1. Since $f(X)$ is countable, we can denote $f(X) = \{r_1, r_2, \cdots\}$ with every $r_n \in \mathbb{R}$. We then denote $A_n = f^{-1}(\{r_n\}) = \{x \in X : f(x) = r_n\}$. We note that $\cup_{n=1}^{\infty} A_n = X$. Since (X, d) is complete, by Baire Category Theorem 2, X is of second category. In other words, it is not of first category, which implies that there must exist an A_k that is not no-where dense.

   Claim: If $f : X \to Y$ is cts then for a closed set $C \subseteq Y$, $f^{-1}(C)$ is closed.

   By contradiction, $f^{-1}(C)$ is not closed. Then, there exist an $(x_n) \subseteq f^{-1}(C)$ where $x_n \to x \notin f^{-1}(C)$. However, f is continuous, so $f(x_n) \to f(x)$. Thus, $f(x) \in C$ since $(f(x_n)) \subseteq C$ and C is closed. This implies that $x \in f^{-1}(C)$, a contradiction. $\square$

   We note that $\{r_k\}$ is a closed set in $\mathbb{R}$. From the claim, it implies that A_k is closed as well. Hence, $Int(\overline{A_k}) = Int(A_k)$ is non-empty. Since $Int(A_k) \subseteq A_k$ and is open, we get that $f(Int(A_k)) = \{r_k\}$, so f is constant on $Int(A_k)$. $\square$

2. (a) For $d(v,B)$, let us denote $i = \inf\{||v-b|| : b \in B\}$. For every $n \in \mathbb{N}$, we note that there exist a $b_1 \in B$ where $i + 1/n < ||v - b_n||$. Hence, we note that $|||v-b_n|| - i| < 1/n$. Thus, since $1/n \to 0$, we get that $||v - b_n|| \to i$. Since $(b_n) \subseteq B$ and B is compact, we note that (b_n) has a converging subsequence $b_{n_k} \to b \in B$ and that $||v - b|| \geq i$. We note that $||v-b|| \leq ||v-b_n|| + ||b_n - b||$. Since our choice of n is arbitrary, $||v - b|| \leq i + 0$. Hence, we get that $||v - b|| = i$, so i is a minimum in $\{||v-b|| : b \in B\}$.

For $d(A,B)$, let us denote $s = \sup\{d(a,B) : a \in A\}$. We select a $a_n \in A$ where $s - 1/n < d(a_n, B)$. Since, $1/n \to 0$, we note that $s - d(a_n, B) < 1/n$ and $d(a_n, B) \to s$. Note that $(a_n) \subseteq A$ and A is compact. This implies that there exists a (a_{n_k}) where $a_{n_k} \to a \in A$. We now consider the following for any given $b \in B$:

$$||a-b|| \leq ||a - a_{n_k}|| + ||a_{n_k} - b||$$

We note since $d(a,B)$ is the minimum, we get that $d(a,B) \leq ||a-b||$. Since this holds true for any $b \in B$, we can select b' where $||a_{n_k} - b'|| = d(a_{n_k}, B)$. Hence:

$$\begin{aligned} d(a,B) &\leq ||a - a_{n_k}|| + d(a_{n_k}, B) \\ d(a,B) - d(a_{n_k}, B) &\leq ||a - a_{n_k}|| \end{aligned}$$

We then consider:

$$||a_{n_k} - b|| \leq ||a_{n_k} - a|| + ||a - b||$$

We can apply a similar logic where $d(a_{n_k}, B) \leq ||a_{n_k} - b||$ and select the $b' \in B$ where $d(a,B) = ||a - b'||$. This gets us:

$$d(a_{n_k}, B) - d(a,B) \leq ||a_{n_k} - a||$$

This implies that $|d(a_{n_k}, B) - d(a,B)| \leq ||a_{n_k} - a||$. Note that $||a_{n_k} - a|| \to 0$, so it follows that $d(a_{n_k}, B) \to d(a,B)$. We then note that $(d(a_{n_k}, B))$ is a subsequence of $(d(a_n, B))$, so $(d(a_{n_k}, B)) \to s$. Limits are unique, so $s = d(a,B)$. Since $s \in \{d(a,B) : a \in A\}$, it is a maximum. $\square$

(b) Consider the sets $A = \{(0,1),(5,1)\}$ and $B = \{(3,1),(4,1)\}$. (Note that on the midterm, we proved that any finite site is compact because any sequence with it must have a constant subsequence). We note that $d(A,B) = ||(0,1) - (3,1)|| = 3$ and $d(B,A) = ||(3,1) - (5,1)|| = 2$. Hence, $d(A,B) \neq d(B,A)$. $\square$

(c) **Property 1:** $D(A,B) = 0$ iff $A = B$.

We assume $A = B$, then $D(A,B) = 0$ because for any $a \in A$, $d(a,B) = 0$ because the minimum value is when we select $b \in B$ where $a = b$. Since $\{d(a,B) : a \in A\} = \{0\}$, the suprenum is 0, so $d(A,B) = 0$. We can follow the same train of logic to get that $d(B,A) = 0$. Thus, $D(A,B) = \max\{0\} = 0$.

For the converse, if $D(A,B) = 0$, then $\max d(A,B), d(B,A) = 0$. Note that the $||x-y|| \geq 0$, so it suffices to state that $d(A,B) \geq 0$. Since 0 is the minimum, $d(A,B) = d(B,A) = 0$. For $d(A,B)$, it implies that for every $a \in A$, the minimum of $\{||a-b|| : b \in B\}$ is 0. This implies that there is a point $b \in B$ where $a = b$. Hence, $A \subseteq B$. Applying the vice-versa gives us $B \subseteq A$ thus $A = B$. $\square$

Property 2: $D(A,B) = D(B,A)$

For the second property, it suffices to note that $D(A,B) = \max\{d(A,B), d(B,A)\}$ and $D(A,B) = \max\{d(B,A), d(A,B)\}$. We note that $\max\{d(A,B), d(B,A)\} = \max\{d(B,A), d(A,B)\}$, so $D(A,B) = D(B,A)$. $\square$

Property 3: $D(A,C) \leq D(A,B) + D(B,C)$ for any arbitrary $A, B, C \in X$.

We first prove the following:

Claim: $d(A,C) \leq d(A,B) + d(B,C)$.

From (a), we note that $d(A,C) = d(a',C)$ for some $a' \in A$ and $d(a',C) = \|a' - c'\|$ for some $c' \in C$. Since $\|a' - c'\|$ is the minimum, for any $c \in C$ we have $\|a' - c'\| \leq \|a' - c\|$. By the same reasoning from (a), we also have $d(a',B) = \|a' - b'\|$ for some $b' \in B$, and $d(b',C) = \|b' - \hat{c}\|$ for some $\hat{c} \in C$. We now put these together with the following inequality:

$$\begin{aligned} d(A,C) &= \|a' - c'\| \\ &\leq \|a' - \hat{c}\| \\ &\leq \|a' - b'\| + \|b' - \hat{c}\| \\ &\leq d(a',B) + d(b',C). \end{aligned}$$

Lastly, since $d(A,B)$ and $d(B,C)$ are maxima from (a), we have $d(A,B) \geq d(a',B)$ and $d(B,C) \geq d(b',C)$. Substituting these gives the desired inequality

$$d(A,C) \leq d(A,B) + d(B,C).$$

$\square$

Since our selection of A, C is arbitrary, we can apply the claim to also get

$$d(C,A) \leq d(C,B) + d(B,A) = d(B,A) + d(C,B).$$

We then note that since $D(A,B)$ is the maximum, $D(A,B) \geq d(A,B)$ and $D(A,B) \geq d(B,A)$. The same holds true for $D(B,C)$. Hence:

$$d(A,C) \leq d(A,B) + d(B,C) \leq D(A,B) + D(B,C),$$

$$d(C,A) \leq d(B,A) + d(C,B) \leq D(A,B) + D(B,C).$$

Since $D(A,C)$ is either $d(A,C)$ or $d(C,A)$, we get that $D(A,C) \leq D(A,B) + D(B,C)$ as desired. $\square$