqrt{2\pi}\right) + \ln\left(\exp\left(\frac{x^2}{2}\right)\right) \geq 0 \longrightarrow \frac{x^2}{2} \geq -\ln\left(0.2\sqrt{2\pi}\right) \longrightarrow x < -1.176 \text{ and } x > 1$$

This result means that the original LRT inequality $L(x) \geq \ln(1.0)$ holds only for values of $x$ outside of the interval $[-1.176, 1.176]$. Being that this derived boundary was found for the detection statistic $T_d(x) = x$, it can be directly related to the original boundaries of the measurement $x$. Mathmatically, the culmination of these boundaries results in the LRT:

$$\text{Choose } H_B \text{ if } T_d(x) \in \mathscr{R}_B \text{ else choose } H_A$$

where $\mathscr{R}_A$ and $\mathscr{R}_B$ are

$$\mathscr{R}_A = \{T_d(x) : T_d(x) < -2.5\} \cup \{T_d(x) : -1.176 < T_d(x) < 1.176\} \cup \{T_d(x) : 2.5 < T_d(x)\}$$

$$\mathscr{R}_B = \{T_d(x) : -2.5 \geq T_d(x) \leq -1.176\} \cup \{T_d(x) : 1.176 \leq T_d(x) \leq 1.5\}$$

more clearly written in set notation as

$$\mathscr{R}_A = (-\infty, 2.5) \cup (-1.176, 1.176) \cup (2.5, \infty)$$

$$\mathscr{R}_B = (-2.5, -1.176) \cup (1.176, 2.5).$$

***2. Determine the probability of error for this decision rule.***

Defining the a posteriori probabilities for hypotheses $H_A$ and $H_B$ as a function of the measurement $x$, the probability of error given this measurment, for a binary detection case, is the a posteriori probability of the least probable hypothesis. This can be stated intuitevly as choosing $H_B$ when the measurement occurs within region $\mathscr{R}_A$ and vice versa.

$$Pr(e|x) = \begin{cases} Pr(x|H_A) & \text{for } x \in \mathscr{R}_B \\ Pr(x|H_B) & \text{for } x \in \mathscr{R}_A \end{cases} \longrightarrow \begin{cases} 1 - Pr(x|H_A) & \text{for } x \in \mathscr{R}_A \\ 1 - Pr(x|H_B) & \text{for } x \in \mathscr{R}_B \end{cases}$$

From this conditional definition of error, the cumulative probability of error, $Pr(e)$, is found by integrating over the entire measurement density function $p(x)$. This is equivalently expressed as the expected value of $Pr(e|x)$ defined below.

$$Pr(e) = \mathbb{E}\{Pr(e|x)\} = \int Pr(e|x)p(x)$$

Using the definitions for $\mathscr{R}_A$ and $\mathscr{R}_B$ derived in part 1 for the detection statistic $T_d(x) = x$, the conditional error is written as

$$Pr(e|x) = \begin{cases} 1 - \left(\frac{1}{\sqrt{2\pi}}\exp\left(\frac{-x^2}{2}\right)\right) & \text{for } -\infty < x < -2.5 \\ 1 - 0.2 & \text{for } -2.5 \le x \le -1.176 \\ 1 - \left(\frac{1}{\sqrt{2\pi}}\exp\left(\frac{-x^2}{2}\right)\right) & \text{for } -1.176 < x < 1.176 \\ 1 - 0.2 & \text{for } 1.176 \le x \le 2.5 \\ 1 - \left(\frac{1}{\sqrt{2\pi}}\exp\left(\frac{-x^2}{2}\right)\right) & \text{for } 2.5 < x < \infty \end{cases}.$$

Through Bayes' theorm in part 1, it was shown that the measurement probability density function was equal to the the sum of the products of each likelihood function and the corresponding a priori probability of the hypothesis. Substituting these values into this definition, $p(x)$ is obtained as

$$p(x) = p(x|H_A)p(H_A) + p(x|H_B)p(H_B) = \begin{cases} \left(\frac{1}{\sqrt{2\pi}}\exp\left(\frac{-x^2}{2}\right)\right)(0.5) + (0.2)(0.5) & \text{for } -2.5 \le x \le 2.5 \\ \left(\frac{1}{\sqrt{2\pi}}\exp\left(\frac{-x^2}{2}\right)\right)(0.5) + 0 & \text{otherwise} \end{cases}.$$

Combinding this representation with the derived formulation of $Pr(e|x)$, the intergal used to find the cumulative probability of error can be written as

$$Pr(e) = \int_{-\infty}^{-2.5} \left(1 - \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right)\right) \left(\frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right)(0.5)\right) dx + \int_{-2.5}^{-1.176} (1 - 0.2) \left(\frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right)(0.5) + (0.2)(0.5)\right) dx$$
$$+ \int_{-1.176}^{1.176} \left(1 - \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right)\right) \left(\frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right)(0.5) + (0.2)(0.5)\right) dx + \int_{1.176}^{2.5} (1 - 0.2) \left(\frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right)(0.5) + (0.2)(0.5)\right) dx$$
$$+ \int_{2.5}^{\infty} \left(1 - \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right)\right) \left(\frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right)(0.5)\right) dx.$$

This intergal exhibits symmetery between the first two and last two terms which allows for the simplification to

$$Pr(e) = 2\int_{-\infty}^{-2.5} \left(1 - \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right)\right) \left(\frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right)(0.5)\right) dx + 2\int_{-2.5}^{-1.176} (1 - 0.2) \left(\frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right)(0.5) + (0.2)(0.5)\right) dx$$
$$+ \int_{-1.176}^{1.176} \left(1 - \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right)\right) \left(\frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right)(0.5) + (0.2)(0.5)\right) dx.$$

Each component of the above intergal is computed in the matlab excerpt below resulting in $Pr(e) = 0.7208$.

```
% intergal from (-infty, -2.5)
f = @(x) 2*((1 - (1/sqrt(2*pi)) .* exp(-x.^2/2)) .* ((1/sqrt(2*pi)) .* exp(-x.^2/2)
* 0.5));
i1 = integral(f, -Inf, -2.5);

% intergal from (-2.5, -1.176)
f = @(x) 2*((1 - 0.2) .* ((1/sqrt(2*pi)) .* exp(-x.^2/2) * 0.5 + (0.2 * 0.5)));
i2 = integral(f, -2.5, -1.176);

% intergal from (-1.176, 1.176)
f = @(x) (1 - (1/sqrt(2*pi)) .* exp(-x.^2/2)) .* ((1/sqrt(2*pi)) .* exp(-x.^2/2) *
0.5 + (0.2 * 0.5));
i3 = integral(f, -1.176, 1.176);

% sum each component
i = i1 + i2 + i3;
fprintf('Pr(e) = %.6f\n', i);
```

```
Pr(e) = 0.720759
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