1. We analyze that $\pi(290) = 61$, so there exist 61 possible prime factors for $\binom{290}{145}$. We see that $290 = 17^2 + 1$ and $\pi(17) = 7$, so there are 7 primes where their square is less than 290.

   These primes are $2, 3, 5, 7, 11, 13, 17$. We evaluate their p-adic valuation for $\binom{290}{145}$ using the same technique in Example 3.7 by converting 290 and 145 to the base of each prime to easily evaluate $\%\ p^k$ for the result in Lemma 3.8 combined with Proposition 3.5 (Legendre's formula).

p	290	145	$\nu_p(\binom{290}{145})$
2	100100010_2	10010001_2	3
3	101202_3	12101_3	2
5	2130_5	1040_5	1
7	563_7	265_7	2
11	244_{11}	122_{11}	0
13	194_{13}	$(11)2_{13}$	1
17	101_{17}	89_{17}	2

   Among the 7, there exists 1 with $\nu_p(\binom{290}{145}) = 3$, 3 with $\nu_p(\binom{290}{145}) = 2$, and 2 with $\nu_p(\binom{290}{145}) = 1$.

   Meanwhile, there exist $\pi(290) - \pi(145) = 27$ primes that are > 145. This implies there exist $61 - 27 - 7 = 27$ primes that are < 145 but whose squares are larger than 290. Thus, for each of these primes p and $k \geq 2$, we get that $\lfloor 290/p^k \rfloor = 0$ and $\lfloor 145/p^k \rfloor = 0$. Hence, by Proposition 3.5, $\nu_p(\binom{290}{145}) = \lfloor 290/p \rfloor - \lfloor 145/p \rfloor - \lfloor 145/p \rfloor$. This is in the form of Lemma 3.8, so $\nu_p(\binom{290}{145}) = 1$ if $290\%p < 145\%p$ and 0 otherwise. We evaluate the 27 primes and their p-adic values with the table as follows.

p	$290\%p$	$145\%p$	$\nu_p\left(\binom{290}{145}\right)$
19	5	12	1
23	14	7	0
29	0	0	0
31	11	21	1
37	31	34	1
41	3	22	1
43	32	16	0
47	8	4	0
53	25	39	1
59	54	27	0
61	46	23	0
67	22	11	0
71	6	3	0
73	71	72	1

p	$290\%p$	$145\%p$	$\nu_p\left(\binom{290}{145}\right)$
79	53	66	1
83	41	62	1
89	23	56	1
97	96	48	0
101	88	44	0
103	84	42	0
107	76	38	0
109	72	36	0
113	64	32	0
127	36	18	0
131	28	14	0
137	16	8	0
139	12	6	0

Among the 27, there exist 9 with $\nu_p(\binom{290}{145}) = 1$.

For the remaining $\pi(290) - \pi(145) = 27$ primes that are greater than 145. From Lemma 3.7, for all such primes p, we see that $\nu_p(\binom{290}{145}) = 1$. Thus, there exist an additional 27 primes with $\nu_p(\binom{290}{145}) = 1$.

In total, among the 61 possible prime factors for $\binom{290}{145}$, there exist 38 with $\nu_p(\binom{290}{145}) = 1$, 3 with $\nu_p(\binom{290}{145}) = 2$, and 1 with $\nu_p(\binom{290}{145}) = 3$. By Corollary 2.11, the number of positive divisors is $(1+1)^{38} \cdot (2+1)^3 \cdot (3+1)^1$. Thus, there exist $2^{40} \cdot 3^3$ positive divisors for $\binom{290}{145}$.

2. a) Assume $o(z) \mid m$. Thus, there exists an integer q where $m = q \cdot o(z)$. By definition, we see that $z^{o(z)} = 1$, thus $z^m = z^{q \cdot o(z)} = (z^{o(z)})^q = 1^q = 1$ as desired.

For the converse, assume $z^m = 1$. By the Division Algorithm, for some integers q and r, we see that $m = q \cdot o(z) + r$ where $0 \le r < o(z)$. Since $z^m = z^{o(z)q} \cdot z^r = 1$ and $z^{o(z)q} = 1^q = 1$, we see that $z^r = 1$. However, if $0 < r$ and $r < o(z)$, it contradicts the minimality of $o(z)$. Thus, r must be 0, which gives us that $m = q \cdot o(z)$ or $o(z) \mid m$ as desired.

b) We denote p as the $\gcd(o(a), k)$, so from Proposition 1.6, $p \mid o(a)$ and $p \mid k$. Thus for some integers q_1, q_2, we get that $o(a) = pq_1$ and $k = pq_2$. Thus, we see that $(a^k)^{q_1} = a^{pq_2q_1} = a^{o(a) \cdot q_2} = 1^{q_2} = 1$.

Now we assume there exists a positive integer $r < q_1$ where $(a^k)^r = 1$. Since $a^{rk} = 1$, from a), we get that $o(a) \mid rk$, so there exists an integer $\hat{r}$ where $\hat{r}o(a) = rk$. By Corollary 2.1 (Bezout's Lemma), there exist integers x, y where $\gcd(o(a), k) = o(a)x + ky$. If we multiply both sides by r, we get that:

$$\begin{aligned}
\gcd(o(a), k)r &= o(a)xr + kyr = o(a)xr + o(a)\hat{r}y = o(a)(xr + \hat{r}y) \\
\gcd(o(a), k)r &= \gcd(o(a), k)q_1(xr + \hat{r}y) \\
r &= q_1(xr + \hat{r}y)
\end{aligned}$$

Thus, $q_1 \mid r$, which implies $q_1 \le r$, a contradiction. This proves q_1 is the smallest positive integer d where $(a^k)^d = 1$, so $o(a^k) = q_1 = \frac{o(a)}{\gcd(o(a),k)}$ as desired.

c) We claim that the order for $-3 \pmod{11^k}$ for any positive integer k is:

$$o_{11^k}(-3) = \begin{cases} 10 & \text{if } k = 1 \\ 10 \cdot 11^{k-2} & \text{if } k \ge 2 \end{cases}$$

To start, we prove by induction for $k \ge 2$ that $3^{5 \cdot 11^{k-2}} \equiv 1 \pmod{11^k}$. For $k = 2$, $3^5 - 1 = 242 = 11^2 \cdot 2$, so $3^5 \equiv 1 \pmod{11^2}$. We then assume the induction hypothesis on k, and for $k + 1$, we get that:

$$3^{5 \cdot 11^{k-1}} - 1 = (3^{5 \cdot 11^{k-2}} - 1)\big((3^{5 \cdot 11^{k-2}})^{10} + (3^{5 \cdot 11^{k-2}})^9 + \cdots + 1\big)$$

Each term in the sum except 1 can be expressed as 3^{5z} for some positive integer z, and since $3^5 \equiv 1 \pmod{11}$, we get that $3^{5z} \equiv 1 \pmod{11}$. Meanwhile, since $1 \equiv 1 \pmod{11}$ and there are 11 terms in the sum, we know that 11 divides the sum and the sum is equal to $11q$ for some positive integer q. Thus, by the inductive hypothesis, $3^{5 \cdot 11^{k-2}} - 1 = 11^k \hat{q}$ for some positive integer $\hat{q}$, so we get $3^{5 \cdot 11^{k-1}} - 1 = 11^k \hat{q} \cdot 11q = 11^{k+1}\hat{q}q$ or $3^{5 \cdot 11^{k-1}} \equiv 1 \pmod{11^{k+1}}$ as desired.

Now that we know for $k \ge 2$ that $3^{5 \cdot 11^{k-2}} \equiv 1 \pmod{11^k}$, we get that $(3^2)^{5 \cdot 11^{k-2}} = ((-3)^2)^{5 \cdot 11^{k-2}} = (-3)^{10 \cdot 11^{k-2}} \equiv 1 \pmod{11^k}$. For $k = 1$, $(-3)^{10} - 1 = 11^2 \cdot 488$, so $(-3)^{10} \equiv 1 \pmod{11}$.

We now prove its minimality. For $k = 1$ and $k = 2$, it suffices to state that for positive integers $1 \le i < 10$, $(-3)^i \not\equiv 1 \pmod{11}$, much less $\pmod{11^2}$. For $k \ge 3$, we first demonstrate that for any positive integer m and with Proposition 4.15 (LTE) that:

$$\begin{aligned}
\nu_{11}\big((-3)^{10 \cdot 11^m} - 1\big) &= \nu_{11}\big((-3)^{10} - 1\big) + \nu_{11}(10 \cdot 11^m) \\
&= 2 + m.
\end{aligned}$$

Thus, if we substitute $m = k - 3$, we get $\nu_{11}\big((-3)^{10\cdot 11^{k-3}} - 1\big) = k - 1$. This implies $11^k \nmid \big((-3)^{10\cdot 11^{k-3}} - 1\big)$ or $(-3)^{10\cdot 11^{k-3}} \not\equiv 1 \pmod{11^k}$. It also implies $o_{11^k}(-3) \nmid 10 \cdot 11^{k-3}$, as if not, then $10 \cdot 11^{k-3} = o_{11^k}(-3)q$ for some positive integer q and $(-3)^{o_{11^k}(-3)\cdot q} \equiv 1^q \pmod{11^k}$, which is a contradiction. Thus, we rule out $10 \cdot 11^{k-3}$

We also note that $(-3)^{5\cdot 11^{k-2}} \equiv -1 \pmod{11^k}$, so $o_{11^k}(-3) \nmid 5 \cdot 11^{k-2}$. Moreover, because $o_{11}(-3) = 10$, by Proposition 5.7 as -3 and 11 are coprime, we see that any exponent e yielding 1 modulo 11^k is also modulo 11, so $10 \mid e$, which rules out 11^{k-2} and $2 \cdot 11^{k-2}$.

Since $o_{11^k}(-3) \mid 10 \cdot 11^{k-2}$ by Proposition 5.7, and we have excluded all proper divisors that could remain, it follows that $o_{11^k}(-3) = 10 \cdot 11^{k-2}$ for $k \geq 3$.

Combining everything that was proven, we get that:

$$\begin{aligned} o_{11^k}(-3) &= \begin{cases} 10 & \text{if } k = 1 \\ 10 & \text{if } k = 2 \\ 10 \cdot 11^{k-2} & \text{if } k \geq 3 \end{cases} \\ &= \begin{cases} 10 & \text{if } k = 1 \\ 10 \cdot 11^{k-2} & \text{if } k \geq 2 \end{cases} \end{aligned}$$

as desired.

3. a) We note that $2^{2^n} \equiv -1 \pmod{p}$, so $2^{2^{n+1}} \equiv 1 \pmod{p}$. However, since $2^{2^n} \not\equiv 1 \pmod{p}$, we note that $o_p(2) \nmid 2^n$, since if it did, then $2^n = o_p(2)q$ for some positive integer q, which implies $2^{2^n} = 2^{o_p(2)q} \equiv 1^q \pmod{p}$, a contradiction. Yet, by Proposition 5.7 and since p and 2 are coprime, we get that $o_p(2) \mid 2^{n+1}$. The only divisor of 2^{n+1} that doesn't divide 2^n is itself, so $o_p(2) = 2^{n+1}$. By Theorem 1.13 (Fermat's Little Theorem), we get that $2^{p-1} \equiv 1 \pmod{p}$. By Proposition 5.7, we see that $2^{n+1} \mid p-1$ or $p \equiv 1 \pmod{2^{n+1}}$ as desired.

b) We note that $5 \cdot 2^7 \equiv -1 \pmod{641}$, so $5^4 \cdot 2^{28} \equiv 1 \pmod{641}$, and $2^4 \equiv -5^4 \pmod{641}$. Thus, we get that $5^4 \cdot 2^{32} \equiv -5^4 \pmod{641}$ or $641 \mid 5^4 \cdot 2^{32} + 5^4 = 5^4(2^{32}+1)$. Since 641 is a prime, $641 \nmid 5^4$, so by Proposition 2.1 (Euclid's Lemma), we get that $641 \mid 2^{32}+1$ as desired.

c) We note that $7 \cdot 2^{14} \equiv -1 \pmod{p}$, so $7^{2^{12}} \cdot 2^{14 \cdot 2^{12}} \equiv (-1)^{2^{12}} = 1 \pmod{p}$. Then we note that $2^{2^{12}} \equiv -1 \pmod{p}$, so $2^{2^{12} \cdot 14} \equiv (-1)^{14} = 1 \pmod{p}$. This implies that $7^{2^{12}} \equiv 1 \pmod{p}$, since we're given 1 and 1 is the only integer x where $1 \cdot x = 1$. Meanwhile, we note that $7^{2^{11}} \cdot 2^{2^{11} \cdot 14} = 2^{2^{12} \cdot 7} \equiv (-1)^{2^{11}} = 1 \pmod{p}$. We also note that $2^{2^{12} \cdot 7} \equiv (-1)^7 = -1 \pmod{p}$, so it implies $7^{2^{11}} \not\equiv 1 \pmod{p}$, as $1 \cdot -1 \neq 1$. Thus, $o_p(7) \nmid 2^{11}$ because if it did, then $2^{11} = o_p(7)q$ for some positive integer q and we get that $7^{o_p(7)q} \equiv 1^q \pmod{p}$, a contradiction. Yet, by Proposition 5.7, since p and 7 are coprime, we get that $o_p(7) \mid 2^{12}$, but the only divisor of 2^{12} that cannot divide 2^{11} is itself. Thus, we get that $o_p(7) = 2^{12}$.

4. a) For $n = 1$, the only divisor of 1 is itself, so $\sum_{d|n} \mu(d) = \mu(1) = 1$.

For $n > 1$, we note that n has a finite number of prime factors, so we denote k as the number of unique prime factors of n or all primes p where $\nu_p(n) \geq 1$. Since the Mobius function converts divisors $d \mid n$ that are not squarefree into 0, they contribute nothing to the sum, so we only focus on the number of divisors that are squarefree, which implies that $\nu_p(d) < 2$ for all prime factors p of n (otherwise $p^2 \mid d$).

Thus, all divisors d of n that are squarefree either have $\nu_p(d) = 0$ or $\nu_p(d) = 1$. We note that each divisor can have up from $0 \leq i \leq k$ distinct prime factors that are also prime factors of n. For each i, the number of divisors with i prime factors is determined by the number of $\nu_p(d) = 1$ it has from k prime factors. Thus, the number of divisors with i unique prime factors is equal to $\binom{k}{i}$. We also note that for all divisors d with i unique prime factors, we get that $\mu(d) = (-1)^i$. Thus:

$$\sum_{d|n} \mu(d) = \sum_{i=0}^{k} \binom{k}{i} \cdot (-1)^i$$

This summation looks familiar to Theorem 3.6 (Binomial Theorem), thus:

$$\sum_{i=0}^{k} \binom{k}{i} \cdot (-1)^i \cdot (1)^{k-i} = (-1+1)^k = 0$$

.

Thus, we proved that for $n > 1$, $\sum_{d|n} \mu(d) = 0$ and for $n = 1$, $\sum_{d|n} \mu(d) = 1$ as desired.

b) We start with the expression that:

$$\sum_{d|n} \mu(d) f(n/d) = \sum_{d|n} \mu(d) \sum_{e|(n/d)} g(e).$$

We note that $d \mid n$ and $e \mid (n/d)$. It follows that there exists an integer q such that $n/d = eq$, or $n = (de)q$, so $de \mid n$. Thus every pair (d, e) corresponds to a divisor of n. Instead of fixing d, we fix e instead, and the possible d for $de \mid n$ are exactly those with $d \mid n/e$. Thus, we get:

$$\sum_{d|n} \mu(d) \sum_{e|(n/d)} g(e) = \sum_{e|n} g(e) \sum_{d|n/e} \mu(d).$$

From a), we know that $\sum_{d|n/e} \mu(d) = 0$ for all $n/e > 1$ and $\sum_{d|n/e} \mu(d) = 1$ for $n/e = 1$. Thus, for $n/e > 1$, they contribute nothing to the sum. Meanwhile, since $n/n = 1$, only $e = n$ matters. Thus:

$$\sum_{e|n} g(e) \sum_{d|n/e} \mu(d) = g(n) \cdot 1 = g(n).$$

Thus, we got $g(n) = \sum_{d|n} \mu(d) f(n/d)$ as desired.