Ackermann Set Theory and ZF

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Abstract

Zermelo-Fraenkel Set Theory (ZF) is considered to be the "standard" set theory for foundation of mathematics. Ackermann proposed an alternative set theory in [1], which is now called *Ackermann Set Theory* (AST). An interesting well-known result is that ZF and a little modified AST is somewhat "the same." In this note we present that variant of AST, the equivalence result, and its significance.

1 That Variant of AST

We will now describe here a variant of Ackermann Set Theory we denote as A^*, which has differences with the original in [1] in that there is one axiom removed and one axiom inserted. A^* is formulated in first-order logic with equality and with a constant V which is interpreted as the set universe, and a binary relation $\in$ which is interpreted as the usual membership relation. Here are the axioms of A^*:

1. (Axiom of Extensionality)
$$\forall z(z \in x \leftrightarrow z \in y) \rightarrow x = y.$$
2. (Ackermann's Schema) Let $\phi(y, z_1, \ldots, z_n)$ be any first-order formula without V and x as free variables. If $x_1, \ldots, x_n \in V$, then
$$\forall y(\phi(y, z_1, \ldots, z_n) \rightarrow y \in V) \rightarrow \exists x(x \in V \wedge \forall y(y \in x \leftrightarrow \phi(y, z_1, \ldots, z_n))).$$
3. (Strong Completeness Axioms for V)
$$(x \in y \vee x \subseteq y) \wedge y \in V \rightarrow x \in V.$$
where $\subseteq$ is the usual subset relation, defined as $x \subseteq y \leftrightarrow \forall z(z \in x \rightarrow z \in y)$.
4. (Axiom of Regularity)
$$x \in V \wedge \exists y(y \in x) \rightarrow \exists y(y \in x \wedge \forall z(z \in x \rightarrow \neg z \in y)).$$

The one axiom removed is called the *Class Construction Axiom Schema* which is just the Axiom Schema of Separation for V, and the one axiom inserted is the Axiom of Regularity above.

2 The Equivalence Result, and its Significance

[WIP] [2] [3]

The main reason for writing this whole note is because this result basically means that for *well-founded* sets, all *unique* set constructions that are

1. definable as a first-order sentence (hence, are "finite"), and
2. "universe-agnostic" (since they do not mention V),

are *exactly* the ones definable using ZF! Informally, for unique set constructions,

"Nice and Finite" = Constructed using ZF.

Let's consider the Axiom of Choice. [WIP]

3 Proofs

[WIP; will take a long time because I am still a beginner in Set Theory.]

References

[1] Wilhelm Ackermann. Zur axiomatik der mengenlehre. *Mathematische Annalen*, 131(4):336–345, Aug 1956.

[2] Azriel Lévy. On ackermann's set theory. *Journal of Symbolic Logic*, 24(2):154–166, 1959.

[3] William N. Reinhardt. Ackermann's set theory equals zf. *Annals of Mathematical Logic*, 2(2):189–249, 1970.