1. (a) Let $A_{n_1} = A_1$ and select an arbitrary $y_1 \in A_{n_1}$. For y_k, select $A_{n_{k+1}}$ where $n_{k+1} > n_k$ and for all $m, n \geq n_{k+1}$, $D(A_m, A_n) < \frac{1}{2^{k+1}}$ and select y_{k+1} where $||y_{k+1} - y_k|| = d(y_k, A_{n_{k+1}})$. We constructed a series (y_k) where $||y_{k+1} - y_k|| \leq D(A_{n_k}, A_{n_{k+1}}) < \frac{1}{2^{k+1}}$. From class, $\frac{1}{2^k}$ converges sufficiently that (y_k) is Cauchy. $\mathbb{R}^2$ is complete, so $y_k \to x \in \mathbb{R}^2$.

We now construct a series (x_n) where we select $x_n \in A_n$ where $||x - x_n|| = d(x, A_n)$, which exists because $d(x, A_n)$ is a minimum. (*) We now note that for every x_n, considering an arbitrary y_k and a point $z \in A_n$ where $||y_k - z|| = d(y_k, A_n)$:

$$\begin{aligned} ||x - x_n|| &\leq ||x - z|| \\ &\leq ||x - y_k|| + ||y_k - z|| \\ &\leq ||x - y_k|| + d(y_k, A_n) \\ &\leq ||x - y_k|| + D(A_{n_k}, A_n) \end{aligned}$$

Let ϵ be given. Select an N where for all $m, n \geq N$, $D(A_m, A_n) < \epsilon/2$. Meanwhile, select a k where $n_k \geq N$ and $||y_k - x|| < \epsilon/2$. This gives us that:

$$||x - x_n|| < \epsilon/2 + \epsilon/2 = \epsilon$$

Hence, $x_n \to x$ and $x \in A$. This proves that A is non-empty. □

(b) Got skill issued :(

(c) Let $\epsilon > 0$ be given. Select a N where $\forall n, m \geq N$, $D(A_n, A_m) < \epsilon/2$. For all $n \geq N$, let x_n be where $d(x_n, A) = d(A_n, A)$ since it is a maxmimum. Let $y_1 = x_n$. For y_k, select $A_{n_{k+1}}$ where $n_{k+1} > n_k$ and for all $m, n \geq n_{k+1}$, $D(A_m, A_n) < \frac{\epsilon}{2^{k+1}}$ and select y_{k+1} where $||y_{k+1} - y_k|| = d(y_k, A_{n_{k+1}})$. We constructed a series (y_k) where $||y_{k+1} - y_k|| \leq D(A_{n_k}, A_{n_{k+1}}) < \frac{\epsilon}{2^{k+1}}$. From class, $\frac{1}{2^k}$ converges sufficiently that (y_k) is Cauchy. $\mathbb{R}^2$ is complete, so $y_k \to x \in \mathbb{R}^2$. From our proof of (a), $x \in A$. We then consider that:

$$\begin{aligned} ||y_1 - x|| &\leq ||y_1 - y_2|| + ||y_2 - y_3|| + \cdots \\ &< \sum_{k=1}^{\infty} \frac{\epsilon}{2^k} = \epsilon \end{aligned}$$

We showed there exist a $x \in A$ where $||y_1 - x|| = ||x_n - x|| < \epsilon$ Since $d(A_n, A) = d(x_n, A) \leq ||x_n - x|| < \epsilon$, we proved that $d(A_n, A) \to 0$. □

(d) Let $\epsilon > 0$ be given. Select a large enough N where $\forall n, m \geq N$, $D(A_n, A_m) < \epsilon/2$. Now, fix any $n \geq N$ and let $a \in A$ where $d(a, A_n) = d(A, A_n)$. Since $a \in A$, there exist a series $(x_n) \to a$ and choose $m \geq N$ where $||x_m - a|| < \epsilon/2$. We now consider that for any $n \geq N$ that

$$\begin{aligned} d(a, A_n) &\leq ||a - x_m|| + d(x_m, A_n) \\ &\leq ||a - x_m|| + D(A_m, A_n) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

Refer to (a) with the part marked with (*) for the justification of the initial inequality. We now note that $d(a, A_n) = d(A, A_n) < \epsilon$. Hence, we proved that $d(A, A_n) \to 0$. □

(e) Let $\epsilon > 0$ be given. Select a large enough N where for all $n \geq N$, both $d(A_n, A) < \epsilon$ and $d(A, A_n) < \epsilon$. This is possible due to our result in (c) and (d). Since $D(A_n, A)$ is either $d(A_n, A)$ or $d(A, A_n)$, this implies that $D(A_n, A) < \epsilon$. By definition of limits, $A_n \to A$. □