1. a) We assume $n \in \mathbb{N}$ is a perfect k-th power, so $n = m^k$ for some $m \in \mathbb{N}$. Thus, for all primes p, $\nu_p(n) = \nu_p(m^k) \iff \nu_p(n) = k \cdot \nu_p(m)$. It follows that $k \mid \nu_p(n)$ as desired.

For the converse, we assume for all primes p, $k \mid \nu_p(n)$. Then for each p, there exist a positive integer q_p where
$\nu_p(n) = kq_p$, so $q_p = \frac{\nu_p(n)}{k}$. We then denote m as:

$$m = \prod_p p^{q_p} = \prod_p p^{\frac{\nu_p(n)}{k}}$$

Since all q_p are positive integers and m is positive, $m \in \mathbb{N}$. Thus:

$$m^k = (\prod_p p^{\frac{\nu_p(n)}{k}})^k = \prod_p p^{\nu_p(n)}$$

.

By Corollary 3.14, $n = m^k$. Since $m \in \mathbb{N}$, n is a perfect k-th power.

b) We assume $x, y \in \mathbb{N}$ are coprime and xy is a perfect k-th power. Since they are coprime, $gcd(x, y) = 1$. By Exercise 3.3, for all primes p, $\nu_p(\gcd(x, y)) = min\{\nu_p(x), \nu_p(y)\} \iff \nu_p(1) = 0$. This guarantees that at least one of $\nu_p(x)$ or $\nu_p(y)$ is 0.

Since $xy = m^k$, from a), $k \mid \nu_p(xy)$ for all primes p. However, since $\nu_p(xy) = \nu_p(x) + \nu_p(y)$ and at least one of them is 0, it follows that if $\nu_p(xy) \neq 0$, then either $\nu_p(x)$ or $\nu_p(y)$ is equal to $\nu_p(xy)$. Since $k|0$ and $k \mid \nu_p(xy)$, we see that $k \mid \nu_p(x)$ and $k \mid \nu_p(y)$. If $\nu_p(xy) = 0$, then both are 0, so $k \mid \nu_p(x)$ and $k \mid \nu_p(y)$. Thus, for all primes p, $k \mid \nu_p(x)$ and $k \mid \nu_p(y)$, so from a), both are perfect k-th powers.

c) If $n^2 \mid a^k - n$, then there exists an integer q where $n^2 q = a^k - n \iff n(nq + 1) = a^k$. By Proposition 2.15, the $\gcd(n, nq + 1) = \gcd(n, (nq + 1) - nq) = 1$. This implies n and $nq + 1$ are co-primes. From $n^2 \mid a^k - n$, we get that $n \mid a^k$, so $a^k/n \in \mathbb{N}$. Since $a^k = n(nq + 1)$, it follows that $nq + 1 = a^k/n$, so $nq + 1 \in \mathbb{N}$. From b), since a^k is a perfect k-th power, $nq + 1$ and n are both co-primes while $nq + 1, n \in \mathbb{N}$, we get that both $nq + 1$ and most importantly n are perfect k-th power as desired.

2. a) From Proposition 4.3, for any prime p and any positive integer n,

$$\nu_p(n!) = \sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor = \sum_{k=1}^{\lfloor \log_p n \rfloor} \left\lfloor \frac{n}{p^k} \right\rfloor = \left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \cdots$$

Thus, considering that we see that $\lfloor \frac{n}{p} \rfloor \leq \nu_p(n!)$ and since $\frac{n}{p} - 1 < \lfloor \frac{n}{p} \rfloor \leq \frac{n}{p}$, we get that:

$$\frac{n}{p} - 1 < \lfloor \frac{n}{p} \rfloor \leq \nu_p(n!)$$
$$\frac{n}{p} - 1 < \nu_p(n!)$$

Meanwhile, since for any $k \in \mathbb{N}$, we get that $\lfloor \frac{n}{p^k} \rfloor \leq \frac{n}{p^k}$ and since k extends to infinity, $\lfloor \frac{n}{p^k} \rfloor = 0 < \frac{n}{p^k}$ for some large k. Thus,

$$\nu_p(n!) < \sum_{k=1}^{\infty} \frac{n}{p^k}$$

For the infinite series, we apply the infinite geometric series formula to get:

$$\nu_p(n!) < \sum_{k=1}^{\infty} \frac{n}{p^k} = n \sum_{k=1}^{\infty} \frac{1}{p^k} = n \cdot \frac{\frac{1}{p}}{1 - \frac{1}{p}} = \frac{n}{p-1}.$$

Thus, we got the bounds for $\nu_p(n!)$ as desired:

$$\frac{n}{p} - 1 < \nu_p(n!) < \frac{n}{p-1}.$$

b) For all primes p, from a), we get that $\nu_p(n!) < \frac{n}{p-1}$. Thus:

$$n! = \prod_{p} p^{\nu_p(n!)} = \prod_{p \leq n} p^{\nu_p(n!)} < \prod_{p \leq n} p^{\frac{n}{p-1}}$$
$$\ln(n!) < \ln\left(\prod_{p \leq n} p^{\frac{n}{p-1}} \right)$$
$$\ln(n!) < n \cdot \sum_{p \leq n} \frac{\ln(p)}{p-1}$$
$$\frac{\ln(n!)}{n} < \sum_{p \leq n} \frac{\ln(p)}{p-1}$$

Among the integers $1, 2, \cdots, n$, at least half of them are $\geq n/2$. If n is even, there exist $n/2$ terms that are $\geq n/2$. If n is odd, there exist $\lfloor n/2 \rfloor + 1$ terms that are $\geq n/2$, which is still $\geq n/2$. Thus, in either case:

$$n! > \left(\frac{n}{2}\right)^{\frac{n}{2}}$$
$$\ln(n!) > \ln\left(\left(\frac{n}{2}\right)^{\frac{n}{2}}\right)$$
$$\ln(n!) > \frac{n}{2} \cdot \big(\ln(n) - \ln(2)\big)$$
$$\frac{\ln(n!)}{n} > \frac{1}{2} \cdot \big(\ln(n) - \ln(2)\big)$$

At last, since we know $\ln(n) \to \infty$, for any arbitrary $M \in \mathbb{R}$, we can always find $\ln(n) > 2M + \ln 2 \iff \frac{\ln n - \ln 2}{2} > M$. Since $\frac{\ln(n!)}{n} < \sum_{p \leq n} \frac{\ln(p)}{p-1}$, we get:

$$\sum_{p \leq n} \frac{\ln(p)}{p-1} > \frac{\ln(n!)}{n} > \frac{\ln n - \ln 2}{2} > M$$

$$\sum_{p \leq n} \frac{\ln(p)}{p-1} > M$$

Since our choice of M was arbitrary, $\sum_{p \leq n} \frac{\ln(p)}{p-1}$ can get arbitrarily large thus $\sum_{p \leq n} \frac{\ln(p)}{p-1} \to \infty$ as desired.

3. a) Since a primorial $p_i\#$ is the product of all primes $\leq p_i$, we first take note that for any prime p that for each primorial $p_i\#$ that:

$$\nu_p(p_i\#) = \begin{cases} 1 & \text{if } p \leq p_i \\ 0 & \text{if } p > p_i \end{cases}$$

We assume N can be written as a product of (not necessarily distinct) primorials $p_1\#, \cdots, p_k\#$. Thus, with what was noted:

$$\nu_p(N) = \sum_{i=1}^{k} \nu_p(p_i\#) = \# \text{ number of } p_i \text{ such that } p_i \geq p.$$

Let $\hat{p}, q$ be primes with $q > \hat{p}$. Since any p_i with $p_i \geq q$ also satisfies $p_i \geq \hat{p}$, the number of $p_i \geq \hat{p}$ is at least that of $p_i \geq q$. This gives us the inequality $\nu_{\hat{p}}(N) \geq \nu_q(N)$ as desired.

For the converse, we assume N where for any primes $p < q$ that $\nu_p(N) \geq \nu_q(N)$. We first note that $N \geq 2$, as the cases $N = 1$ and $N = 0$ are trivial: they cannot be expressed by any products of primes, much less primorials. For every prime $p \leq N$, and denoting the finite number of them as k, we order them as $p_1 < p_2 < p_3 < \cdots < p_k$ such that any prime $p > p_k$ will also be $p > N$. We then denote the sequence a_i for $1 \leq i \leq k$ by

$$a_i = \begin{cases} \nu_{p_i}(N) - \nu_{p_{i+1}}(N), & \text{if } i < k, \\ \nu_{p_k}(N), & \text{if } i = k. \end{cases}$$

For any $n < k$, since $p_{n+1} > p_n$ and $\nu_{p_n}(N) \geq \nu_{p_{n+1}}(N)$, we have $a_n \geq 0$. Thus, we use it to denote the product of primorials for a positive integer M as

$$M = \prod_{i=1}^{k} (p_i\#)^{a_i}.$$

For any prime number p, we see that

$$\nu_p(M) = \sum_{i=1}^{k} a_i \cdot \nu_p(p_i\#).$$

Since, as noted, for any $p > p_i$, $\nu_p(p_i\#) = 0$: if $p > p_k$ then all $\nu_p(p_i\#) = 0$ and consequently $\nu_p(M) = 0$. Meanwhile, if $p \leq p_k$ then $p \leq N$, thus p occurs among the p_i as p_j, and any $i \geq j$ has $\nu_p(p_i\#) = 1$, therefore

$$\nu_p(M) = \sum_{i=j}^{k} a_i.$$

Expanding the a_i gives:

$$\nu_p(M) = \nu_{p_k}(N) + (\nu_{p_{k-1}}(N) - \nu_{p_k}(N)) + \cdots + (\nu_{p_{j+1}}(N) - \nu_{p_{j+2}}(N)) + (\nu_{p_j}(N) - \nu_{p_{j+1}}(N)),$$

$$\nu_p(M) = (\nu_{p_k}(N) - \nu_{p_k}(N)) + (\nu_{p_{k-1}}(N) - \nu_{p_{k-1}}(N)) + \cdots + (\nu_{p_{j+1}}(N) - \nu_{p_{j+1}}(N)) + \nu_{p_j}(N),$$

$$\nu_p(M) = \nu_{p_j}(N) = \nu_p(N).$$

For all $p \leq p_k \leq N$, we see that $\nu_p(M) = \nu_p(N)$. Meanwhile, for all $p > p_k$, $\nu_p(M) = 0$. Since any $p > p_k$ is also $p > N$, we get that $\nu_p(N) = 0 = \nu_p(M)$. Hence, for all primes p, $\nu_p(M) = \nu_p(N)$. Thus, since prime factorization is unique, and both are positive integers, we have

$$M = \prod_p p^{\nu_p(N)} = N.$$

Since $M = N$ and M is a product of primorials, N is also a product of primorials as desired.

b) By Proposition 4.3, for any prime p, $\nu_p(2025!)$ can be denoted as:

$$\nu_p(2025!) = \sum_{k=1}^{\infty} \lfloor \frac{2025}{p^k} \rfloor$$

For every k, for a prime q where $q > p$, then $q^k > p^k$, so $\frac{2025}{q^k} < \frac{2025}{p^k} \iff \lfloor \frac{2025}{q^k} \rfloor \leq \lfloor \frac{2025}{p^k} \rfloor$. Thus, also noting that $\lfloor \frac{2025}{q^k} \rfloor = 0$ and $\frac{2025}{p^k} \rfloor = 0$ for a large enough k, which makes their infinite sums finite, we see that:

$$\sum_{k=1}^{\infty} \lfloor \frac{2025}{p^k} \rfloor \geq \sum_{k=1}^{\infty} \lfloor \frac{2025}{q^k} \rfloor$$
$$\nu_p(2025!) \geq \nu_q(2025!)$$

From a), since $q > p$ and $\nu_p(2025!) \geq \nu_q(2025!)$, it implies that 2025! can be expressed as a product of (not necessarily distinct) primorials.

c) We note that $2024 = 2^3 \cdot 11 \cdot 23$. Thus, since the smallest prime before 11 is 7 and the smallest prime before 23 is 19, we get that $11\#/7\# = 11$ and $23\#/19\# = 23$. Thus:

$$2024 = (2\#)^3 \cdot \frac{11\#}{7\#} \cdot \frac{23\#}{19\#}$$

Thus, if $A = 23\# \cdot 11\# \cdot (2\#)^3$ and $B = 19\# \cdot 7\#$, we get that $2024 = A/B$ as desired.

4. a) We claim that:

$$s_2(n) = n - \nu_2(n!)$$

To start, given a positive integer n, if n = 0, then $s_2(0) = 0$ and $v_2(0!) = 0$, so $0 - 0 = 0$. For $n \geq 1$, it can be expressed as a sum of powers of 2 that:

$$n = \sum_{i=0}^{\lfloor \log_2(n) \rfloor} b_i \cdot 2^i \quad b_i \in \{0, 1\}$$

The sum only goes up to $\lfloor \log_2(n) \rfloor$ because $2^{\lfloor \log_2(n) \rfloor + 1} > n$. We also see that b_k represents the value of k-th digit in base 2. Since we wish to find the sum of it, we need to find the relation for b_k, and we start by:

$$n \ \% \ 2^{k+1} = \sum_{i=0}^{\lfloor \log_2(n) \rfloor} b_i \cdot 2^i \ \% \ 2^{k+1}$$

$$n \ \% \ 2^{k+1} = \sum_{i=0}^{k} b_i \cdot 2^i$$

This result is achieved because 2^{k+1} divides any terms 2^i where $i \geq k+1$, so only the $i \leq k$ remains. We then isolate b_k as follows:

$$n \ \% \ 2^{k+1} = b_k \cdot 2^k + \sum_{i=0}^{k-1} b_i \cdot 2^i$$

$$\frac{n \ \% \ 2^{k+1}}{2^k} = b_k + \frac{\sum_{i=0}^{k-1} b_i \cdot 2^i}{2^k}$$

Since $\sum_{i=0}^{k-1} 2^i < 2^k$ and b_i is at most 1, the fraction on the right hand side must be < 1. Since b_i is either 0 or 1, if we apply the floor function on both sides, we get that:

$$\lfloor \frac{n \ \% \ 2^{k+1}}{2^k} \rfloor = b_k$$

With this, since $s_2(n)$ is the sum of all such digit from 0 to $\lfloor \log_2(n) \rfloor$, we can now denote that:

$$s_2(n) = \sum_{k=0}^{\lfloor \log_2(n) \rfloor} b_k$$

$$s_2(n) = \sum_{k=0}^{\lfloor \log_2(n) \rfloor} \lfloor \frac{n \ \% \ 2^{k+1}}{2^k} \rfloor$$

We start by manipulating the terms. By Remark 2.3, since $n \ \% \ 2^{k+1}$ is the remainder when n divides by 2^{k+1}, it is equal to $n - \lfloor \frac{n}{2^{k+1}} \rfloor 2^{k+1}$. Thus:

$$\begin{aligned} \lfloor \frac{n \ \% \ 2^{k+1}}{2^k} \rfloor &= \lfloor \frac{n - \lfloor \frac{n}{2^{k+1}} \rfloor 2^{k+1}}{2^k} \rfloor \\ &= \lfloor \frac{n}{2^k} - 2 \lfloor \frac{n}{2^{k+1}} \rfloor \rfloor \end{aligned}$$

Since $2 \lfloor \frac{n}{2^{k+1}} \rfloor$ is an integer, we can take it outside of the floor function and get:

$$\lfloor \frac{n}{2^k} \rfloor - 2\lfloor \frac{n}{2^{k+1}} \rfloor$$

Returning to the summation, we get as follows:

$$\begin{aligned}
\sum_{k=0}^{\lfloor \log_2 n \rfloor} \left\lfloor \frac{n}{2^k} \right\rfloor - 2\left\lfloor \frac{n}{2^{k+1}} \right\rfloor &= \left\lfloor \frac{n}{2^0} \right\rfloor - 2\left\lfloor \frac{n}{2^1} \right\rfloor + \left\lfloor \frac{n}{2^1} \right\rfloor - 2\left\lfloor \frac{n}{2^2} \right\rfloor + \cdots + \left\lfloor \frac{n}{2^{\lfloor \log_2 n \rfloor}} \right\rfloor - 2\left\lfloor \frac{n}{2^{\lfloor \log_2 n \rfloor + 1}} \right\rfloor \\
&= \left\lfloor \frac{n}{1} \right\rfloor - \sum_{k=1}^{\lfloor \log_2 n \rfloor} \left\lfloor \frac{n}{2^k} \right\rfloor - 2\left\lfloor \frac{n}{2^{\lfloor \log_2 n \rfloor + 1}} \right\rfloor
\end{aligned}$$

Since $n < 2^{\lfloor \log_2 n \rfloor + 1}$, the last term of the subtraction is equal to 0, which gives us the desired result as the summation is now equal to the equation of $\nu_2(n!)$ from Proposition 4.3:

$$\begin{aligned}
s_2(n) &= n - \sum_{k=1}^{\lfloor \log_2 n \rfloor} \left\lfloor \frac{n}{2^k} \right\rfloor \\
&= n - \nu_2(n!)
\end{aligned}$$

b) We prove that $a = 1979$ and $b = 47$ satisfy the criteria. For the first criteria, $1979 + 47 = 2026 \geq 2026$. Note that for $1979! \cdot 47!$:

$$1979! \cdot 47! = 1978! \cdot 46! \cdot 47 \cdot 1979$$

Thus, since $1978 + 46 = 2024$, we see that:

$$\frac{2024!}{1979! \cdot 47!} = \frac{2024!}{1978! \cdot 46!} / (1979 \cdot 47) = \frac{\binom{2024}{1978}}{47 \cdot 1979}$$

Thus, for each prime p, $\nu_p(2024!/(1979! \cdot 47!)) = \nu_p(\binom{2024}{1978}) - \nu_p(1979) - \nu_p(47)$. By Corollary 4.8, we get that $\nu_p(\binom{2024}{1978}) \geq 0$. Since both 47 and 1979 are primes, if $p \neq 47$ *and* $p \neq 1979$, then $\nu_p(1979) = 0$ and $\nu_p(47) = 0$, thus $\nu_p(\binom{2024}{1978}) - \nu_p(1979) - \nu_p(47) = \nu_p(\binom{2024}{1978}) \geq 0$.

If $p = 47$, by Proposition 4.3, we get that:

$$\begin{aligned}
\nu_{47}(2024!) &= \left\lfloor \frac{2024}{47^1} \right\rfloor + \left\lfloor \frac{2024}{47^2} \right\rfloor = 43 + 0 = 43 \\
\nu_{47}(1978!) &= \left\lfloor \frac{1978}{47^1} \right\rfloor + \left\lfloor \frac{1978}{47^2} \right\rfloor = 42 + 0 = 42 \\
\nu_{47}(46!) &= \left\lfloor \frac{46}{47^1} \right\rfloor = 0
\end{aligned}$$

$$\nu_{47}(\binom{2024}{1978}) = \nu_{47}(2024!) - \nu_{47}(1978!) - \nu_{47}(46!) = 43 - 42 - 0 = 1$$

Thus, we see that $\nu_p(\binom{2024}{1978}) - \nu_p(1979) - \nu_p(47) = 1 - 0 - 1 = 0 \geq 0$.

If $p = 1979$, by Proposition 4.3, we get that:

$$\begin{aligned}
\nu_{1979}(2024!) &= \left\lfloor \frac{2024}{1979} \right\rfloor + \left\lfloor \frac{2024}{1979^2} \right\rfloor = 1 + 0 = 1 \\
\nu_{1979}(1978!) &= \left\lfloor \frac{1978}{1979} \right\rfloor = 0 \\
\nu_{1979}(46!) &= \left\lfloor \frac{46}{1979} \right\rfloor = 0
\end{aligned}$$

$$\nu_{1979}(\binom{2024}{1978}) = \nu_{1979}(2024!) - \nu_{1979}(1978!) - \nu_{1979}(46!) = 1 - 0 - 0 = 1$$

Thus, we see that $\nu_p(\binom{2024}{1978}) - \nu_p(1979) - \nu_p(47) = 1 - 1 - 0 = 0 \geq 0$. Thus, for all primes p, $\nu_p(\binom{2024}{1978}) - \nu_p(1979) - \nu_p(47) = \nu_p(2024!) - \nu_p(1979! \cdot 47!) \geq 0$. Hence, by Corollary 3.15, it implies that $1979! \cdot 47! \mid 2024!$, which satisfies the second criteria.