Fractal Tangency

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Abstract

Fractals are fascinating objects, but are still considered as 'pathological' because most of the time, analytical techniques do not work on those. For example, the well-known "inscribed square problem" is still unsolved for most fractal-like curves [2]. In this short note we attempt to add something else to say to fractals by carefully distinguishing them from 'smooth' curves. For now, this study is restricted to the continuum, but we provide some ideas for generalization to other sets.

1 Intuition

For a start, paths to a point are enough to determine 'fractality'. For example, we can say that a surface is 'fractal' if a path passing through a point inside that surface is fractal. When restricted to continuous curves in the continuum ($\mathbb{R}^n$), fractals can be thought of as exactly the nowhere differentiable functions. We somewhat follow that line of thought, but please remember a definition of the derivative: that it is the slope of the *tangent line*. This means that differentiable curves will eventually look like a straight line when 'zoomed in' to a point.

On the other hand, fractals are often viewed as 'self-similar' (although it should not always be [1]!). Now observe that a straight line looks like itself when zoomed in. Thus, straight line is self-similar. In a sense, the straight line is to fractals as 0 is to positive natural numbers!

Since the derivative already captures *tangency*, we can modify this to define *fractal tangency*, and distinguish fractals from 'smooth' curves. The following sections just develop a formalization to this idea.

2 Definitions

Let C_n (for $n \geq 1$) be the set of all continuous functions $f : [0, 1] \to \mathbb{R}^n$ such that $f(0) = (0)^n$ and $f(x) \neq (0)^n, \forall x \neq 0$, where $(0)^n$ is the origin on $\mathbb{R}^n$. Aside from the Cartesian coordinates, we make use of a variant of *Hyper-spherical coordinates* (we shorten it as *HSC* here). Let the set of HSCs be $S_n = \mathbb{R} \times [0, \pi]^{n-1}$ ($S_1 = \mathbb{R}$).

Definition 2.1. *Let $T_n : S_n \to \mathbb{R}^n$ (again, for $n \geq 1$) be the transformation from Hyper-spherical coordinates back to Cartesian coordinates such that:*

- $T_1(r) = r$.
- $T_{n+1}(r, \theta_1, \dots, \theta_n) = (\sin\theta_n \cdot T_n(r, \theta_1, \dots, \theta_{n-1}), |r| \cos\theta_n)$.

Exercise 2.1 (aka author is lazy)**.** *Prove that T_n is bijective except at origin either by explicit construction of T_n^{-1}, or by showing surjection and injection. Please utilize induction if possible.*

Definition 2.2. ***Quasi-multiplication on HSC*** *is the binary operation $\cdot : S_n \times S_n \to S_n$ such that*

$$(r, \theta_1, \dots, \theta_{n-1}) \cdot (s, \phi_1, \dots, \phi_{n-1}) = (rs, \pi - |\pi - (\theta_1 + \phi_1)|, \dots, \pi - |\pi - (\theta_{n-1} + \phi_{n-1})|).$$

Definition 2.3. ***Quasi-division on HSC*** *is the binary operation $/ : S_n \times S_n \to S_n$ such that*

$$(r, \theta_1, \dots, \theta_{n-1})/(s, \phi_1, \dots, \phi_{n-1}) = (r/s, |\theta_1 - \phi_1|, \dots, |\theta_{n-1} - \phi_{n-1}|).$$

Notice the operations on angles. The idea is that if the sum/difference of angles is outside of $[0, \pi]$, we put it back to $[0, \pi]$ by 'remeasuring' it from angle 0.

3 Equivalence Relation

Now we can define the main concept to distinguish the rough from the smooth:

Definition 3.1. *Two continuous paths to the origin $f, g \in C_n$ have the 'same tangency at origin' (we notate as $f \sim_n g$) if there exists a 'hyper-rotation' $r = (1, \theta_1, \ldots, \theta_{n-1}) \in S_n$ such that*

$$\lim_{x \to 0^+} \frac{r \cdot T_n^{-1}(f(x))}{T_n^{-1}(g(x))} = (1, (0)^{n-1}).$$

Theorem 3.1. *$\sim_n$ is an equivalence relation on C_n.*

Proof. For reflexivity, the r for $f \sim_n f$ would be $(1, (0)^{n-1})$. For symmetry, given the r for $f \sim_n g$, the r' for $g \sim_n f$ would be $(1, (0)^{n-1})/r$. Lastly for transitivity, given the r for $f \sim_n g$ and r' for $g \sim_n h$, the r'' for $f \sim_n h$ would be $r \cdot r'$. □

Corollary 3.2. *$\sim_n$ partitions C_n into the set of equivalence classes $C_n / \sim_n$.*

One such class is the class L_n of 'smooth' paths to a point, or more informatively, the class of paths with the *tangent line* at origin.

Definition 3.2. *For a continuous path to the origin $f \in C_n$, $f \in L_n$ iff $f \sim_n (x, (0)^{n-1})$.*

Now we can finally distinguish 'fractals', and define the title of this note:

Definition 3.3. *A continuous path to the origin $f \in C_n$ has **fractal tangency** at origin iff $f \notin L_n$.*

4 Other Ideas

Here are some other ideas/comments for further study of the concepts outlined in previous sections:

- The limit in Definition 3.1 can be simplified to

  $$\lim_{x \to 0^+} \frac{T_n^{-1}(f(x))}{T_n^{-1}(g(x))} = (1, \theta_1, \ldots, \theta_{n-1}).$$

  This suggests a weakening of the equivalence relation: the limit should now be $(r, \theta_1, \ldots, \theta_{n-1})$ for any $r \neq 0$. However, at the time of writing, we do not have a good 'geometric' interpretation for this.

- Of course there are fractals beyond C_n like the well-known Sierpiński carpet! The first idea is that paths on *sets* can be defined as $f : [0, 1] \to 2$. Fortunately there are no notions of continuity, HSC, and hyper-rotation anymore. Then the next idea is to modify the ϵ part of the ϵ-δ definition: instead of $|f(x) - L| < \epsilon$, we simply have $f(x) \oplus L = 0$, where $\oplus$ is the XOR operation. Now the only perfectly 'smooth' paths are $f_0(x) = 0$ and $f_1(x) = 1$, for all $x \in [0, 1]$. The last idea is to define a suitable (quasi-)multiplication and (quasi-)division: maybe the XOR operation can also be used for that. Hopefully we can now define the analogue for Definition 3.1. These ideas should generalize to other sets aside from $\mathbb{R}^n$ and 2.

References

[1] 3Blue1Brown. Fractals are typically not self-similar. `https://www.youtube.com/watch?v=gB9n2gHsHN4`.

[2] Mark J. Nielsen. Figures inscribed in curves: A short tour of an old problem. `https://www.webpages.uidaho.edu/~markn/squares/`.