

1. (a) We note that  $F$  is a field because  $f(x)$  is an irreducible polynomial in  $\mathbb{F}_p$ . Due to the inclusion of constants in  $\mathbb{F}_p[x]$  and that  $\deg f \geq 1$ , we get that  $\mathbb{F}_p \subset F$ . Hence,  $f(x) = a_0 + a_1x + \cdots + a_dx^d$  with  $a_i \in \mathbb{F}_p$ . We note that:

$$f(\beta)^p = (a_0 + a_1\beta + \cdots + a_d\beta^d)^p = a_0^p + a_1^p\beta^p + \cdots + a_d^p\beta^{dp}$$

Because of  $1 \in \mathbb{F}_p \subset F$ ,  $\text{char}(\mathbb{F}_p) = p = \text{char}(F)$ , and that  $F$  is a field, the result follows from Frobenius Map of  $(a+b)^p \mapsto a^p + b^p$ . Hence, we note that  $f(\beta)^p = f(\beta^p) = 0^p = 0$ . This concludes that  $\beta^p$  is a root of  $f(x)$  in  $F$ .  $\square$

- (b) We note that since  $\alpha = [x]$ , we get that  $f(\alpha) = [f(x)] = 0$ . Hence  $\alpha$  is a root of  $f(x)$  in  $F$ . From (a), it follows that  $\{\alpha, \alpha^p, \dots, \alpha^{p^{d-1}}\}$  are all roots of  $f(x)$ .

**Claim:**  $\{\alpha, \alpha^p, \dots, \alpha^{p^{d-1}}\}$  are distinct

By contradiction, there exist  $0 \leq i < j \leq d-1$  where  $\alpha^{p^i} = \alpha^{p^j}$ . Hence,  $\alpha^{p^j} - \alpha^{p^i} = (\alpha^{p^{j-i}} - \alpha)^{p^i} = 0$  by applying the Frobenius Map in  $\text{char } p$  of  $a^{p^i} + b^{p^i} \mapsto (a+b)^{p^i}$ . Thus, we get that  $[x^{p^{j-i}} - x] = 0$  as we recall that  $\alpha = [x]$ . This implies that  $f(x) \mid x^{p^{j-i}} - x$ . By Theorem 9.23,  $x^{p^{j-i}} - x$  is the product of all monic irreducible polynomials in  $\mathbb{F}_p[x]$  of degree dividing  $j-i$ . However,  $j-i < d$ , so  $d \nmid j-i$ , so  $f(x)$  is not a monic irreducible factor of  $x^{p^{j-i}} - x$ , which gives us a contradiction. Hence, they must be distinct.  $\square$

Since we have found  $d$  distinct roots for a monic polynomial  $f(x)$ , by Corollary 9.7, it follows that  $\prod_{i=0}^{d-1} (x - \alpha^{p^i}) \mid f(x)$ . Since  $f(x)$  is monic and the product has degree  $d$ , we get that  $\prod_{i=0}^{d-1} (x - \alpha^{p^i}) = f(x)$  as desired.  $\square$

- (c) By Proposition 9.21, defining a homomorphism of  $\varphi : F \rightarrow E$  is the same as giving a root  $\beta$  of  $f(x)$  in  $E$ , which exists as defined. By the First Isomorphism Theorem, the natural map of  $\phi : F/\ker(\varphi) \rightarrow \text{im}(\varphi)$  is an isomorphism. Note that  $F$  is a field, so any homomorphism is injective, which implies that only  $\varphi(0) = 0$ . Thus, the  $\ker(\varphi) = \{0\}$ , which further implies that  $F/\ker(\varphi) = F$ . We note  $\text{im}(\varphi)$  is finite from being isomorphic to a finite field and also a subring of a field  $E$ , so it is a finite integral domain thus a subfield of  $E$ . Since  $\phi$  is an isomorphism, note that  $\{\phi(\alpha), \phi(\alpha^p), \dots, \phi(\alpha^{p^{d-1}})\} \subset E$  must also be all distinct.

We then note that since  $\text{char}(E) = p$ ,  $\mathbb{F}_p$  is a subfield of  $E$ . Since all of  $f(x)$ 's coefficients lie in  $\mathbb{F}_p$  and that  $\phi(1) = 1$ , for any  $a \in \mathbb{F}_p$ ,  $\phi(a) = a$ . Hence,  $\phi(f(x)) = \phi(a_0 + a_1x + \cdots + a_dx^d) = a_0 + a_1\phi(x) + \cdots + a_d\phi(x)^d = f(\phi(x))$ . For any root  $\alpha^{p^i} \in F$ , we note that  $\phi(f(\alpha^{p^i})) = \phi(0) = 0 = f(\phi(\alpha^{p^i}))$ . Hence,  $\{\phi(\alpha), \phi(\alpha^p), \dots, \phi(\alpha^{p^{d-1}})\}$  are all roots of  $f(x) \in E$ . Since  $f(x)$  has  $d$  distinct roots, by Corollary 9.7, it must be in the form  $c(x - \phi(\alpha)) \cdots (x - \phi(\alpha^{p^{d-1}}))$ , so it splits completely.  $\square$

2. (a) We assume  $o(\alpha) = m$ . Since  $|\mathbb{F}_{p^n}^\times| = p^n - 1$ , we get that  $\alpha^{p^n-1} = 1$ , so  $m \mid p^n - 1$  as desired.

We assume  $m \mid p^n - 1$ . Then for some  $\beta \in \mathbb{F}_{p^n}$  by Theorem 10.7, we get that  $o(\beta) = |\mathbb{F}_{p^n}^\times| = p^n - 1$ . This implies that  $\mathbb{F}_{p^n} = \{0, \beta, \dots, \beta^{p^n-1}\}$ . Since  $m \mid p^n - 1$ , there exist some integer  $q$  where  $mq = p^n - 1$ . Since  $q \leq p^n - 1$ ,  $\beta^q \in \mathbb{F}_{p^n}$ . Note that  $(\beta^q)^m = \beta^{p^n-1}$ , so  $o(\beta^q) \mid m$ . Suppose  $o(\beta^q) < m$  and let  $d = o(\beta^q)$ , this implies that  $\beta^{qd} = 1$ . However,  $qd < qm = p^n - 1$ , so this contradicts  $o(\beta)$ 's minimality. Hence, it must be that  $o(\beta^q) = m$ .  $\square$

- (b) **Claim:**  $\Phi_m(x)$  splits completely in  $\mathbb{F}_{p^d}$ .

Since  $d = o_m(p)$ , we get that  $m \mid p^d - 1$ . By the result of the proof from Theorem 10.7, if  $m \mid |\mathbb{F}_{p^d}^\times|$ , then the number of elements in  $\mathbb{F}_{p^d}$  with order exactly  $m$  is  $\phi(m)$ . Hence, for each of the  $\phi(m)$  elements  $\beta$  with  $o(\beta) = m$ , we get that  $\Phi_m(\beta) = 0$  from HW7(b).  $\Phi_m(x)$  is degree  $\phi(m)$  with  $\phi(m)$  distinct roots. By Corollary 9.7,  $\Phi_m(x)$  is divisible by the product of  $\phi(m)$  linear polynomials, so it splits completely in  $\mathbb{F}_{p^d}$ .  $\square$

**Claim:** Irreducible polynomial factors of  $\Phi_m(x)$  in  $\mathbb{F}_p$  has degree  $d$ .

Let  $f(x)$  be an irreducible polynomial factor of  $\Phi_m(x)$  in  $\mathbb{F}_p$  with degree  $k$ . Since  $f(x)$  is a factor of  $\Phi_m(x)$ , which splits completely in  $\mathbb{F}_{p^d}$ .  $f(x)$  must also split completely and shares roots with  $\Phi_m(x)$ . Earlier, we showed that all roots of  $\Phi_m(x)$  has order  $m$ . Hence, there is a root  $\alpha$  of  $f(x)$  where  $o(\alpha) = m$ .

By Theorem 9.23,  $f(x)$  is factor of  $x^{p^k} - x$  in  $\mathbb{F}_p$  since  $k \mid k$ . Since there is a natural way to view  $\mathbb{F}_p$  in  $\mathbb{F}_{p^d}$ , we note that  $f(x)$  is a factor of  $x^{p^k} - x$  in  $\mathbb{F}_{p^d}$  and consequently,  $\alpha^{p^k} - \alpha = 0$  and that  $\alpha^{p^k-1} = 1$ . Since  $o(\alpha) = m$ ,  $p^k \equiv 1 \pmod{m}$  and  $d \mid k$ .

Meanwhile, from Theorem 9.21, there exist a homomorphism  $\mathbb{F}_p[x]/(f(x)) \rightarrow \mathbb{F}_{p^d}$  is the same as giving the root  $\alpha$  in  $\mathbb{F}_{p^d}$ . Since  $\mathbb{F}_p[x]/(f(x))$  is a field with order  $p^k$ , from Theorem 10.1 (d), this homomorphism implies that  $k \mid d$ . Since  $k \mid d$  and  $d \mid k$ , we get that  $k = d$  as desired.  $\square$

Since  $\Phi_m(x)$  is a monic polynomial, it can be written as a product of irreducible polynomials as quoted from Example 9.15. Since each irreducible polynomial factor is degree  $d$  and  $\Phi_m(x)$  is degree  $\phi(m)$ , it must be that there exists  $\phi(m)/d$  irreducible polynomial factors for  $\Phi_m(x)$ .  $\square$

- (c) We assume  $p$  is not a square mod  $q$ . We first note that  $\phi(m) = q - 1$ . Meanwhile,  $\left(\frac{p}{q}\right) = -1 \equiv p^{(q-1)/2} \pmod{q}$ . This implies  $d \nmid (q-1)/2$ . We also note that  $1 \equiv p^{q-1} \pmod{q}$ , so  $d \mid q - 1$ . Since  $dr = q - 1$ , it follows that  $d(r/2) = (q-1)/2$ .  $(r/2) \notin \mathbb{Z}$  or else  $d \mid (q-1)/2$ . This implies that  $r$  is odd. Since  $q - 1$  is even from  $q$  being an odd prime, it follows that  $d$  must be even.

We assume  $d$  is even and  $r$  is odd. Since  $rd = q - 1$ , we note that  $r/2 \notin \mathbb{Z}$  and that  $d \nmid (q-1)/2$ . This implies that  $p^{(q-1)/2} \not\equiv 1 \pmod{q}$ . However,  $p^{q-1} \equiv 1 \pmod{q}$  by Fermat's Little Theorem, so  $p^{(q-1)/2} \equiv -1 \pmod{q}$ . By Corollary 11.3,  $p$  is a quadratic non-residue mod  $q$ , so  $p$  is not a square mod  $q$ .  $\square$

3. (a) For any  $a_i \in F$ , we re-organize  $f(x)$  to  $(x - a_i) \prod_{\substack{k=1 \\ k \neq i}}^n (x - a_k)$ . We note that:

$$f'(x) = (x - a_i) \left( \prod_{\substack{k=1 \\ k \neq i}}^n (x - a_k) \right)' + 1 \cdot \prod_{\substack{k=1 \\ k \neq i}}^n (x - a_k) = \prod_{\substack{k=1 \\ k \neq i}}^n (x - a_k)$$

Within the product of  $f(a_i)$ , there exists  $i - 1$  instances where  $i > k$ . We correct this by applying  $(-1)^{i-1} f'(a_i)$  to get that:

$$(-1)^{i-1} f'(a_i) = \prod_{k=1}^{i-1} (a_k - a_i) \cdot \prod_{k=i+1}^n (a_i - a_k)$$

We then note that for any  $1 \leq i < j \leq n$ ,  $(a_i - a_j)$  is a factor in the products of  $(-1)^{i-1} f'(a_i)$  and  $(-1)^{j-1} f'(a_j)$ . Hence, we note that the product of all such  $(-1)^{i-1} f'(a_i)$  is equal to:

$$\begin{aligned} \prod_{i=1}^n (-1)^{i-1} f'(a_i) &= (-1)^{0+\dots+n-1} \prod_{i=1}^n f'(a_i) \\ &= (-1)^{n(n-1)/2} \prod_{1 \leq i < j \leq n} (a_i - a_j)^2 \\ &= \Delta f \end{aligned}$$

as desired.  $\square$

- (b) Let  $f(x) = x^8 + x^2 + 1$ . We note that  $f'(x) = 8x^7 + 2x = 2x(4x^6 + 1)$ , assuming  $\text{char}(F) \neq 2$  or else  $f'(x) = 0$ . We then note that if  $f(x)$  splits completely in  $F$ , it must have 8 roots because  $\deg f = 8$ . We apply the altered calculation of the discriminant from (a) and consider that  $(-1)^{(8 \cdot 7/2)} = (-1)^{28} = 1$ . Thus:

$$\Delta f = \prod_{i=1}^8 f'(a_i) = 2^8 \cdot \prod_{i=1}^8 a_i \cdot \prod_{i=1}^8 4a_i^6 + 1$$

**Claim (Vieta):** If a polynomial  $f(x)$  splits completely and is monic. Then  $f(0)$  is equal to product of its roots times  $(-1)^{\deg f}$ .

Let us assume  $f(x) = (x - a_1) \cdots (x - a_k)$  with degree  $k$ . Then  $f(0) = (-a_1) \cdots (-a_k) = (-1)^k \cdot a_1 \cdots a_k$ .  $\square$

With Vieta, we note that  $f(0) = 1$ , so by Vieta, we get that  $(-1)^8 \prod_{i=1}^8 a_i = \prod_{i=1}^8 a_i = 1$ . Thus, we can simplify  $\Delta f$  into:

$$\Delta f = 2^8 \cdot \prod_{i=1}^8 4a_i^6 + 1$$

We also then note that for any root  $a_i$ ,  $a_i^8 + a_i^2 + 1 = 0$ , so  $a_i^6 = -(1 + a_i^{-2})$ . Thus:

$$\begin{aligned}
\Delta f &= 2^8 \cdot \prod_{i=1}^8 -4(1 + 1 \cdot a_i^{-2}) + 1 \\
&= 2^8 \cdot \prod_{i=1}^8 -4 - 4 \cdot a_i^{-2} + 1 \\
&= 2^8 \cdot \prod_{i=1}^8 -a_i^{-2} \prod_{i=1}^8 3a_i^2 + 4
\end{aligned}$$

We note that  $\prod_{i=1}^8 -a_i^{-2} = (\prod_{i=1}^8 a_i)^{-2} = 1^{-2} = 1$ , so:

$$\Delta f = 2^8 \cdot \prod_{i=1}^8 3a_i^2 + 4$$

We then note that  $f(x)$  is an even polynomial as all of its powers are even. Hence,  $f(-x) = f(x)$ . This means that the negation of each of the roots are also roots. Hence, we can pair each of the roots  $a_1, \dots, a_8$  into  $b_1, \dots, b_4, -b_1, \dots, -b_4$ , which gives us the follow since  $(-b_i)^2 = b_i^2$ :

$$\Delta f = 2^8 \cdot \left( \prod_{i=1}^4 3b_i^2 + 4 \right)^2$$

We now denote a new polynomial  $g(x) = x^4 + x + 1 \in F[x]$ . Because  $f(x) = (x^2 - b_1^2) \cdots (x^2 - b_4^2)$  and that  $f(x) = g(x^2)$ , we get that  $g(x) = (x - b_1^2) \cdots (x - b_4^2)$ . Hence, it splits completely and it has 4 roots.

Hence, let  $h(x) = 3^4 \cdot (((x - 4)/3)^4 + ((x - 4)/3) + 1)$  and  $h(x) \in F[x]$ . Note that  $\text{char}(F) \neq 3$ , so we can divide by 3. We also note that:

$$\begin{aligned}
h(x) &= 3^4 \cdot ((x - 4)/3 - b_1^2) \cdots ((x - 4)/3 - b_4^2) \\
&= ((x - 4) - 3b_1^2) \cdots ((x - 4) - 3b_4^2) \\
&= (x - (3b_1^2 + 4)) \cdots (x - (3b_4^2 + 4))
\end{aligned}$$

Thus,  $h(x)$  is monic and splits completely. We expand to get  $h(x) = (w - 4)^4 + 27(w - 4) + 81$ . We then calculate  $h(0) = 256 - 108 + 81 = 229$ . We apply Vieta to get that  $(-1)^4 \prod_{i=1}^4 3b_i^2 + 4 = 229$  as they are roots of  $h(x)$ . Hence, we substitute it in and get that for  $\text{char}(F) \neq 2, 3$ :

$$\Delta f = 2^8 \cdot 229^2$$

For  $\text{char}(F) = 2$ ,  $f'(x) = 0$ , so  $\Delta f = 0$ . That said,  $2^8 \cdot 229^2 = 0$  in  $\text{char}(2)$  anyways.  $\square$

- (c) To start, we prove the  $f(x) = x^8 + x^2 + 1$  is reducible in  $\mathbb{F}_p[x]$  for any prime  $p \neq 3$ . By contradiction,  $f(x)$  is irreducible in  $\mathbb{F}_p[x]$  for some prime  $p \neq 3$ . Let  $p$  be one such prime. Since  $f(x)$  is a monic irreducible polynomial in  $\mathbb{F}_p[x]$ , by Q1(b), we can factor  $f(x)$  in  $\mathbb{F}_p[x]/(f(x))$  as:

$$f(x) = \prod_{i=0}^{8-1} (x - \alpha^{p^i})$$

We also note that:

$$\Delta f = \prod_{0 \leq i < j \leq 7} (\alpha^{p^i} - \alpha^{p^j})^2$$

From (b), since  $\text{char}(\mathbb{F}_p) \neq 3$ ,  $\Delta f = 2^8 \cdot 229^2$ . However, this looks very similar to the  $\beta$  defined in HW1(d) where:

$$\beta = \prod_{0 \leq i < j \leq 7} (\alpha^{p^i} - \alpha^{p^j})$$

Hence,  $\beta = 2^4 \cdot 229$ . Since  $2^4 \cdot 229 = 2^4 \cdot 229 \cdot 1$ ,  $\beta \in \mathbb{F}_p$ . However, from HW1(d),  $\beta \in \mathbb{F}_p$  if and only if  $\deg f$  is odd. Clearly,  $\deg f$  is not odd, so we reached a contradiction. Thus,  $f(x)$  is reducible in  $\mathbb{F}_p[x]$  for any prime  $p \neq 3$ .

We now consider the case  $p = 3$ . We note that:

$$\begin{aligned} (x^2 - 1)(x^6 + x^4 + x^2 + 2) &= (x^8 + x^6 + x^2 + 2x^2) - (x^6 + x^4 + x^2 + 2) \\ &= x^8 + x^2 - 2 \\ &= x^8 + x^2 + 1 \end{aligned}$$

Hence, if  $p = 3$ ,  $f(x)$  is reducible in  $\mathbb{F}_3[x]$ . □