My Pet Set Theory: $\mathsf{AST{+}NWF}$

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Abstract

In this note we present a variant of Ackermann Set Theory (AST) with easy construction of non-well-founded sets (NWF). Motivations for using up our time for this are also laid out.

1 The Theory

We will now describe a variant of Ackermann Set Theory [1] we denote as $\mathsf{AST{+}NWF}$. $\mathsf{AST{+}NWF}$ is formulated in first-order logic with equality and with a constant V which is interpreted as the set universe, and a binary relation $\in$ which is interpreted as the usual membership relation.

Definition 1.1 (Super-Completeness of V)**.** *Let $SC(V)$ be the statement*

$$\forall x, y(x \in V \land (y \in x \lor y \subseteq x) \to y \in V)$$

where $\subseteq$ is the usual subset relation, defined as $x \subseteq y \leftrightarrow \forall z(z \in x \to z \in y)$.

Now here are the axioms of $\mathsf{AST{+}NWF}(V)$:

1. Axiom of Extensionality $\mathsf{Ext}(V)$:

$$\forall x, y(x \in V \land y \in V \land \forall z(z \in x \leftrightarrow z \in y) \to x = y).$$

2. Ackermann Schema $\mathsf{Ack}(V)$: Let $\phi(y, z_1, \ldots, z_n)$ be any unary first-order formula where the all the free variables $z_1, \ldots, z_n \in V$ and $\neq V$. Then

$$(SC(V) \to \forall y(\phi(y, \ldots) \to y \in V)) \to \exists x(x \in V \land \forall y(y \in x \leftrightarrow \phi(y, \ldots))).$$

3. **Back** Ackermann Schema $\mathsf{BAck}(V)$: Let p be an $(m+1)$-ary predicate. Let

$$\phi(y, z_1, \ldots, z_m, z_{m+1}, \ldots, z_{(k-1)m}, z_{(k-1)m+1}, \ldots, z_{km}, a_1, \ldots, a_n)$$

be any unary first-order formula where all the free variables (all except y) are $\in V$ and $\neq V$, and there are k instances of p in ϕ:

$$\begin{aligned} &p(y, z_1, \ldots, z_m) \\ &p(y, z_{m+1}, \ldots, z_{2m}) \\ &\vdots \\ &p(y, z_{(k-1)m+1}, \ldots, z_{km}). \end{aligned}$$

Then if

$$\forall x \forall y(y \in x \leftrightarrow \phi(y, \ldots)) \to p(x, b_1, \ldots, b_m)$$

and

$$\forall x(\bigwedge_{i=0}^{k-1} p(x, z_{im+1}, \ldots, z_{(i+1)m}) \to x \in V) \to (SC(V) \to \forall y(\phi(y, \ldots) \to y \in V)),$$

then

$$\bigwedge_{i=0}^{k-1} \Big(\exists! x(p(x, z_{im+1}, \ldots, z_{(i+1)m})) \land \\ \forall x(p(x, z_{im+1}, \ldots, z_{(i+1)m}) \to (x \in V \land \forall c_1, \ldots, c_m(\bigvee_{j=1}^{m} c_j \neq z_{im+j} \to \neg p(x, c_1, \ldots, c_m))))\Big).$$

2 Motivations

There are several motivations for the "design" of the theory.

The first motivation is the focus on *set constructions* instead of sets and/or proper classes. Because of this, we decided to:

- Adopt AST in the first place. The Ackermann schema captures the idea that "natural"/"uncontroversial" set constructions are 1) definable as a first-order sentence (hence, are "finite"), and 2) "universe-agnostic" (since those do not mention V). This reminds us while studying infinities that our "full" descriptions of objects are always finite.

  On the contrary, consider the Axiom of Choice (AC), a well-known axiom independent to ZF. The sets it constructs are not unique, hence it is said to postulate existence of sets without defining it, unlike other ZF (set construction) axioms [2, Chapter 5]. However, if we just permit "lengths" of any ordinal to ϕ in Ackermann Schema, AC can now "produce" unique sets again! To see this, let S be a set of sets, then an appropriate ϕ for AC would simply be

  $$y = a_0 \vee y = a_1 \vee y = a_2 \vee \ldots$$

  where $a_0 \in s_0, a_1 \in s_1, a_2 \in s_2, \ldots$ and $S = \{s_0, s_1, s_2, \ldots\}$. Nevertheless, stronger axioms like Choice can be added to our theory via relativization to V wherein every instance of $\forall x \varphi$ in the statement is replaced by $\forall x(x \in V \rightarrow \varphi)$ and every instance of $\exists x \varphi$ is replaced by $\exists x(x \in V \wedge \varphi)$.

- Remove the Class Construction Schema for our AST+NWF. Note that Class Construction Schema *is* Separation Schema for V and that Separation Schema immediately follows from Ackermann Schema by setting ϕ to $y \in a \wedge \varphi$ for $a \in V$.

The next motivation is to make the universe(s) "as closed as possible". Because of this, we decided to:

- Restrict Extensionality to sets ($x \in V$) only.

- Put super-completeness inside Ackermann schemas instead of it being an axiom on its own. This is done so that when we work on multiple universes $V_0 \subset V_1 \subset \ldots$ where AST-NWF$(V_0) \wedge$ AST-NWF$(V_0) \wedge \ldots$, no additional sets in V_0 will be shown to exist through the higher universes $V_1, \ldots$, and *not* through Ackermann schemas for V_0.

Now the Back Ackermann Schema is formulated for easy contruction of non-well-founded sets, but it looks spooky and extremely complicated. Nevertheless, the intuition is actually simple:

> If potential non-well-founded sets (the instances of p in ϕ) are used to successfully construct another version of p (via ϕ, in Ackermann Schema sense) *assuming that those are all sets*, then those are indeed sets, and *no other version of p should ever be equal to those.*

For example, to construct a set such that $x = \{x, a\}$ where $a \in V$, the m of p is simply 1 (as in $p(y)$), and $\phi(y, \ldots)$ is simply $p(y) \vee y = a$.

Lastly, the last highlight in the above quote, the statement "no other version of p should ever be equal to those", needs explanation. Our main motivation for this is the "intuitive" set-theoretic definition of ordered pair $(x, y) = \{(0, x), (1, y)\}$. For this to satisfy the ordered pair property $(x_0, y_0) = (x_1, y_1) \leftrightarrow x_0 = x_1 \wedge y_0 = y_1$ in our theory, the newly proved non-well-founded sets $(0, x)$ and $(1, y)$ should *always* be unequal. Since by the nature of non-well-founded sets we cannot "view inside" p, there is freedom on equalities between the newly proved non-well-founded sets. Hence the statement can be seen as either a "cheat", or the most natural generalization of the always inequality between $(0, x)$ and $(1, y)$.

References

[1] Wilhelm Ackermann. Zur axiomatik der mengenlehre. *Mathematische Annalen*, 131(4):336–345, Aug 1956.

[2] Thomas Jech. *Set Theory: The Third Millennium Edition.* Springer, 2003.