# The Diagonal Argument, Clarified and Generalized

poypoyan

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**Abstract**

The diagonal argument is a key ingredient in proving both the uncountability of $2^\omega$ (which is a celebrated Cantor's usage of the argument) and the same-named diagonal lemma (which is used in proving Gödel's first incompleteness theorem). In this short note we present a simple formulation of the diagonal argument that is applicable to both results, and hopefully to a lot more.

## 1 Introduction

Well the abstract introduced already. We think that doing this formulation "exercise" provided us with a clearer understanding of the concepts surrounding diagonal lemma and Cantor's diagonal argument. Now let's get into the main thing!

## 2 Diagonal Argument

**Lemma 2.1.** *Let $A, B, C$ be sets such that:*

1. *$C$ is a set of functions from $A \to B$.*
2. *There exists a bijection $f : A \to C$.*

*Then there exists a function $d : A \to B$ such that $d(x) = (f(x))(x)$.*

*Proof.* Trivial! For every $x \in A$, $f(x)$ is unique in $C$. Letting $c = f(x)$, $c(x)$ is unique in $B$ because $c : A \to B$. $\square$

## 3 Cantor's Diagonal Argument

We follow Cantor's original diagonal argument for the uncountability of $2^\omega$ in [1], which he further used to show the uncountability of the set of real numbers $\mathbb{R}$. We'll interpret $2^\omega$ as the set of *all* functions from $\omega \to 2$.

The proof starts by assuming that there is a bijection $f : \omega \to 2^\omega$. This $f$ is actually the indexing of elements of $2^\omega$, more commonly shown as the "listing" of infinite binary sequences. Applying Lemma 2.1 to $A = \omega$, $B = 2$, $C = 2^\omega$, and $f$ gives the diagonal function $d : \omega \to 2$. What we really need is $d' : \omega \to 2$ such that $d'(x) = 1 - d(x)$; $d'$ is more commonly shown as the "diagonal" sequence constructed from the one's complement of the diagonal bits in the list. Now one may notice that for all $c \in 2^\omega$, $c \neq d'$ because $c(f^{-1}(c)) \neq d'(f^{-1}(c))$. Hence $d' \notin 2^\omega$, but this is in contradiction to the definition of $2^\omega$. Therefore the initial assumption of existence of bijection $f : \omega \to 2^\omega$ is false, and this completes the proof.

One may notice that a variant of the more general *Cantor's theorem*, $2^A > A$ for any set $A$, can be proven similarly.

## 4 Gödel-Carnap Diagonal Lemma

We'll just do an informal sketch. Setup first: let $\Gamma$ be a logic on a language $\mathcal{L}$, $N \subset \mathcal{L}$ be the set of all "objects of study" of $\Gamma$, $P \subset \mathcal{L}$ be the set of all "definable" propositions, and $C \subset \mathcal{L}$ be the set of all definable predicates with only one free variable $x$. For example, if $\Gamma = \mathsf{PA}$, (Classical) Peano Arithmetic, then the strings "$\mathsf{S0}$", "$\mathsf{SS0}$" $\in N$, and "$x + \mathsf{S0} = \mathsf{SS0}$" $\in C$. In an abuse of notation, while $c \in C$ is a string, it can also be treated as a function $c : N \to P$. For example in $\mathsf{PA}$, "$x + \mathsf{S0} = \mathsf{SS0}$"("$\mathsf{S0}$") = "$\mathsf{S0} + \mathsf{S0} = \mathsf{SS0}$".

One requirement of the lemma is the existence of the so-called *Gödel numbering* which is an injective (total) function $g : \mathcal{L} \to N$. We can now apply the diagonal argument: applying Lemma 2.1 to $A = \{x \in N \mid g^{-1}(x) \in C\}$, $B = P$, $C$ is itself, and $f = g^{-1}$ restricted to $A$, gives the diagonal function $d : A \to B$ such that $d(x) = (g^{-1}(x))(x)$. We can now present a strong variant of the lemma:

**Lemma 4.1.** *If $d$ is definable in $\mathcal{L}$ (in short, $d \in C$) and for all $c_1, c_2 \in C$, $c_1 \circ g \circ c_2 \in C$, then for every $c \in C$, there exists a predicate $x \in C$ such that $x = c \circ g \circ x$.*

*Proof.* Define a new predicate $h = c \circ g \circ d$. This is in $C$ because both $c, d \in C$. We can now define $x = h \circ g \circ h$. This works because $h \circ g \circ h = c \circ g \circ d \circ (g \circ h) = c \circ g \circ (g^{-1}(g \circ h))(g \circ h) = c \circ g \circ (h \circ g \circ h)$. $\square$

This proof is based from Section 2.2 of [3] [1] which ultimately came from [2]. The name of lemma is not due to the resemblance to the statement, but to the usage of a diagonal function in its proof!

Lastly, the reason we say that the presented lemma is "strong" is that the more commonly presented property of $x$ is something like $\Gamma \vdash \forall n : x(n) \leftrightarrow (c \circ g \circ x)(n)$ [2], and surely that is weaker than $x = c \circ g \circ x$.

# References


[1] Georg Cantor. Ueber eine elementare frage der mannigfaltigketislehre. *Jahresbericht der Deutschen Mathematiker-Vereinigung*, 1:75–78, 1891.

[2] R. G. Jeroslow. Redundancies in the hilbert-bernays derivability conditions for gödel's second incompleteness theorem. *The Journal of Symbolic Logic*, 38(3):359–367, 1973.

[3] Saeed Salehi. On the diagonal lemma of gödel and carnap. *The Bulletin of Symbolic Logic*, 26(1):80–88, 2020.


[1] The "$\varrho$" there is defined as $g \circ d$ here.

[2] Note though that the weaker property requires that " $\forall$ ", " $\leftrightarrow$ " $\in \mathcal{L}$.