

1. (a) If  $p = 2$ , we note that  $3 \equiv 1 \pmod{2}$ , so 3 is a quadratic residue mod 2 since 1 is a square. If  $p = 3$ , then  $3 \equiv 0 \pmod{3}$  since 0 is a square.

Hence, it remains to prove the claim if  $p$  is an odd prime where  $p \geq 5$ . Let us assume 3 is a quadratic residue mod  $p$ . Hence,  $\left(\frac{3}{p}\right) = 1$ . We consider 2 cases on the basis of Theorem 11.4 (c).

**Case 1:** If  $p \equiv -1 \pmod{4}$ . Then,  $\left(\frac{3}{p}\right) = -\left(\frac{p}{3}\right) = 1$ . Thus,  $\left(\frac{p}{3}\right) = -1$ . This implies that  $p^{(3-1)/2} = p \equiv -1 \pmod{3}$ . Since 3 and 4 are co-prime, we get that  $p \equiv -1 \pmod{12}$ .

**Case 2:** If  $p \not\equiv -1 \pmod{4}$ . Then,  $\left(\frac{3}{p}\right) = \left(\frac{p}{3}\right) = 1$ . This implies that  $p^{(3-1)/2} = p \equiv 1 \pmod{3}$ . Since  $p$  is an odd prime, we note that  $p \not\equiv 2 \pmod{4}$  or  $p \not\equiv 0 \pmod{4}$ . This leaves that  $p \equiv 1 \pmod{4}$ . Since 3 and 4 are co-prime, we get that  $p \equiv 1 \pmod{12}$ .

Considering all the cases, we get that  $\left(\frac{3}{p}\right) = 1$  if  $p = 2, 3$  or  $p \equiv \pm 1 \pmod{12}$ . For the converse, we assume that  $p \equiv \pm 1 \pmod{12}$ . For  $p \equiv -1 \pmod{12}$ , it implies that  $p \equiv -1 \equiv 3 \pmod{4}$  and  $p \equiv -1 \pmod{3}$ . Since  $3 \equiv 3 \pmod{4}$  by Theorem 11.4 (c), we note that  $\left(\frac{3}{p}\right) = -\left(\frac{p}{3}\right)$ . Further more,  $-1$  is not a quadratic residue mod 3, so  $-\left(\frac{p}{3}\right) = -(-1) = 1$ . Meanwhile, if  $p \equiv 1 \pmod{12}$ , then  $p \equiv 1 \pmod{3}$ , so  $p$  is a quadratic residue mod 3. In both cases, 3 is a quadratic residue.  $\square$

- (b) For  $p = 2$ , we note that  $-3 \equiv -1 \equiv 1 \pmod{2}$ , so it is a quadratic residue. Meanwhile for  $p = 3$ ,  $-3 \equiv 0 \pmod{3}$ , so it is also a quadratic residue. Hence, we concern ourselves with the case where  $p \geq 5$ . Let us assume that  $-3$  is a quadratic residue mod  $p$ . This implies that  $\left(\frac{-3}{p}\right) = \left(\frac{-1}{p}\right)\left(\frac{3}{p}\right) = 1$ . In other words, either both  $\left(\frac{-1}{p}\right)$  and  $\left(\frac{3}{p}\right)$  must be equal. Hence:

$$\left(\frac{-1}{p}\right) = \left(\frac{3}{p}\right)$$

Applying Theorem 11.4 (a), we get that:

$$(-1)^{(p-1)/2} = \left(\frac{3}{p}\right)$$

We now consider Theorem 11.4 (c), and note that

$$\left(\frac{3}{p}\right) \cdot \left(\frac{p}{3}\right) = (-1)^{(p-1)/2 \cdot (3-1)/2} = (-1)^{(p-1)/2}$$

We know that  $\left(\frac{3}{p}\right) \neq 0$  because  $p$  is prime and  $p \neq 3$ , so:

$$\left(\frac{3}{p}\right) = \left(\frac{p}{3}\right) \cdot (-1)^{(p-1)/2}$$

Substituting the value with the earlier equation gives:

$$\begin{aligned} (-1)^{(p-1)/2} &= \left(\frac{p}{3}\right) \cdot (-1)^{(p-1)/2} \\ 1 &= \left(\frac{p}{3}\right) \end{aligned}$$

This implies that  $p$  is a square mod 3. In  $\mathbb{F}_3$ , only 0 and 1 are squares. We can rule out  $p = 0$  because  $p \neq 3$ . Hence, we proved that if  $-3$  is a quadratic residue for  $p \geq 5$ , then  $p \equiv 1 \pmod{3}$ . For the converse, assume that  $p \equiv 1 \pmod{3}$ . By Theorem 11.4 (c), we note that:

$$\left(\frac{3}{p}\right)\left(\frac{p}{3}\right) = (-1)^{(p-1)/2 \cdot (3-1)/2} = (-1)^{(p-1)/2}$$

Note that  $\left(\frac{p}{3}\right) = 1$ , so  $\left(\frac{3}{p}\right) = (-1)^{(p-1)/2}$ . We now examine  $\left(\frac{-3}{p}\right) = \left(\frac{-1}{p}\right)\left(\frac{3}{p}\right) = (-1)^{(p-1)/2} = 1$ . This implies that  $-3$  is a quadratic residue mod  $p$ . □

- (c) By contradiction, we assume that  $p \neq 2$  and  $p \not\equiv 1 \pmod{4}$ . This implies that  $p$  is an odd prime where  $2 \nmid (p-1)/2$ . We note that  $x^2 \equiv -y^2 \pmod{p}$ . This implies that  $-y^2$  is a quadratic residue mod  $p$ , so either  $\left(\frac{-y^2}{p}\right) = 0$  or  $1$ . If  $\left(\frac{-y^2}{p}\right) = 1 = \left(\frac{-1}{p}\right)\left(\frac{y}{p}\right)\left(\frac{y}{p}\right)$ , we note that  $\left(\frac{-1}{p}\right) = -1$  since  $(p-1)/2$  is odd. This implies that  $\left(\frac{y}{p}\right)^2 = -1$ . The values of Legendre symbols squared is either  $0$  or  $1$ , so  $-1$  is impossible thus a contradiction. Meanwhile, if  $\left(\frac{-y^2}{p}\right) = 0$ , then with  $\left(\frac{-1}{p}\right) = -1$ ,  $\left(\frac{y}{p}\right) = 0$ . This implies  $p \mid y$ , but if  $y \neq 0$ , then  $x^2 + y^2 \geq y^2 > p$ , a contradiction, and if  $y = 0$ , then  $p = x^2$ , contradicting  $p$  being prime. In all cases, we reach a contradiction thus  $p = 2$  or  $p \equiv 1 \pmod{4}$ . □

- (d) By contradiction, we assume that  $p \neq 3$  and  $p \not\equiv 1 \pmod{3}$ . This leaves that  $p \equiv -1 \pmod{3}$ . Now we consider that:

$$\begin{aligned} x^2 - xy + y^2 &= p \\ 4x^2 - 4xy + 4y^2 &= 4p \\ (2x - y)^2 + 3y^2 &= 4p \end{aligned}$$

We now consider that:

$$(2x - y)^2 + 3y^2 \equiv 4p \pmod{3}$$

In  $\mathbb{F}_3$ ,  $1 = 4$  and  $3 = 0$ . Hence:

$$(2x - y)^2 \equiv p \pmod{3}$$

However, this implies that  $(2x - y)^2 \equiv -1 \pmod{3}$ . This is a contradiction because  $-1$  is not a square in  $\mathbb{F}_3$  because  $\left(\frac{-1}{3}\right) = (-1)^{(3-1)/2} = -1$ . □

2. (a) By contradiction, assume  $n$  is even then  $n = 2k$  for some integer  $k$ . We then note that  $2^n - 1 = 4^k - 1$ . Note that  $4 \equiv 1 \pmod{3}$ , so  $4^k \equiv 1 \pmod{3}$ . Thus,  $3 \mid 4^k - 1$  thus  $3 \mid 2^n - 1$ . However, we note that  $3 \nmid 3^n - 1$ , contradiction.  $\square$
- (b) We note that  $3^n \equiv 1 \pmod{p}$ . This implies that  $3^n$  is a quadratic residue mod  $p$ , so  $\left(\frac{3^n}{p}\right) = 1$ . We then note that we can split  $\left(\frac{3^n}{p}\right)$  into  $n$  products of  $\left(\frac{3}{p}\right)$ . However,  $n$  is odd, so  $\left(\frac{3}{p}\right) \neq 0, -1$ . This leaves that  $\left(\frac{3}{p}\right) = 1$ . By 1(a), this implies that  $p = 2, 3$  or  $p \equiv \pm 1 \pmod{12}$ . However,  $p \neq 2$  because 2 is even. Meanwhile,  $p \neq 3$  because  $3 \nmid 3^n - 1$ . This leaves that  $p \equiv \pm 1 \pmod{12}$  as desired.  $\square$
- (c) For  $n = 1$ , we note that  $1 \mid 2$ , so it works. We now assume that  $n > 1$ . Since from (a), we get that  $n$  is odd, so we can express  $n = 2k + 1$  for some integer  $k$ . Hence,  $2^n - 1 = 2 \cdot 4^k - 1$ . We then note that  $4^2 \equiv 4 \pmod{12}$ . This implies that for  $k \geq 1$ , we get that  $4^k \equiv 4 \pmod{12}$  and that  $2 \cdot 4^k - 1 \equiv 2 \cdot 4 - 1 \equiv 7 \pmod{12}$ . However, we note that since  $n > 1$  and is odd,  $2^n - 1 \geq 7$  and is also odd. This implies that  $2^n - 1$  can be prime factorized into entirely of odd primes of  $p_1 p_2 \cdots p_j$ . Note that any odd prime  $p_i$  has  $p_i \mid 2^n - 1$ , so it is also  $p_i \mid 3^n - 1$ . From (b), this implies that  $p_i \equiv \pm 1 \pmod{12}$ . This is a contradiction because  $p_1 p_2 \cdots p_j \equiv \pm 1 \pmod{12}$  and that  $7 \not\equiv \pm 1 \pmod{12}$ . Thus, it must be that  $n = 1$ .  $\square$

3. (a) We first note that if  $p \equiv 1 \pmod{3}$ , from 1(b), we note that  $-3 \equiv a^2 \pmod{p}$  for some integer  $a$ . We then note that if  $a$  is even, we note that  $p - a$  is odd and that  $p - a \equiv a \pmod{p}$ . Hence this conversion allows us to assume there must exist an odd  $a$ . Since  $a$  is odd, it is in the form  $a = 2k + 1$  for some integer  $k$ . This gives:

$$\begin{aligned}(2k + 1)^2 &\equiv -3 \pmod{p} \\ 4k^2 + 4k + 1 + 3 &\equiv 0 \pmod{p} \\ 4(k^2 + k + 1) &\equiv 0 \pmod{p}\end{aligned}$$

Since  $p \equiv 1 \pmod{3}$ , we may assume  $p \neq 2$  thus an odd prime. Hence,  $p \nmid 4$  and so  $4 \neq 0 \in \mathbb{F}_p$ . From the above expression, we get that  $4(k^2 + k + 1) = 0$  in  $\mathbb{F}_p$ . Since  $\mathbb{F}_p$  is an integral domain, it must be that  $k^2 + k + 1 = 0$  in  $\mathbb{F}_p$ . We note that the formal derivative of this polynomial is  $2x + 1$ . We aim to prove that  $2k + 1 \neq 0$  in  $\mathbb{F}_p$ . By contradiction,  $2k + 1 = 0$  in  $\mathbb{F}_p$ . We get that  $k = -2^{-1}$  in  $\mathbb{F}_p$  (Note that  $p \neq 2$ , so  $2^{-1}$  exists). Hence:

$$\begin{aligned}4k^2 + 4k + 4 &= 0 \\ 4(-2^{-1})^2 + 4(-2^{-1}) + 4 &= 0 \\ 1 - 2 + 4 &= 0 \\ 3 &= 0\end{aligned}$$

Since  $p \equiv 1 \pmod{3}$ , we note that  $p \neq 3$ . This is a contradiction, so it must be that  $2k + 1 \neq 0$  in  $\mathbb{F}_p$ . We satisfied the conditions for Corollary 11.16, so there exists a  $a \in \mathbb{Z}_p$  where  $a^2 + a + 1 = 0$  in  $\mathbb{Z}_p$ . In other words, by taking the  $k$ -th coordinate of  $a$  as  $a_k$ , we note that  $a_k^2 + a_k + 1 = 0$  in  $\mathbb{Z}/p^k\mathbb{Z}$ . Hence, there is a solution for  $x^2 + x + 1 \equiv 0 \pmod{p^k}$ .  $\square$

- (b) Let us assume  $p$  is a prime where  $p \equiv 2 \pmod{3}$ .

**Claim:** If  $a$  is an integer where  $p \nmid a$ , then there exists an integer solution  $x$  for  $x^3 \equiv a \pmod{p}$ .

We first note that  $p = 3k + 2$  for some integer  $k$ . By Fermat's Little Theorem, we note that  $a^{p-1} \equiv 1 \pmod{p}$  and  $a^p \equiv a \pmod{p}$ . Multiplying the two gives that  $a^{2p-1} \equiv a \pmod{p}$ . We then substitute  $p$  to get that  $a^{6k+3} \equiv (a^{(2k+1)})^3 \equiv a \pmod{p}$ . We note that  $a^{(2k+1)}$  is the solution, so we conclude the proof.  $\square$

We now note that  $19 \equiv 1 \pmod{3}$ , so since 19 is a prime, we get that  $p \nmid 19$ . Hence, by the claim, there is an integer solution  $x$  to  $x^3 \equiv 19 \pmod{p}$ . In other words, there is a solution to  $x^3 - 19 \equiv 0 \pmod{p}$ , or that  $x^3 - 19 = 0$  in  $\mathbb{F}_p$ . The formal derivative for  $x^3 - 19$  is  $3x^2$ . We aim to prove that  $3x^2 \neq 0$  in  $\mathbb{F}_p$ . By contradiction,  $3x^2 = 0$  in  $\mathbb{F}_p$ . We first note that  $\mathbb{F}_p$  is an integral domain and that  $3 \neq p$  thus  $3 \neq 0$  in  $\mathbb{F}_p$ , so  $x^2 = 0$  and consequently,  $x = 0$  in  $\mathbb{F}_p$ . However, this implies that  $x^3 \equiv 0^3 \equiv 19 \pmod{p}$ , a contradiction as  $p \nmid 19$ . Thus, it must be that  $3x^2 \neq 0$  in  $\mathbb{F}_p$ .

By Corollary 11.16, there exists a  $a \in \mathbb{Z}_p$  where  $a^3 - 19 = 0$  in  $\mathbb{Z}_p$ . In other words, by taking the  $k$ -th coordinate of  $a$  as  $a_k$ , we note that  $a_k^3 - 19 = 0$  in  $\mathbb{Z}/p^k\mathbb{Z}$ . Hence, there is a solution for  $x^3 - 19 \equiv 0 \pmod{p^k}$ .  $\square$

- (c) Let us use  $7 \in \mathbb{Z}_3$  and denote  $f(x) = x^3 - 19$ . We note that  $f(7) = 7^3 - 19 = 324$  and that  $f'(7) = 3 \cdot (7)^2 = 147$  in  $\mathbb{Z}_3$ . Note that using the definition from HW6 6 (c):

$$324 = 0 + 0 + 0 + 0 + 1 \cdot 3^4 + 1 \cdot 3^5 + 0 + \dots$$

$$147 = 0 + 1 \cdot 3^1 + 1 \cdot 3^2 + 2 \cdot 3^3 + 1 \cdot 3^4 + 0 + \dots$$

This implies that  $\nu_3(324) = 4$  and  $\nu_3(147) = 1$ . In other words,  $\nu_3(f(7)) > 2\nu_3(f'(7))$ . We can apply Hensel's Lemma and get that there exist a  $\alpha \in \mathbb{Z}_3$  where  $f(\alpha) = 0$ . Take the  $k$ -th coordinate of  $\alpha$  as  $a_k$  and we see that  $a_k^3 - 19 = 0$  in  $\mathbb{Z}/3^k\mathbb{Z}$ . This translates to that there exist an integer solution to  $x^3 - 19 \equiv 0 \pmod{3^k}$ .  $\square$

4. (a) We denote  $f(x) = x^{46} + 69x + 23$ . We first note that both  $23 \mid 69$  and  $23 \mid 23$ . Thus, we note that in  $\mathbb{F}_{23}[x]$ ,  $x^{46} + 69x + 23 = x^{46}$ . By contradiction, let us assume that the polynomial is reducible and that  $f(x) = g(x)h(x)$  in  $\mathbb{Z}[x]$ . This implies that  $g(x)h(x) = x^{46}$  in  $\mathbb{F}_{23}[x]$ .

**Claim:** At least  $g(x)$  or  $h(x)$  is equal to perfect powers, meaning polynomials in the form  $x^k$  in  $\mathbb{F}_{23}[x]$ .

By contradiction, they are both equal to the form  $h(x) = x^m a_{m-1} x^{m-1} + \dots + a_r x^r$  and  $g(x) = x^n + b_{n-1} x^{n-1} + \dots + b_s x^s$  where all  $a_r$  and  $b_r$  are non-zeros in  $\mathbb{F}_{23}$ . We then get  $g(x)h(x) = x^{m+n} + \dots + b_s a_r x^{r+s}$ . Since  $\mathbb{F}_{23}$  is an integral domain, we get that  $b_s a_r x^{r+s}$  is a non-zero term, which implies  $f(x)$  cannot be equal to a perfect power, a contradiction. If only one of them is equal to a perfect power, where we assume  $h(x) = x^m a_{m-1} x^{m-1} + \dots + a_r x^r$  and  $g(x) = x^n$  where  $a_r$  is non-zero in  $\mathbb{F}_{23}$ . This means  $h(x)g(x) = x^{m+n} + \dots + a_r x^{r+n}$  and the same contradiction follows with the non-zero term  $a_r x^{r+n}$ . Hence, both  $g(x)$  and  $h(x)$  are equivalent to perfect powers in  $\mathbb{F}_{23}[x]$ .  $\square$

Since  $g(x) = x^m$  and  $h(x) = x^n$  in  $\mathbb{F}_{23}$ , we note that  $g(0) = 0$  and  $h(0) = 0$  in  $\mathbb{F}_{23}$ . This implies that in  $\mathbb{Z}$ ,  $23 \mid g(0)$  and  $23 \mid h(0)$ , which implies  $23^2 \mid f(0)$ . However, we note that  $f(0) = 23$  and  $23^2 \nmid 23$ , a contradiction. Thus, it must be that  $f(x)$  is irreducible in  $\mathbb{Z}[x]$ .  $\square$

- (b) We consider  $f(x) = x^{46} + 69x + 2025$ . Hence, we note that  $f(0) = 2025$  and  $f'(0) = 46 \cdot 0^{45} + 69 = 69$  in  $\mathbb{Z}_3$ . Note that using the definition from HW6 6 (c):

$$\begin{aligned} 2025 &= 0 + 0 + 0 + 0 + 1 \cdot 3^4 + 2 \cdot 3^5 + 2 \cdot 3^6 + 0 + \dots \\ 69 &= 0 + 2 \cdot 3^1 + 1 \cdot 3^2 + 2 \cdot 3^3 + 0 + \dots \end{aligned}$$

This implies that  $\nu_3(2025) = 4$  and  $\nu_3(69) = 1$ . In other words,  $\nu_3(f(0)) > 2\nu_3(f'(0))$ . We apply Hensel's Lemma and get that there exists an  $\alpha \in \mathbb{Z}_3$  where  $f(\alpha) = 0$ . In other words, we found such root.  $\square$