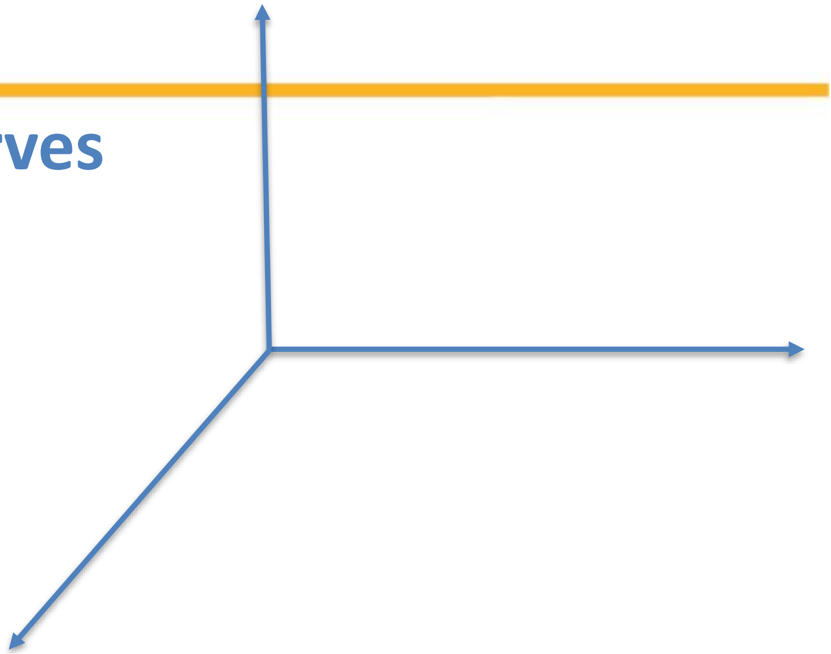
SJSU SAN JOSÉ STATE UNIVERSITY

DATA 220: Mathematical Methods for Data Analytics

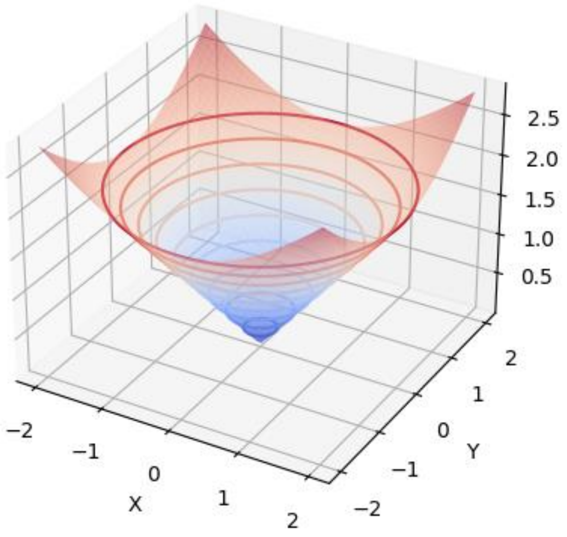
Dr. Mohammad Masum

Multivariate Calculus - sketching graphs and level curves

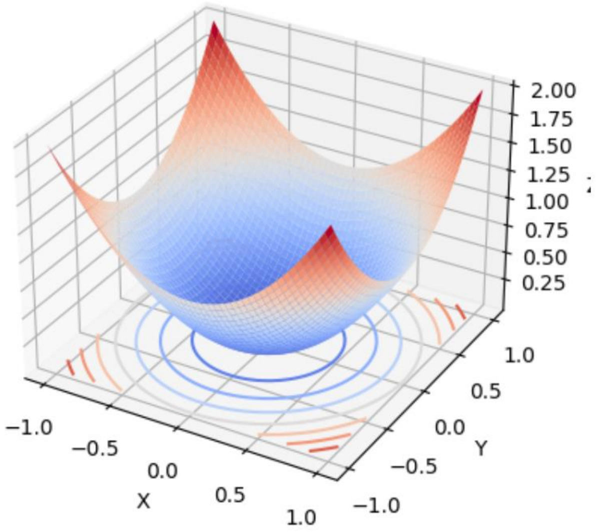
- Plot a 3 dimensional point
- Find distance between two 3-dimensional points
- What is level curves and how to draw



Cone and its Level Curve

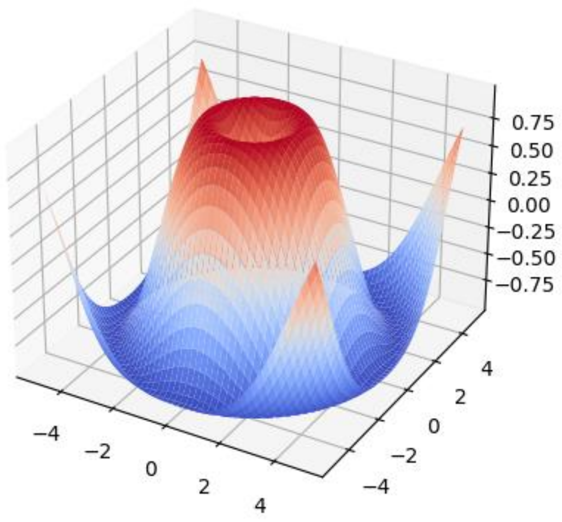


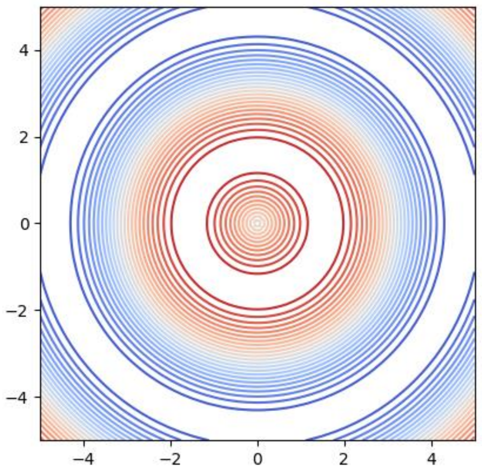
Sphere and its level curves



Multivariate Calculus - sketching graphs and level curves

$$f(x,y) = \sin(\sqrt{(x^2 + y^2)})$$

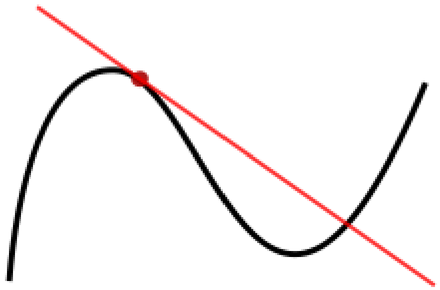


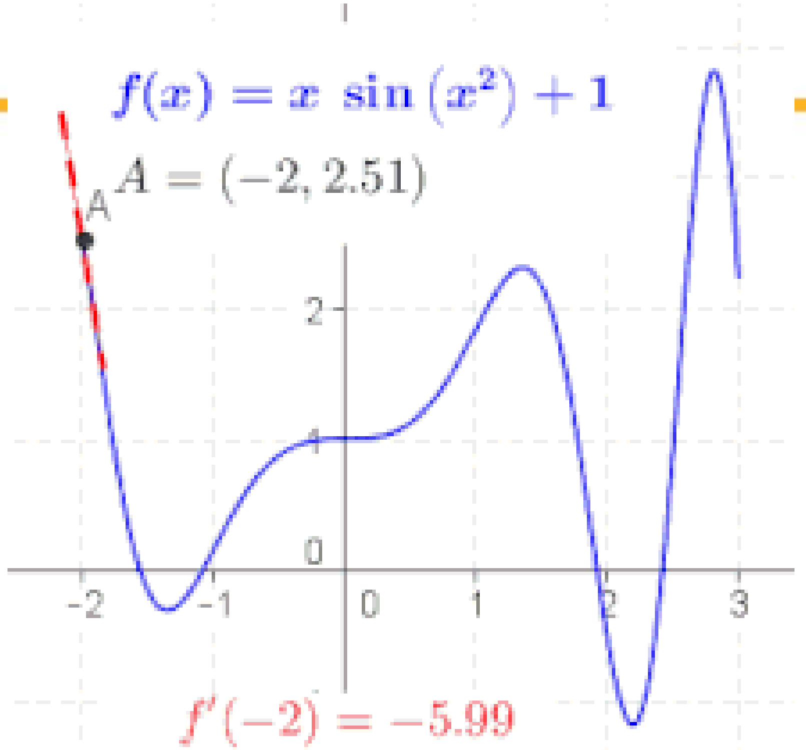


SJSU SAN JOSÉ STATE UNIVERSITY

Derivatives

- If a function of a single variable is differentiable at a chosen input value, then the derivative of the function at that point is defined as the slope of the tangent line to the graph of the function at that point





Derivate at different point of the function & the derivative = $\sin(x^2) + 2x^2 \cos(x^2)$

SJSU SAN JOSÉ STATE UNIVERSITY

Multivariate Calculus - partial derivatives

- We can not take derivatives for multivariate function as we do for single variable function
- In multivariable function, we must account for all the variables in function, which introduces partial derivatives
  - A partial derivative of a multivariable function is its derivative w.r.t. one of those variables hold other variables constant

Example

$$f(x) = 6x^3 + x^2 \quad - 5x$$
$$f(x, y) = x^2 + y^2 + xy + 4x$$

Find the value of the first order partial derivatives at the point (2,1) for $f(x, y) = x^2 + y^2 + xy + 4x$

SJSU SAN JOSÉ STATE UNIVERSITY

Multivariate Calculus - partial derivatives

Partial derivatives in three dimensions-

Find the value of the first order partial derivatives of

$$\boldsymbol{f(x, y, z) = x^2 + y^2 + z^2 + x^2y^2z^2}$$

Multivariate Calculus - partial derivatives

- Higher order partial derivatives
  - We know how to take the higher order partial derivatives for single variable
  - We extend it for multivariable function

$$
\begin{tikzcd}[column sep=small]
 & & & f(x,y) \arrow[dll] \arrow[drr] & & & \\
 & \frac{\partial f}{\partial x} \arrow[dl] \arrow[dr] & & & & \frac{\partial f}{\partial x} \arrow[dl] \arrow[dr] & \\
\frac{\partial^2 f}{\partial x^2} & & \frac{\partial^2 f}{\partial x \partial y} & = & \frac{\partial f}{\partial y \partial x} & & \frac{\partial f}{\partial y^2}
\end{tikzcd}
$$

Example: Find the second order partial derivatives of the multivariate function

$$f(x,y) = 5x^2y$$

Multivariate Calculus - chain rule

- Chain rule expresses the derivative of the composition function of two differentiable functions f and g in terms of the derivatives f and g. If $h = f \circ g$ is the function such that $h = f\big(g(x)\big)\ \forall\ x$ then the chain rule:

$$h'(x) = f'(g(x))g'(x).$$

or, equivalently,

$$h' = (f \circ g)' = (f' \circ g) \cdot g'.$$

https://en.wikipedia.org/wiki/Chain_rule

Previously, we did partial derivative for multivariable functions

$$f(x, y, z) = 2xyz$$

f is dependent variable while other three variables x, y, z are independent variables; we have partial derivative

$\frac{\partial f}{\partial x}$ = 2yz ; $\frac{\partial f}{\partial y}$ = 2xz ; $\frac{\partial f}{\partial z}$ = 2xy

we can introduce intermediate variables between the dependent and independent variables

$$f(x, y, z) = 2xy$$

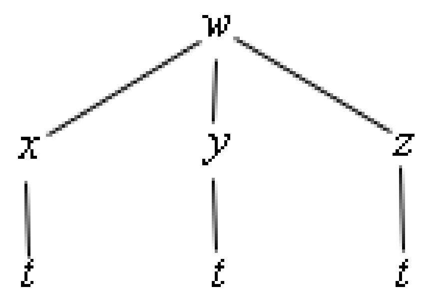
Where, $x = t; y = t^2; z = t^3$

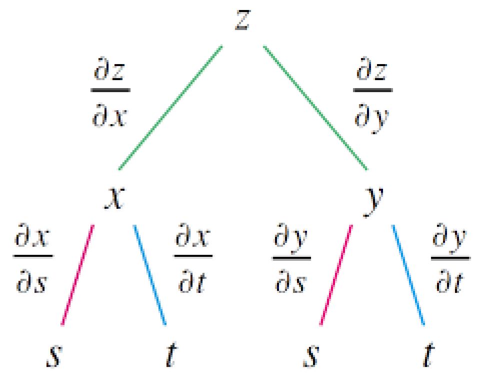
Now, f is dependent variable; x, y, z can become intermediate variables and parameter t is independent variable

Multivariate Calculus - chain rule

Draw variable dependency graph

- Case I: one independent variable
- Case II: multiple independent variables





The number of independent variables dictate the number derivative we need.

Case I: $\dfrac{dw}{dt} = \dfrac{\partial w}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial w}{\partial y}\dfrac{\partial y}{\partial t} + \dfrac{\partial w}{\partial z}\dfrac{\partial z}{\partial t}$

Case 2: $\dfrac{\partial z}{\partial s} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial s}$ $\qquad \dfrac{\partial z}{\partial t} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial t}$

https://www.usna.edu/Users/oceano/raylee/SM223/Ch14_5_Stewart(2016).pdf

Multivariate Calculus - chain rule

Use chain rule to find the partial derivatives of the multivariable function

$$w = x^2 y + x$$

Where,

$$x = 1 + t$$
$$y = 2 + t^2$$

Case I: $\dfrac{dw}{dt} = \dfrac{\partial w}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial w}{\partial y}\dfrac{\partial y}{\partial t} + \dfrac{\partial w}{\partial z}\ \dfrac{\partial z}{\partial t}$

Multivariate Calculus - chain rule

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Use chain rule to find the partial derivatives of the multivariable function

$$z = \ln r + r^2 \sin\theta$$

Where,

$$r = 3s^2 - t$$

$$\theta = 2t^2 - \frac{4}{s^2}$$

Multivariate Calculus - Gradient vectors

Multivariate Calculus – Optimization

- The process of optimization – find the absolute highest and lowest points in a function
  - For higher dimensional space, optimization is to find highest and lowest points in along every dimensional surface
- As for 2-dimensional, for three-dimensional space-
  - Find the first derivative → find critical points → second derivative test to determine maxima and minima
  - With multivariable function those points can be saddle points

Multivariate Calculus - Optimization

After finding the critical points using first derivative ($f'(x) = 0$), second derivative test determines the local extrema (minima/ maxima) points. If the function f is twice differentiable at a critical point:

- $\boldsymbol{f''(x) < 0}$ ➔ **local maxima**
- $\boldsymbol{f''(x) > 0}$ ➔ **local minima**
- $\boldsymbol{f''(x) = 0}$ ➔ **test is inconclusive**

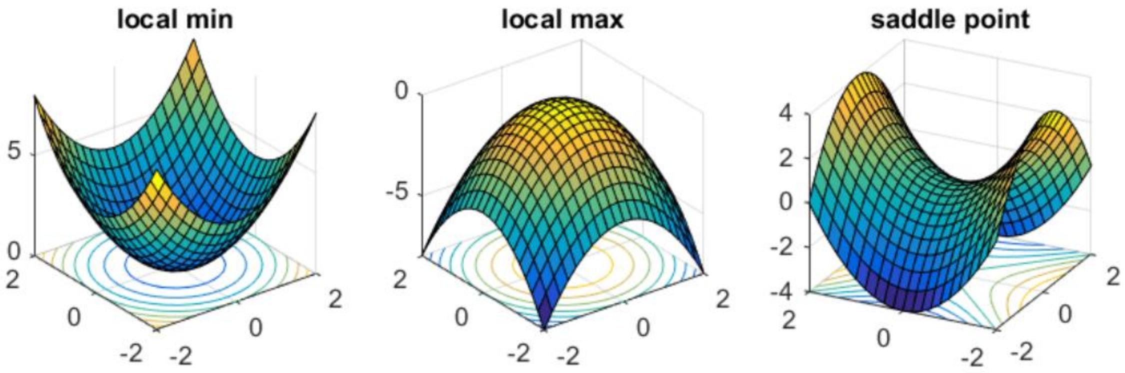
SJSU SAN JOSÉ STATE UNIVERSITY

Multivariate Calculus - Optimization

Critical values of a two variable function $f(x, y)$ is defined to be a point at which both partial derivatives are zero

$$\frac{\partial f}{\partial x} = 0$$

$$\frac{\partial f}{\partial y} = 0$$



We can use the second derivative test to classify the critical points

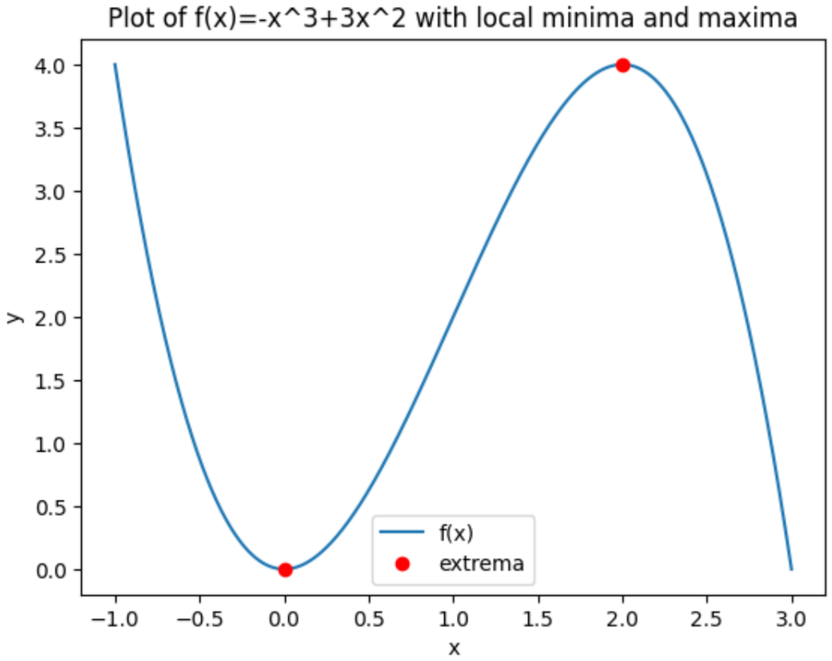
$$D(x, y) = f_{xx} f_{yy} - f_{xy}^2 ,$$

$$\begin{gathered} if \;\; D(x,y) < 0 \;\; then\ critical\ points\ are\ saddle\ points \\ if \;\; D(x,y) = 0 \;\; then\ the\ test\ is\ inconclusive \\ if \;\; D(x,y) > 0 \;\; then\ critical\ points\ are\ extrema \end{gathered}$$

$$\begin{gathered} If\ D(x,y) > 0 \;\, \&\ f_{xx} > 0 \Rightarrow \;\; critical\ point\ is\ local\ minimum \\ If\ D(x,y) > 0 \;\, \&\ f_{xx} < 0 \Rightarrow \;\; critical\ point\ is\ local\ maximum \end{gathered}$$

Multivariate Calculus - Optimization

Q:Find the critical values of the function
$$f(x) = -x^3 + 3x^2$$



Multivariate Calculus - Optimization

Q:Find the critical values of the function

$$f(x, y) = 4x^2 - 2xy + 2y^2 - 18$$

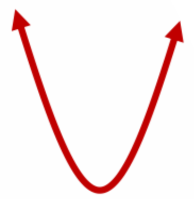
Multivariate Calculus - Optimization

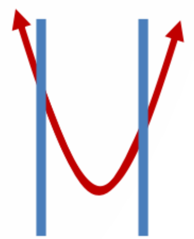
Q: for which values of a and b, (-8,-5) is the critical point of the function

$$\boldsymbol{f(x,y) = x^2 + y^2 - ax + by}$$

Multivariate Calculus – Constrained Optimization

- Constrained optimization- to maximize or minimize an objective function subject to some constraints
- Constraints can be anything that limits the feasible region of the optimization problem, such as inequalities, equalities, or bounds
- Example: a company wants to maximize its profits subject to constraints on its production capacity and resources
- In this case, the objective function is the profit function, and the constraints are the production capacity and resource availability
- Constrained optimization problems can be solved using various techniques
    - Lagrange multipliers

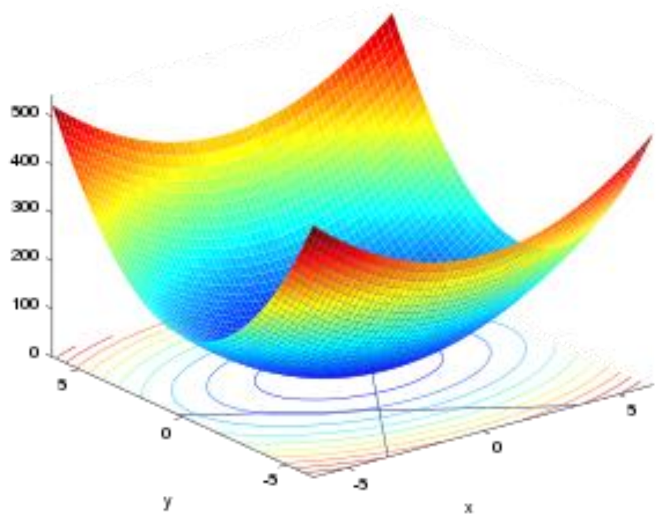


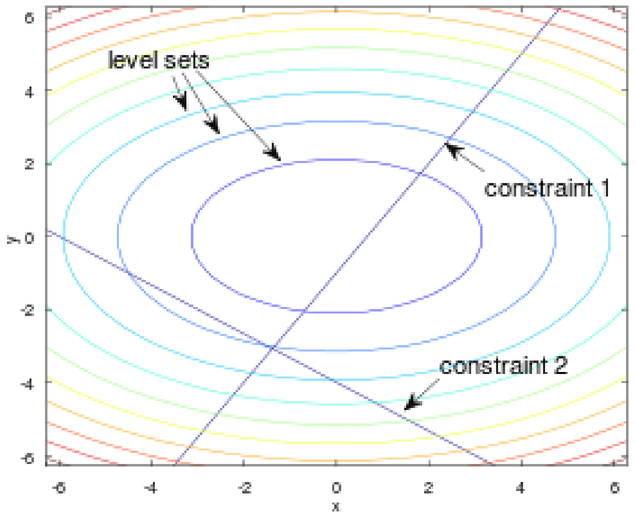


https://web.stanford.edu/group/sisl/k12/optimization/MO-unit3-pdfs/3.1introandgraphical.pdf

Multivariate Calculus – Lagrange Multipliers

- Lagrange multipliers is a method to find extrema of function subject to equality or inequality constraints





Multivariate Calculus – Lagrange Multipliers

- Lagrange multipliers is a method to find extrema of function subject to equality or inequality constraints

single constraint

Q: Find the extrema of the function, subject to the given constraint

$$f(x, y) = x^2 + y^2 + 100$$

When $2x + y = 6$