1. (a) Let $\epsilon > 0$ be given, there exist a $L \in \mathbb{R}$ where for $n \geq L$ that $|s_n - s| < \frac{\epsilon}{2}$. Thus for any $n \geq L$, we denote $\epsilon_n = s_n - s$ and $|\epsilon_k| < \frac{\epsilon}{2}$. We then also note that for b_n that:

$$b_n = \frac{s_1 + s_2 + s_3 + \cdots + s_n}{n} = \frac{s_1 + \cdots + s_L}{n} + \frac{s_{L+1} + \cdots + s_n}{n}$$

Since $s_1 + s_2 + \cdots + s_L$ is a constant for any n since $n \geq L$, let M denote the sum. We then apply our notion of ϵ_n combined with there exist $n - L$ terms from $L + 1$ to n to get that:

$$\begin{aligned} |b_n - s| &= |\frac{M}{n} + \frac{s_{L+1} + \cdots + s_n}{n} - \frac{n}{n}s| \\ &= |\frac{M}{n} + \frac{(n-L)s}{n} + \frac{\epsilon_{L+1} + \cdots + \epsilon_n}{n} - \frac{n}{n}s| \\ &= |\frac{M - Ls}{n} + \frac{\epsilon_{L+1} + \cdots + \epsilon_n}{n}| \end{aligned}$$

We then apply the triangle inequality and recalling that $|\epsilon_k| < \frac{\epsilon}{2}$ to get that:

$$\begin{aligned} |b_n - s| &\leq |\frac{M - Ls}{n}| + |\frac{\epsilon_{L+1} + \cdots + \epsilon_n}{n}| \\ &\leq \frac{|M - Ls|}{n} + \frac{|\epsilon_{L+1}| + \cdots + |\epsilon_n|}{n} \\ &\leq \frac{|M - Ls|}{n} + \frac{|n - L|}{n} \cdot \frac{\epsilon}{2} \end{aligned}$$

Since $\frac{|M-Ls|}{n}$ is the constant $\frac{|M-Ls|}{n}$ multiplied by $\frac{1}{n}$ which converges to 0, by the Limit Laws, we can find an $N > L$ where for any $n \geq N$, we get that $\frac{|M-Ls|}{n} < \frac{\epsilon}{2}$. Since $\frac{|n-L|}{n} < 1$, it follows that that $|b_n - s| \leq \frac{|M-Ls|}{n} + \frac{|n-L|}{n} \cdot \frac{\epsilon}{2} < \epsilon$. Thus, we get $|b_n - s| < \epsilon$ for all $n \geq N$ as desired.

(b) We construct a_n by the rational r_n where $x - \frac{1}{n} < r_n < x$. By the density of $\mathbb{Q}$, r_n is guaranteed, so we assign $a_n = r_n$. Thus since $a_n < x < x + \frac{1}{n}$, we get that $x - \frac{1}{n} < a_n < x + \frac{1}{n}$, so $|a_n - x| < \frac{1}{n}$. Since $\frac{1}{n} \to 0$, let $\epsilon > 0$ be given, there exist a $N \in \mathbb{R}$ where for all $n \geq N$, $|a_n - x| < \frac{1}{n} < \epsilon$ thus we get $|a_n - x| < \epsilon$ as desired.

2. (a) Since $1 > L$, let $\epsilon = \frac{1-L}{2}$, there exist a natural number M where for all $n \geq M$, $|\frac{s_{n+1}}{s_n} - L| < \epsilon$. This implies that $L - \epsilon < \frac{s_{n+1}}{s_n} < L + \epsilon$, so observe that $|\frac{s_{n+1}}{s_n}| < L + \frac{1-L}{2} < 1$. We then denote that $a = L + \frac{1-L}{2}$ thus $0 < a < 1$.

We then note that for $n > M$:

$$\prod_{i=M}^{n-1} |\frac{s_{i+1}}{s_i}| = |\frac{s_{M+1}}{s_M}| \cdot |\frac{s_{M+2}}{s_{M+1}}| \cdots |\frac{s_{n-1}}{s_{n-2}}| \cdot |\frac{s_n}{s_{n-1}}| = |\frac{s_n}{s_M}|$$

Since there exist $n - M$ terms between $M \leq i \leq n-1$ and all of them are $< a$, we get that:

$$\begin{aligned} |\frac{s_n}{s_M}| &< a^{n-M} \\ |s_n| &< a^{n-M}|s_M| \end{aligned}$$

We now prove the convergence. Let $\epsilon > 0$ be given, select $N = \log_a(\frac{\epsilon a^M}{|s_M|}) + M$. Thus, for all $n \geq N$, we get that $n > \log_a(\frac{\epsilon a^M}{|s_M|})$. Since $0 < a < 1$, the function $\log_a$ is decreasing, so we reverse the inequalities to get that:

$$\begin{aligned} n &> \log_a(\frac{\epsilon a^M}{|s_M|}) \\ a^n &< \frac{\epsilon a^M}{|s_M|} \\ a^{n-M}|s_M| &< \epsilon \end{aligned}$$

Since $n > M$, we can apply the inequality for $|s_n|$ for all $n \geq N$ to get that:

$$\begin{aligned} |s_n| &< a^{n-M}|s_M| < \epsilon \\ |s_n - 0| &< \epsilon \end{aligned}$$

Thus, $s_n \to 0$.

Proposition (proven in class) Let $(a_n) \subseteq (0, \infty)$, then $\frac{1}{a_n} \to 0 \iff a_n \to \infty$

(b) We denote the series $t_n = \frac{1}{|s_n|}$. Since $|\frac{s_{n+1}}{s_n}| \to L$, $|\frac{t_{n+1}}{t_n}| = |\frac{\frac{1}{s_{n+1}}}{\frac{1}{s_n}}| = \frac{1}{|\frac{s_{n+1}}{s_n}|} \to \frac{1}{L}$ by the limit laws. Since $L > 1$, it follows that $\frac{1}{L} < 1$, so part a) applies to get $t_n \to 0$. Thus, by the Proposition, it implies that $\frac{1}{t_n} = |s_n| \to \infty$ as desired.