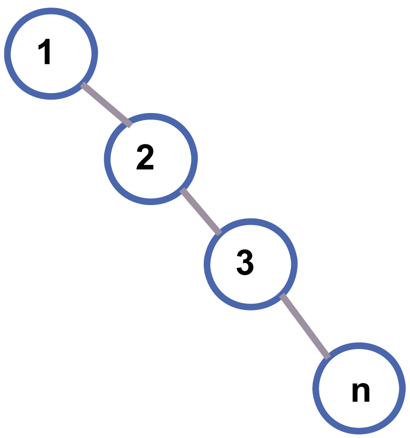
Advanced Topics – More Trees

Yi-Shin Chen
Institute of Information Systems and Applications
Department of Computer Science
National Tsing Hua University
yishin@gmail.com

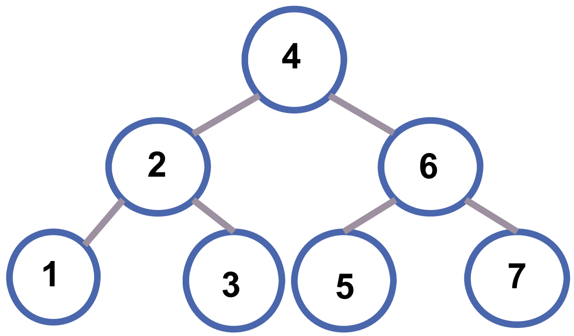
Binary Search Tree

- All BST operations are O(*h*)
  - h = height of BST

- Worst case h=n
  - Insert keys 1, 2, … n



- Best case h=logn
  - Insert keys : 4, 2, 6, 1, 3, 5, 7



How to Keep a Balanced BST

- AVL Trees
- B Trees
  - Multiway search trees
- B^+ Trees

Height Balanced Trees

- An empty tree is height balanced.
- If T is a non-empty binary tree with $\boldsymbol{T_L}$ and $\mathrm{T_R}$
  - As its left and right subtrees respectively
- Balance factor

  $$bf(\boldsymbol{T}) = height(\boldsymbol{T_L}) - height(\boldsymbol{T_R})$$

- $\boldsymbol{T}$ is height balanced iff
  1) $\boldsymbol{T_L}$ and $\boldsymbol{T_R}$ are height balanced.
  2) $|bf(\boldsymbol{T})| \leq 1$

AVL Trees

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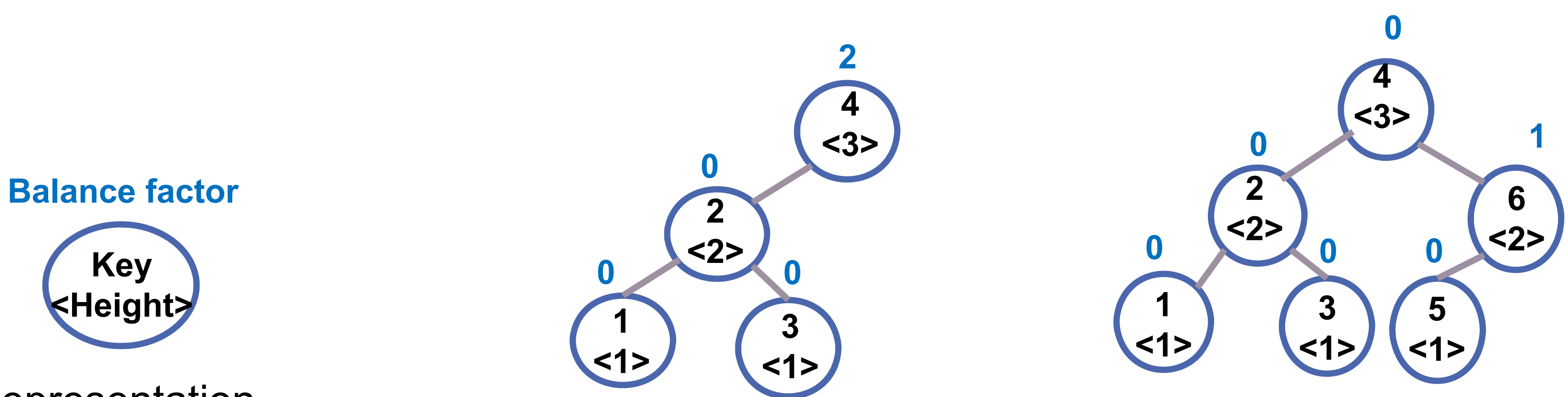
AVL Trees

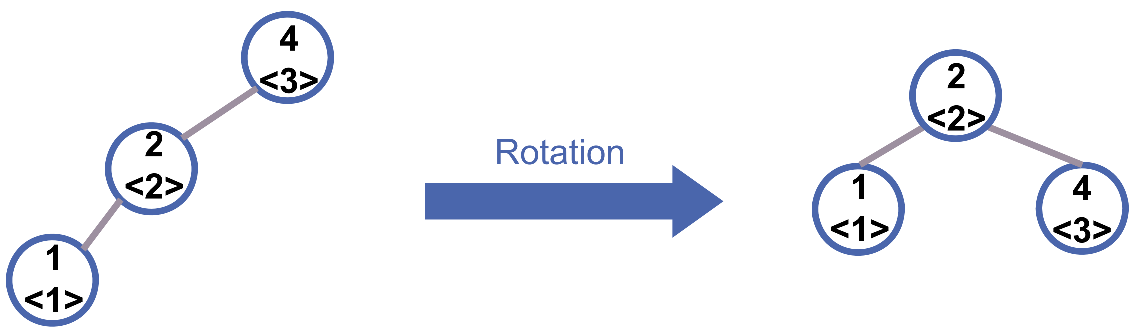
- AVL tree is a *height-balanced* binary search tree
- Each node in an AVL tree stores the current node height
  - For calculating the balance factor



Representation

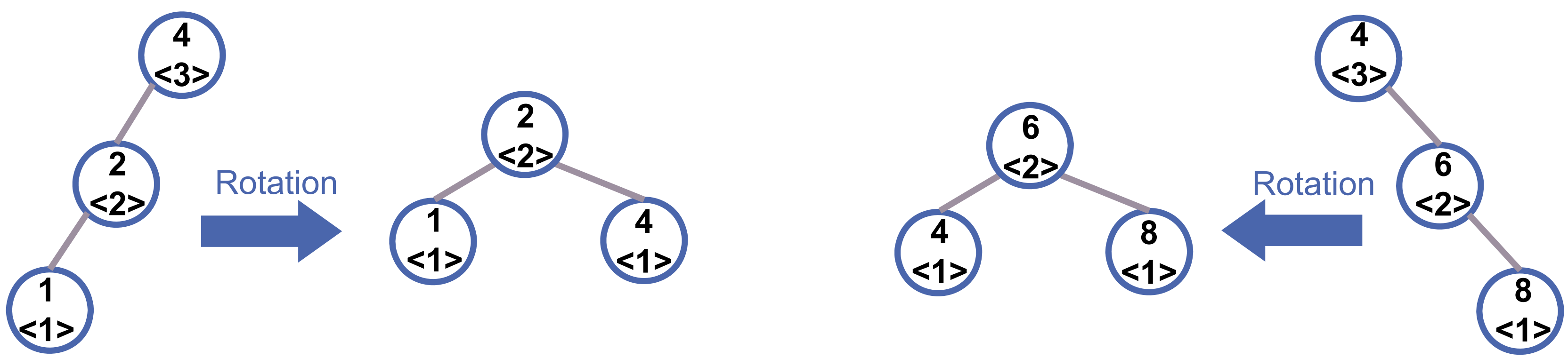
Rebalancing

- During BST insertion/deletion operations,
  - if balance factor >1 or <–1, activate rebalance process
- Rebalancing process
  - Fix unbalanced situations using “rotations”



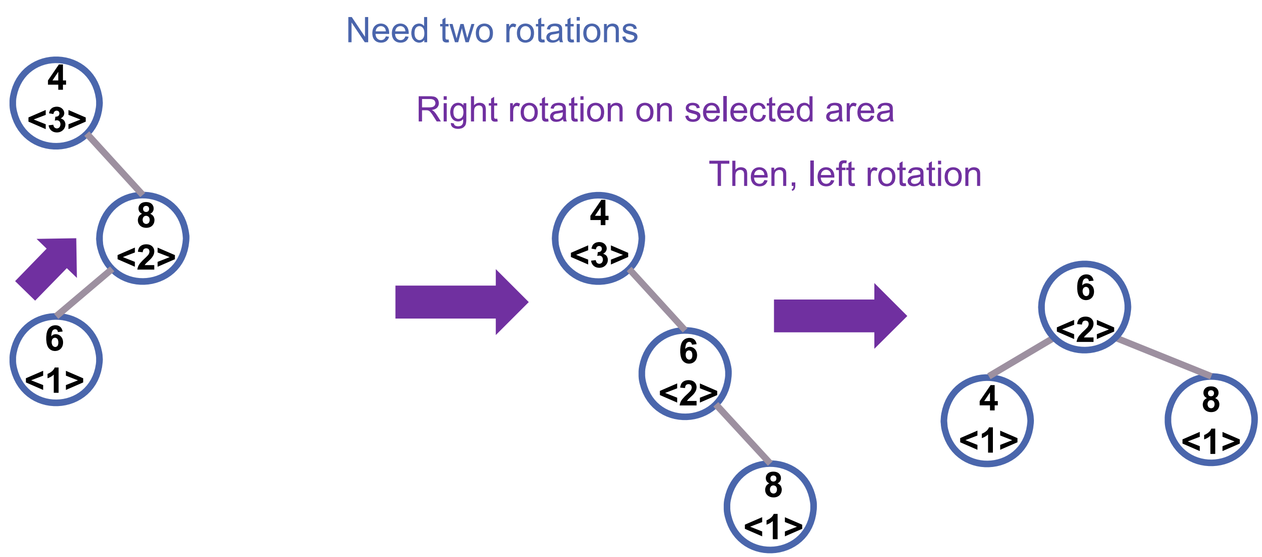
Rebalancing Operations

- Right rotation
- Left rotation



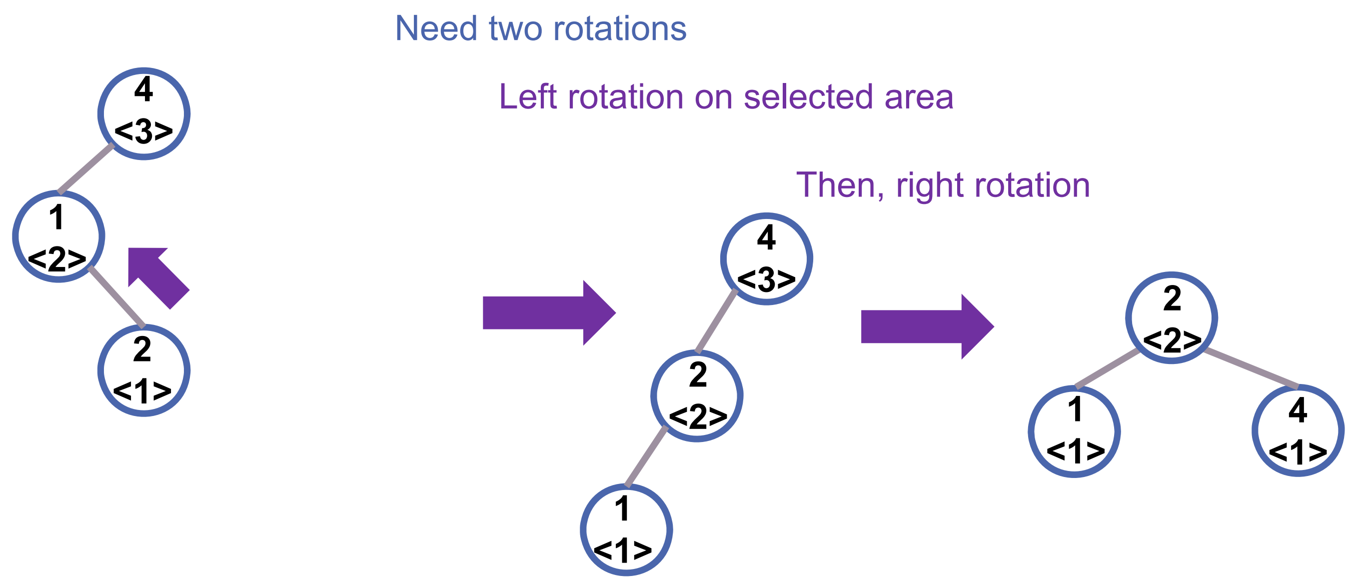
Rebalancing Operations

- Two rotations for inside cases



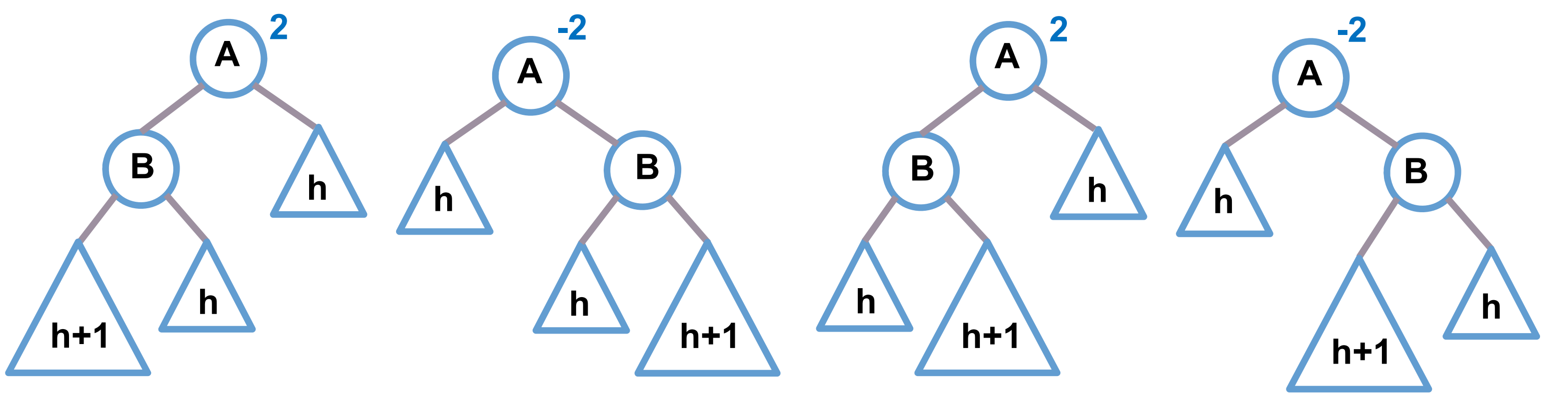
Rebalancing Operations

- Two rotations for inside cases



Unbalanced Situations

- There are 4 kinds of unbalanced situations:
  - 2 outside cases: require single rotation (LL, RR)
  - 2 inside cases: require two rotations (LR, RL)



2 outside cases

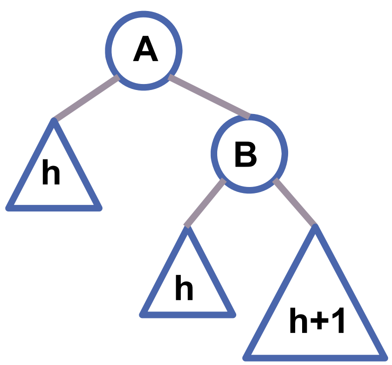
2 inside cases

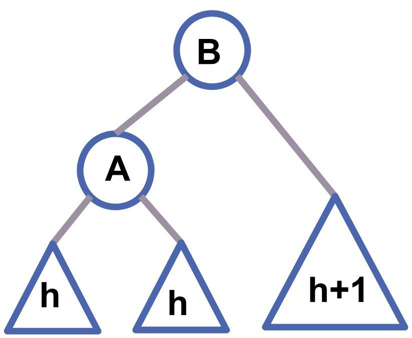
Outside Cases - LL Rotation

- Right rotation (LL Rotation)
  - The new node is inserted in the left subtree of the left subtree of A

Outside Cases - RR Rotation

- Left rotation (RR Rotation)
  - RR Rotation: The new node is inserted in the right subtree of the right subtree of A



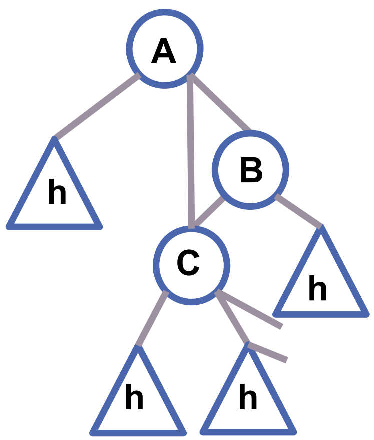


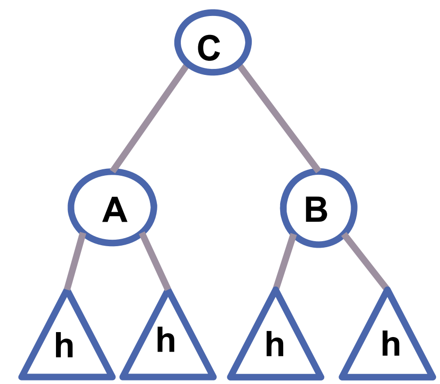
Inside Cases - LR Rotation

- LR Rotation
  - The new node is inserted in the right subtree of the left subtree of A
  - Left rotation + Right rotation

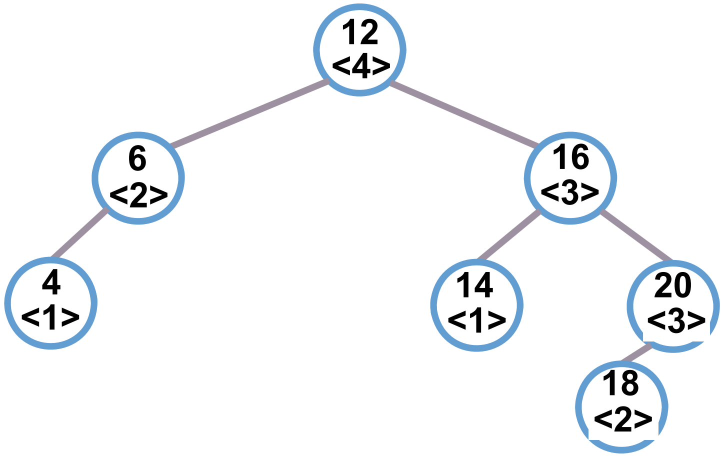
Inside Cases - RL Rotation

- RL Rotation
  - The new node is inserted in the left subtree of the right subtree of A
  - Right rotation + left rotation

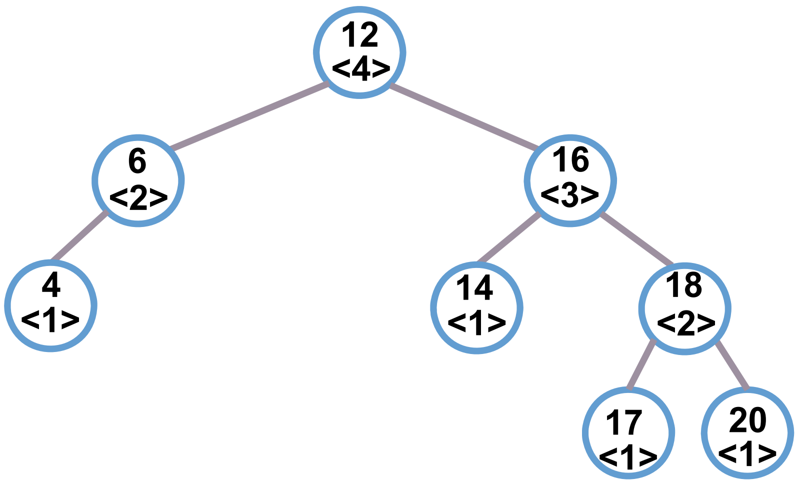




Example AVL Tree: Insert 17

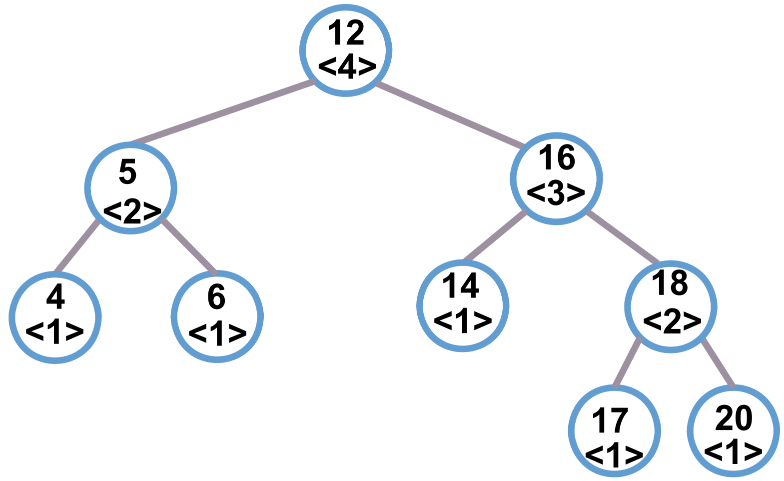


Example AVL Tree: Insert 17

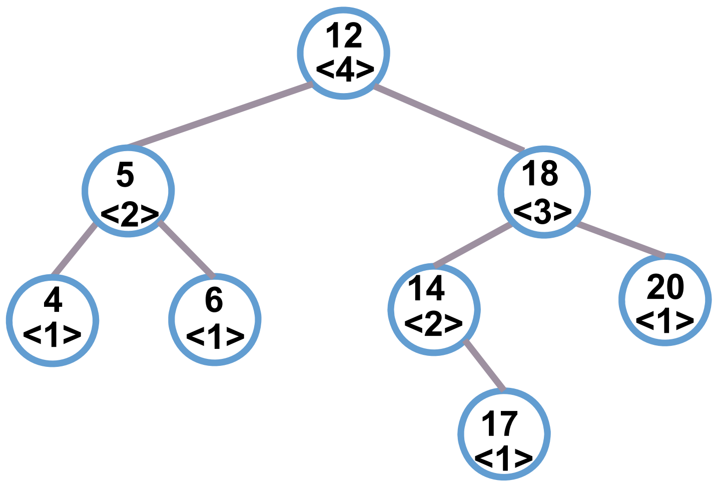


Back to Deletion Example

Example AVL Tree: Delete 16



Example AVL Tree: Delete 16



ADT: AVL Tree

```
template < class T > class AVLTree;

template < class T >
Class TreeNode {
friend class AVLTree <T>;
private:
    T data;
    int height;
    void updateHeight();
    int bf();
    TreeNode<T>* left, right;
};

template <class T>
Class AVLTree{
public:
       // Constructor
       AVLTree(void) {root=NULL;}

       // Tree operations here…

private:
      TreeNode<T> *root;
};
```

AVL Tree Insert/delete

```
template < class T >
TreeNode<T>* AVLTree<T>::insert(TreeNode<T> *node, T data)
{
    // BST Insert
    // ...

    // rebalance from node to root
    node->updateHeight();
    return rebalance( node );
}

template < class T >
TreeNode<T>* AVLTree<T>::delete(TreeNode<T> *node, T data)
{
    // BST Delete
    // ...

    // rebalance from node to root
    node->updateHeight();
    return rebalance( node );
}
```

AVL Tree Rebalance

```
template < class T >
TreeNode<T>* AVLTree<T>::rebalance(TreeNode<T> *node){
    // LL Rotation
    if ( node->bf()>1 && node->left->bf()>=0 ){
        return rightRotate( node );
    }
    // RR Rotation
    if ( node->bf()<-1 && node->right->bf()<=0 ){
        return leftRotate( node );
    }
    // LR Rotation
    if ( node->bf()>1 && node->left->bf()<0 ){
        node->left = leftRotate( node->left );
        return rightRotate( node );
    }
    // RL Rotation
    if ( node->bf()<-1 && node->right->bf()>0 ){
        node->right = rightRotate( node->right );
        return leftRotate( node );
    }
}
```

AVL Tree Left/Right Rotation

```
template < class T >
TreeNode<T>* AVLTree<T>::leftRotate(TreeNode<T> *node)
{
    TreeNode<T>* node_r = node->right;
    TreeNode<T>* node_rl = node_r->left;
    node_r->left = node;
    node->right = node_rl;
    node->UpdateHeight();
    node_r->UpdateHeight();
    return node_r;
}

template < class T >
TreeNode<T>* AVLTree<T>::rightRotate(TreeNode<T> *node)
{
    TreeNode<T>* node_l = node->left;
    TreeNode<T>* node_lr = node_l->right;
    node_l->right = node;
    node->left = node_lr;
    node->UpdateHeight();
    node_l->UpdateHeight();
    return node_l;
}
```

B-Tree

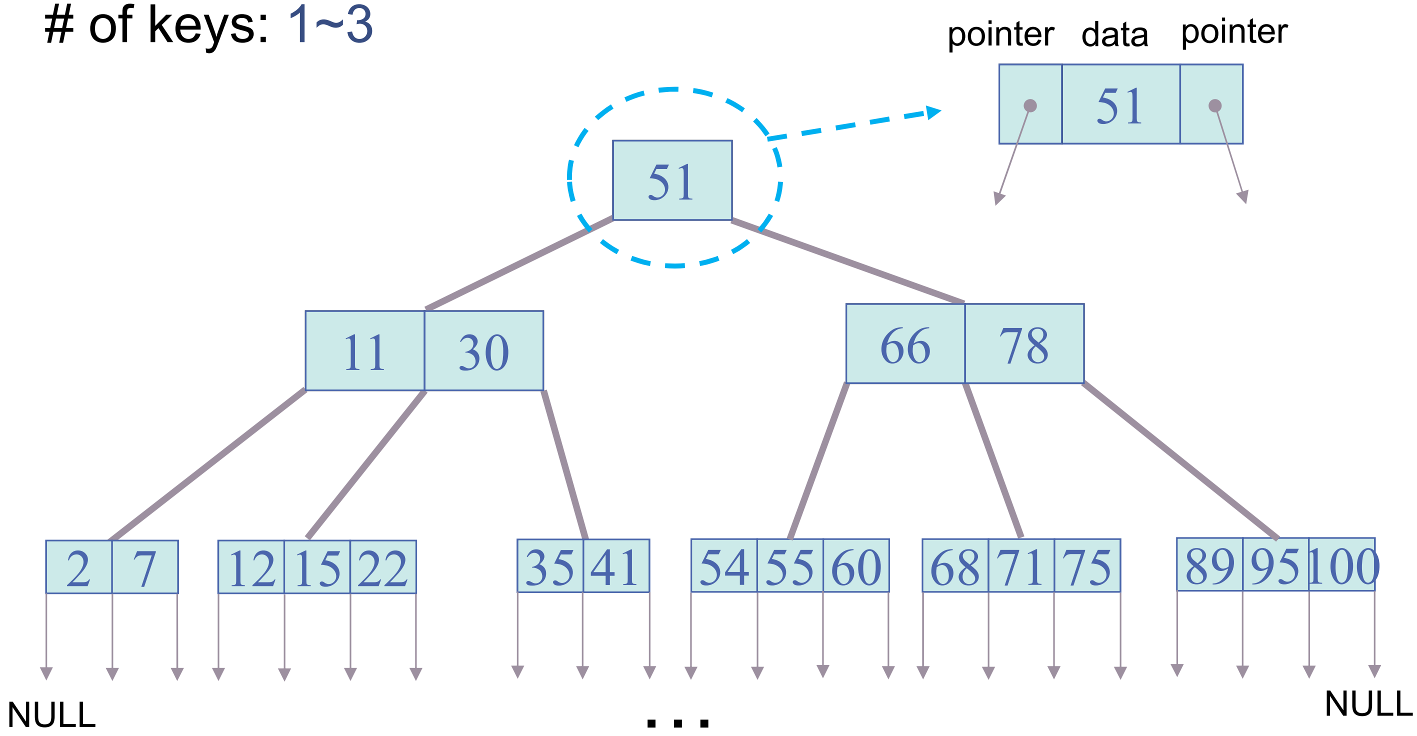
[redacted]

B-tree: Definition

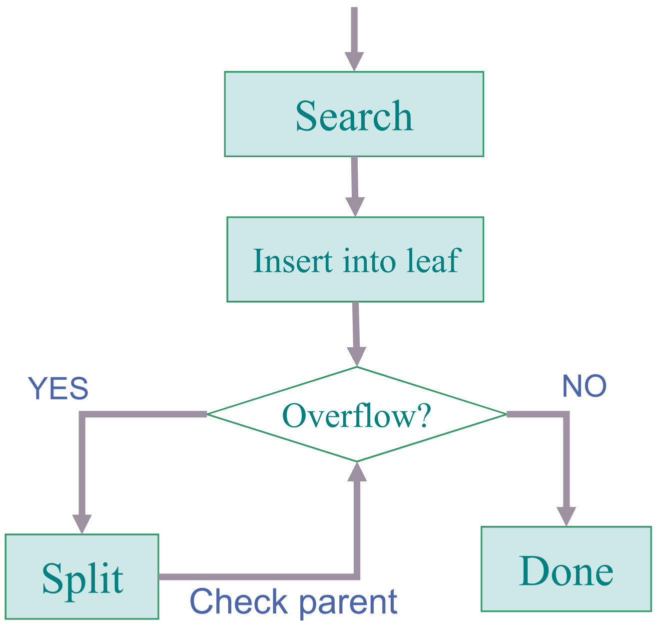
- A B-tree of order *m* is a height-balanced tree , where each node may have up to *m* children, and in which:
  1. All leaves are on the same level
  2. No node can contain more than *m* children
  3. All nodes except the root have at least $\left\lceil \frac{m}{2} \right\rceil$-1 keys
  4. The root is either a leaf node, or it has from 2 to m children

B-tree: Example

m = 4
# of keys: 1~3



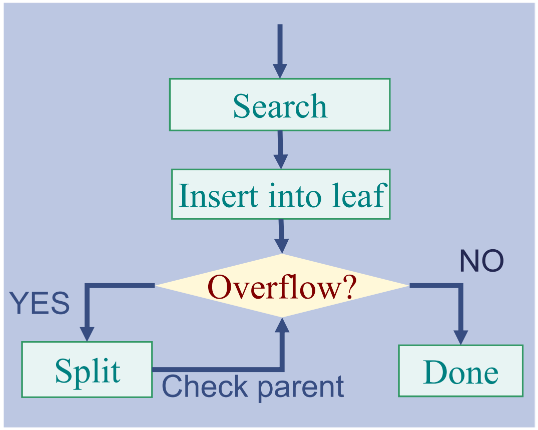
B-tree: Insert

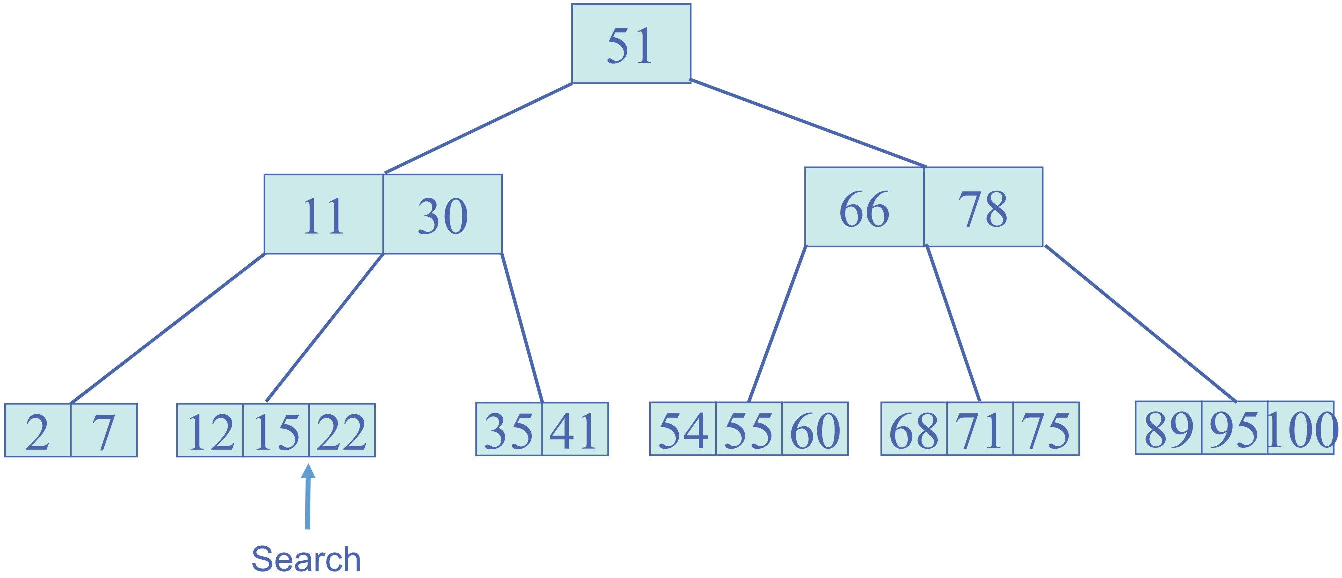


- Search
- Insert the new key into a leaf
- If the leaf overflows
  - Split the leaf into two and push up the middle key to the leaf's parent
  - If the parent overflows
    - Split the parent into two and push up the middle key again
  - This strategy might have to be repeated all the way until arriving the root
  - If necessary, the root is split in two and the middle key is pushed up to a new root, making the tree one level higher

Example of Insertion

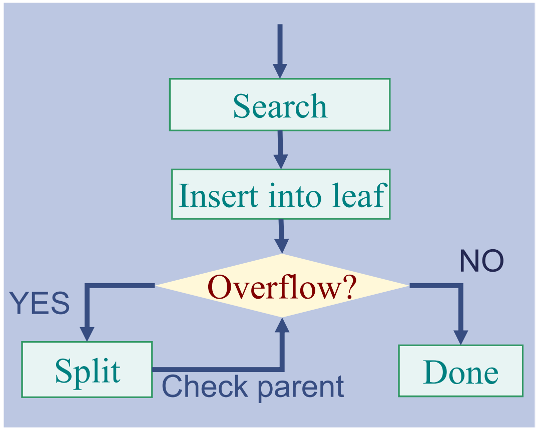
Insert 20

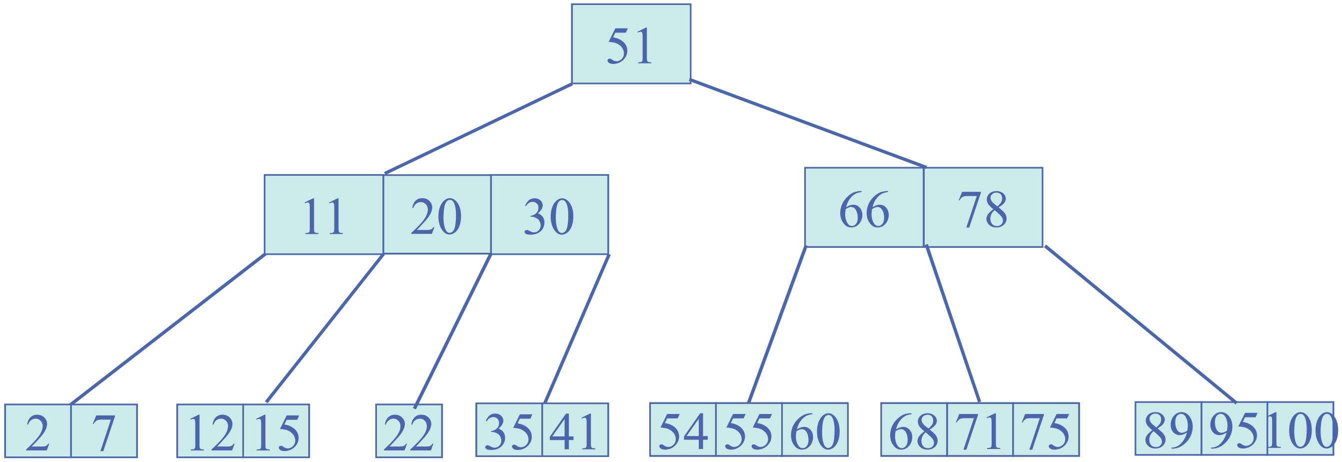




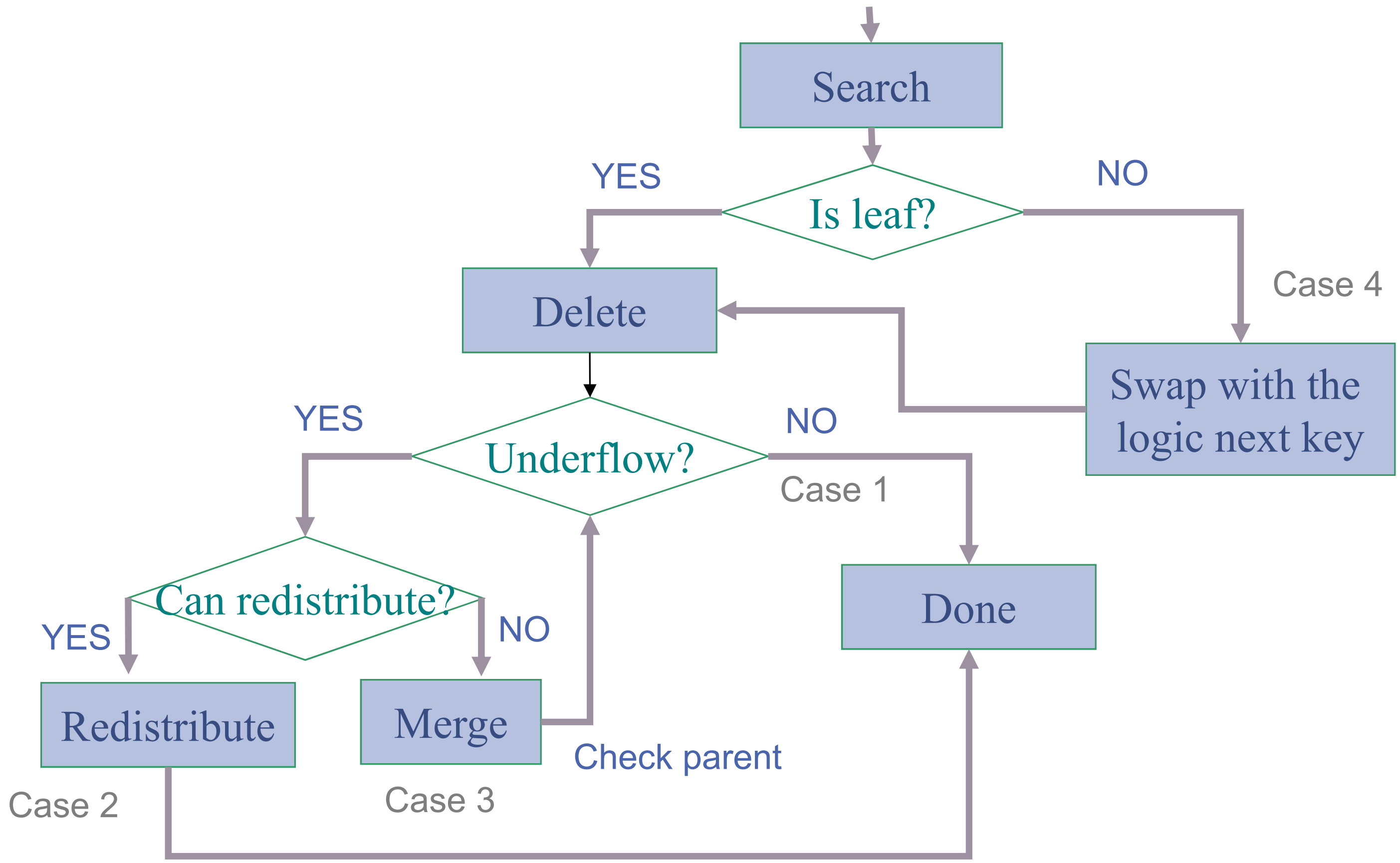
Example of Insertion

Insert 20



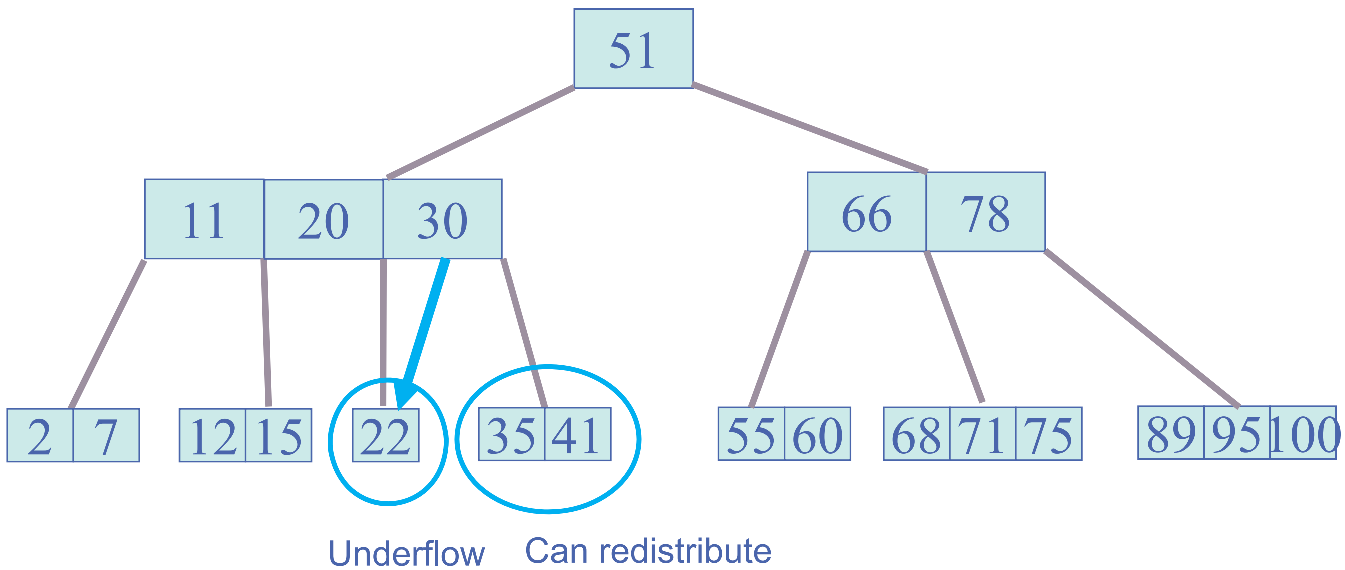


B-tree: Flow Chart of Deletion



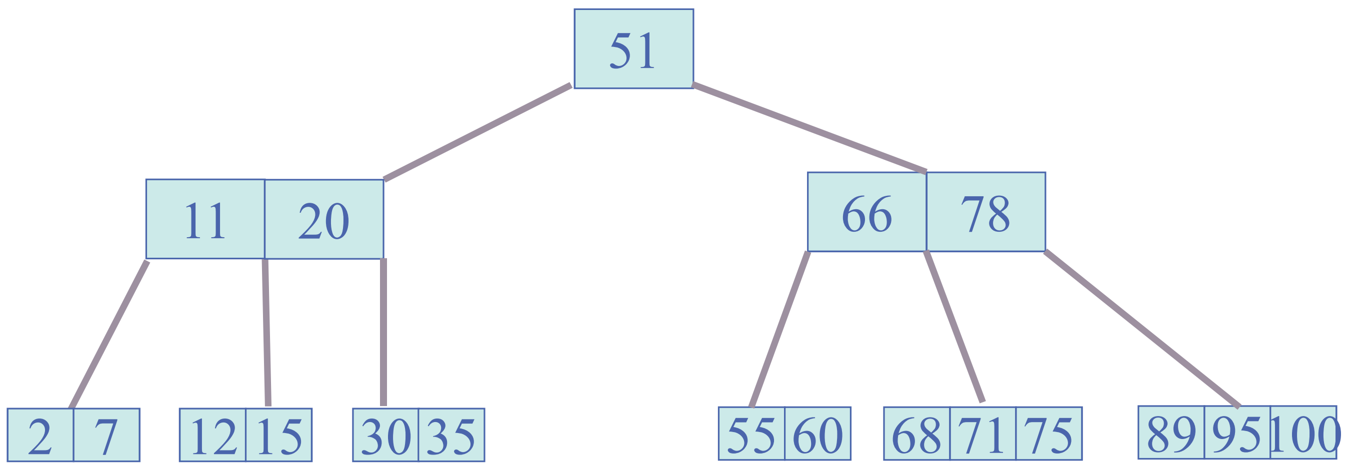
Example of Deleting A Leaf

Case 2 (Redistribute): delete 22



Example of Deleting Leaf

Case 3 (Merge): delete 41



B-tree: Deleting Leaf

- Leaf

```
Delete the key
If the number of keys is valid after deletion
        O.K.
else      //underflow
        If any sibling node has keys more than  ⌈m/2⌉-1
                Redistribute key from siblings
    else
                Merge nodes into one node
                Check if parent is underflow
```

B-tree: Deleting Non-leaf

- Non-leaf
  Swap the key with the logic next key (i.e. the first key in the leftmost leaf of the right subtree)
  Call Delete leaf

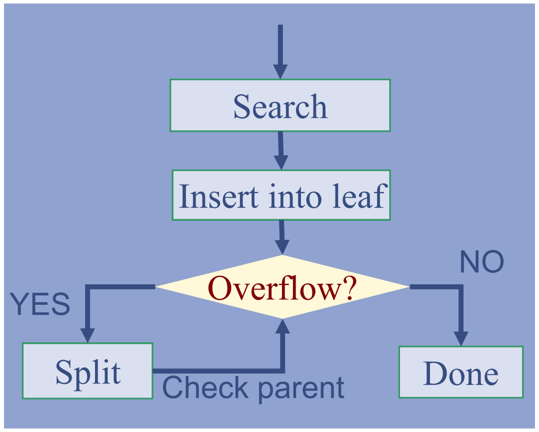
P.S. Alternatively, we can choose the logic previous key
(i.e. the last key in the rightmost leaf of the left subtree)
to swap

B^+ Tree

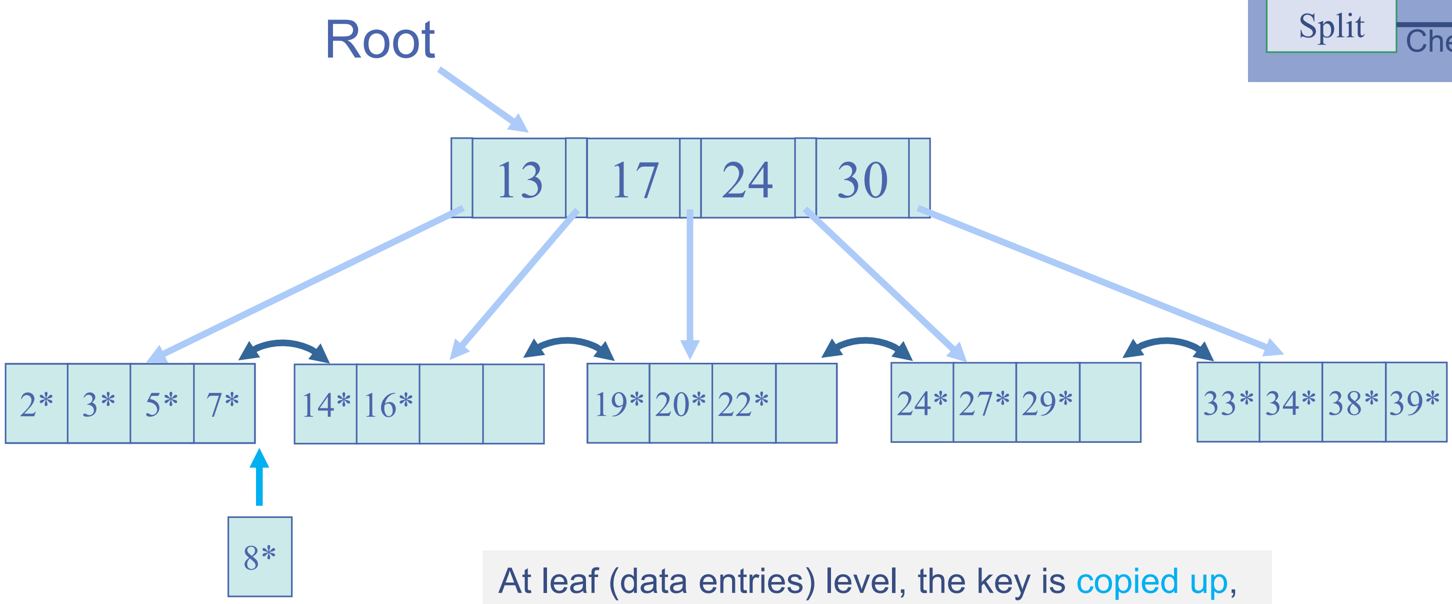
B^+-tree: Informal Definition

- A variation of B-tree
- Two kinds of nodes
  - Index (non-leaf) nodes
    - Store keys
    - Guide the search for a record in a leaf node
  - Data (leaf) nodes
    - Store data records
      - Real data files or data pointers
    - Linked list (sequence set)

Example of Insertion

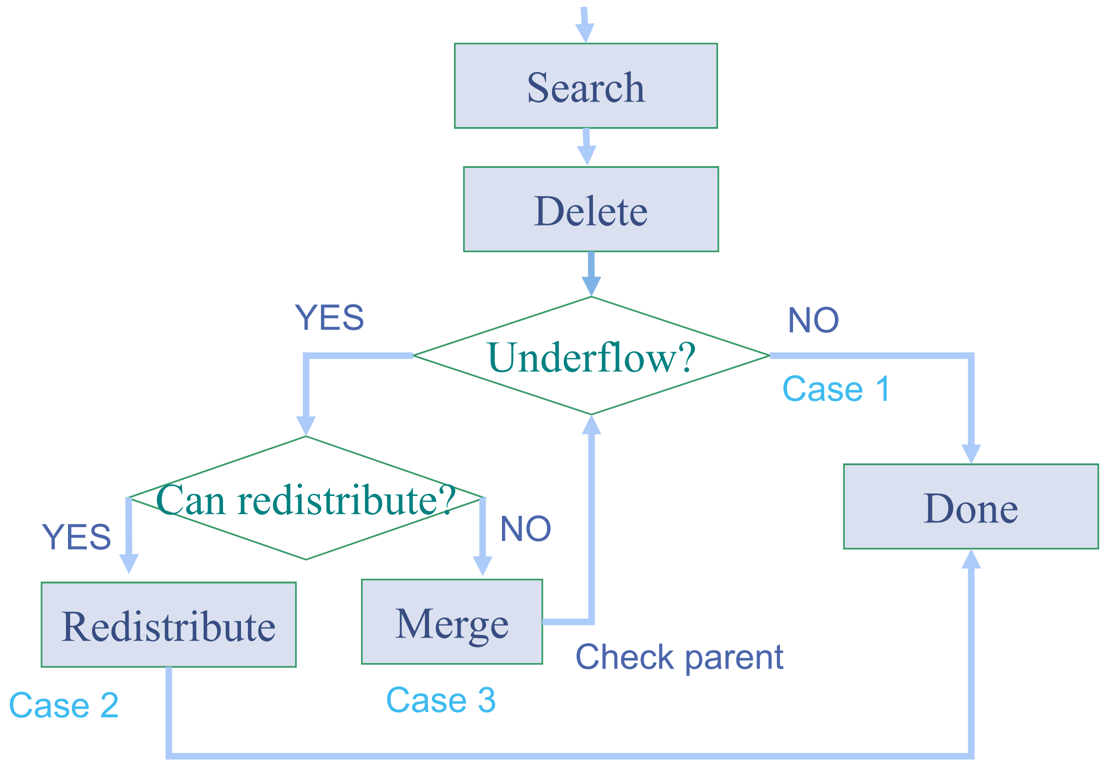


Insert 8*



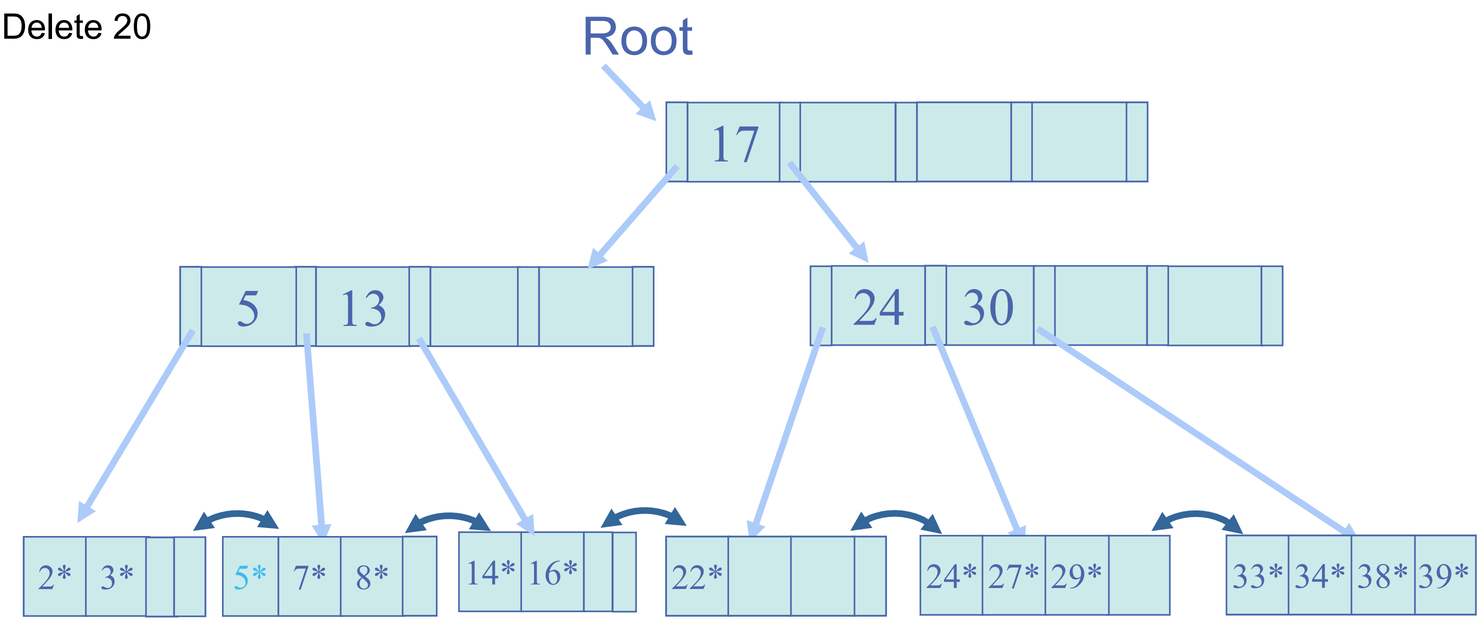
At leaf (data entries) level, the key is copied up, which is different from B-tree

B^{+}-tree: Flow Chart of Deletion



Example of Deletion

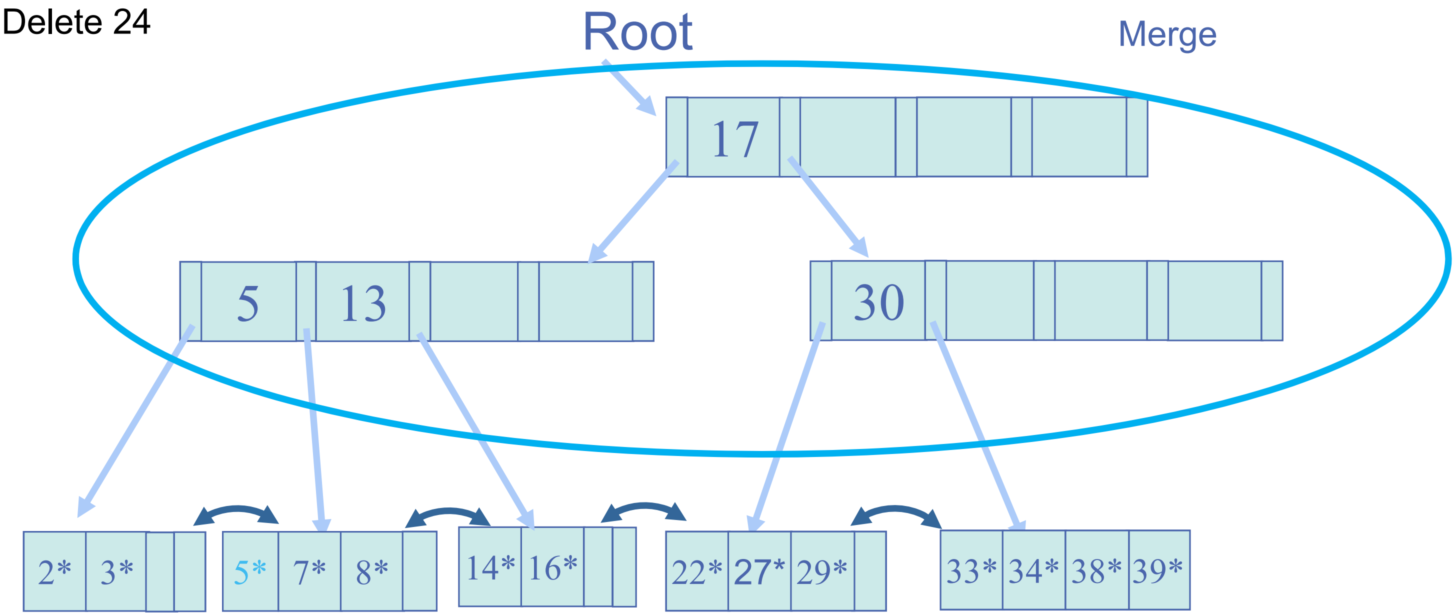
Case 2 Redistribution in leaf-level
Delete 20



At leaf (data entries) level, the key is copied up, which is different from B-tree

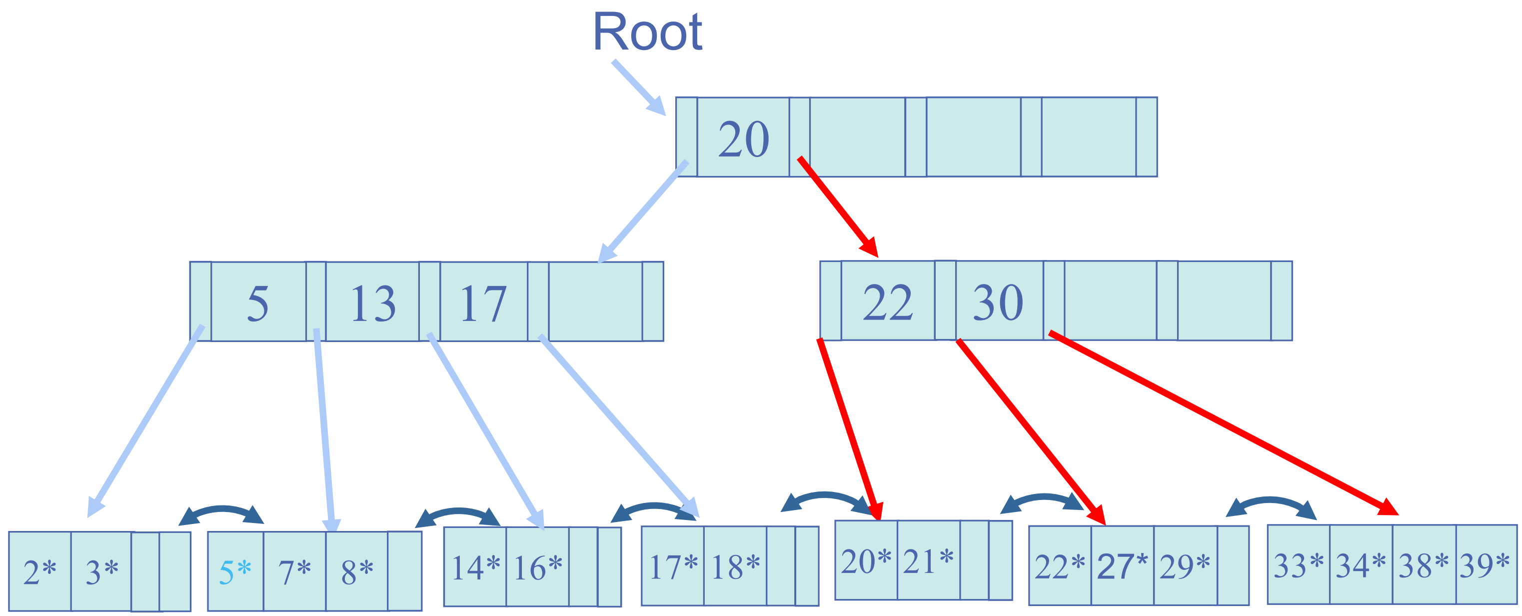
Example of Deletion

Case 3 Merge in leaf-level and non-leaf-level
Delete 24



Example of Deletion

Case 4 redistribution in non-leaf-level



Difference between B-tree and B^{+}-tree

- In a B-tree, pointers to data records exist at all levels of the tree
- In a B+-tree, all pointers to data records exists at the leaf-level nodes
- A B+-tree can have less levels (or higher capacity of search values) than the corresponding B-tree