Probability & Statistics

(I) Discrete Random Variable

Q.1 (1) $P(x) = \frac{3}{4}\left(\frac{1}{4}\right)^x$, $x = 0, 1, 2, \ldots$

we know that,
sum of all $P(x)$ should be equal to 1
for all $x = 0, 1, 2, \ldots$

$\Rightarrow \sum_{i=0}^{\infty} \quad 0 + \left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^1 + \left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^2 + \ldots$

$= \frac{3}{4}\left[1 + \frac{1}{4} + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 + \ldots\right]$

$= \frac{3}{4}\left[\frac{1}{(1-1/4)}\right]$ (∵ sum of infinite GP $\Rightarrow a/1-r$)

$= \frac{3}{4}\left[\frac{4}{3}\right] = 1.$

$\Rightarrow P(x)$ is a valid PMF of discrete r.v.

(2) $P(X=2) = \frac{3}{4}\left(\frac{1}{4}\right)^2 = \frac{3}{4} \times \frac{1}{4} \times \frac{1}{4} = \frac{3}{64}$

(3) $P(X \leq 2) = P(X=0) + P(X=1) + P(X=2)$

$= \frac{3}{4}\left(\frac{1}{4}\right)^0 + \frac{3}{4}\left(\frac{1}{4}\right)^1 + \frac{3}{4}\left(\frac{1}{4}\right)^2$

$= \frac{3}{4}\left[1 + \frac{1}{4} + \frac{1}{16}\right] = \frac{63}{64}$

Q.2 (1) X can potentially take all positive integers
i.e $R_X = \{1, 2, 3, 4, \ldots\}$
So, $P_X(k) = P(X=k)$ for $k = 1, 2, 3, \ldots$
i.e,

$P_X(1) = P(X=1) = p$

$P_X(2) = P(X=2) = (1-p)p.$

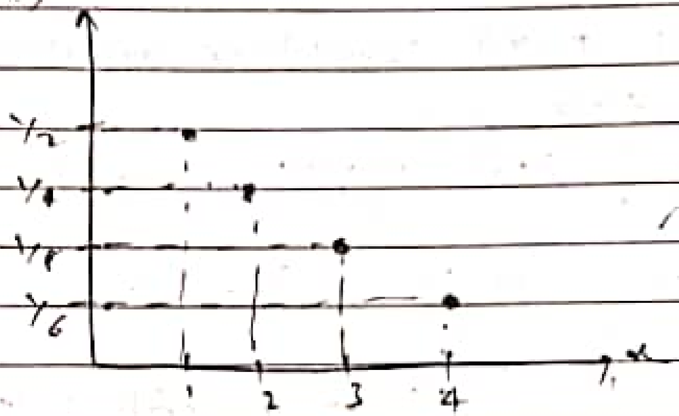
$P_X(k) = P(Y=k) = (1-p)^{k-1} \cdot p$

Thus, we can write PMF of X as:—

$$P_X(x) = \begin{cases} (1-p)^{x-1} \cdot p, & x = 1,2,3,4,\ldots \\ 0, & \text{otherwise} \end{cases}$$

PMF(x)

(x)



(2) CDF of x:—

$$F(k) = P(X \leq k)$$
$$= P(X=1) + P(X=2) + P(X=3) + \cdots$$
$$= \sum_{k'=1}^{k'} P(X=k')$$
$$= \sum_{k'=1}^{k} p(1-p)^{k'-1}$$
$$= 1 - p(1-p)^{k}\left[1 + (1-p) + (1-p)^2 + \cdots\right]$$
$$= 1 - p(1-p)^{k} \left[\frac{1}{1-(1-p)}\right]$$
$$\Rightarrow F(k) = 1 - (1-p)^{k}$$

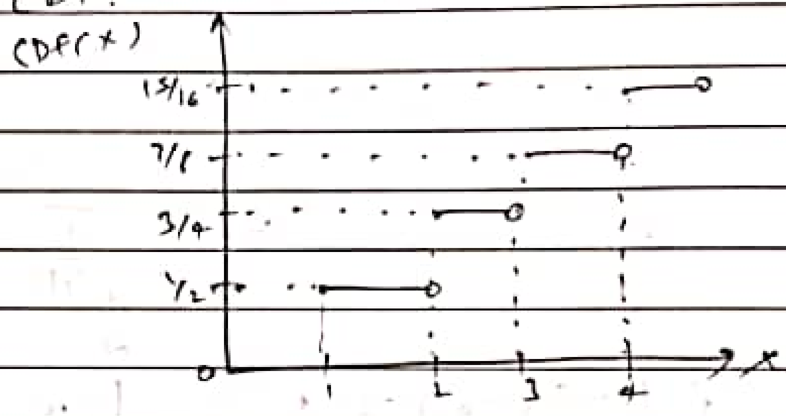
As coin is fair so, $p = [= \frac{1}{2}$.

So, $p_X(x) = \begin{cases} \left(\frac{1}{2}\right)^{x-1}\left(\frac{1}{2}\right)^{1} & , x=1,2,3,\ldots \\ 0 & , \text{otherwise} \end{cases}$

$= \begin{cases} \left(\frac{1}{2}\right)^{x} & , x=1,2,3,\ldots \\ 0 & , \text{otherwise} \end{cases}$

Similarly, $F_X(x) = 1 - \left(1-\frac{1}{2}\right)^x = 1 - \left(\frac{1}{2}\right)^x$

Sketch of CDF:-



③ $P(1 < X \leq 4)$

$= P(X=2) + P(X=3) + P(X=4)$

$= \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 = \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{7}{16}.$

Q.3 Given: $P_{error} = 0.01$

Using Binomial distribution:

we are given: $p = 0.01$, $q = 1 - 0.01 = 0.99$, $n = 10$

So,

$P(X>1) = 1 - P(X \leq 1)$

$= 1 - [P(X=0) + P(X=1)]$

$= 1 - \left[{}^{10}C_0 (0.01)^0 (0.99)^{10} + {}^{10}C_1 (0.01)^1 (0.99)^9\right]$

$= 1 - [0.9044 + 0.09135] = 0.00425.$

By poison distribution:—

poission avg $(\lambda) = np = 10(0.01) = 0.1$

S,

$$P(X>1) = 1 - P(X \le 1)$$

$$= 1 - [P(X=0) + P(X=1)]$$

$$= 1 - \left[\frac{e^{-0.1} \cdot (0.1)^0}{0!} + \frac{e^{-0.1} \cdot (0.1)^1}{1!}\right]$$

$$= 1 - \left[e^{-0.1} + 0.1 e^{-0.1}\right]$$

$$= 1 - 0.9048 - 0.0905 = 0.0047$$

$$\Rightarrow P(X>1) = 0.0047.$$

DOMS Page No.
Date / /

Q.4 $X = N(0,\sigma^2)$

$$f(x \mid X>0) = \frac{f_X(x)}{P(X>0)}$$

Since,

X is Normal distribution with mean 0, it is symmetric about $(0,0)$

$$\Rightarrow P(X>0) = P(X<0) = \frac{1}{2}$$

So,

$$f(x \mid X>0) = \frac{\frac{1}{\sqrt{2\pi}\,\sigma}\exp\left(-\frac{x^2-0}{2\sigma^2}\right)}{1/2}$$

$$= \frac{1}{\sigma}\sqrt{\frac{2}{\pi}}\, e^{-x^2/2\sigma^2}$$

$$\therefore E[X \mid X>0] = \frac{1}{\sigma}\sqrt{\frac{2}{\pi}}\int_0^\infty x \cdot \exp\left(\frac{-x^2}{2\sigma^2}\right)dx$$

Let $-\frac{1}{2}\left(\frac{x^2}{\sigma^2}\right) = t$ and $\frac{-x}{\sigma^2}\,dx = dt$

$$\Rightarrow dx = \frac{-dt}{x} \qquad = \frac{\sigma^2}{\sigma}\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-t}\,dt$$

$$= -\sigma\sqrt{\frac{2}{\pi}}\left[e^{t}\right]_0^\infty$$

So, $E[X \mid X>0] = \sigma\sqrt{\frac{2}{\pi}}$

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$$E[x^2 | x>0] = \frac{1}{\sigma}\sqrt{\frac{2}{\pi}} \int_0^\infty x^2 \exp\left(\frac{-x^2 - 0}{2\sigma^2}\right) dx$$

$$Var(X | X>0) = E[x^2 / x>0] - (E[X | X>0])^2$$

$$= \frac{1}{\sigma}\sqrt{\frac{2}{\pi}} \int_0^\infty x^2 \exp\left(\frac{-x^2}{2\sigma^2}\right) dx - \frac{2\sigma^2}{\pi}$$

Q.5 Property of Memoryless function :-

$$P(X > t+s / X > s) = P(X > t)$$

$$P(X > t+s) = P(X > s) P(X > t+s | X > s)$$

$$= P(X > s) \; P(X > t)$$

$$\therefore P(X > t+s) = P(X > s) \cdot P(X > t)$$

Let $G(x) = P(X > x)$

$$G(t+s) = G(t) \cdot G(s)$$

where $G(s)$ is a memory function.

If

$$s = t \Rightarrow G(2t) = \cancel{G(t^2)}\, G(t)^2$$

$$s = 2t \Rightarrow G(3t) = G(t)^3$$

$$\vdots$$

$$\Rightarrow G(nt) = G(t)^n \quad , n \geq 0.$$

If,

$$G(t/2) = G(t)^{1/2}$$

So, we can say that : $G\left(\frac{m}{n}t\right) = G(t)^{m/n}$ $, m, n \geq 0$

$\Rightarrow G(xt) = G(t)^x$ for all real $x > 0$

Suppose $t = 1$.

$$G(x) = G(1)^x$$

$$\Rightarrow G(x) = e^{\log_e G(1)^x} = e^{x \log_e G(1)}.$$

Let, $\log_e G(1) = -\lambda$

$\Rightarrow G(x) = e^{-\lambda x}$

which is a exponential function.

And only Exponential function can satisfy memoryless equation.

(6) ① we know that $\int_{-\infty}^{\infty} f_X(x) = 1$.

$$\Rightarrow \int_{-\infty}^{\infty} k e^{-|\lambda x|} dx = 1.$$

$$\Rightarrow k \int_{-\infty}^{0} e^{\lambda x} dx + k \int_{0}^{\infty} e^{-\lambda x} dx = 1.$$

$$\Rightarrow k \left[\frac{1}{\lambda} \left[e^{\lambda x} \right]_{-\infty}^{0} + \frac{1}{\lambda} \left[e^{-\lambda x} \right]_{0}^{\infty} \right] = 1.$$

$$\Rightarrow \frac{k}{\lambda} (1 + 1) = 1. \quad \Rightarrow \boxed{k = \frac{\lambda}{2}}$$

② CDF i.e $F_X(x) = \int f_X(x)\, dx$.

Here we have 2 cases

Case I : $x \geq 0$

case II : $x < 0$

for $x \geq 0$:

$$F_X(x) = \int_{-\infty}^{0} k e^{-|\lambda x|} dx = \int_{-\infty}^{0} k e^{\lambda x} dx + \int_{0}^{x} k e^{-\lambda x} dx.$$

$$= \frac{\lambda}{2} \left[\frac{1}{\lambda} \left(e^{\lambda x} \right]_{-\infty}^{0} \right] + \frac{\lambda}{2} \left[\frac{1}{\lambda} \left[-e^{-\lambda x} \right]_{0}^{x} \right]$$

$$= \frac{\lambda}{2} \left[\frac{1}{\lambda} \right] + \frac{\lambda}{2} \left[\frac{1}{\lambda} \left(-e^{-\lambda x} + 1 \right) \right] = \frac{1}{2} + \left(\frac{-e^{-\lambda x} + 1}{2} \right)$$

$$= 1 - \frac{e^{-\lambda x}}{2} \quad \ldots \; x \geq 0.$$

Page No.
Date / /

Case II : $x < 0$

$$F_X(x) = \int_{-\infty}^{x} \frac{\lambda}{2} e^{-\lambda|x|} dx = \frac{\lambda}{2}\int_{-\infty}^{x} e^{\lambda x} dx$$

$$= \frac{\lambda}{2}\cdot\frac{1}{\lambda}\left[e^{\lambda x}\right]_{-\infty}^{x} = \boxed{\frac{1}{2}e^{\lambda x}}$$

$\Rightarrow$ CDF of x :-

$$F_X(x) = \begin{cases} \frac{1}{2}e^{\lambda x}, & x < 0 \\ 1 - \frac{e^{-\lambda x}}{2}, & x \geq 0 \end{cases}$$

③ $E[X] = \int_{-\infty}^{\infty} x\cdot\frac{\lambda}{2}e^{-\lambda|x|} = \frac{\lambda}{2}\int_{-\infty}^{0} x e^{\lambda x}dx + \frac{\lambda}{2}\int_{0}^{\infty} x\cdot e^{-\lambda x}dx$

$$= \frac{\lambda}{2}\left[\frac{xe^{\lambda x}}{\lambda} - \frac{1}{\lambda}\int e^{\lambda x}\right]_{-\infty}^{0} + \frac{\lambda}{2}\left[\frac{xe^{-\lambda x}}{-\lambda} + \frac{1}{\lambda}\int e^{-\lambda x}\right]_{0}^{\infty}$$

$$= \frac{\lambda}{2}\left[\frac{xe^{\lambda x}}{\lambda}\right]_{-\infty}^{0} - \frac{1}{\lambda^2}\left[e^{\lambda x}\right]_{-\infty}^{0} + \frac{\lambda}{2}\left[\frac{xe^{-\lambda x}}{-\lambda}\right]_{0}^{\infty} - \frac{1}{\lambda^2}\left[e^{-\lambda x}\right]_{0}^{\infty}$$

$$= \frac{\lambda}{2}(0) - \frac{1}{\lambda^2}(1-0) + 0 - \frac{1}{\lambda^2}(-1) = 0.$$

$$Var(X) = E[X^2] - (E[X])^2$$

$$= \int_{-\infty}^{\infty} x^2\cdot\frac{\lambda}{2}e^{-\lambda|x|}dx - 0.$$

$$Var(X) = \frac{\lambda}{2}\int_{-\infty}^{0} x^2 e^{\lambda x}dx + \frac{\lambda}{2}\int_{0}^{\infty} x^2 e^{-\lambda x}dx. = \boxed{\frac{2}{\lambda^2}}$$

Q.7 X : outcomes of rolling 4 sided die

Y : outcomes of rolling 6 sided die.

$$Z = \left(\frac{X+Y}{2}\right)$$

(i)

X	1	2	3	4
P(X)	1/4	1/4	1/4	1/4

So,

$$E[X] = \frac{1}{4} + \frac{2}{4} + \frac{3}{4} + \frac{4}{4} = \frac{10}{4} = \frac{5}{2}$$

$$E[X^2] = \frac{1}{4} + \frac{4}{4} + \frac{9}{4} + \frac{16}{4} = \frac{15}{2}$$

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Date

Page

$$Var(x) = E[x^2] - (E[x])^2$$
$$= \frac{15}{2} - \left(\frac{5}{2}\right)^2 = \frac{5}{4}$$

(ii)

y	1	2	3	4	5	6
$p(y)$	1/6	1/6	1/6	1/6	1/6	1/6

$$E[y] = \frac{1}{6} + \frac{2}{6} + \frac{3}{6} + \frac{4}{6} + \frac{5}{6} + \frac{6}{6} = \frac{7}{2}$$

$$E[y^2] = \frac{1}{6} + \frac{4}{6} + \frac{9}{6} + \frac{16}{6} + \frac{25}{6} + \frac{36}{6} = \frac{91}{6}.$$

So,

$$Var(y) = E[y^2] - (E[y])^2$$
$$= \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{35}{12}.$$

So,

$$Var(z) = \frac{1}{4}(Var(x) + Var(y))$$
$$= \frac{1}{4}\left(\frac{5}{4} + \frac{35}{12}\right) = \frac{25}{24}$$

$\Rightarrow P(X > 1) = 0.0047.$

Q.8 Two r.v X & Y are independent if $P(x \cap y) = P(x) \cap P(y)$.

So, $P((x=1) \cap (y=1)) = \frac{1}{18}$ $\longrightarrow$ (1)

But,

~~$P(x=1) \cdot P(y=1) =$~~

$P(x=1) = \frac{1}{18} + \frac{1}{9} + \frac{1}{6} = \frac{3}{18} + \frac{3}{18} = \frac{1}{3}.$

$P(y=1) = \frac{1}{18} + \frac{1}{9} + \frac{1}{6} = \frac{1}{3}.$

So, $P(x=1) \cdot P(y=1) = \frac{1}{3} \times \frac{1}{3}$ $\longrightarrow$ (2)

$\Rightarrow$ eq^2 (1) is not equal to (2)

$\Rightarrow$ They are not independent r.v

Date
Page

Q.4 ① for valid PMF, Summation of all probabilities should be 1.

$\Rightarrow P(x=1,y=0)+P(x=2,y=0)+P(x=3,y=0)+P(x=4,y=0)+P(x=5,y=0)+P(x=1,y=1)+P(x=2,y=1)+P(x=3,y=2)+P(x=4,y=1)+P(x=5,y=1)$

$= 0.05+0.2+0.1+0.04+0.01+0.01+0.09+0.15+0.20+0.15$

$= 1$

$\Rightarrow$ This is a valid PMF.

② (i) $P(Y=1 / X \geq 3)$

$$= \frac{P((Y=1) \cap (x \geq 3))}{P(X \geq 3)}$$

$$= \frac{P(y=1, x=3)+P(y=1, x=4)+P(y=1, x=5)}{P(X \geq 3)}$$

So, $P(X \geq 3) = P(X=3, y=0)+P(X=3, y=1)+P(X=4, y=0)+P(X=4, y=1)+P(X=5, y=0)+P(X=5, y=1)$

$= \cancel{0.55}\ 0.65$

$$\Rightarrow \frac{\cancel{0.15}\ 0.15+0.20+0.15}{0.65}$$

$$= \frac{0.50}{0.65} = 0.769.$$

(ii) $P(Y=0, X\geq 3)$

$= P(Y=0, X=3) + P(Y=0, X=4) + P(Y=0, X=5)$

$= 0.1 + 0.04 + 0.01 = 0.15.$

(3) Marginal PMF of X :—

$P_X(1) = P(1,0) + P(1,1) = 0.06$

$P_X(2) = P(2,0) + P(2,1) = 0.29$

$P_X(3) = P(3,0) + P(3,1) = 0.25$

$P_X(4) = P(4,0) + P(4,1) = 0.24$

$P_X(5) = P(5,0) + P(5,1) = 0.16.$

$\Rightarrow$ ~~marginal of X : $P_X(x)$~~

~~$= 0.06 + 0.29 + 0.25 + 0.24 + 0.16$~~

$$P_X(x) = \begin{cases} 0.06 & , x=1 \\ 0.29 & , x=2 \\ 0.25 & , x=3 \\ 0.24 & , x=4 \\ 0.16 & , x=5 \end{cases}$$

(4) Marginal PMF of Y :—

$P_Y(0) = 0.05 + 0.2 + 0.1 + 0.04 + 0.01 = 0.4$

$P_Y(1) = 0.01 + 0.09 + 0.15 + 0.20 + 0.15 = 0.6$

$$P_Y(y) = \begin{cases} 0.4 & , y=0 \\ 0.6 & , y=1 \\ 0 & , \text{otherwise} \end{cases}$$

(5) ~~$E[X|Y=1]$~~ =

classmate

Date

Page

(5) $E[X|Y] = \sum x_i \, P_{X|Y}(x_i / y_i)$.

so,

$E[X \mid Y=1] = 1 \cdot P(1/1) + 2 \cdot P(2/1)$

$+ 3 \cdot P(3/1) + 4 \cdot P(4/1) + 5 \cdot P(5/1)$

$= 1 \cdot \left(\frac{0.01}{0.6}\right) + 2\left(\frac{0.09}{0.6}\right) + 3\left(\frac{0.15}{0.6}\right)$

$+ 4\left(\frac{0.20}{0.6}\right) + 5\left(\frac{0.15}{0.6}\right) = \frac{73}{20}$.

$= 3.65$.

$$= \frac{7!}{s^8} \quad \left(\because \Gamma n = (n-1)!\right)$$

Q.10 Given, $x \sim \text{Unif}(0,1) \Rightarrow E[x] = 1/2$

$y \sim \text{Uniform}(0,1) \Rightarrow E(y) = 1/2$

(1) Also, x & y are Independent

$$\Rightarrow E[xy] = E(x) \times E[y]$$

$$= \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

(2) $E[e^{x+y}] = E[e^x] \times E[e^y]$

$$E[e^x] = \int_0^1 e^x . p(x)\, dx = \int_0^1 e^x\, dx = (e-1)$$

$$E[e^y] = \int_0^1 e^y . p(y)\, dy = \int_0^1 e^y\, dy = (e-1)$$

$$\Rightarrow E[e^{x+y}] = (e-1)^2.$$

③ $E[x^2+y^2+xy] = E[x^2]+E[y^2]+E[xy]$

In general, $E[x^n] = \int_a^b x^n \cdot \frac{1}{(b-a)}$

$$= \frac{1}{(b-a)} \left(\frac{x^{n+1}}{n+1}\right)_a^b.$$

$$= \frac{1}{(b-a)} \times \frac{1}{(r+1)} \times (b^{r+1} - a^{r+1})$$

So, $E[x^2] = \frac{1}{(1-0)} \times \frac{1}{(2+1)} (1^{2+1} - 0^{2+1}) = \frac{1}{3}.$

Simillarly, $E[y^2] = \frac{1}{3}.$

So, $E[xy] = \int_0^1 \int_0^1 xy \times \left(\frac{1}{1-0}\right) dy\, dx$

$$= \int_0^1 \left(\frac{xy^2}{2}\right)_0^1 dx = \left[\frac{x^2}{4}\right]_0^1 = \frac{1}{4}$$

$\Rightarrow E[x^2]+E[y^2]+E[xy] = \frac{1}{3}+\frac{1}{3}+\frac{1}{4}$

so,

$$E[x^2+y^2+xy] = \frac{1}{3}+\frac{1}{3}+\frac{1}{4} = \frac{11}{12}.$$

$$= \frac{1}{4}\left(\frac{5}{4} + \frac{35}{12}\right) = \frac{25}{24}$$

Q.11 $Z = 7 + X + Y$

$W = 1 + Y$,

Joint PDF of Z and W.

$X = Z - 7 - Y$, $Y = W - 1$

So, $X = Z - 7 - W + 1$

$$\Rightarrow \boxed{X = Z - W - 6}$$

$$f_{XY}(x,y) = f_X(x) \cdot f_Y(y)$$

Date
Page

$$= \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \cdot \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$$

$$= \frac{1}{2\pi} e^{\frac{-x^2-y^2}{2}}$$

So, $f_{zw}(z,w) = \frac{1}{2\pi} e^{-\left[\frac{(z-w-6)^2+(w-1)^2}{2}\right]} \cdot |J|$

where $|J|$ is Jacobian

So, $|J| = \begin{vmatrix} \frac{\partial h_1}{\partial z} & \frac{\partial h_1}{\partial w} \\ \frac{\partial h_2}{\partial z} & \frac{\partial h_2}{\partial w} \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1$

$$f_{zw}(z,w) = \frac{1}{2\pi} e^{-\left[\frac{(z-w-6)^2+(w-1)^2}{2}\right]} \cdot 1$$

So, $f_{zw}(z,w) = \frac{1}{2\pi} e^{-\left[\frac{(z-w-6)^2+(w-1)^2}{2}\right]}$

Q.12 $f_{xy}(x,y) = \begin{cases} \frac{x^2+y}{3}, & -1 \le x \le 1, 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$

(i) Conditional PDF $f_{x|y}(x|y)$

$$f_{x|y}(x|y) = \frac{f_{xy}(x,y)}{f_y(y)}$$

So, $f_Y(y) = \int_{-1}^{1} \left(x^2 + \frac{y}{3}\right) dx$

$$= \left[\frac{x^3}{3} + \frac{xy}{3}\right]_{-1}^{1} = \left(\frac{1}{3} + \frac{y}{3}\right) - \left(\frac{-1}{3} - \frac{y}{3}\right)$$

$$= \frac{1}{3} + \frac{y}{3} + \frac{1}{3} + \frac{y}{3} = \frac{2+2y}{3}$$

$$f_{X|Y}(x|y) = \frac{x^2 + y/3}{\frac{2+2y}{3}}$$

$$\Rightarrow \boxed{f_{X|Y}(x|y) = \frac{x^2 + \frac{y}{3}}{\frac{2+2y}{3}}} \Rightarrow \frac{3x^2+y}{2+2y}.$$

(ii) $P(X>0 | Y=y)$. Does this depend on y?

$$P(X>0 | Y=y) = \int_0^1 f_{X|Y}(x|y)\, dx$$

$$= \int_0^1 \frac{x^2 + \frac{y}{3}}{\frac{2+2y}{3}} dx = \frac{3}{2+2y} \int_0^1 \left(x^2 + \frac{y}{3}\right) dx$$

$$= \frac{3}{2+2y}\left[\frac{x^3}{3} + \frac{xy}{3}\right]_0^1 = \frac{1}{2+2y}(1+y-0)$$

$$= \frac{(1+y)}{2(1+y)} = \frac{1}{2}$$

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classmate
Date
Page

$\Rightarrow$ It is independent of Y.

iii) Are X & Y independent?

$$f_X(x) = \int_0^1 \left(x^2 + \frac{y}{3}\right) dy = \left[x^2 y + \frac{y^2}{6}\right]_0^1$$

$$= x^2 + \frac{1}{6}.$$

$$f_Y(y) = \frac{2+2y}{3} \quad (\because \text{from eq}^n\ 2)$$

So,

$$f_X(x) \cdot f_Y(y) = \left(x^2 + \frac{1}{6}\right)\left(\frac{2+2y}{3}\right)$$

$$\neq f_{XY}(x,y).$$

$\therefore$ X and Y are not independent.

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11 Q.13 (1) $I = \Gamma(7/2)$

$$I = \frac{5}{2}\Gamma\left(\frac{5}{2}\right) \quad \because \Gamma n = (n-1)\Gamma(n-1)$$

$$I = \frac{5}{2} \times \frac{3}{2} \times \Gamma\left(\frac{3}{2}\right) = \frac{5}{2} \times \frac{3}{2} \times \frac{1}{2}\Gamma\frac{1}{2}$$

So, $I = \frac{5}{2} \times \frac{3}{2} \times \frac{1}{2} \times \sqrt{\pi}$ $\left(\because \Gamma\frac{1}{2} = \sqrt{\pi}\right)$

So, $I = \frac{15}{8} \times \sqrt{\pi}$

(2) $I = \int_0^\infty x^7 e^{-5x}\, dx$

So, $I = \frac{\Gamma 8}{5^8}$ $\left(\text{as } \int_0^\infty x^{\alpha-1} e^{-\lambda x}\, dx = \frac{\Gamma \alpha}{\lambda^\alpha}\right)$

$= \frac{7!}{5^8}$ $\left(\because \Gamma n = (n-1)!\right)$

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Q.14 Given L = length of needle

d = distance between parallel lines

where $L < d$.

Let x = distance from centre of needle to closest parallel line

θ = Actual angle between needle and one of the parallel lines.

So, Uniform PDF of x $= \begin{cases} 2/d, & 0 \le x \le d/2 \\ 0, & \text{otherwise} \end{cases}$

$x=0$ represents a needle that is centered directly on a line and $x=d/2$ represents a needle that is perfectly centred bet? two lines.

So,

$$\text{PDF of } \theta = \begin{cases} 2/\pi, & 0 \le \theta \le \pi/2 \\ 0, & \text{otherwise} \end{cases}$$

Here,

$\theta = 0 \Rightarrow$ Needle is parallel to marked lines.

$\theta = \pi/2 \Rightarrow$ Needle is perpendicular to marked lines.

The two R.V's; x and θ are independent,

$$\therefore \text{Joint PDF} = \begin{cases} \frac{4}{d\pi}, & 0 \le x \le d/2, 0 \le \theta \le \frac{\pi}{2} \\ 0, & \text{otherwise} \end{cases}$$

So,

needle crosses a line if $x \le \frac{l}{2}\sin\theta$

Probability is given by integrating by the joint PDF.

$$\text{So, } P = \int_{\theta=0}^{\pi/2} \int_{x=0}^{(l/2\sin\theta)} \frac{4}{d\pi}\, dx\, d\theta$$

So,

$$P = \frac{4}{d\pi} \int_{\theta=0}^{\pi/2} \frac{l}{2}\sin\theta \cdot d\theta$$

$$\Rightarrow P = \frac{2L}{d\pi}\left[-\cos\theta\right]_0^{\pi/2}$$

$$= \frac{2L}{d\pi}\left[-0-(-1)\right] = \frac{2L}{d\pi}$$

$$\Rightarrow \boxed{\text{Probability} = \frac{2L}{d\pi}}$$

Q. 15 $X \sim N(\mu_x, \sigma_x^2)$, $Y \sim N(\mu_y, \sigma_y^2)$

Let,

$$Z = X + Y$$

$$\Rightarrow Z \sim N(\mu_x + \mu_y, \sigma_x^2 + \sigma_y^2)$$

So, $f_Z(z) = \int_{-\infty}^{\infty} f_Y(z-x) f_X(x)\, dx$

$$\because f_X(x) = \frac{1}{\sqrt{2\pi}\sigma_x} e^{-\frac{(x-\mu_x)^2}{2\sigma_x^2}}$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma_y} \cdot e^{-\frac{(y-\mu_y)^2}{2\sigma_y^2}}$$

So, $f_Z(z) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_y} e^{\frac{-(z-x-\mu_y)^2}{2\sigma_y^2}} \cdot \frac{1}{\sqrt{2\pi}\sigma_x} e^{\frac{-(x-\mu_x)^2}{2\sigma_x^2}} d$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\times\sqrt{2\pi}\sigma_x\sigma_y} \exp\left(\frac{-\sigma_x^2(z-x-\mu_y)^2 + \sigma_y^2(x-\mu\right.}{2\sigma_x^2\sigma_y^2}$$

$$= \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma_x\sigma_y} \exp\left[\frac{x^2(\sigma_x^2+\sigma_y^2) - 2x(\sigma_x^2(z-\mu_y) + \sigma_y^2\mu_x) + \sigma_x^2(z^2+\mu_y^2 - 2z\mu_y) + \sigma_y^2\mu_x^2}{2\sigma_x^2\sigma_y^2}\right]$$

Let, $\sigma_z = \sqrt{\sigma_x^2+\sigma_y^2}$

So,

$$f_z(z) = \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma_x\sigma_y} \cdot \exp\left[-\frac{x^2 - 2x\sigma_x^2(z-\mu_y) + \sigma_y^2\mu_x + \sigma_y^2\mu_x^2 + \sigma_x^2(z^2+\mu_y^2-2z\mu_y)}{2\left(\frac{\sigma_x\sigma_y}{\sigma_z}\right)^2}\right]$$

So, This equation is the Normal density ~~f~~ distribution on x & so the integral evaluates to 1.

so,

$$f_z(z) = \frac{1}{\sqrt{2\pi}\sigma_z} e^{\left[\frac{-(z-(\mu_x+\mu_y))^2}{2\sigma_z^2}\right]}.$$

Q.16 $X_1 \sim N(2,3)$ $\therefore \mu_{x_1} = 2$, $\sigma_{x_1}^2 = 3$

$X_2 \sim N(1,4)$ $\mu_{x_2} = 1$, $\sigma_{x_2}^2 = 4$.

(i) $Y = 2X_1 + 3X_2$

$\because X_1$ & X_2 are independend.

$\Rightarrow Y$ is also independend. $Y \sim N(7,48)$

So, $\mu_y = 2\mu_{x_1} + 3\mu_{x_2}$

$= 2(2) + 3(1) = 7.$

(ii) $Var(Y) = 4V(X_1) + 9V(X_2)$

$= 4\times 3 + 9\times 4 = 48.$

(i) $Y = X_1 - X_2$

Since X_1 & X_2 are independent
$\Rightarrow$ y is also independent

$\therefore \mu_Y = \mu_{X_1} - \mu_{X_2}$

$= 2 - 1 = 1$

(ii) $Var(Y) = V(X_1 - X_2)$

$= Var(X_1) + Var(-X_2)$

$= Var(X_1) + Var(X_2)$

$= 3 + 4$

$= 7.$