1. (a) To prove that f is differentiable on $\mathbb{R}$, we consider the two cases.

   Case 1: For $x \neq 0$, we note that it is differentiable over x^2, $\sin(x)$, and $\frac{1}{x}$. Since $f(x) = x^2 \sin(\frac{1}{x})$, we can compute the $f'(x)$ with the product rule and chain rule to get as follows:

   $$f'(x) = 2x \sin(\frac{1}{x}) + x^2 \cos(\frac{1}{x})(-\frac{1}{x^2})$$
   $$= 2x \sin(\frac{1}{x}) - \cos(\frac{1}{x})$$

   For $x \neq 0$, $f'(x)$ produces a real value, so f is differentiable at x.

   Case 2: For $x = 0$, we note that $f(0) = 0$. Thus:

   $$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{f(x)}{x} = \lim_{x \to 0} x \sin(\frac{1}{x})$$

   We note that the range of $\sin(x)$ is $[-1, 1]$, so if we consider $\lim_{x \to 0^-} x$ and $\lim_{x \to 0^-} -x$, for any $x \to 0^-$, we get that $x \leq x \sin(1/x) \leq -x$. Hence, by Squeeze Theorem, since both $\lim_{x \to 0^-} x = 0$ and $\lim_{x \to 0^-} -x = 0$, we get that $\lim_{x \to 0^-} x \sin(1/x) = 0$. For $x \to 0^+$, we get that $-x \leq x \sin(1/x) \leq x$. We can apply a similar argument to get that $\lim_{x \to 0^+} x \sin(1/x) = 0$. Thus, $f'(0) = 0$ and f is differentiable at $x = 0$.

   Since we proved f is different on any $x \in \mathbb{R}$, f is diffenetiable on $\mathbb{R}$. We also note that for $x \in \mathbb{R}$:

   $$f'(x) = \begin{cases} 2x \sin(\frac{1}{x}) - \cos(\frac{1}{x}), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

   (b) By contradiction, $f'(x)$ is continuous. Thus, at $x = 0$, for $\epsilon = 0.5$, there exists a $\delta > 0$ where for $x \in \mathbb{R}$ that is $|x| < \delta$, we get that $|f'(x)| < \epsilon$. Select an $N \in \mathbb{N}$ where $N2\pi > 1/\delta$ then denote $x = 1/N2\pi$. Note that $|1/N2\pi| < \delta$ and that:

   $$|f'(1/N2\pi)| = |2(1/N2\pi) \sin(N2\pi) - \cos(N2\pi)|$$
   $$= |0 - 1|$$
   $$= 1$$

   Clearly, $1 > \epsilon$, so we arrived at a contradiction. Thus, f' is not continuous at $x = 0$, so f' is not continuous.

2. (a) Note that since $|f(0)| \leq |0|^\alpha = 0$, we get that $f(0) = 0$. Thus:

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{f(x)}{x}$$

For $x \to 0$, we note that:

$$|\frac{f(x)}{x}| \leq |\frac{x^\alpha}{x}| \implies -\frac{x^\alpha}{x} \leq \frac{f(x)}{x} \leq \frac{x^\alpha}{x}$$

Since $\alpha > 1$, we get that $\alpha - 1 > 0$, so $\lim_{x \to 0} x^{\alpha-1} = 0$ and $\lim_{x \to 0} -x^{\alpha-1} = 0$. Hence, by Squeeze Theorem, we get that $\lim_{x \to 0} f(x)/x = 0$, so $f'(0) = 0$. Since $f'(0)$ exists, f is differentiable at 0.

(b) We first note that:

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{f(x)}{x}$$

For $x \to 0$, we also get that:

$$|\frac{f(x)}{x}| \geq \frac{|x|^\beta}{|x|} = |x|^{\beta-1}$$

Since $0 < \beta < 1$, we note that $-1 < \beta - 1 < 0$. Thus, $|x|^{\beta-1}$ is the same as $1/|x|^{-(\beta-1)}$. For $x \to 0$, $|x|^{-(\beta-1)}$ can get arbitrarily small, so $1/|x|^{-(\beta-1)} \to \infty$. However, this implies that for $x \to 0$, $|f(x)/x| \to \infty$. The values $f(x)/x$ near 0 are unbounded, so there does not exist a finite value for $\lim_{x \to 0} f(x)/x$. Thus, $f'(0)$ cannot exist, so f is not differentiable at $x = 0$.