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```
In [2]: import numpy as np
import matplotlib.pyplot as plt
import pandas as pd

import statsmodels.api as sm
```

## Exercise 1: Univariate Problems

Code the bisection algorithm for any function  $f(x)$

```
In [65]: def bisection(function, low, high, epsilon = 1e-4, sigma = 1e-4, numIter = 100
0):
    """
    Finds a root of function in an interval [low, high] such that function(low) < 0 and function(high) > 0

    Arguments:

    function - continuous input function
    low - lower bound of domain
    high - upper bound of domain
    epsilon - distance between refined low and high
    sigma - function value at mid point
    numIter - maximum number of iterations

    Returns:

    mid - root of function such that f(mid) ~ 0 in interval [low, high]
    """
    if function(low) * function(high) < 0:
        for i in range(numIter):
            # check whether function(low) and function(high) have different signs
            mid = (low + high) / 2

            # shorten search domain by moving to mid point either from below or above
            if function(low) * function(mid) > 0:
                low = mid
            else:
                high = mid

            # check epsilon convergence or sigma convergence
            if high - low <= epsilon * (1 + abs(low) + abs(high)) or abs(function(mid)) <= sigma:
                return mid
            else:
                print('function(low) * function(high) > 0 => Define a better interval')
    )
```

## Use bisection to compute the zeros of the functions

1.  $f(x) = x^3 + 4 - \frac{1}{x}$
2.  $f(x) = -\exp(-x) + \exp(-x^2)$

```
In [24]: def function1(x):
    return x**3 + 4 - 1/x

def function2(x):
    return -np.exp(-x) + np.exp(-x**2)
```

```
In [50]: mid1 = bisection(function1, -2, -1)
mid2 = bisection(function2, 1/2, 2)

print('ROOT: {} <=> Bisection on function1 in an interval [-2, -1]'.format(mid
1))
print('ROOT: {} <=> Bisection on function2 in an interval [1/2, 2]'.format(mid
2))
```

ROOT: -1.663330078125 <=> Bisection on function1 in an interval [-2, -1]

ROOT: 1.000244140625 <=> Bisection on function2 in an interval [1/2, 2]

```

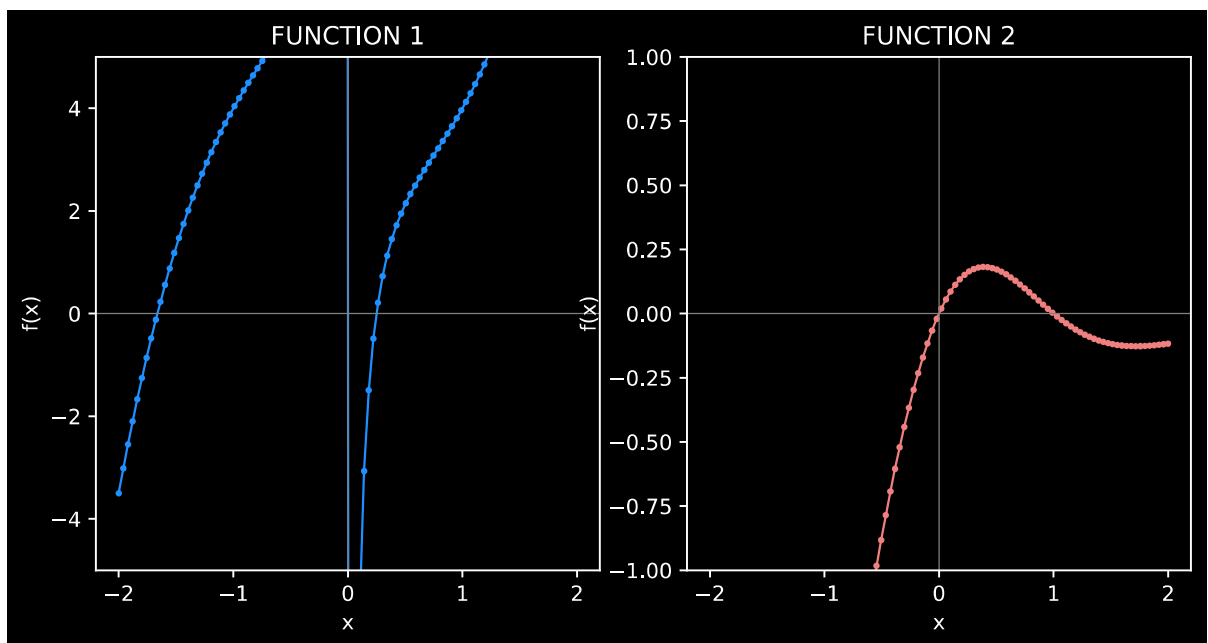
In [64]: x = np.linspace(-2, 2, 100)
y1 = function1(x)
y2 = function2(x)

fig, (ax1, ax2) = plt.subplots(1, 2, figsize=(8, 4))
fig.tight_layout()
ax1.plot(x, y1, color = 'dodgerblue', marker = 'o', markersize = 2, linewidth = 1)
ax1.axhline(y = 0, color = 'gray', linewidth = 0.5)
ax1.axvline(x = 0, color = 'gray', linewidth = 0.5)
ax1.title.set_text('FUNCTION 1')
ax1.set_xlabel('x')
ax1.set_ylabel('f(x)')
ax1.set_ylim(-5, 5)

ax2.plot(x, y2, color = 'lightcoral', marker='o', markersize = 2, linewidth = 1)
ax2.axhline(y = 0, color = 'gray', linewidth = 0.5)
ax2.axvline(x = 0, color = 'gray', linewidth = 0.5)
ax2.title.set_text('FUNCTION 2')
ax2.set_xlabel('x')
ax2.set_ylabel('f(x)')
ax2.set_ylim(-1, 1)

```

Out[64]: (-1.0, 1.0)



**Code the secant method for any function  $f(x)$**

```
In [73]: def secant(function, x1, x2, tol = 1e-4, numIter = 1000):
        """
        Finds a root of function given two points x1 and x2

        Arguments:

        function - continuous input function
        x1 - initial point 1
        x2 - initial point 2
        tol - breakpoint tolerance
        numIter - maximum number of iterations

        Returns:
        z - root of function
        """
        fx1 = function(x1)
        fx2 = function(x2)

        for i in range(numIter):
            z = (x1 * fx2 - x2 * fx1) / (fx2 - fx1)

            if abs(function(z)) <= tol * (1 + function(z)):
                return z

            x1, x2 = x2, z
            fx1, fx2 = fx2, function(z)

        else:
            print('numIter exceeded. Current z is {}'.format(z))
```

## Use secant to compute the zeros of the functions

1.  $f(x) = x^2 + 10 - \frac{1}{x}$
2.  $f(x) = \exp(-x^2)$

```
In [68]: def function3(x):
        return x**2 + 10 - 1/x

        def function4(x):
            return np.exp(-x**2)
```

```
In [75]: z3 = secant(function3, 0.1, 1)
        z4 = secant(function4, 1/2, 2)

        print('ROOT: {} <=> Secant on function3 given x1 = 0.1 and x2 = 1'.format(z3))
        print('ROOT: {} <=> Secant on function4 in an interval [1/2, 2]'.format(z4))
```

```
ROOT: 0.099990030050935535 <=> Secant on function3 given x1 = 0.1 and x2 = 1
ROOT: 3.063995107434631 <=> Secant on function4 in an interval [1/2, 2]
```

```

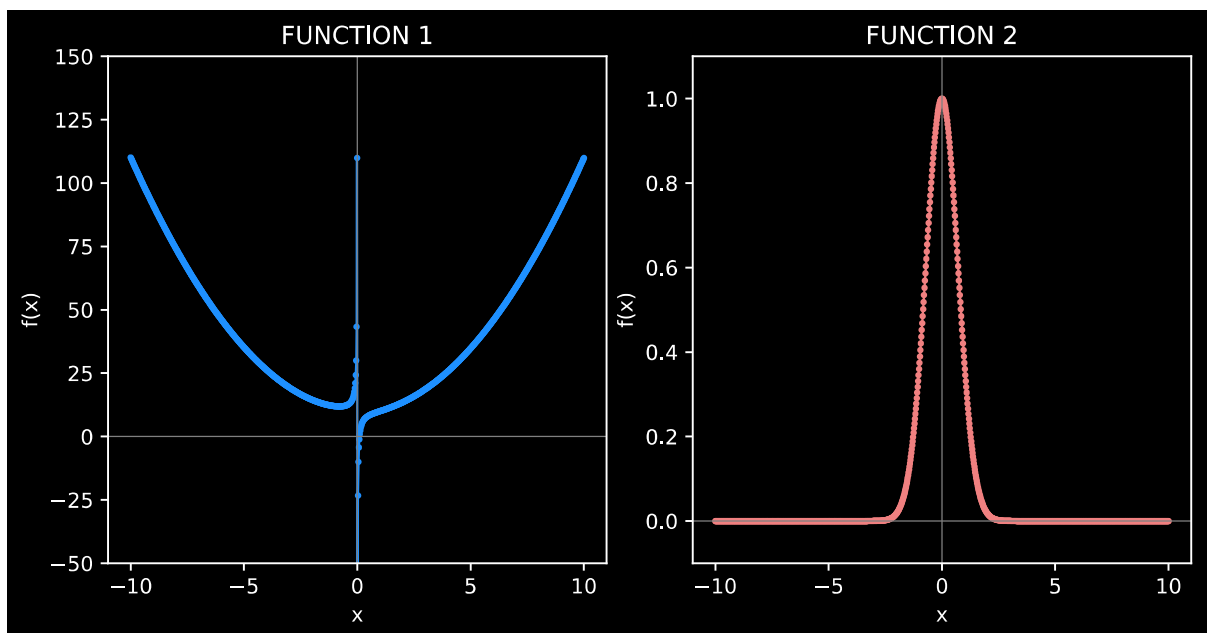
In [98]: x3 = np.linspace(-10, 10, 1000)
x4 = np.linspace(-10, 10, 1000)
y3 = function3(x)
y4 = function4(x)

fig, (ax1, ax2) = plt.subplots(1, 2, figsize=(8, 4))
fig.tight_layout()
ax1.plot(x3, y3, color = 'dodgerblue', marker = 'o', markersize = 2, linewidth = 1)
ax1.axhline(y = 0, color = 'gray', linewidth = 0.5)
ax1.axvline(x = 0, color = 'gray', linewidth = 0.5)
ax1.title.set_text('FUNCTION 1')
ax1.set_xlabel('x')
ax1.set_ylabel('f(x)')
ax1.set_ylim(-50, 150)

ax2.plot(x4, y4, color = 'lightcoral', marker='o', markersize = 2, linewidth = 1)
ax2.axhline(y = 0, color = 'gray', linewidth = 0.5)
ax2.axvline(x = 0, color = 'gray', linewidth = 0.5)
ax2.title.set_text('FUNCTION 2')
ax2.set_xlabel('x')
ax2.set_ylabel('f(x)')
ax2.set_ylim(-0.1, 1.1)

```

Out[98]: (-0.1, 1.1)



## Revisit the demand-supply example of problem set 1

$$\text{D: } p = q - bq$$

$$\text{S: } p = c + dq^\phi$$

where  $p$  is the price,  $q$  is the quantity,  $a, b, c, d, \phi$  are some parameters.

1. Write it as a univariate problem

$$bq + dq^\phi - (a - c) = 0$$

```
In [100]: def supplyDemand(q, a = 3, b = 1/2, c = 1, d = 1, phi = 1/2):
          return b * q + d * q**phi - (a - c)
```

1. Parametrize the model with  $a = 3, b = 0.5, c = d = 1, \phi = 1/2$ . Compute the solution analytically.

Let  $\tilde{q} = q^{\frac{1}{2}}$  hence

$$\frac{1}{2}\tilde{q}^2 + \tilde{q} - 2 = 0$$

Solutions to this quadratic equations are

$$\tilde{q}_1 = \sqrt{5} - 1 \quad \tilde{q}_2 = -\sqrt{5} - 1$$

Hence, the only solution  $q_1$  is given by

$$q_1 = (\sqrt{5} - 1)^2 = 6 - 2\sqrt{5}$$

1. Compute the solutions with your bisection algorithm

```
In [112]: q = bisection(supplyDemand, 1, 4)
```

```
In [113]: print('ROOT: {} <=> Bisection on supplyDemand in interval [1, 3]'.format(q))
```

```
ROOT: 1.5277099609375 <=> Bisection on supplyDemand in interval [1, 3]
```

## Exercise 2: A Contribution to the Empirics of Economic Growth

1. Load the data set and delete countries with missing values

```
In [55]: df = pd.read_excel('../Helpers/MRW92QJE-data.xls', header = 0)
df = df.dropna()
df.head(10)
```

Out[55]:

	country number	country name	Non- oil	intermediate	oecd	gdp/adult 1960	gdp/adult 1985	growth gdp	growth working age pop	l/y	st
0	1	Algeria	1	1	0	2485.0	4371.0	4.8	2.6	24.1	
1	2	Angola	1	0	0	1588.0	1171.0	0.8	2.1	5.8	
2	3	Benin	1	0	0	1116.0	1071.0	2.2	2.4	10.8	
3	4	Botswana	1	1	0	959.0	3671.0	8.6	3.2	28.3	
4	5	Burkina Faso	1	0	0	529.0	857.0	2.9	0.9	12.7	
5	6	Burundi	1	0	0	755.0	663.0	1.2	1.7	5.1	
6	7	Cameroon	1	1	0	889.0	2190.0	5.7	2.1	12.8	
7	8	Central African Republic	1	0	0	838.0	789.0	1.5	1.7	10.5	
8	9	Chad	1	0	0	908.0	462.0	-0.9	1.9	6.9	
9	10	Congo, Peoples Republic	1	0	0	1009.0	2624.0	6.2	2.4	28.8	

1. Generate sub-samples for non-oil countries, intermediate countries and OECD countries

```
In [62]: nonOil = df[df['Non-oil'] == 1]
intermediate = df[df['intermediate'] == 1]
oecd = df[df['oecd'] == 1]
```

1. For each sub-sample compute the regression coefficients and respective standard error for the following regression model

$$\begin{aligned}
 \log(\text{gdp1985}_j) - \log(\text{gdp1960}_j) = & \beta_0 + \beta_1 \log(\text{gdp1960}_j) \\
 & + \beta_2 \log\left(\frac{\text{investment}}{\text{gdp}_j}\right) \\
 & + \beta_3 \log(\text{popgrowth}_j + g + \delta) \\
 & + \beta_4 \log(\text{schoolenrol}_j) + \epsilon_j
 \end{aligned}$$

where  $g + \delta = 0.05$ .



```
In [82]: results = []
gdelta = 0.05
for data in [nonOil, intermediate, oecd]:

    y = np.log(data['gdp/adult 1985']) - np.log(data['gdp/adult 1960']) # Log
    y = np.array(y).reshape(-1, 1)

    intercept = np.ones(len(y))
    x1 = np.log(data['gdp/adult 1960'])
    x2 = np.log(data['I/y'])
    x3 = np.log(data['growth working age pop']) + gdelta
    x4 = np.log(data['school'])
    X = np.matrix([intercept, x1, x2, x3, x4])

    model = sm.OLS(y, X.T)
    results.append(model.fit())
```

```
In [78]: print('RESULTS FOR NON-OIL')
         results[0].summary()
```

RESULTS FOR NON-OIL

Out[78]: OLS Regression Results

<b>Dep. Variable:</b>	y	<b>R-squared:</b>	0.491
<b>Model:</b>	OLS	<b>Adj. R-squared:</b>	0.470
<b>Method:</b>	Least Squares	<b>F-statistic:</b>	22.47
<b>Date:</b>	Mon, 01 Feb 2021	<b>Prob (F-statistic):</b>	5.27e-13
<b>Time:</b>	17:36:18	<b>Log-Likelihood:</b>	-26.378
<b>No. Observations:</b>	98	<b>AIC:</b>	62.76
<b>Df Residuals:</b>	93	<b>BIC:</b>	75.68
<b>Df Model:</b>	4		
<b>Covariance Type:</b>	nonrobust		

	coef	std err	t	P> t	[0.025	0.975]
<b>const</b>	1.0639	0.458	2.325	0.022	0.155	1.973
<b>x1</b>	-0.2993	0.062	-4.824	0.000	-0.423	-0.176
<b>x2</b>	0.5209	0.086	6.031	0.000	0.349	0.692
<b>x3</b>	-0.1403	0.068	-2.050	0.043	-0.276	-0.004
<b>x4</b>	0.2340	0.059	3.960	0.000	0.117	0.351

<b>Omnibus:</b>	2.156	<b>Durbin-Watson:</b>	2.148
<b>Prob(Omnibus):</b>	0.340	<b>Jarque-Bera (JB):</b>	1.674
<b>Skew:</b>	-0.121	<b>Prob(JB):</b>	0.433
<b>Kurtosis:</b>	3.593	<b>Cond. No.</b>	118.

Notes:

[1] Standard Errors assume that the covariance matrix of the errors is correctly specified.

```
In [79]: print('RESULTS FOR INTERMEDIATE')
results[1].summary()
```

## RESULTS FOR INTERMEDIATE

Out[79]: OLS Regression Results

<b>Dep. Variable:</b>	y	<b>R-squared:</b>	0.472
<b>Model:</b>	OLS	<b>Adj. R-squared:</b>	0.442
<b>Method:</b>	Least Squares	<b>F-statistic:</b>	15.66
<b>Date:</b>	Mon, 01 Feb 2021	<b>Prob (F-statistic):</b>	3.38e-09
<b>Time:</b>	17:36:21	<b>Log-Likelihood:</b>	-14.145
<b>No. Observations:</b>	75	<b>AIC:</b>	38.29
<b>Df Residuals:</b>	70	<b>BIC:</b>	49.88
<b>Df Model:</b>	4		
<b>Covariance Type:</b>	nonrobust		

	coef	std err	t	P> t	[0.025	0.975]
<b>const</b>	1.5984	0.496	3.223	0.002	0.609	2.588
<b>x1</b>	-0.3760	0.068	-5.547	0.000	-0.511	-0.241
<b>x2</b>	0.5364	0.102	5.284	0.000	0.334	0.739
<b>x3</b>	-0.1432	0.067	-2.130	0.037	-0.277	-0.009
<b>x4</b>	0.2743	0.080	3.431	0.001	0.115	0.434

<b>Omnibus:</b>	6.148	<b>Durbin-Watson:</b>	2.477
<b>Prob(Omnibus):</b>	0.046	<b>Jarque-Bera (JB):</b>	9.311
<b>Skew:</b>	-0.156	<b>Prob(JB):</b>	0.00951
<b>Kurtosis:</b>	4.698	<b>Cond. No.</b>	125.

Notes:

[1] Standard Errors assume that the covariance matrix of the errors is correctly specified.

```
In [80]: print('RESULTS FOR OECD')
         results[2].summary()
```

RESULTS FOR OECD

Out[80]: OLS Regression Results

<b>Dep. Variable:</b>	y	<b>R-squared:</b>	0.702
<b>Model:</b>	OLS	<b>Adj. R-squared:</b>	0.632
<b>Method:</b>	Least Squares	<b>F-statistic:</b>	10.02
<b>Date:</b>	Mon, 01 Feb 2021	<b>Prob (F-statistic):</b>	0.000235
<b>Time:</b>	17:36:23	<b>Log-Likelihood:</b>	13.213
<b>No. Observations:</b>	22	<b>AIC:</b>	-16.43
<b>Df Residuals:</b>	17	<b>BIC:</b>	-10.97
<b>Df Model:</b>	4		
<b>Covariance Type:</b>	nonrobust		

	coef	std err	t	P> t	[0.025	0.975]
<b>const</b>	2.4839	0.774	3.209	0.005	0.851	4.117
<b>x1</b>	-0.4009	0.072	-5.552	0.000	-0.553	-0.249
<b>x2</b>	0.3702	0.177	2.092	0.052	-0.003	0.744
<b>x3</b>	-0.1414	0.061	-2.306	0.034	-0.271	-0.012
<b>x4</b>	0.2359	0.150	1.572	0.134	-0.081	0.552

<b>Omnibus:</b>	1.681	<b>Durbin-Watson:</b>	1.740
<b>Prob(Omnibus):</b>	0.431	<b>Jarque-Bera (JB):</b>	1.164
<b>Skew:</b>	0.555	<b>Prob(JB):</b>	0.559
<b>Kurtosis:</b>	2.810	<b>Cond. No.</b>	234.

Notes:

[1] Standard Errors assume that the covariance matrix of the errors is correctly specified.