1. (a) Assume f is continuous, and by contradiction, there exist an $x \in f^{-1}(\text{Int(A)})$, where $x \notin \text{Int}(f^{-1}(A))$. Thus, there exists an $(a_n) \notin f^{-1}(A)$, where $a_n \in (x - \frac{1}{n}, x + \frac{1}{n})$. This implies $(a_n) \to x$, but for each a_n, $f(a_n) \notin A$ and $f(a_n) \to f(x)$ due to the continuity of f. Meanwhile, Int(A) is open, so there exists $\epsilon > 0$ where $(f(x) - \epsilon, f(x) + \epsilon) \subseteq Int(A)$. This is a contradiction because there exists an $N \in \mathbb{N}$ where $|f(a_n) - f(x)| < \epsilon$ for $n \geq N$ while $f(a_n) \notin A \supseteq \text{Int(A)}$. Thus, $(f(x) - \epsilon, f(x) + \epsilon) \not\subseteq \text{Int}(A)$.

For the converse, assume for all $A \subseteq \mathbb{R}$ that $f^{-1}(\text{Int(A)}) \subseteq \text{Int}(f^{-1}(A))$. This also implies that for all open sets $U \subseteq \mathbb{R}$ that $f^{-1}(U) \subseteq \text{Int}(f^{-1}(U))$, as $\text{Int}(U) = U$. We prove $f^{-1}(U)$ is relatively open in $\mathbb{R}$. By contradiction, it is not relatively open, so there exists an $x \in f^{-1}(U)$ s.t. for all $n \in \mathbb{N}$, there exists an $a_n \in \mathbb{R}$ with $|a_n - x| < \frac{1}{n}$ and $a_n \notin f^{-1}(U)$. We then note that since $x \in \text{Int}(f^{-1}(U))$, so there exists a $\delta > 0$ where $(x - \delta, x + \delta) \subseteq f^{-1}(U)$. However, there exists an $N \in \mathbb{N}$, for $n \geq N$ where $|a_n - x| < \delta$, so there exists an $a_n \in (x - \delta, x + \delta)$, which implies $(x - \delta, x + \delta) \not\subseteq f^{-1}(U)$. This is a contradiction. Thus, $f^{-1}(U)$ is relatively open in $\mathbb{R}$. Since all open sets $U \subseteq \mathbb{R}$ have their pre-image being relatively open in $\mathbb{R}$, it implies f must be continuous from a proposition proven in class.

(b) Let $f(x) = x^2$, and $A = [0, 1]$. We then denote $x = 0$, so $f(x) = 0$. We note that $x \in \text{Int}(f^{-1}(A))$ since $(-\frac{1}{2}, \frac{1}{2}) \subseteq f^{-1}(A)$. However, $x \notin f^{-1}(\text{Int}(A))$ because $\text{Int}(A) = (0, 1)$ and $0 \notin (0, 1)$. Hence, we proved that $f^{-1}(\text{Int}(A)) \neq \text{Int}(f^{-1}(A))$.

2. (a) Note that $f(x) = f(x+d) = f(x+d+d)$ and $f(x-d-d) = f(x-d) = f(x)$. We can repeat either of them indefinitely to get that $f(x) = f(x+kd)$ for $k \in \mathbb{Z}$. For any $x \in \mathbb{R}$, we note that $x = \lfloor x/d \rfloor d + r$ where $0 \leq r < d$, so $f(x) = f(r)$. Thus, we get that $f(\mathbb{R}) = f([0,d])$. Let $R := f([0,d])$, we note that $[0,d]$ is a closed and bounded, hence compact. By EVT, we get that there exist $x_{\min}, x_{\max} \in [0,d]$ s.t. $f(x_{\min}) = \min(R)$ and $f(x_{\max}) = \max(R)$. Thus, f attains its maximum and minimum values.

   (b) Let $a, b \in [0,1]$ where $a < b$ and $a, b \neq 0, 1$.

   Case 1: If $f(0) = f(1)$, we note this is impossible because f is one-to-one.

   Case 2: If $f(0) < f(1)$, we get that by IVT that $f([0,a]) = A = [a_1, a_2]$ and $f([b,1]) = B = [b_1, b_2]$. Since f is one-to-one and $[0,a] \cap [b,1] = \emptyset$, $A \cap B = \emptyset$. Thus, either $a_2 < b_1$ or $b_2 < a_1$. Since $f(0) \in A$ and $f(1) \in B$, we get that:

$$a_1 \leq f(0) \leq a_2 < b_1 \leq f(1) \leq b_2$$

   Thus, since $f(a) \in A$ and $f(b) \in B$, $f(a) < f(b)$.

   Case 3: If $f(0) > f(1)$, denote the same A and B and we get that:

$$b_1 \leq f(1) \leq b_2 < a_1 \leq f(0) \leq a_2$$

   Thus, since $f(a) \in A$ and $f(b) \in B$ and $b_2 < a_1$, $f(a) > f(b)$

   Since $0 < 1$, we can extend this to all $a, b \in [0,1]$ where $a < b$. Thus, f is strictly monotone on $[0,1]$.