My Pet Set Theory: $\mathsf{AST+NWF}$

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Abstract

In this note we present a variant of Ackermann Set Theory (AST) with easy construction of non-well-founded sets (NWF). Motivations for using up our time for this are also laid out.

1 The Theory

We will now describe a variant of Ackermann Set Theory [1] we denote as $\mathsf{AST+NWF}$. $\mathsf{AST+NWF}$ is formulated in first-order logic with equality and with a constant V which is interpreted as the set universe, and a binary relation $\in$ which is interpreted as the usual membership relation.

Definition 1.1 (Super-Completeness of V)**.** *Let $SC(V)$ be the statement*

$$\forall x, y(x \in V \wedge (y \in x \vee y \subseteq x) \rightarrow y \in V)$$

where $\subseteq$ is the usual subset relation, defined as $x \subseteq y \leftrightarrow \forall z(z \in x \rightarrow z \in y)$.

Now here are the axioms of $\mathsf{AST+NWF}(V)$:

1. Axiom of Extensionality $\mathsf{Ext}(V)$:

$$\forall x, y(x \in V \wedge y \in V \wedge \forall z(z \in x \leftrightarrow z \in y) \rightarrow x = y).$$

2. Ackermann Schema $\mathsf{Ack}(V)$: Let $\phi(y, z_0, \ldots, z_{n-1})$ be any unary first-order formula where all the free variables $z_0, \ldots, z_{n-1} \in V$ and $\neq V$. Then

$$(SC(V) \rightarrow \forall y(\phi(y, \ldots) \rightarrow y \in V)) \rightarrow \exists x(x \in V \wedge \forall y(y \in x \leftrightarrow \phi(y, \ldots))).$$

3. **Non-well-founded** Ackermann Schema $\mathsf{NWFAck}(V)$: Let $\phi(y, z_0, \ldots, z_{n-1})$ be any unary first-order formula where all the free variables $z_0, \ldots, z_{n-1} \in V$ and $\neq V$, and there are different instances of ϕ: $\phi(y_0, \ldots_0), \phi(y_1, \ldots_1)$, $\ldots$ in ϕ[1], where $y_0, y_1, \ldots$ are all bound variables. Then if

$$(\bigwedge_i \forall y_i(\phi(y_i, \ldots_i) \rightarrow y_i \in V)) \rightarrow (SC(V) \rightarrow \forall y(\phi(y, \ldots) \rightarrow y \in V)),$$

then

$$\exists x(x \in V \wedge \forall y(y \in x \leftrightarrow \phi(y, \ldots))) \wedge \forall y, c_0, \ldots, c_{n-1}(\phi(y, z_0, \ldots, z_{n-1}) \wedge \phi(y, c_0, \ldots, c_{n-1}) \rightarrow \bigwedge_{j=0}^{n-1} c_j = z_j)).$$

2 Motivations

There are several motivations for the "design" of the theory.

The first motivation is the focus on *set constructions* instead of sets and/or proper classes. Because of this, we decided to:

[1] Yes, this is a recursive definition!

- Adopt AST in the first place. The Ackermann schema captures the idea that "natural"/"uncontroversial" set constructions are 1) definable as a first-order sentence (hence, are "finite"), and 2) "universe-agnostic" (since those do not mention V). This reminds us while studying infinities that our "full" descriptions of objects are always finite.

  On the contrary, consider the Axiom of Choice (AC), a well-known axiom independent to ZF. The sets it constructs are not unique, hence it is said to postulate existence of sets without defining it, unlike other ZF (set construction) axioms [2, Chapter 5]. However, if we just permit "lengths" of any ordinal to ϕ in Ackermann Schema, AC can now "produce" unique sets again! To see this, let S be a set of sets, then an appropriate ϕ for AC would simply be

$$y = a_0 \vee y = a_1 \vee y = a_2 \vee \ldots$$

  where $a_0 \in s_0, a_1 \in s_1, a_2 \in s_2, \ldots$ and $S = \{s_0, s_1, s_2, \ldots\}$. Nevertheless, stronger axioms like Choice can be added to our theory via relativization to V wherein every instance of $\forall x \varphi$ in the statement is replaced by $\forall x(x \in V \to \varphi)$ and every instance of $\exists x \varphi$ is replaced by $\exists x(x \in V \wedge \varphi)$.

- Remove the Class Construction Schema for our AST+NWF. Note that Class Construction Schema *is* Separation Schema for V and that Separation Schema immediately follows from Ackermann Schema by setting ϕ to $y \in a \wedge \varphi$ for $a \in V$.

The next motivation is to make the universe(s) "as closed as possible". Because of this, we decided to:

- Restrict Extensionality to sets ($x \in V$) only.

- Put super-completeness inside Ackermann schemas instead of it being an axiom on its own. This is done so that when we work on multiple universes $V_0 \subset V_1 \subset \ldots$ where AST-NWF(V_0) $\wedge$ AST-NWF(V_0) $\wedge \ldots$, no additional sets in V_0 will be shown to exist through the higher universes $V_1, \ldots$, and *not* through Ackermann schemas for V_0.

Now the Non-well-founded Ackermann Schema is formulated for easy contruction of non-well-founded sets (obviously), but it is a beast! Nevertheless, the intuition is actually simple:

> If instances of ϕ are used to define itself, and we assume that all that satisfy those instances of ϕ are all sets, then ϕ can indeed construct sets, and the elements of the constructed set are *unique through the given parameters.*

For example, to construct a set such that $x = \{x, a\}$ where $a \in V$, $\phi(y, a)$ is defined as

$$\forall x(\forall y_0(y_0 \in x \leftrightarrow \phi(y_0, a)) \to y = x \vee y = a).$$

The last highlight in the above quote, the statement "unique through the given parameters", needs explanation. Our main motivation for this is the "intuitive" set-theoretic definition of ordered pair $(x, y) = \{(0, x), (1, y)\}$. For this to satisfy the ordered pair property $(x_0, y_0) = (x_1, y_1) \leftrightarrow x_0 = x_1 \wedge y_0 = y_1$ in our theory, the sets $(0, x)$ and $(1, y)$ should *always* be unequal. Since by the nature of non-well-founded sets we cannot "view inside" ϕ, there is freedom on equalities of elements across non-well-founded sets. Hence the statement can be seen as either a "cheat", or the most natural generalization of the always inequality between $(0, x)$ and $(1, y)$.

Lastly, the nature of recursive defining of ϕ in the schema needs further consideration. Maybe AST+NWF is not a first-order theory anymore because recursive definition of predicates causes its signature to have "incrementing" quantity of predicates. I think that this shows that the concept of "definition" itself can be tricky in formal logic, and hence its "effects" on theories and logics also need further consideration.

References

[1] Wilhelm Ackermann. Zur axiomatik der mengenlehre. *Mathematische Annalen*, 131(4):336–345, Aug 1956.

[2] Thomas Jech. *Set Theory: The Third Millennium Edition*. Springer, 2003.