1. Let V be a vector space over a field $\mathbb{F}$. Let $b, c \in V$ and a be their additive inverse. Clearly, $a + b = 0$ and $a + c = 0$, so inversely, b, c are the additive inverses of a. With $1 \in \mathbb{F}$, we note that $(1 + (-1))a = a + (-1 \cdot a) = 0$. We now consider $a + b + (-1 \cdot a) = (-1 \cdot a)$ and $a + c + (-1 \cdot a) = (-1 \cdot a)$. By the communative property of vector space, we get $a + (-1 \cdot a) + b = b = (-1 \cdot a)$ and $a + (-1 \cdot a) + c = c = (-1 \cdot a)$. Thus, we get that $b = (-1 \cdot a) = c$, so $b = c$. Hence, additive inverses are unique.

2. We first prove that $\{w_1, w_2, ..., w_n\}$ provides a scalar representation of any $b \in V$ over $\mathbb{F}$. By the definition of basis, there exists scalars $b_1, ..., b_n$ where:

$$\sum_{i=1}^{n} b_i v_i = b$$

Hence, it follows that

$$\sum_{i=1}^{n} b_i \left(\sum_{j=1}^{n} a_j^i w_j \right) = b$$

$$\sum_{j=1}^{n} \left(\sum_{i=1}^{n} a_j^i b_i \right) w_j = b$$

We now prove that this repsentation is unique for b. We denote the current representation as $\beta_1, ..., \beta_n$. We then denote another scalar representation of b from $\{w_1, w_2, ..., w_n\}$ as $\delta_1, ..., \delta_n$. Suppose there exist an i where $\delta_i \neq \beta_i$. It then follows that:

$$\sum_{i=1}^{n} (\beta_i - \delta_i) w_i = b - b = 0$$

We created a scalar representation of $0 \in V$ from $\{w_1, w_2, ..., w_n\}$. Note that there exist a $(\beta_i - \delta_i) \neq 0$. Let us select v_1 and notice that:

$$\sum_{i=1}^{n} (a_i^1 + (\beta_i - \delta_i)) w_i = v_1 + 0 = v_1$$

There exist an i where $a_i^1 + (\beta_i - \delta_i) \neq a_i$ as $(\beta_i - \delta_i) \neq 0$. We created another scalar representation for v_1 from $\{w_1, w_2, ..., w_n\}$. This contradicts that their scalar representation is unique. Hence, it must be that there does not exist an i where $\delta_i \neq \beta_i$. In other words, it must be that $\beta_i = \delta_i$ and scalar representation of b from $\{w_1, w_2, ..., w_n\}$ is unique. Thus, it is a basis of V over $\mathbb{F}$

3. (a) No. For example, for P^1, a basis would be $\{x+1, x\}$. We note that the transformed set $\{x, x\} = \{x\}$ is not a basis because no linear combination can create $1 \in P^1$.

   (b) For any f, g in $P^{n,0}$, we note that $(f+g)(0) = 0 + 0 = 0$, so $f + g \in P^{n,0}$ and is closed under addition. Furthermore, for any scalar $c \in \mathbb{C}$, $(c \cdot f)(0) = c \cdot 0 = 0$. Hence, $c \cdot f \in P^{n,0}$ and is closed under scalar multiplication. Lastly, it contains 0 from P^n. Since $P^{n,0} \subseteq P^n$, by the subspace test, it is a vector space.

   (c) We denote the set of the derivatives from B as B'. We first prove that B' is spanning system for P^{n-1}. Let $f \in P^{n-1}$ and define it as $f = \sum_{i=0}^{n-1} a_i x^i$ with $a_i \in \mathbb{C}$. We then denote $\hat{f} = \sum_{i=0}^{n-1} \frac{a_i}{i+1} x^{i+1}$. It should be noted that $\hat{f}(0) = 0$ and $\hat{f}' = f$. Since $\hat{f}' \in P^{n,0}$, there exist a linear combination with scalars α_i where:

$$\hat{f} = \alpha_1 f_1 + \dots + \alpha_n f_n$$

   From the additive property of derivatives, we note that:

$$\hat{f}' = f = \alpha_1 f_1' + \dots + \alpha_n f_n'$$

   Hence, B' is a spanning system for P^{n-1}. We now prove that B' is also linearly independent. Let us assume it is not. Then there exists a set of scalars β_i where:

$$0 = \beta_1 f_1' + \dots + \beta_n f_n' \quad \text{where} \quad \exists i \text{ that } \quad \beta_i \neq 0$$

   In this case, let us denote $g = \beta_1 f_1 + \dots + \beta_n f_n$. By the additive property of derivatives, we note that $g' = \beta_1 f_1' + \dots + \beta_n f_n' = 0$. This implies that $g(x) = c$ for $c \in \mathbb{C}$. If $c = 0$, B has a non-trival representation of 0 thus not a basis, a contradiction. If $c \neq 0$, then $g(0) \neq 0$, which implies that $P^{n,0}$ is not closed under addition, another contradiction. Hence, it implies that our earlier assumption is incorrect, so there does not exist a $\beta_i \neq 0$.

   We proved B' to be a spanning system and linearly independent for P^{n-1}. By a proposition in class, this implies it is a basis for P^{n-1}

4. Using the operations of addition and scalar multiplication provided, we aim to prove V_C satisfies the 7 axioms for a vector space. Let $\vec{v_1} + i\vec{v_2}, \vec{u_1} + i\vec{u_2}, \vec{w_1} + i\vec{w_2} \in V_C$ and define the scalars $\alpha, \beta, \beta_1, \beta_2 \in \mathbb{C}$. Let $\alpha = a + bi$ and $\beta = c + di$.

   1. **Communitivity:** Since addition in V is commutative, we have:

$$\begin{aligned}(\vec{v_1} + i\vec{v_2}) + (\vec{u_1} + i\vec{u_2}) &= (\vec{v_1} + \vec{u_1}) + i(\vec{v_2} + \vec{u_2}) \\ &= (\vec{u_1} + \vec{v_1}) + i(\vec{u_2} + \vec{v_2}) \\ &= (\vec{u_1} + i\vec{u_2}) + (\vec{v_1} + i\vec{v_2})\end{aligned}$$

   2. **Associativity:** Since addition in V is associative, we have:

$$\begin{aligned}[(\vec{v_1} + i\vec{v_2}) + (\vec{u_1} + i\vec{u_2})] + (\vec{w_1} + i\vec{w_2}) &= [(\vec{v_1} + \vec{u_1}) + i(\vec{v_2} + \vec{u_2})] + (\vec{w_1} + i\vec{w_2}) \\ &= [(\vec{v_1} + \vec{u_1}) + \vec{w_1}] + i[(\vec{v_2} + \vec{u_2}) + \vec{w_2}] \\ &= [\vec{v_1} + (\vec{u_1} + \vec{w_1})] + i[\vec{v_2} + (\vec{u_2} + \vec{w_2})] \\ &= (\vec{v_1} + i\vec{v_2}) + [(\vec{u_1} + \vec{w_1}) + i(\vec{u_2} + \vec{w_2})] \\ &= (\vec{v_1} + i\vec{v_2}) + [(\vec{u_1} + i\vec{u_2}) + (\vec{w_1} + i\vec{w_2})]\end{aligned}$$

   3. **Zero Vector:** We define the 0 to be from $0 \in V$ as $0 + 0i \in V_C$. We note that:

$$\begin{aligned}(\vec{v_1} + i\vec{v_2}) + (0 + i0) &= (\vec{v_1} + 0) + i(\vec{v_2} + 0) \\ &= \vec{v_1} + i\vec{v_2}\end{aligned}$$

   4. **Additive Inverse:** We define the additive inverse for $\vec{v_1} + i\vec{v_2}$ as $(-\vec{v_1}) + i(-\vec{v_2})$:

$$\begin{aligned}(\vec{v_1} + i\vec{v_2}) + [(-\vec{v_1}) + i(-\vec{v_2})] &= (\vec{v_1} - \vec{v_1}) + i(\vec{v_2} - \vec{v_2}) \\ &= 0 + i0\end{aligned}$$

   5. **Multiplicative Identity:** Let $1 \in \mathbb{C}$ be the complex scalar $1 + 0i$.

$$\begin{aligned}1(\vec{v_1} + i\vec{v_2}) &= (1 + 0i)(\vec{v_1} + i\vec{v_2}) \\ &= [1\vec{v_1} - 0\vec{v_2}] + i[1\vec{v_2} + 0\vec{v_1}] \\ &= \vec{v_1} + i\vec{v_2}\end{aligned}$$

   6 **Multiplicative Associativity:**

$$\begin{aligned}(\beta_1\beta_2)(\vec{v_1} + i\vec{v_2}) &= [(ac - bd) + i(ad + bc)](\vec{v_1} + i\vec{v_2}) \\ &= [(ac - bd)\vec{v_1} - (ad + bc)\vec{v_2}] + i[(ac - bd)\vec{v_2} + (ad + bc)\vec{v_1}] \\ &= [a(c\vec{v_1} - d\vec{v_2}) - b(d\vec{v_1} + c\vec{v_2})] + i[a(d\vec{v_1} + c\vec{v_2}) + b(c\vec{v_1} - d\vec{v_2})] \\ &= (a + bi)[(c\vec{v_1} - d\vec{v_2}) + i(c\vec{v_2} + d\vec{v_1})] \\ &= \beta_1[\beta_2(\vec{v_1} + i\vec{v_2})]\end{aligned}$$

   7. **Distributive Properties:**

$$\begin{aligned}(\alpha + \beta)(\vec{v_1} + i\vec{v_2}) &= [(a + c) + i(b + d)](\vec{v_1} + i\vec{v_2}) \\ &= [(a + c)\vec{v_1} - (b + d)\vec{v_2}] + i[(a + c)\vec{v_2} + (b + d)\vec{v_1}] \\ &= [(a\vec{v_1} - b\vec{v_2}) + (c\vec{v_1} - d\vec{v_2})] + i[(a\vec{v_2} + b\vec{v_1}) + (c\vec{v_2} + d\vec{v_1})] \\ &= [(a\vec{v_1} - b\vec{v_2}) + i(a\vec{v_2} + b\vec{v_1})] + [(c\vec{v_1} - d\vec{v_2}) + i(c\vec{v_2} + d\vec{v_1})] \\ &= \alpha(\vec{v_1} + i\vec{v_2}) + \beta(\vec{v_1} + i\vec{v_2})\end{aligned}$$

$$\begin{aligned}
\alpha[(\vec{v_1} + i\vec{v_2}) + (\vec{u_1} + i\vec{u_2})] &= \alpha[(\vec{v_1} + \vec{u_1}) + i(\vec{v_2} + \vec{u_2})] \\
&= [a(\vec{v_1} + \vec{u_1}) - b(\vec{v_2} + \vec{u_2})] + i[a(\vec{v_2} + \vec{u_2}) + b(\vec{v_1} + \vec{u_1})] \\
&= [(a\vec{v_1} - b\vec{v_2}) + (a\vec{u_1} - b\vec{u_2})] + i[(a\vec{v_2} + b\vec{v_1}) + (a\vec{u_2} + b\vec{u_1})] \\
&= [(a\vec{v_1} - b\vec{v_2}) + i(a\vec{v_2} + b\vec{v_1})] + [(a\vec{u_1} - b\vec{u_2}) + i(a\vec{u_2} + b\vec{u_1})] \\
&= \alpha(\vec{v_1} + i\vec{v_2}) + \alpha(\vec{u_1} + i\vec{u_2})
\end{aligned}$$

I hope I never have to do this again.