Graphs

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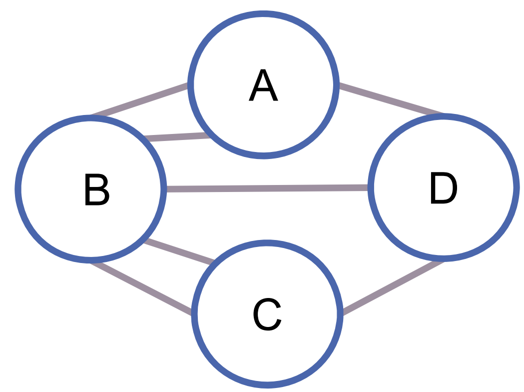
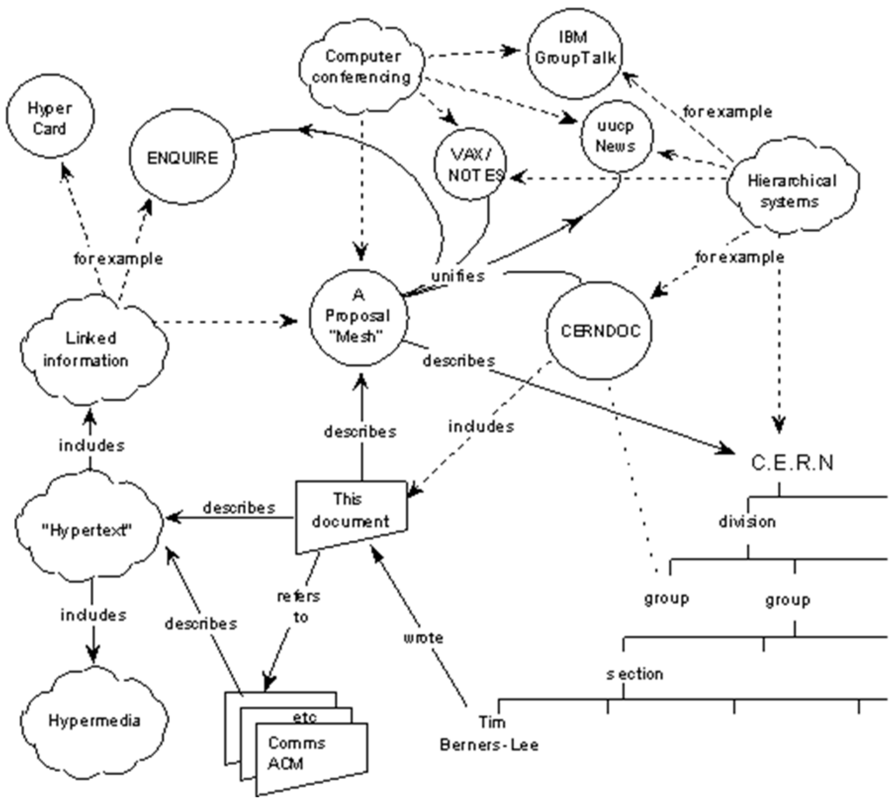
The Use of Graphs

Konigsberg Bridge Problem

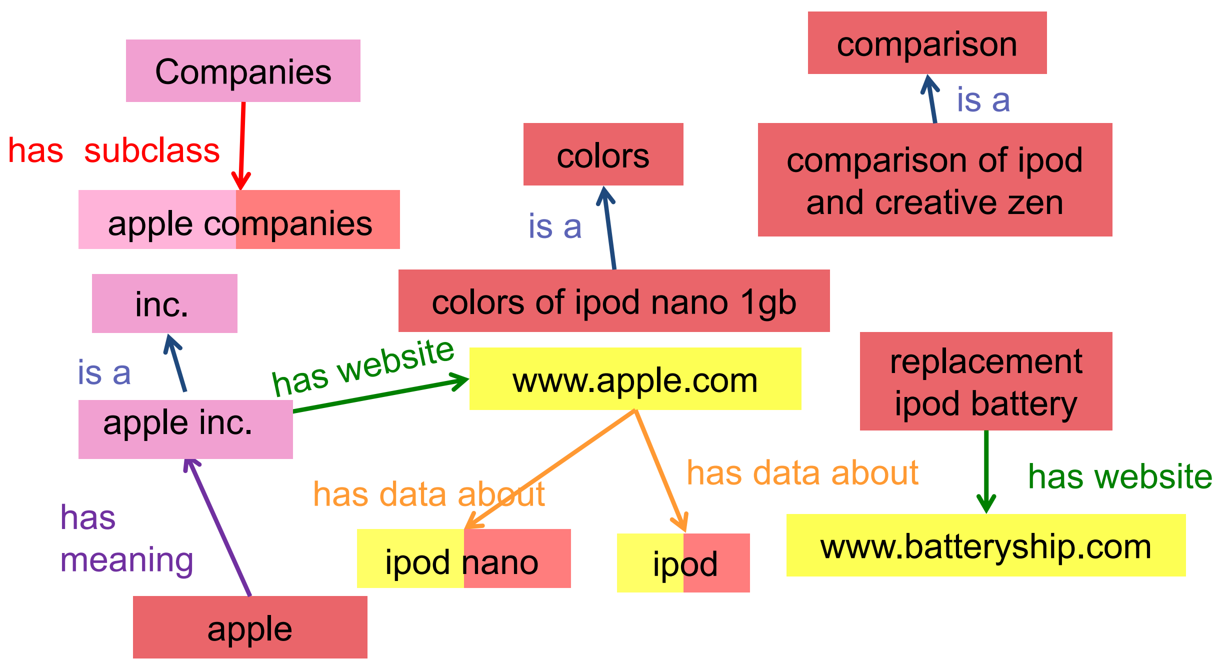
- The first record (1736)
  - Solved by Euler (1707-1783)
- Problem: Walk across all the bridges exactly once
- Formulate as a graph
- Prove: possible
  - Iff the degree of each vertex is even



World Wide Web



Ontology



Companies
has subclass
apple companies
inc.
is a
apple inc.
has website
has meaning
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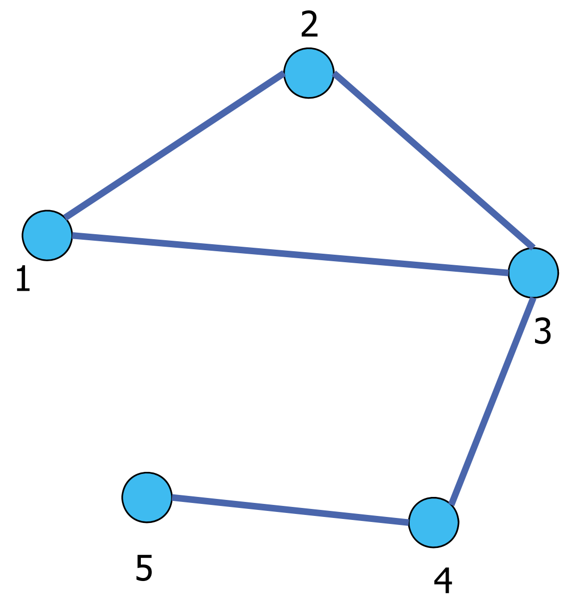
colors of ipod nano 1gb
www.apple.com
has data about
ipod nano
has data about
ipod
comparison
is a
comparison of ipod and creative zen
replacement ipod battery
has website
www.batteryship.com

Graph Concept

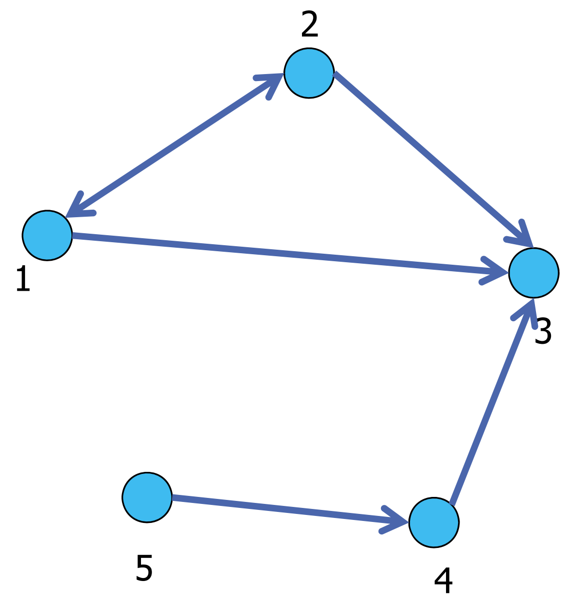
Undirected Graph

- Graph G=(V,E)
  - V = set of vertices
  - E = set of edges
- Undirected graph
  - E={(1,2),(1,3),(2,3),(3,4),(4,5)}



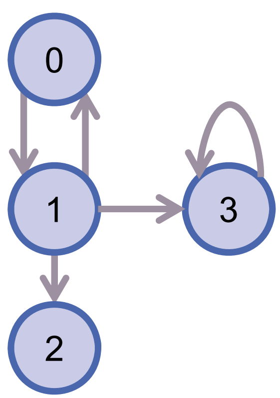
Directed Graph

- Graph G=(V,E)
  - V = set of vertices
  - E = set of edges
- Directed graph
  - <u,v> ≠ <v,u>
  - <u,v> → u is tail and v is head of edge
  - E={<1,2>,<2,1>,<2,3>,<1,3>,<4,3>,<5,4>}

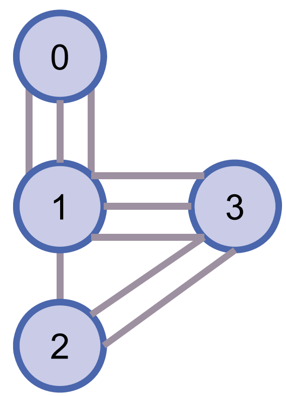


Restrictions

- ***Self edges*** and ***self loops*** are not permitted
  - Edges of the form (v, v) and <v, v> are not legal
- A graph may not have multiple occurrences of the same edge (***multigraph***)



Graph with self edge



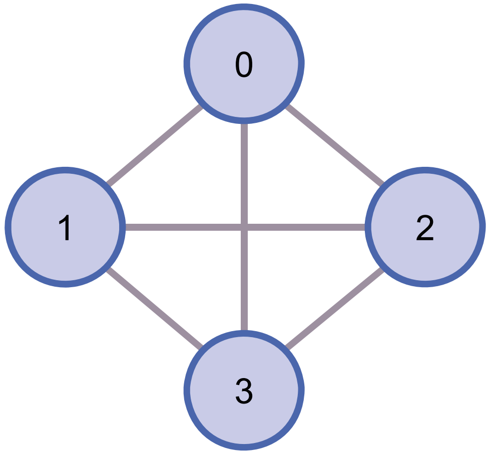
Multigraph

Terminology

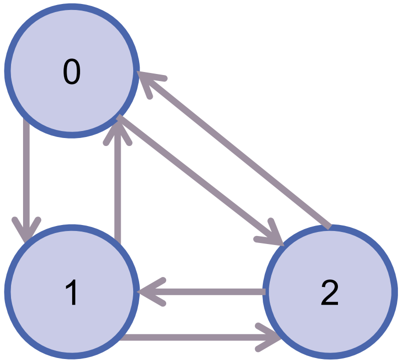
- For a graph with n vertices, the maximum # of edges:
  - $n(n-1)/2$ for undirected graph
  - $n(n-1)$ for directed graph
- Vertices u and v are **adjacent** if (u,v) $\in$ E
  - Edge (u,v) is **incident** on vertices u and v
- <u,v>, u is **adjacent to** v and v is **adjacent from** u
  - Edge <u,v> is **incident** on vertices u and v

Complete Graph

- Complete undirected graph
  - Graph with n vertices has exactly $n(n-1)/2$ edges

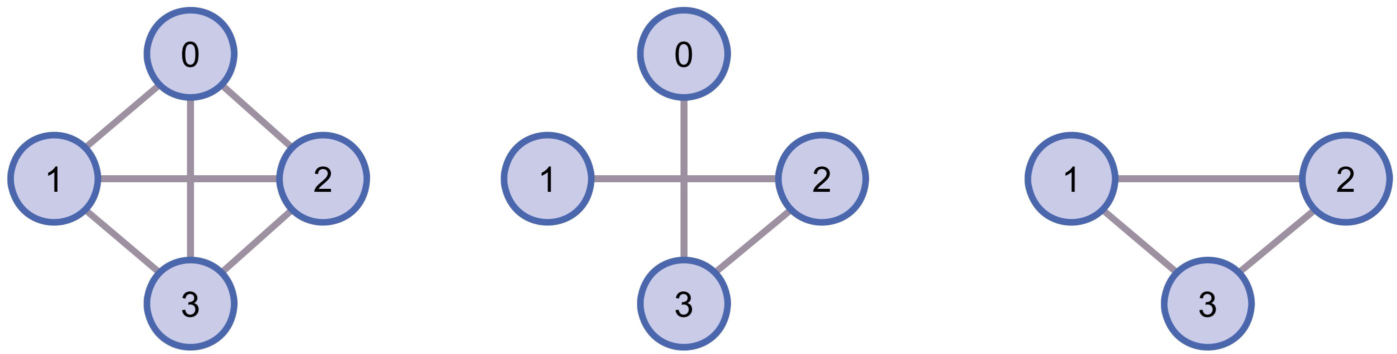


- Complete directed graph
  - Graph with n vertices has exactly $n(n-1)$ edges



Subgraph

- G' is a subgraph of G
  - $V(G') \subseteq V(G)$ and $E(G') \subseteq E(G)$.



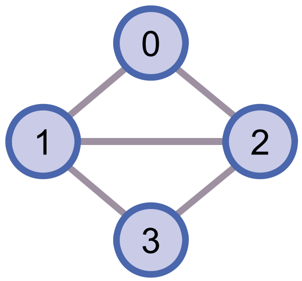
Graph

Subgraph

Subgraph

Path

- A path from **u** to **v**
  - A sequence of vertices $\mathbf{u}, \mathbf{i_1}, \mathbf{i_2}, \ldots, \mathbf{i_k}, \mathbf{v}$
  - $(\mathbf{u}, \mathbf{i_1}), (\mathbf{i_1}, \mathbf{i_2}), \ldots, (\mathbf{i_k}, \mathbf{v})$ are edges in graph
- Simple path:
  - A path in which all vertices are distinct
    - Except possibly the first and the last



Sequence	Path?	Simple path?
0,1,3,2		
0,2,0,1		
0,3,2,1		

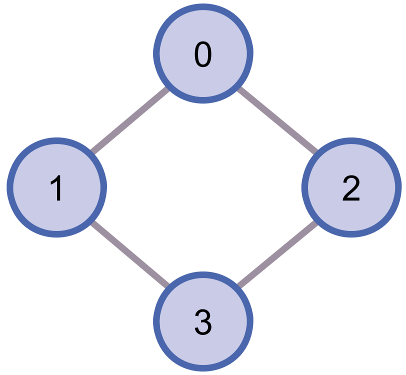
Cycle

- A cycle is a simple path
  - The first and the last vertices are the same

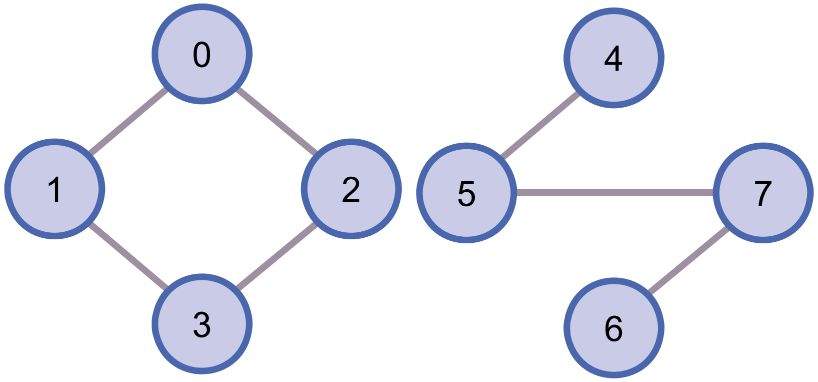
- Notes: if the graph is a directed graph:
  - Directed path
  - Directed simple path
  - Directed cycle

Connected

- Undirected graph G is said to be connected
  - iff there is a path for every pair of distinct vertices



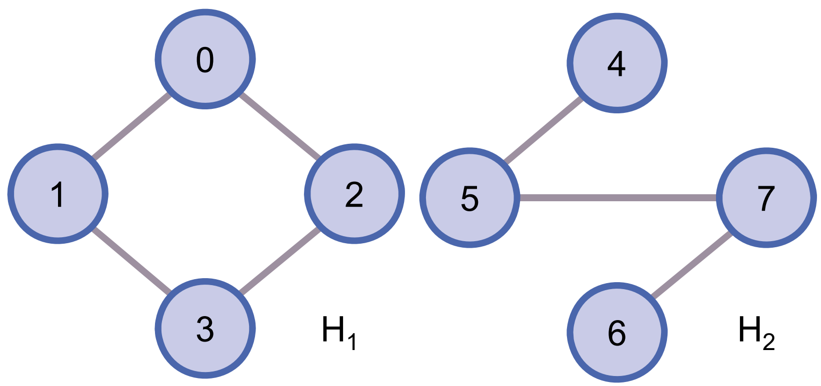
Connected graph



Not a connected graph

Connected Component

- A maximal connected subgraph

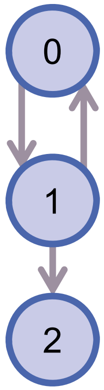


Graph with two connected components

- Tree:
  - A connected acyclic graph

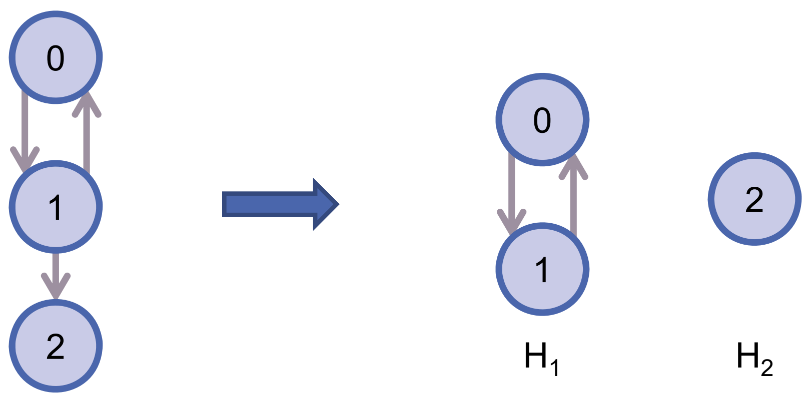
Strongly Connected

- Directed graph G is strongly connected
  - iff there is a directed path for every pair of distinct vertices



Strongly Connected Component

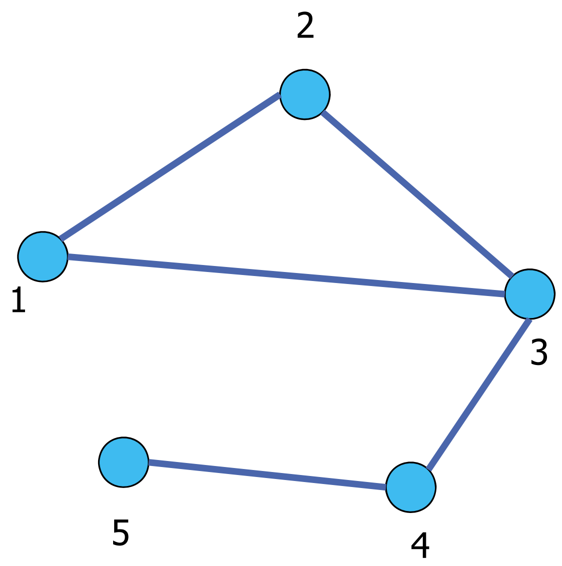
- A maximal subgraph that is strongly connected



Two strongly connected components

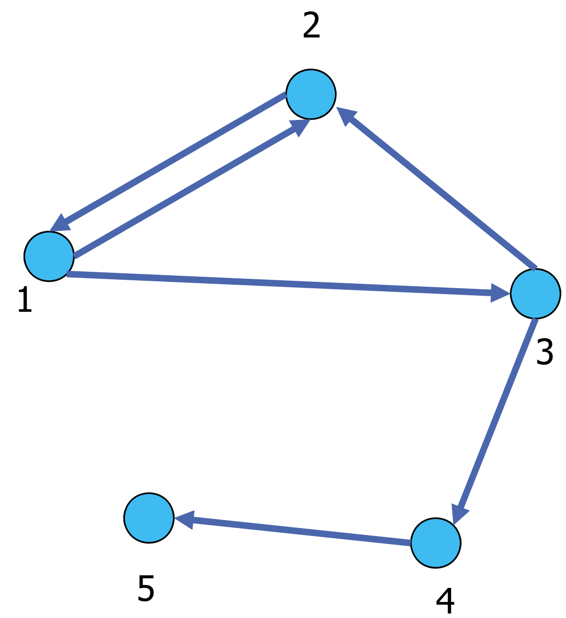
Degree of Undirected Graph

- degree d(i) of node i
  - number of edges a node i involved
- degree sequence
  - [d(1),d(2),d(3),d(4),d(5)]
  - [2,2,3,2,1]



Degree of Directed Graph

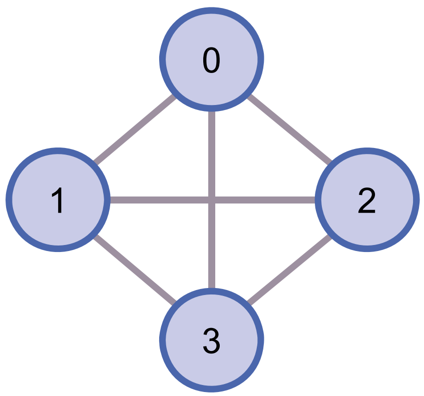
- in-degree $d_{in}(i)$ of node i
  - number of edges pointing
- out-degree $d_{out}(i)$ of node i
  - number of edges leaving node i
- Degree of v = in-degree + out-degree
- in-degree sequence
  - [1,2,1,1,1]
- out-degree sequence
  - [2,1,2,1,0]

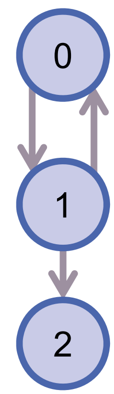


Graph Representation

Adjacency Matrix

- A two dimensional array
  - a[i][j] = 1 iff the edge (i,j) or <i,j> is in E(G)

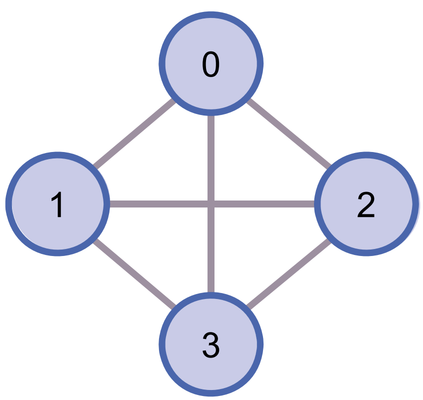


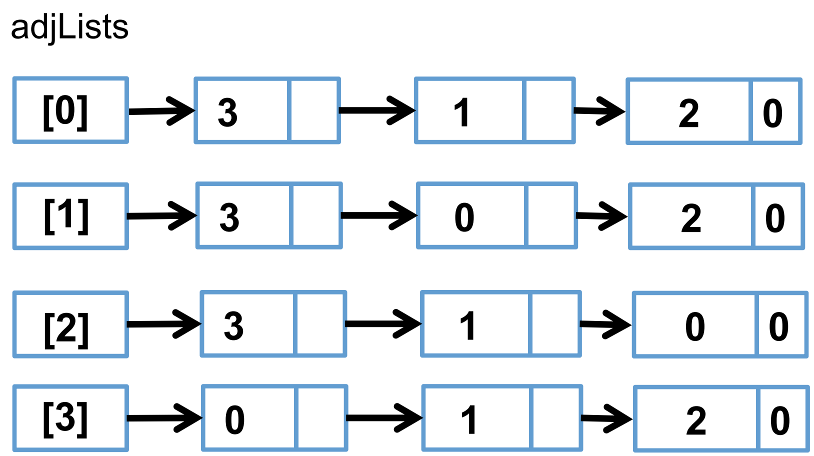


- Waste of memory when a graph is sparse
  - Storage $O(n^2)$

Adjacency Lists in Undirected Graph

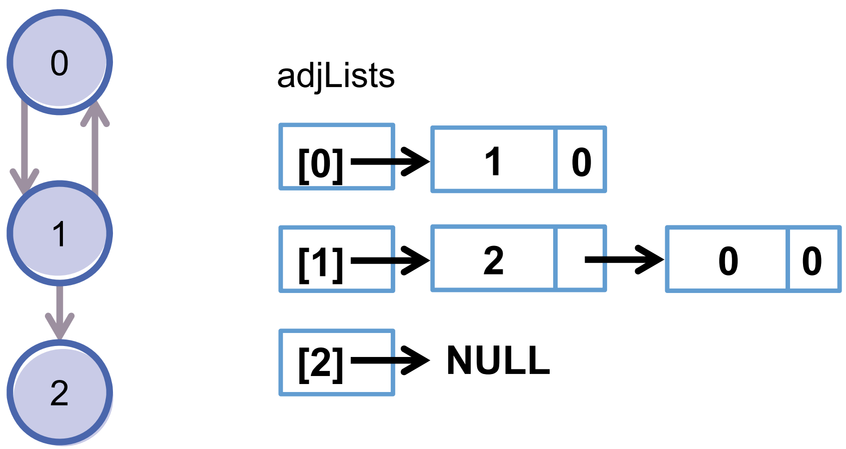
- Use a chain to represent each vertex and its **adjacent** vertices





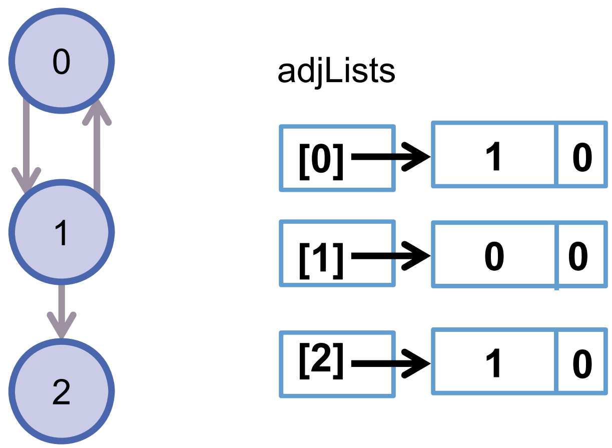
Adjacency Lists in Directed Graph

- Use a chain to represent each vertex and its adjacent to vertices
  - Length of list = Out-degree of v



Inverse Adjacency Lists in Directed Graph

- Use a chain to represent each vertex and its adjacent from vertices
  - Length of list = In-degree of v



Weighted Edges

- Edges of a graph sometimes have weights associated with them
  - Distance from one vertex to another.
  - Cost of going from one vertex to an adjacent vertex.
- Use additional field in each edge to store the weight
- A graph with weighted edges is called a network

ADT: Graph

```
class Graph
{// object: A nonempty set of vertices and a set of undirected edges.
public:
  virtual ~Graph() {}                        // virtual destructor
  bool IsEmpty() const{return n == 0};       // return true iff graph has no vertices
  int NumberOfVertices() const{return n}; // return the # of vertices
  int NumberOfEdges() const{return e};       // return the # of edges
  virtual int Degree(int u) const = 0;       // return the degree of a vertex
  virtual bool ExistsEdge(int u, int v) const = 0; // check the existence of edge
  virtual void InsertVertex(int v) = 0;            // insert a vertex v
  virtual void InsertEdge(int u, int v) = 0;       // insert an edge (u, v)
  virtual void DeleteVertex(int v) = 0;            // delete a vertex v
  virtual void DeleteEdge(int u, int v) = 0;       // delete an edge (u, v)
  // More graph operations…
protected:
  int n; // number of vertices
  int e; // number of edges
};
```

Implementation Notes

- Several assumptions:
  - Data type of edge weight is **double**
    - Or represented as a template parameter
- Operations are **independent** of specific graph representation
- The **iterator** is used to visit adjacent vertices

Example: LinkedGraph

```
void Graph::foo(void){
  // use iterator to visit adjacent vertices of v
  for (each vertex w adjacent to v)…
}
```

```
class LinkedGraph : public Graph
{
public:
  // constructor
  LinkedGraph(const int vertices = 0) : n(vertices), e(0){
    adjLists = new Chain<int>[n];
  }
  // more customized operations…
private:
  Chain<int> *adjLists         // adjacency lists
};
```

Elementary Graph Operations

Graph Operations

- Graph traversal
  - Depth-first search
  - Breadth-first search
- Connected components
- Spanning trees

Depth-First Search (DFS)

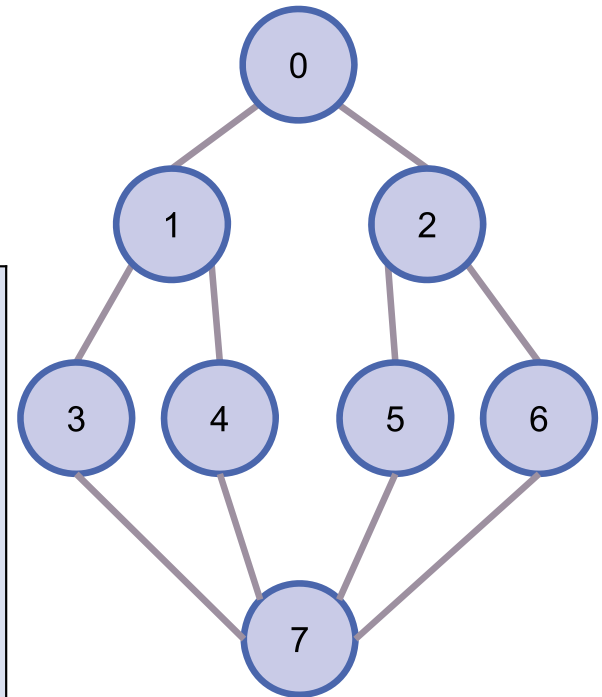
- Starting from a vertex v
  - Visit the vertex v => DFS(v)
  - For each vertex w adjacent to v, if w is not visited yet, then visit w => DFS(w).
  - If a vertex u is reached such that all its adjacent vertices have been visited, we go back to the last visited vertex.
- The search terminates when no unvisited vertex can be reached from any of the visited vertices.

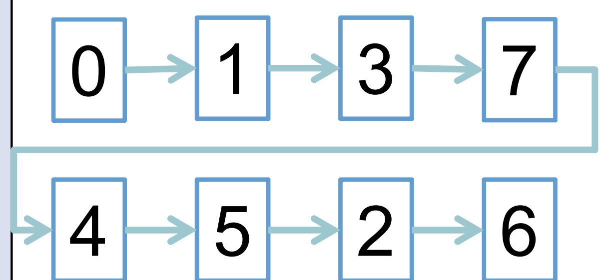
Real Life Examples of DFS

- School life
  - Elementary school → high school → college → master → work
- Games
  - E.g., tic-tac-toe
- Maze-solving

Non-Recursive DFS

```
void Graph::DFS(int v){
   visited = new bool[n]; // this is a data member of Graph
   fill(visited, visited+n, false);
   Stack<int> s;          // declare and init a stack
   s.Push(v);
   while(!s.IsEmpty()){
      v = s.Top(); s.Pop();
      if(!visited[v]){
         output(v);
         visited[v]=true;
         for(each vertex w adjacent to v)
           if(!visited[w]) s.Push(w);
      }
   }
}
```





Note that there are other possibilities

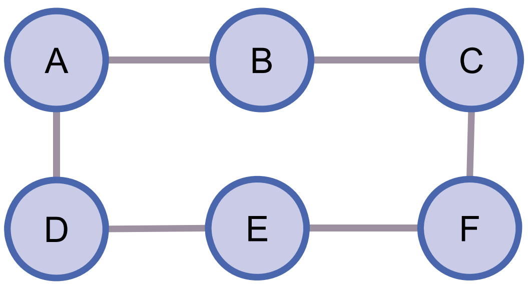
Recursive DFS

```
void Graph::DFS(void){
   visited = new bool[n]; // this is a data member of Graph
   fill(visited, visited+n, false);
   DFS(0); // start search at vertex 0
   delete [] visited;
}

void Graph::DFS(const int v){
   // visit all previously unvisited vertices that are adjacent to
v
   output(v);
   visited[v]=true;
   for(each vertex w adjacent to v)
     if(!visited[w]) DFS(w);
}
```

DFS Complexity

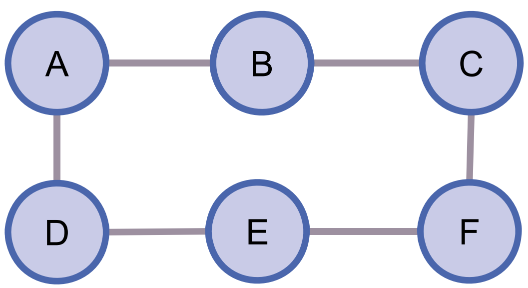
- Adjacency matrix
  - Time to determine all adjacent vertices: O(n)
  - At most n vertices are visited: $O(n \times n) = O(n^2)$

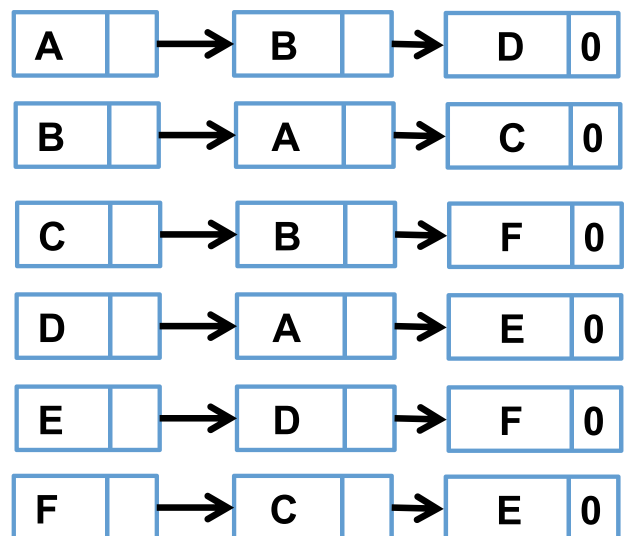


	A	B	C	D	E	F
A	0	1	0	1	0	0
B	1	0	1	0	0	0
C	0	1	0	0	0	1
D	1	0	0	0	1	0
E	0	0	0	1	0	1
F	0	0	1	0	1	0

DFS Complexity

- Adjacency lists
  - There are n+2e chain nodes
  - Each node in the adjacency lists is examined at most once.
  - Total time complexity = O(n+e)





Breadth-First Search (BFS)

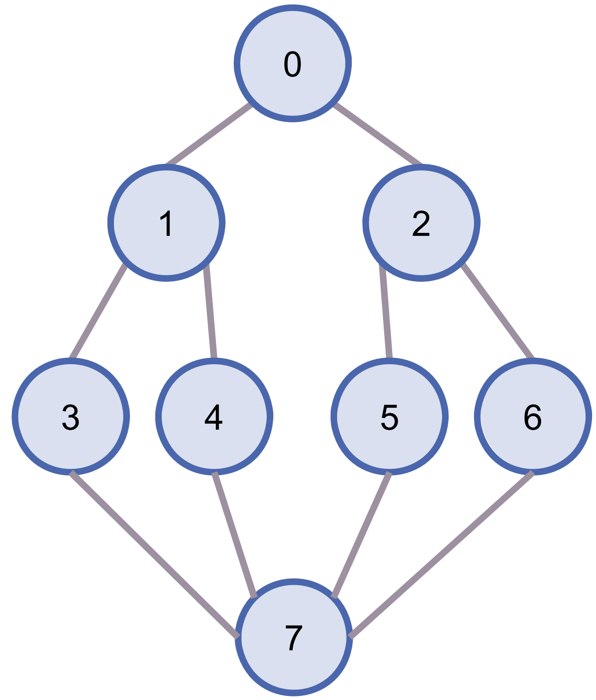
- Starting from a vertex v
  - Visit the vertex v
  - Visit all unvisited vertices adjacent to v
  - Unvisited vertices adjacent to these newly visited vertices are then visited

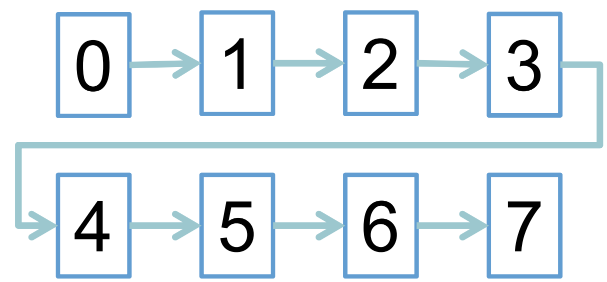
Real Life Examples of BFS

- Network Broadcasting: broadcast messages to all connected devices
- Social Networks
  - Meet your friend's friends before hopping to another networks
- Some "rare" students in school:
  - They would like to try many clubs before studying
  - They cannot stay with one partner
- Attention deficit hyperactivity disorder, ADHD

BFS: Implementation

```
void Graph::BFS(int v){
   visited = new bool[n]; // this is a data member of Graph
   fill(visited, visited+n, false);
   Queue<int> q;          // declare and init a queue
   q.Push(v);
   visited[v]=true;
   while(!q.IsEmpty()){
      v = q.Front(); q.Pop();
      if(!visited[v]){ output(v);
         for(each vertex w adjacent to v){
            if(!visited[w]){
               q.Push(w);
               visited[w]=true; } }
      }
   }
   delete [] visited;
}
```





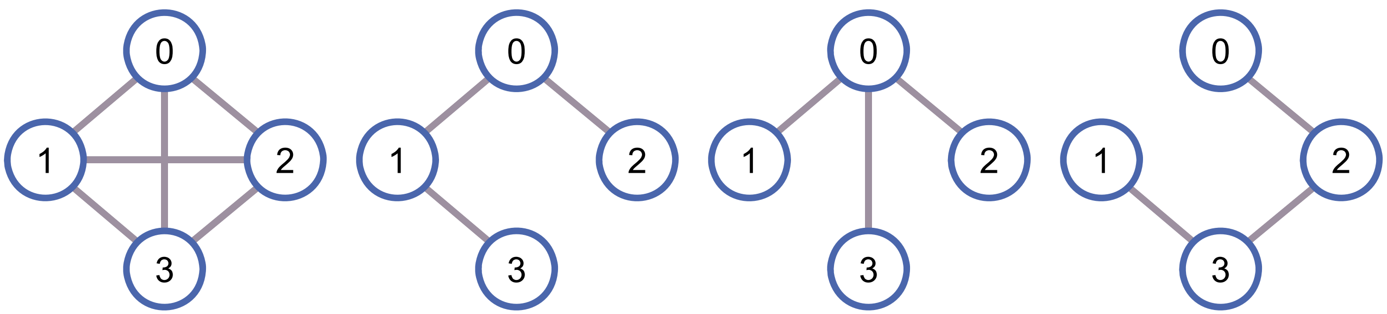
Time complexity is the same as DFS

Connected Components

- Determine whether a graph is connected
  - Call DFS or BFS once
  - Check if there is any unvisited vertices
  - Yes, then the graph is not connected.
- How to identify connected components
  - Call DFS or BFS repeatedly
  - Each call will output a connected component
  - Start next call at an unvisited vertex

Spanning Trees

- Any tree consists of solely of edges in E(G) and including all vertices of V(G)
  - Number of tree edges is n-1.
  - Add a non-tree edge will create a cycle

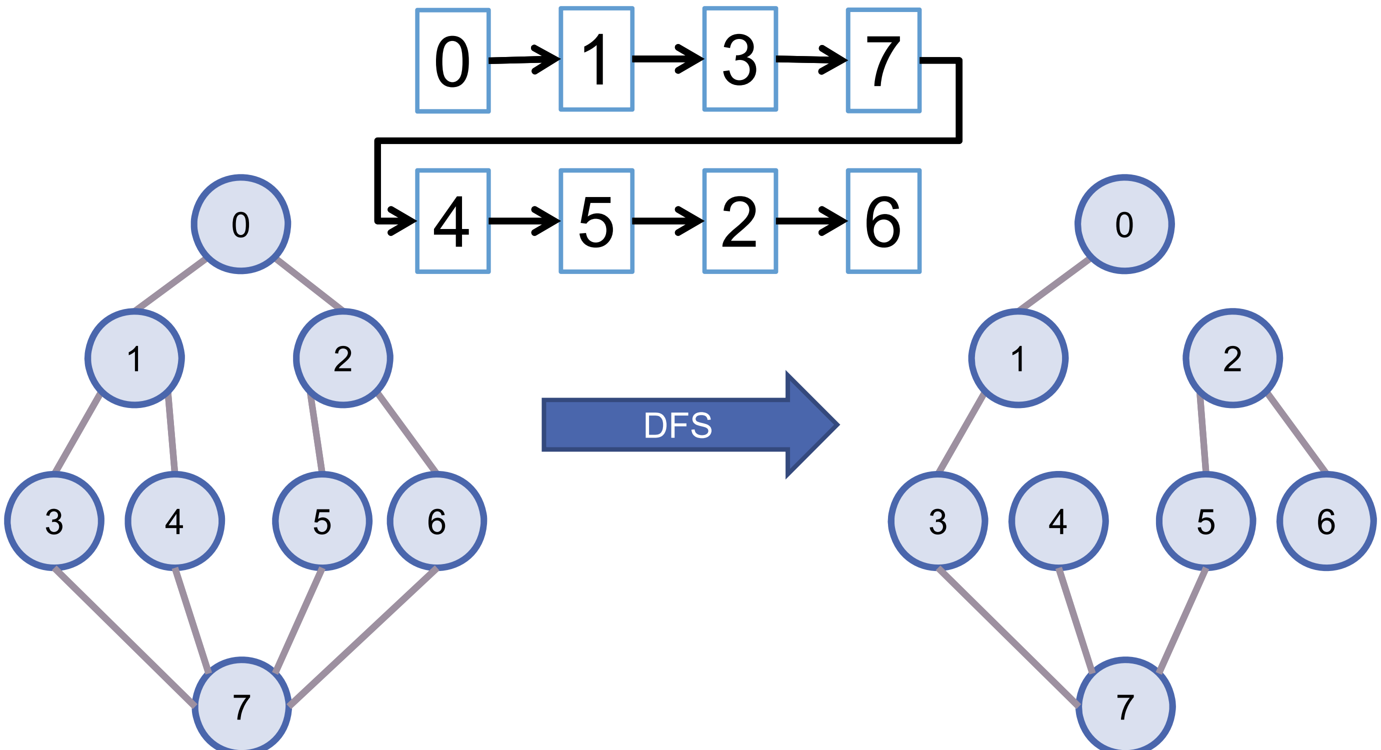


Complete graph

Possible spanning trees

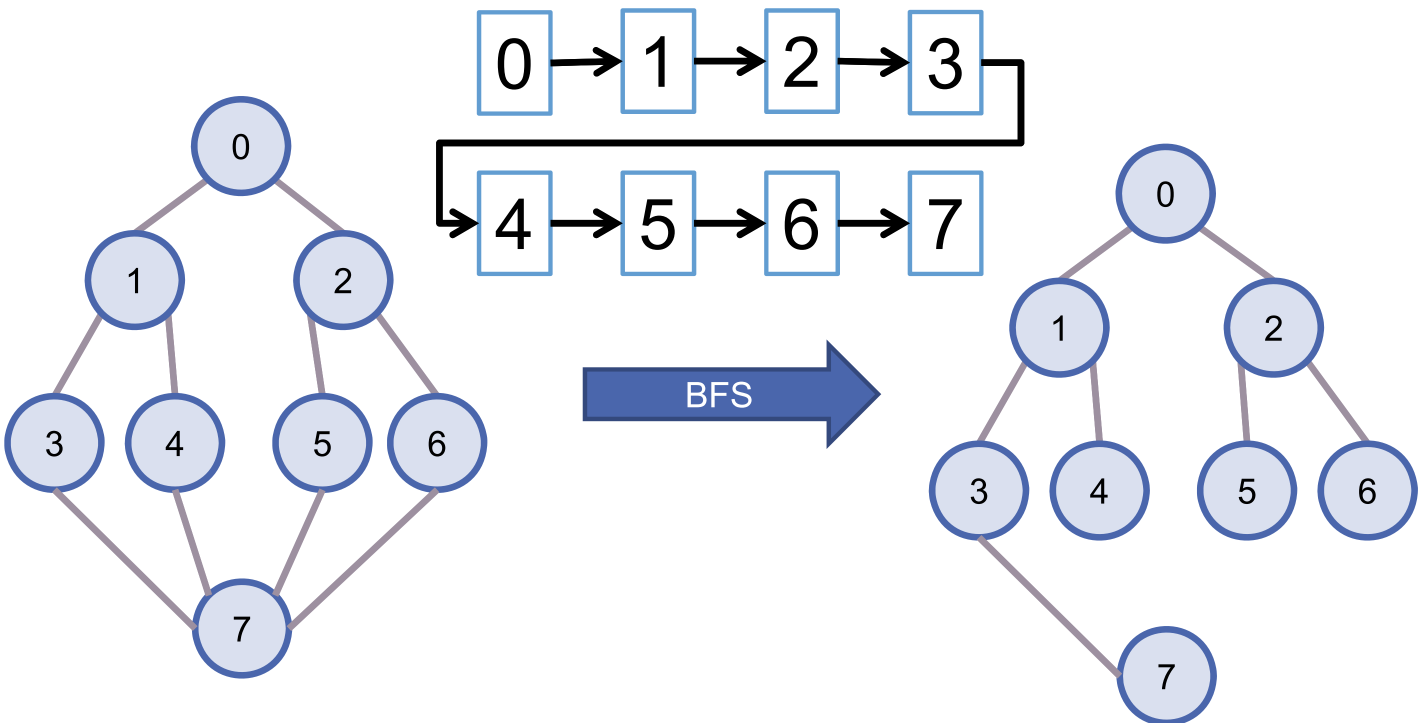
DFS Spanning Tree

- Tree edges are those edges met during the traversal



BFS Spanning Tree

- Tree edges are those edges met during the traversal



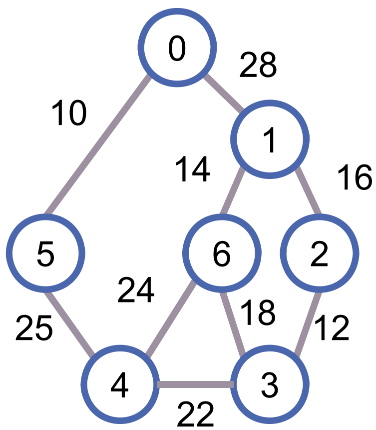
Self-Study Topics

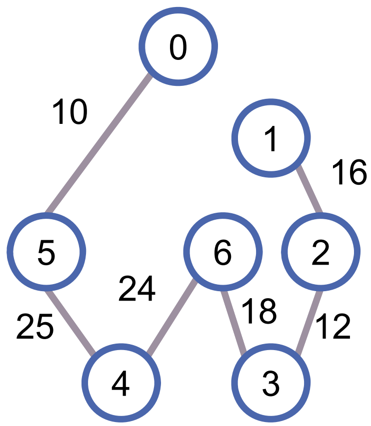
- Graph representations
  - Sequential lists
  - Adjacency multilists
- Graph operation
  - Biconnected components

Minimum-Cost Spanning Trees

Minimum-Cost Spanning Trees

- For a weighted undirected graph
  - Find a spanning tree with least cost of the sum of the edge weights
- Three greedy algorithms:
  - Kruskal's algorithm
  - Prims's algorithm
  - Sollin's Algorithm

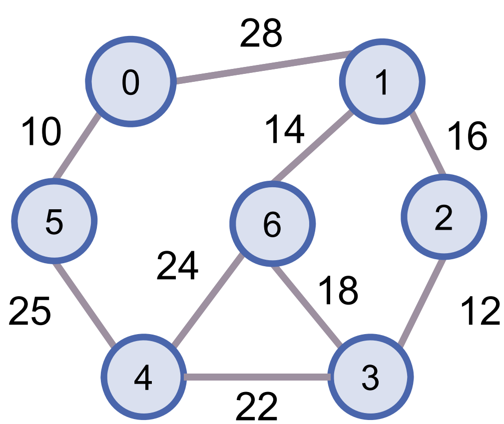




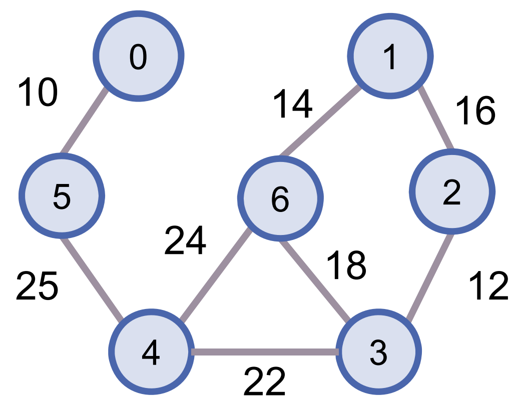
Spanning tree
with cost 105

Kruskal's Algorithm

- Idea: Add edges with minimum cost one at a time
  - Step 1: Find an edge with minimum cost
  - Step 2: If it creates a cycle, discard the edge
  - Step 3: Repeat step 1 and 2 until we find n-1 edges



Connected graph



Spanning tree with cost 99

Kruskal's Algorithm

```
Kruskal's algorithm
1.  T = ϕ
2.  While((T contains less than n-1 edges)&&(E is not empty)){
3.    choose an edge (v,w) from E of lowest cost;
4.    delete (v,w) from E
5.    if((v,w) does not create a cycle) add (v,w) to T;
6.    else discard (v,w)
7.  }
8.  If(T contains less than n-1 edges)
9.    cout << "there is no spanning tree!" <<endl;
```

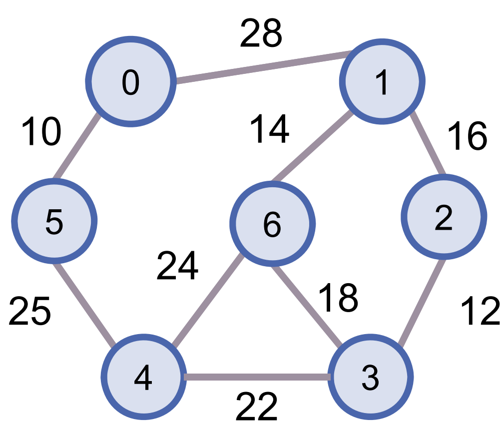
- Step 3 & 4: use **min heap** to store edge cost.
- Step 5: use **set representation** to group all vertices in the same connected component into a set.
  - For an edge (v,w) to be added, if vertices are in the same set, discard the edge, else merge two sets

Time Complexity

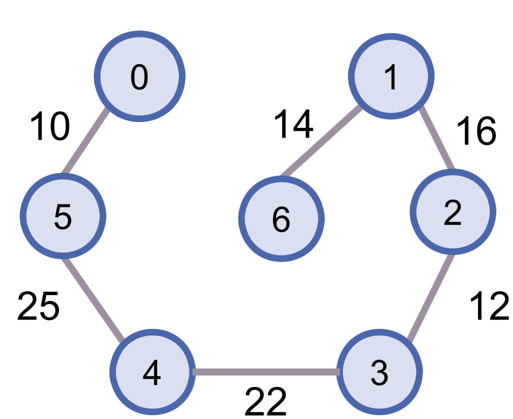
- Min heap:
  - Step 3&4 : O(log e)
- Set:
  - Step 5: O(a(e))
- At most execute e-1 rounds:
  - $(e-1)\cdot(\log e + a(e)) = O(e \log e)$

Prim's Algorithm

- Idea: Add edges with minimum edge weight to tree
  - The set of selected edges form a tree
  - Step 1: Start with a tree T contains a single arbitrary vertex
  - Step 2: Add a least cost edge (u,v) to T, $T \cup (u, v)$ is still a tree
  - Step 3: Repeat step 2 until T contains n-1 edges



Connected graph



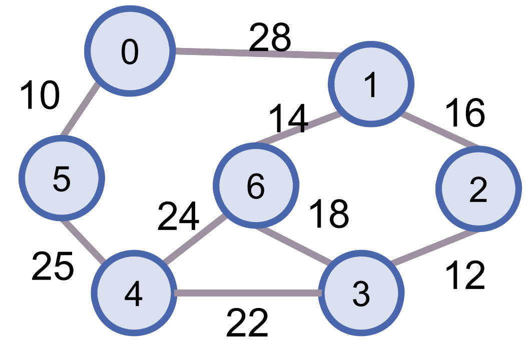
Spanning tree with cost 99

Prim's Algorithm

```
Prim's algorithm
1. V(T) = {0} // start with vertex 0
2. for(T=φ ; T contains less than n-1 edges; add (u,v) to T){
3.   Let (u,v) be a least cost edge such that u⊆V(T) and v⊄V(T);
4.   if(there is no such edge) break;
5.   add v to V(T);
6. }
7. If(T contains fewer than n-1 edges)
8.    cout << "there is no spanning tree!" <<endl;
```

- Step 3: use a **near-to-tree** data structure
  - Create an array to record the nearest distance of vertices to T
  - Only vertices not in V(T) and adjacent to T are recorded

Running Example



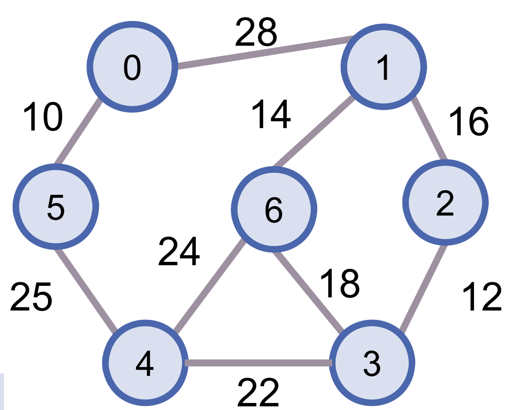
near-to-tree	0	1	2	3	4	5	6
V(T)={0}	*	28	∞	∞	∞	10	∞
V(T)={0,5}	*	28	∞	∞	25	*	∞
V(T)={0,5,4}	*	28	∞	22	*	*	24
V(T)={0,5,4,3}	*	28	12	*	*	*	18
V(T)={0,5,4,3,2}	*	16	*	*	*	*	18
V(T)={0,5,4,3,2,1}	*	*	*	*	*	*	14
V(T)={0,5,4,3,2,1,6}							

Time Complexity

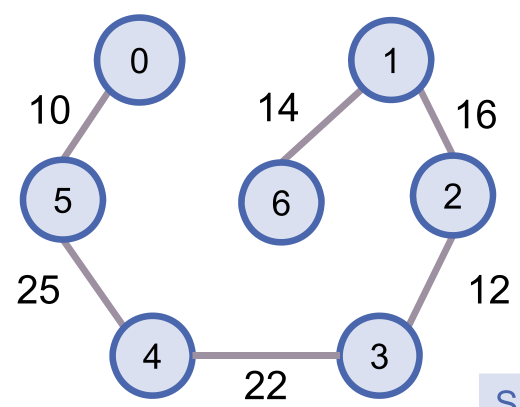
- Near-to-tree
  - Step 3 : O(n)
- At most execute n rounds: $O(n^2)$

Sollin’s (Borůvka’s) Algorithm

- Idea: Select several edges at each stage
  - Step 1: Start with a forest that has n spanning trees
  - Step 2: Select one minimum cost edge for each tree
  - Step 3: Delete multiple copies of selected edges and if two edges with the same cost connecting two trees, keep only one of them
  - Step 4: Repeat until we obtain only one tree.



Connected graph



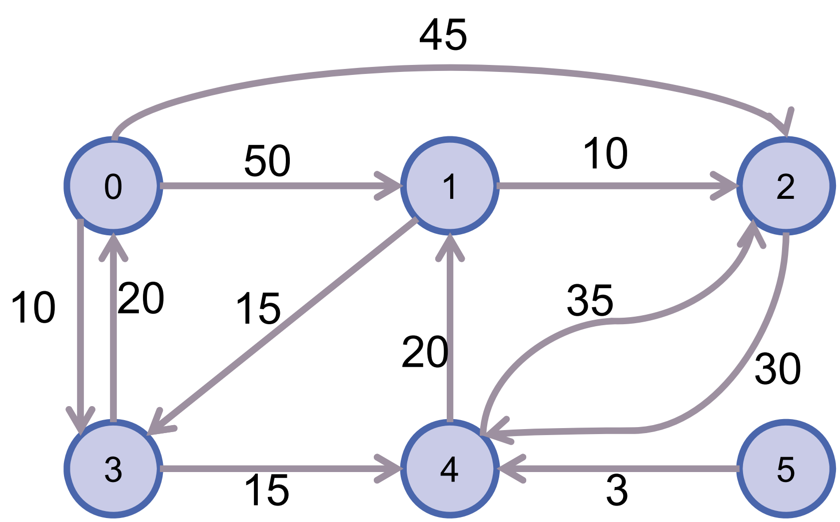
Spanning tree with cost 99

Shortest Paths

[redacted]

Single Source Shortest Paths

- Single source/all destinations problem
  - Given a digraph with nonnegative edge costs and a source vertex *v*, compute a shortest path from *v* to each of the remaining vertices



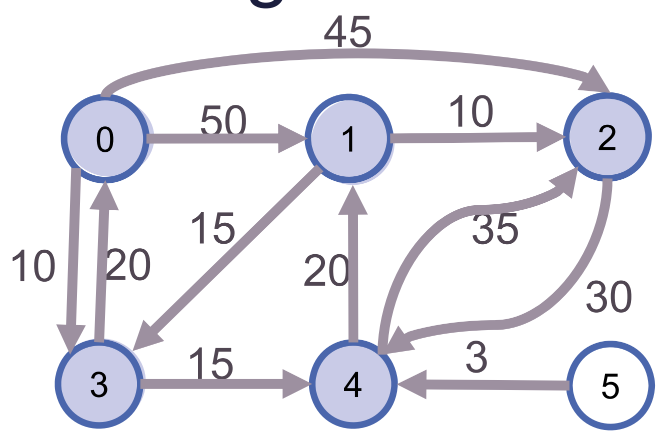
Shortest Paths From 0

Path	Length
1) 0, 3, 4, 1	45
2) 0, 2	45
3) 0, 3	10
4) 0, 3, 4	25

Dijkstra's Algorithm

- Use a set S:
  - Store the vertices whose shortest path have been found
- An array *dist*:
  - Store the shortest distances from source v to all visited vertices
  - When a new vertex *w* is visited, update *dist*
    - $dis[w] = min(dist[u] + length(\langle u, w \rangle), dist[w])$
    - u is the previously visited vertex adjacent to w

Example for Dijkstra's Algorithm



Visited	0	1	2	3	4	5
{0}	0_0	50_0	45_0	10_0	∞	∞
{0, 3}	0_0	50_0	45_0	10_0	25_3	∞
{0, 3, 4}	0_0	45_4	45_0	10_0	25_3	∞
{0, 3, 4, 1}	0_0	45_4	45_0	10_0	25_3	∞
{0, 3, 4, 1, 2}	0_0	45_4	45_0	10_0	25_3	∞

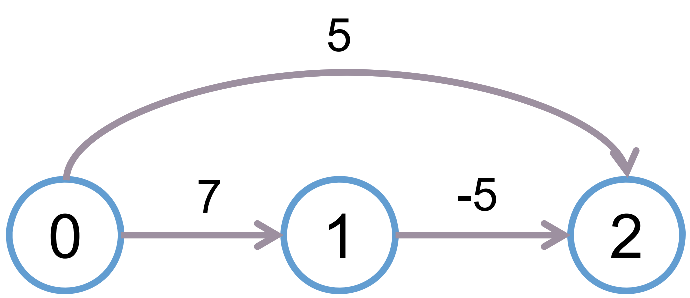
Dijkstra's Algorithm

```
void MatrixWDigraph::ShortestPath(const int n, const int v)
{ // dist[j], 0 ≤ j < n, stores the shortest path from v to j
  // length[i][j] stores length of edge <i, j>
  for(int i=0; i<n; i++){ s[i]=false; dist[i]=length[v][i];}
  s[v] = true;
  dist[v] = 0;
  // find n - 1 paths starting from v
  for(int i=0; i<n-1 ;i++){
    // Choose a vertex u, such that dist[u]
    // is minimum and s[u] = false
    int u = Choose(n);
    s[u] = true;
    for(int w=0; w<n; w++)
    if(!s[w] && dist[u] + length[u][w] < dist[w])
       dist[w] = dist[u] + length[u][w];
   } // end of for (i = 0; ...)
 }
```

Time complexity: $O(n^2)$

Digraph with Negative Costs

- This algorithm can apply to digraph with negative cost edges
  - Restriction: The digraph MUST NOT contain cycles of negative length



Digraph with a negative cost edge

All-Pairs Shortest Paths

- Apply single source shortest path to each of n vertices
- Floyd-Warshall's algorithm
  - Dynamic programming approach

Dynamic Programming

- Divide-and-conquer approach
- Usually applied to optimization problems
- Improve the inefficient problems
  - The same recursive call is called repeatedly

Dynamic Programming

- Divide-and-conquer approach
- Usually applied to optimization problems
- Improve the inefficient problems
  - The same recursive call is called repeatedly
- Used when sub-problems share sub-sub-problems
- “Programming” refers to a tabular method
  - Remember the solutions
- Compute solution in a bottom-up fashion

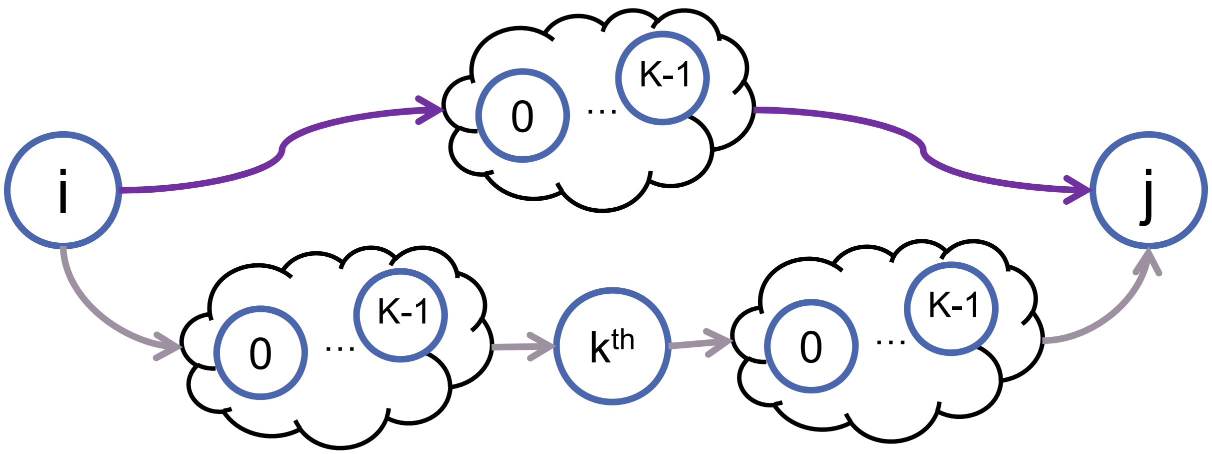
All-Pairs Shortest Paths

- Apply single source shortest path to each of n vertices
- Floyd-Warshall's algorithm
  - Dynamic programming approach
  - $A^{-1}[i][j]$: the length[i][j]
  - $A^{k}[i][j]$: the length of the shortest path from i to j going through no intermediate vertex of index greater than k
  - $A^{k}[i][j] = \min\{A^{k-1}[i][j], A^{k-1}[i][k] + A^{k-1}[k][j]\}$, $k \geq 0$
  - $A^{n-1}[i][j]$: the length of the shortest i-to-j path in G

Intuition of Floyd-Warshall's Algorithm

- There are only two possible paths for $A^k[i][j]$
  - The path dose not pass the k^{th} vertex
  - The path dose pass the k^{th} vertex

- $A^k[i][j] = \min\{A^{k-1}[i][j], A^{k-1}[i][k] + A^{k-1}[k][j]\}, k \geq 0$



Floyd-Warshall's Algorithm

```
void MatrixWDigraph::AllLengths(const int n)
{// length[n][n] stores edge length between
 // adjacent vertices
 // a[i][j] stores the shortest path from i to j
 for (int i = 0; i<n; i++)
  for (int j = 0; j<n; j++)
      a[i][j]= length[i][j];

 // path with top vertex index k
 for (int k= 0; k<n; k++)
  // all other possible vertices
  for (int i= 0; i<n; i++)
   for (int j= 0; j<n; j++)
    if((a[i][k]+a[k][j])<a[i][j])
      a[i][j] = a[i][k] + a[k][j];
}
```

Floyd-Warshall's Algorithm

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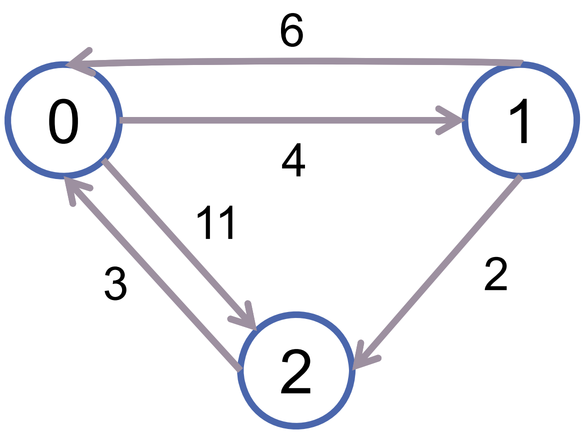
    if((a[i][k]+a[k][j])<a[i][j])
      a[i][j] = a[i][k] + a[k][j];
}
```

$A^0 \sim A^{n-1}$

Example of Floyd-Warshall's Algorithm



A^2	**0**	**1**	**2**
0	0	4	6
1	5	0	2
2	3	7	0

Floyd-Warshall's Algorithm

```
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 for (int k= 0; k<n; k++)
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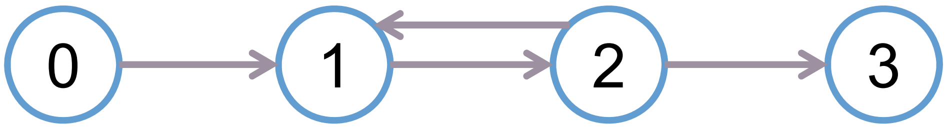
  for (int i= 0; i<n; i++)
   for (int j= 0; j<n; j++)
    if((a[i][k]+a[k][j])<a[i][j])
      a[i][j] = a[i][k] + a[k][j];
}
```

Time complexity: $O(n^3)$

Transitive Closure

- Determine if there is a path from i to j in a graph with unweighted edges
  - Only positive path lengths → transitive closure
  - Only nonnegative path lengths → reflexive transitive closure
- The transitive closure matrix A^+:
  - $A^+[i][j] = 1$ if there is a path of length > 0 from i to j in the graph; otherwise, $A^+[i][j] = 0$
- The reflexive transitive closure matrix A^*:
  - $A^*[i][j] = 1$ if there is a path of length >= 0 from i to j in the graph; otherwise, $A^*[i][j] = 0$
- Use Floyd-Warshall's algorithm!
  - $A^k[i][j] = A^{k-1}[i][j]$ || ($A^{k-1}[i][k]$ && $A^{k-1}[k][j]$);

Transitive Closure Example



A^+	0	1	2	3
0	0	1	1	1
1	0	1	1	1
2	0	1	1	1
3	0	0	0	0

Transitive closure matrix

A^*	0	1	2	3
0	1	1	1	1
1	0	1	1	1
2	0	1	1	1
3	0	0	0	1

Reflexive transitive closure matrix

Activity Networks

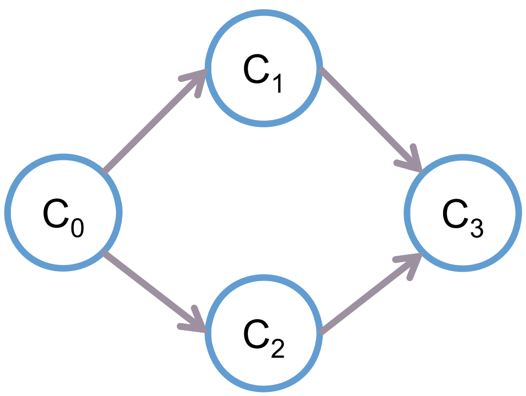
Activities

- Projects can be subdivided into several subprojects
  - Each subproject is called activity
  - The completion of activities $\rightarrow$ the completion of the project

- Activity-on-Vertex (AOV) Networks
  - A digraph G with the vertices represent tasks or activities and the edges represent precedence relations between tasks
  - Predecessor: Vertex i is a predecessor of vertex j, iff there is a directed path from vertex i to vertex j

AOV: Topological order

- A **linear ordering** of the vertices of a graph:
  - For any two vertices i and j, if i is a predecessor of j in the network, then i precedes j in the linear ordering



C_0 -> C_1 -> C_2 -> C_3 ( )

C_0 -> C_2 -> C_1 -> C_3 ( )

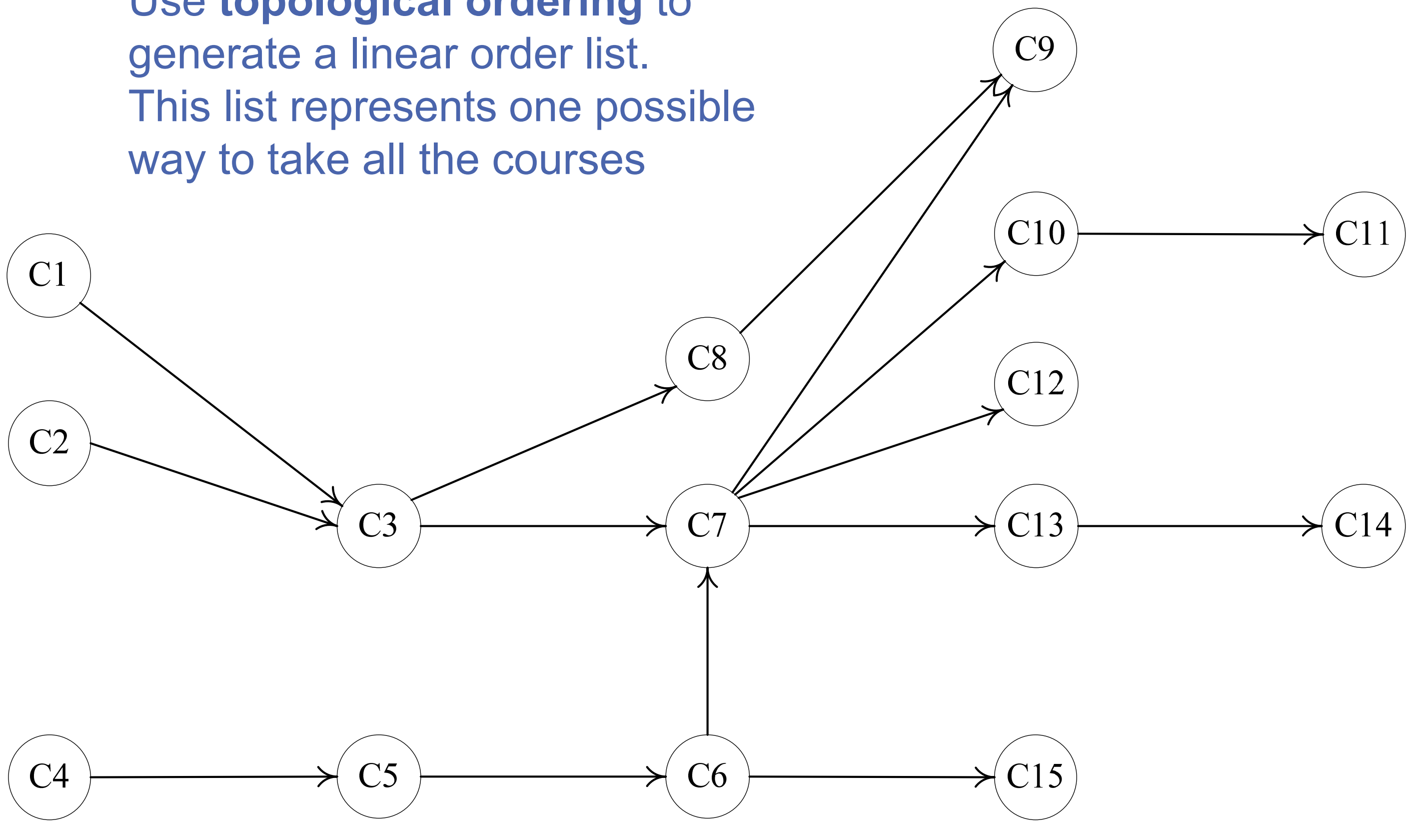
C_0 -> C_2 -> C_3 -> C_1 ( )

Courses Need for CS Degree

Course No.	Course	Prerequisites
C1	Programming I	None
C2	Discrete Mathematics	None
C3	Data Structures	C1, C2
C4	Calculus I	None
C5	Calculus II	C4
C6	Linear Algebra	C5
C7	Analysis of Algorithms	C3, C6
C8	Assembly Language	C3
C9	Operating Systems	C7, C8
C10	Programming Languages	C7
C11	Compiler Design	C10
C12	Artificial Intelligence	C7
C13	Computational Theory	C7
C14	Parallel Algorithms	C13
C15	Numerical Analysis	C5

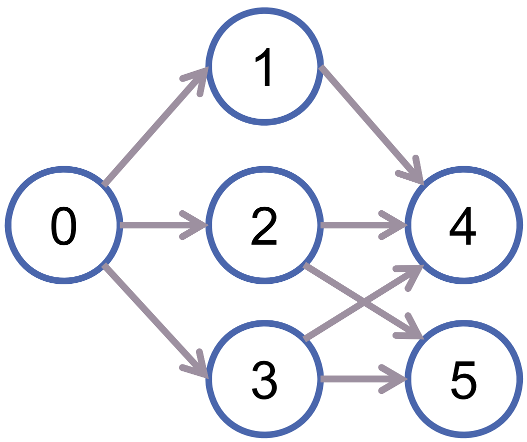
AOV Network of Courses

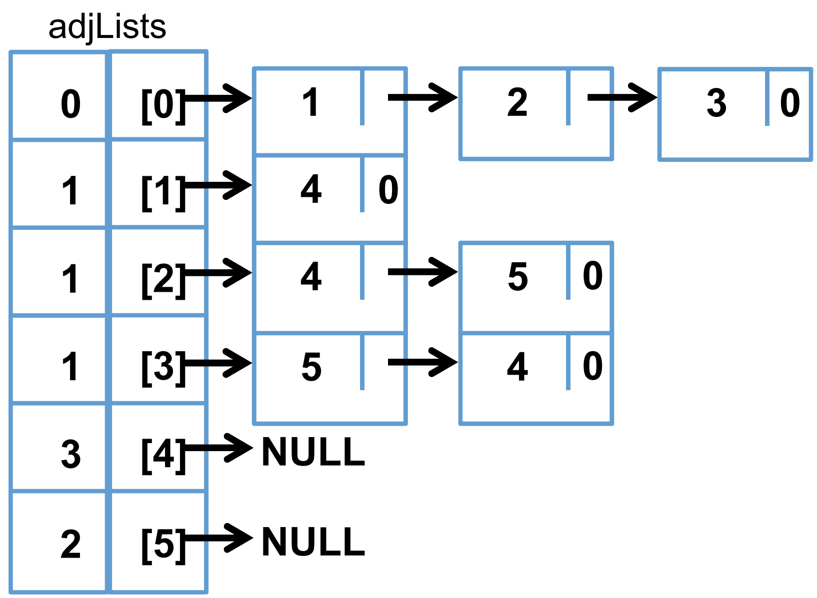
Use **topological ordering** to generate a linear order list. This list represents one possible way to take all the courses



Topological Ordering

- Iteratively pick a vertex v that has no predecessors
  - Use a field "count" to record the "in-degree" value of each vertex





Self-Study Topics

- Single source shortest path
  - Bellman-Ford's algorithm (Digraph with negative edge costs)
- Activity-on-Edge (AOE) Networks
  - Critical path analysis