Problem Set 5

Due date: Feb 19, 2021

Instructions: Hand-in is in electronic form via email. Please create a single PDF-file containing all your answers and results. Show the names and student numbers of the group members on top. Maximum 4 students can work in a group. Make use of figures and tables. The code should be well documented and readable. Have a look at Latex-Example_with_Matlab (on Olat) for how to include Matlab code in a Latex file. Pay attention to the use of mcode package.

Exercise 1: Consumption Savings Problem

Consider the following life-cycle problem

$$
\begin{aligned}
& \max_{\{c_t\}_{t=0}^{T}} \sum_{t=0}^{T} \beta^t u(c_t) \\
\text{s.t. } & u(c_t) = \frac{c_t^{1-\theta} - 1}{1-\theta} \\
& c_t + a_{t+1} = a_t(1+r) + w_t \\
& a_0 \text{ given and } \; a_{T+1} \geq 0.
\end{aligned}
$$

- Characterize the analytical solution for any β, r, w_t.
- Compute the numerical solution setting $w_0 = 10, a_0 = 0, \;\; w_t = 0$, for $t \geq 1$, $\beta = 0.99$, $r = 0.05$ and any γ.
- How do you deal with the constraint $c_t > 0$ for all t?

Exercise 2: The Consumption-Savings Problem with Human Capital

Consider the 2-period consumption-savings problem with human capital:

- c_t, k_t, h_t: cons., stock of physical and human capital, $t = 1, 2$
- i_k, i_h: investment in physical and human capital
- $\delta_k, \delta_h \in [0, 1]$: depreciation rates for physical and human capital
- $k_2 = (1-\delta_k)k_1 + i_k$ and $h_2 = ((1-\delta_h)h_1 + i_h)^\eta$
- $k_2 \geq 0$: borrowing constraint on physical capital
- $i_h \geq 0$: irreversibility constraint on human capital
- Optimization problem (substituting in the physical and human capital accumulation functions):

$$
\begin{aligned}
& \max_{c_1, c_2, i_k, i_h} \left\{ \frac{c_1^{1-\gamma} - 1}{1-\gamma} + \beta \, \frac{c_2^{1-\gamma} - 1}{1-\gamma} \right\} \\
\text{s.t.} \quad & (1-\delta_k)\, k_1 + i_k \geq 0 \\
& i_h \geq 0 \\
& c_1 + i_k + i_h - \Big(r_k \, k_1 + r_h \, h_1 \Big) = 0 \\
& c_2 - \Big((1+r_k) \, ((1-\delta_k)\, k_1 + i_k) + r_h \, ((1-\delta_h)\, h_1 + i_h)^\eta \Big) = 0
\end{aligned}
$$

- CRRA utility with risk aversion γ, discount factor β

- r_k, r_h: return on physical and human capital
- $\eta \in (0,1)$: decreasing marginal returns to human capital investment
- Parametrization:

Table 1: Parameters

γ	β	r_k	r_h	η	δ_k	δ_h
2.00	0.96	0.10	1.40	0.80	0.05	0.05

1. Derive the Kuhn-Tucker conditions of this optimization problem.
2. Use `Matlab` routine of your choice to solve the above problem for the following initial endowments
   i) $k_1 = 1, h_1 = 5$
   ii) $k_1 = 1, h_1 = 1$
   iii) $k_1 = 1, h_1 = 0.2$
3. Interpret your results. Which constraints are binding?

Exercise 3: Interpolation of Simple Function

Read the chapter on interpolation in the book by Miranda and Fackler (2002) to get acquainted with their toolbox. Download their toolbox and save it on your disk. Add the location of the toolbox to Matlab's Path. For instructions how to do this refer to `https://de.mathworks.com/help/matlab/matlab_env/add-remove-or-reorder-folders-on-the-search-path.html`.

- Interpolate the function
$$f(x) = \frac{1}{1+25x^2}$$
on $x \in [-1,1]$ using Chebychev polynomials for n equidistant nodes. Increase n from 5 to 15 and plot the residual by comparing the solution with the solution for some large n. What do you find?
Hint: MF assigns Chebychev nodes to Chebychev polynomials. To combine Chebychev polynomials with equidistant nodes, you
  - define the function space for Chebychev polynomials, `fspace`
  - define a vector of equidistant nodes, `x`
  - calculate the matrix of basis functions, `B`, at these nodes by `B=funbas(fspace,x)`.
  - you next calculate the function values at `x`
  - and finally get the polynomial coefficients by `c=B\y`.
- Repeat the exercise using Chebychev nodes. What do you find?
- Repeat the exercise using splines with equidistant nodes. What do you find?

Exercise 4: A Simple Portfolio Choice Problem

This exercise helps you to get a better understanding of constrained optimization problems. We will deal with a simple portfolio choice problem. The household has an initial endowment of wealth w_0 and $w_1 = w_0(1 + r^p)$ is the terminal wealth for some portfolio return r^p . This portfolio return depends on the investment in one risky and one risk-free asset and may be written as

$$r^p = \alpha^f r^f + \alpha r,$$

where r is the return on the risky asset. We assume that the share invested in the risky asset, α, is constrained:

$$\underline{\alpha} \leq \alpha \leq \overline{\alpha}.$$

Furthermore, we assume that there are only two possible realizations of the return on the risky asset, r_{low} and r_{high}, which are realized with probabilities p and $1-p$. We assume throughout that $r^f = 0.02$, $r_{low} = -0.08$, and $r_{high} = 0.12$. The objective function is given by $\mathbb{E}u(w_1)$. We will again assume a constant relative risk aversion (CRRA) utility function given by

$$u(w_1) = \frac{1}{1-\gamma} w_1^{1-\gamma}$$

- The portfolio shares α^f and α must satisfy

  $$\alpha^f + \alpha = 1.$$

  It is now straightforward to write the portfolio return as

  $$r^p = r^f + \alpha(r - r^f).$$

- We substitute out $w_1 = w_0(1 + r^p)$.

The maximization problem now writes as

$$\begin{aligned} &\max_{\alpha} \mathbb{E}\left[\frac{1}{\phi}\left(w_0(1 + r^f + \alpha(r - r^f))\right)^{\phi}\right] \\ &\text{s. t.} \quad \underline{\alpha} \leq \alpha \leq \overline{\alpha} \end{aligned}$$

for $\phi = 1 - \gamma$.

1. Is the problem convex? If so, what does it imply for the solution method?
2. Assume an unconstrained problem, i.e., $\underline{\alpha} = -\infty$ and $\overline{\alpha} = \infty$.
   a) Show that the optimal portfolio share is independent of initial wealth. This is an important result in portfolio theory due to Merton (1969) and Samuelson (1969).
   b) Solve the problem again for $(\phi = -3, p = 0.1)$ and plot α against the value of the objective function on the interval $[\alpha^\star - 1, \alpha^\star + 1]$. Provide an economic interpretation.
3. Now consider the constrained optimization problem with $\underline{\alpha} = 0$ and $\overline{\alpha} = 1$.
   a) Calculate the solution for $(\phi = -3, p = 0.1)$ using the Matlab solvers `fminbnd` and `fmincon`. Why do the two solvers yield different results?
   b) Compare your results to part 2 and discuss.

References

Merton, R. C. (1969): "Litetime Portfoliio Selection under Uncertainty: The Continuous-Time Case," The Review of Economics and Statistics, 51(3), 247–257.

Miranda, M. J. and P. L. Fackler (2002). Applied Computational Economics and Finance. Cambridge: MIT Press.

Samuelson, P. A. (1969): "Lifetime Portfolio Selection by Dynamic Stochastic Programming," Review of Economics and Statistics, 51(3), 239–246.