My Pet Set Theory: $\mathsf{AST{+}NWF}$

poypoyan

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Abstract

In this note we present a variant of Ackermann Set Theory (AST) with easy construction of non-well-founded sets (NWF). Motivations for using up our time for this are also laid out.

1 The Theory

We will now describe a variant of Ackermann Set Theory [1] we denote as $\mathsf{AST{+}NWF}$. $\mathsf{AST{+}NWF}$ is formulated in first-order logic with equality and with a constant V which is interpreted as the set universe, and a binary relation $\in$ which is interpreted as the usual membership relation.

Definition 1.1 (Super-Completeness of V)**.** *Let $SC(V)$ be the statement*

$$\forall x, y(x \in V \land (y \in x \lor y \subseteq x) \to y \in V)$$

where $\subseteq$ is the usual subset relation, defined as $x \subseteq y \leftrightarrow \forall z(z \in x \to z \in y)$.

Now here are the axioms of $\mathsf{AST{+}NWF}(V)$:

1. Axiom of Extensionality $\mathsf{Ext}(V)$:

$$\forall x, y(x \in V \land y \in V \land \forall z(z \in x \leftrightarrow z \in y) \to x = y).$$

2. Ackermann Schema $\mathsf{Ack}(V)$: Let $\phi(y, z_0, \ldots, z_{n-1})$ be a unary first-order formula where all the constants $z_0, \ldots, z_{n-1} \in V$ and $\neq V$. Then

$$(SC(V) \to \forall y(\phi(y, \ldots) \to y \in V)) \to \exists x(x \in V \land \forall y(y \in x \leftrightarrow \phi(y, \ldots))).$$

3. **Non-well-founded** Ackermann Schema $\mathsf{NWFAck}(V)$: Let $\phi(y, z_0, \ldots, z_{n-1})$ be a unary first-order formula where all the constants $z_0, \ldots, z_{n-1} \in V$ and $\neq V$, and it is of the form

$$\forall x_0, \ldots((\forall y_0(y_0 \in x_0 \leftrightarrow \phi(y_0, \ldots_0)) \land \ldots) \to \ldots)$$

where all those instances of ϕ[1] are also *defined*, and in the last ellipsis, the x's only occur in equality relation (e.g., $y = x_0$). Then if

$$(\bigwedge_i \forall y_i(\phi(y_i, \ldots_i) \to y_i \in V)) \to (SC(V) \to \forall y(\phi(y, \ldots) \to y \in V)),$$

then

$$\exists x(x \in V \land \forall y(y \in x \leftrightarrow \phi(y, \ldots))) \\ \land \forall y, c_0, \ldots, c_{n-1}((\phi(y, z_0, \ldots, z_{n-1}) \leftrightarrow \phi(y, c_0, \ldots, c_{n-1})) \to \bigwedge_{j=0}^{n-1} c_j = z_j).$$

[1] Yes, this is a recursive definition!

2 Motivations

There are several motivations for the "design" of the theory.

The first motivation is the focus on *set constructions* instead of sets and/or proper classes. Because of this, we decided to:

- Adopt AST in the first place. The Ackermann schema captures the idea that "natural"/"uncontroversial" set constructions are 1) definable as a first-order sentence (hence, are "finite"), and 2) "universe-agnostic" (since those do not mention V). This reminds us while studying infinities that our "full" descriptions of objects are always finite.

  On the contrary, consider the Axiom of Choice (AC), a well-known axiom independent to ZF. The sets it constructs are not unique, hence it is said to postulate existence of sets without defining it, unlike other ZF (set construction) axioms [2, Chapter 5]. However, if we just permit "lengths" of any ordinal to ϕ in Ackermann Schema, AC can now "produce" unique sets again! To see this, let S be a set of sets, then an appropriate ϕ for AC would simply be

  $$y = a_0 \vee y = a_1 \vee y = a_2 \vee \ldots$$

  where $a_0 \in s_0, a_1 \in s_1, a_2 \in s_2, \ldots$ and $S = \{s_0, s_1, s_2, \ldots\}$. Nevertheless, stronger axioms like Choice can be added to our theory via relativization to V wherein every instance of $\forall x \varphi$ in the statement is replaced by $\forall x(x \in V \rightarrow \varphi)$ and every instance of $\exists x \varphi$ is replaced by $\exists x(x \in V \wedge \varphi)$.

- Remove the Class Construction Schema for our AST+NWF. Note that Class Construction Schema *is* Separation Schema for V and that Separation Schema immediately follows from Ackermann Schema by setting ϕ to $y \in a \wedge \varphi$ for $a \in V$.

The next motivation is to make the universe(s) "as closed as possible". Because of this, we decided to:

- Restrict Extensionality to sets ($x \in V$) only.

- Put super-completeness inside Ackermann schemas instead of it being an axiom on its own. This is done so that when we work on multiple universes $V_0, V_1, \ldots$ such that $V_0 \in V_1 \in \ldots$, and $\mathsf{AST}+\mathsf{NWF}(V_0) \wedge \mathsf{AST}+\mathsf{NWF}(V_1) \wedge \ldots$, no additional sets in V_0 will be shown to exist through the higher universes (i.e., $V_1, \ldots$).

Now the Non-well-founded Ackermann Schema is formulated for easy contruction of non-well-founded sets (obviously), but it is a beast! Nevertheless, the intuition is actually simple:

> If instances of ϕ are used to define itself, and we assume that all that satisfy those instances of ϕ are all sets, then ϕ can indeed construct a set, and the constructed set is *unique through the given parameters*.

For example, to construct a set such that $x = \{x, a\}$ where $a \in V$, $\phi(y, a)$ is defined as

$$\forall x_0(\forall y_0(y_0 \in x_0 \leftrightarrow \phi(y_0, a)) \rightarrow y = x_0 \vee y = a).$$

Notice that this satisfies the prescribed form for ϕ. This is motivated by what we'll call the *black box principle*: elements and subsets of a non-well-founded set should *never* be used to build that set, for two reasons: 1) to prevent constructing properties "so loose" that well-founded sets can satisfy it (i.e., ω is a set such that $\bigcup x = x$), and 2) to prevent inconsistency (i.e., a set such that $2^x = x$ contradicts Cantor's theorem). Next, the statement "all those instances of ϕ are also *defined*" means that $x(\emptyset) = \{x(\{\emptyset\}), x(\emptyset)\}$ for example, will not be defined unless $x(\{\emptyset\})$ is defined first.

The last highlight in the above quote, the statement "unique through the given parameters", needs explanation. Our main motivation for this is the "intuitive" set-theoretic definition of ordered pair $(x, y) = \{(0, x), (1, y)\}$. For this to satisfy the ordered pair property $(x_0, y_0) = (x_1, y_1) \leftrightarrow x_0 = x_1 \wedge y_0 = y_1$ in our theory, the sets $(0, x)$ and $(1, y)$ should *always* be unequal. Since by the Opacity Principle we cannot "view inside" ϕ, there is freedom on equalities of elements across non-well-founded sets. Hence the statement can be seen as either a "cheat", or the most natural generalization of the always inequality between $(0, x)$ and $(1, y)$.

Lastly, the nature of recursive defining of ϕ in the schema needs further consideration. Maybe AST+NWF is not a first-order theory anymore because recursive definition of predicates causes its signature to have "incrementing" quantity of predicates.

3 What About Replacement?

All set construction axioms of ZF (relativized to V) are straightforward to prove in AST, except the Axiom Schema of Replacement. When Reinhardt in [3] proved that ZF can be derived from the original AST plus Regularity, the whole proof (which we don't understand) really boils down to proving Replacement! Regularity is used in the proof, hence that work only applies to well-founded sets.

Here's the Replacement Schema, relativized to V:

$$\forall w(w \in a \to \exists! y(y \in V \land \varphi(w, y, \ldots))) \to \exists x(x \in V \land \forall y(y \in x \leftrightarrow y \in V \land \exists w(w \in a \land \varphi(w, y, \ldots))))$$

where $a \in V$ and the other free variables of φ are either elements of V *or equal to* V. What about our variant? We'll leave this as a conjecture/"to-do" for now. We lean towards "no" (our theory cannot prove Replacement), but we haven't found a counterexample either.

Nonetheless, we have a "weaker" variant of Replacement at home:

$$\begin{aligned}&\forall w(w \in a \to \exists y(y \in V \land \forall v(v \in y \leftrightarrow \varphi(v, w, \ldots))))\\ &\qquad\qquad \to \exists x(x \in V \land \forall y(y \in x \leftrightarrow \exists w(w \in a \land \forall v(v \in y \leftrightarrow \varphi(v, w, \ldots)))))\end{aligned}$$

where $a \in V$ and the other free variables of φ are always elements of V *and not equal to* V. The relation between this schema and the original Replacement is akin to the relation between the first-order induction schema and the second-order induction axiom in arithmetic. In the antecedent, the weaker variant demands "explicit" existence and uniqueness of sets, whereas the original only demands *the* existence and uniqueness of sets, regardless of how those came about.

References

[1] Wilhelm Ackermann. Zur axiomatik der mengenlehre. *Mathematische Annalen*, 131(4):336–345, Aug 1956.

[2] Thomas Jech. *Set Theory: The Third Millennium Edition*. Springer, 2003.

[3] William N. Reinhardt. Ackermann's set theory equals zf. *Annals of Mathematical Logic*, 2(2):189–249, 1970.