Ackermann Set Theory and ZF

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Abstract

Zermelo-Fraenkel Set Theory (ZF) is considered to be the "standard" set theory for foundation of mathematics. Ackermann proposed an alternative set theory in [1], which is now called *Ackermann Set Theory* (AST). An interesting well-known result is that ZF and a little modified AST is somewhat "the same." In this note we present that variant of AST, the equivalence result, and its significance.

1 That Variant of AST

We will now describe here a variant of Ackermann Set Theory we denote as A^*, which has differences with the original in [1] in that there is one axiom removed and one axiom inserted. A^* is formulated in first-order logic with equality and with a constant V which is interpreted as the set universe, and a binary relation $\in$ which is interpreted as the usual membership relation. Here are the axioms of A^*:

1. (Axiom of Extensionality)
$$\forall z(z \in x \leftrightarrow z \in y) \to x = y.$$
2. (Ackermann's Schema) Let $\phi(y, z_1, \ldots, z_n)$ be any first-order formula without V and x as free variables. If $x_1, \ldots, x_n \in V$, then
$$\forall y(\phi(y, z_1, \ldots, z_n) \to y \in V) \to \exists x(x \in V \land \forall y(y \in x \leftrightarrow \phi(y, z_1, \ldots, z_n))).$$
3. (Strong Completeness Axioms for V)
$$(x \in y \lor x \subseteq y) \land y \in V \to x \in V.$$
where $\subseteq$ is the usual subset relation, defined as $x \subseteq y \leftrightarrow \forall z(z \in x \to z \in y)$.
4. (Axiom of Regularity)
$$x \in V \land \exists y(y \in x) \to \exists y(y \in x \land \forall z(z \in x \to \neg z \in y)).$$

The one axiom removed is called the *Class Construction Axiom Schema* which is just the Axiom Schema of Separation for V, and the one axiom inserted is the Axiom of Regularity above.

2 The Equivalence Result, and its Significance

[WIP] [2] [3]

The main reason for writing this whole note is because this result basically means that for *well-founded* sets, all *unique* set constructions that are

1. definable as a first-order sentence (hence, are "finite"), and
2. "universe-agnostic" (since they do not mention V),

are *exactly* the ones definable using ZF! Informally, for unique well-founded set constructions,

$$\text{“Nice and Finite”} = \text{Constructed using ZF}.$$

Let's consider the Axiom of Choice. [WIP]

Lastly, it is this focus on set constructions instead of proper classes that we decided to remove the Class Construction Axiom Schema for our A^*. Note that the Axiom of Separation immediately follows from Ackermann's Schema by setting ϕ to $y \in a \land \varphi$ for $a \in V$.

3 Proofs

[WIP; will take a long time because I am not much knowledgeable in Set Theoretic methods.]

References

[1] Wilhelm Ackermann. Zur axiomatik der mengenlehre. *Mathematische Annalen*, 131(4):336–345, Aug 1956.

[2] Azriel Lévy. On ackermann's set theory. *Journal of Symbolic Logic*, 24(2):154–166, 1959.

[3] William N. Reinhardt. Ackermann's set theory equals zf. *Annals of Mathematical Logic*, 2(2):189–249, 1970.