Unimodal inversion sequences and related pattern classes

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15th International Conference on Permutation Patterns
Reykjavik University, Iceland
June 26-30, 2017

Joint work with Walter Stromquist

Back to Iceland

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2017: It is 93206 steps from Landmannalaugar to Thórsmörk.

(Source: Legs)

Inversion sequences

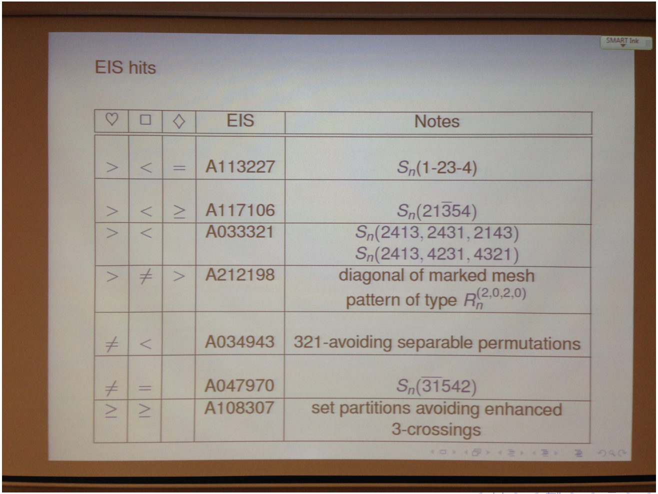
Definition

An inversion sequence (subexcedant sequence, reversal of Lehmer code) is a sequence $e = e_1 e_2 \dots e_n$ ($n \geqslant 0$) such that $e_i \in [0, i-1]$ for all i.

Let $\mathrm{SE}_n = \prod_{i=1}^{n} [0, i-1]$ denote the set of all inversion sequences of length n, and let $\mathrm{SE} = \cup_{n \geqslant 0} \mathrm{SE}_n$.

Patterns and inversion sequences

At Permutation Patterns 2014, Savage initiated the study of patterns in inversion sequences.



EIS hits

♡	□	◇	EIS	Notes
$>$	$<$	$=$	A113227	$S_n(1\text{-}23\text{-}4)$
$>$	$<$	$\geq$	A117106	$S_n(21\bar{3}54)$
$>$	$<$		A033321	$S_n(2413, 2431, 2143)$ $S_n(2413, 4231, 4321)$
$>$	$\neq$	$>$	A212198	diagonal of marked mesh pattern of type $R_n^{(2,0,2,0)}$
$\neq$	$<$		A034943	321-avoiding separable permutations
$\neq$	$=$		A047970	$S_n(\overline{31}542)$
$\geq$	$\geq$		A108307	set partitions avoiding enhanced 3-crossings

Unimodal inversion sequences

A033321 in OEIS counts inversion sequences avoiding subsequences $e_i e_j e_k$ such that $e_i > e_j < e_k$. Since $e_1 = 0$, this is equivalent to unimodality of e, i.e.

$$e_1 \leqslant \cdots \leqslant e_m > e_{m+1} \geqslant \cdots \geqslant e_n$$

for some $m \leqslant n$. Denote the set of unimodal inversion sequences of length n by USE_n and let $\mathrm{USE} = \cup_{n \geqslant 0} \mathrm{USE}_n$.

Head of e, $\mathrm{head}(e) = e_1 \dots e_m$ is the longest nondecreasing prefix of e. Let $h(e) = |\mathrm{head}(e)| = m \geqslant 2$.

Tail of e, $\mathrm{tail}(e) = e_{h(e)+1} \dots e_n$ (possibly empty).

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The counting sequence

Mansour, Shattuck, 2017:

A033321 also counts 9 pattern avoiding classes, $\mathrm{Av}(T_i)$, $1 \leqslant i \leqslant 9$, where

$$\begin{array}{lll} T_1 = \{2134, 3124, 4123\}, & T_2 = \{1234, 1324, 1423\}, & T_3 = \{2134, 1324, 1423\} \\ T_4 = \{3124, 2314, 2413\}, & T_5 = \{3214, 2314, 2413\}, & T_6 = \{2143, 1243, 1342\} \\ T_7 = \{1243, 1342, 1432\}, & T_8 = \{2143, 3142, 4132\}, & T_9 = \{2143, 2413, 3142\}. \end{array}$$

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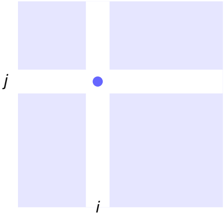
Augmentation

π



$\pi(i,j)$

$$\mathrm{Aug}(\pi,i,j)=\{\pi(i,k)\ :\ 1\leqslant k\leqslant|\pi|+1,\ k\neq j\}$$

Example

$123(1,2)=2134$, $123(1,3)=3124$, $123(1,4)=4123$,
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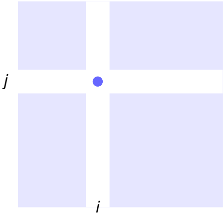

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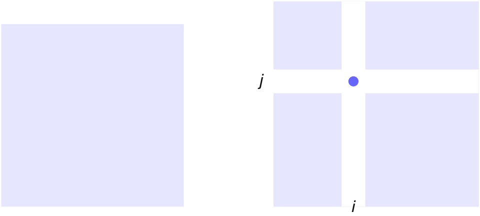

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Recall: Wilf-equivalence of the T_i's is equivalent to Wilf-equivalence of the corresponding mesh patterns (π_i, R_i).

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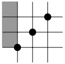
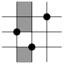
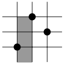
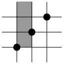
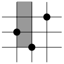
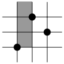
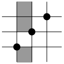
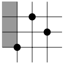
$T_1 =$ , $T_4 =$ , $T_6 =$ ,

$T_2 =$ , $T_5 =$ , $T_7 =$ ,

$T_3 =$ , $T_9 =$, $T_8 =$ .

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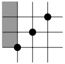
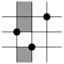
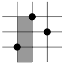
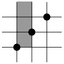
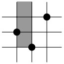
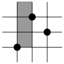
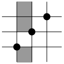
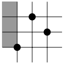
$T_1 =$ , $T_4 =$ , $T_6 =$ ,

$T_2 =$ , $T_5 =$ , $T_7 =$ ,

$T_3 =$ , $T_8 =$ .

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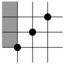
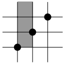
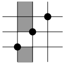
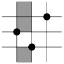
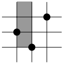
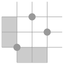
$T_1 =$  ,

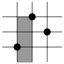
$T_2 =$  ,

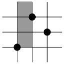
$T_3 =$  ,

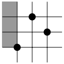
$T_4 =$  ,

$T_5 =$  ,

$T_9 =$  ,

$T_6 =$  ,

$T_7 =$  ,

$T_8 =$  .

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Encoding permutations by subexcedant sequences

We would like to find bijections $f_i : \mathrm{SE}_n \to \mathfrak{S}_n$ such that $f_i(\mathrm{USE}_n) = \mathrm{Av}(T_i)$. To do that, we first define the bijections g_{132} and g_{123} from the monotone (nondecreasing) strings in SE_n to $\mathrm{Av}(132)$ and $\mathrm{Av}(123)$.

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g_{132} and g_{123}

- For pattern 132: For each $i = 1, \ldots, n$, let $\pi^{(i)} = \pi^{(i-1)}(1, e_i + 1)$. In other words, at the i-th step, $1 \leqslant i \leqslant n$, insert $e_i + 1$ on the left. Then $\pi = \pi^{(n)} = g_{132}(e)$ and e_i is the number of inversions starting from π_{n-i+1}.
- For pattern 123: Let $\pi^{(1)} = 1$, and for each $i = 2, \ldots, n$, let

$$\pi^{(i)} = \begin{cases} \pi^{(i-1)}(1, e_i) & \text{if } e_i > e_{i-1}, \\ \pi^{(i-1)}(1, i) & \text{if } e_i = e_{i-1}. \end{cases}$$

Then $\pi = \pi^{(n)} = g_{123}(e)$ and non-right-to-left-maxima of π are the distinct nonzero entries of e.

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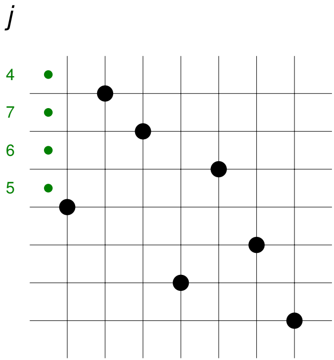
A step in g_{123} algorithm



$g_{123}(0002224j)$

The bijections f_1 and f_8

We can define bijections f_1 and f_8 as follows.

- For set $T_8 = \mathrm{Aug}(132, 1, 1)$: Start with $g_{132}(\mathrm{head}(e))$ and insert $e_{h(e)+1} + 1, \ldots, e_n + 1$ successively on the left to obtain $\pi = f_8(e)$. Then e is just the inversion code of $\pi = f_8(e)$ as for pattern 132.
- For set $T_1 = \mathrm{Aug}(123, 1, 1)$: Start with $g_{123}(\mathrm{head}(e))$ and insert $e_{h(e)+1} + 1, \ldots, e_n + 1$ successively on the left to obtain $\pi = f_1(e)$. Then $\mathrm{tail}(e)$ is the inversion code of the prefix of π ending on the rightmost "1" in an occurrence of 123 in π.

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f-insertion point

Idea for f_2 and f_3: instead of inserting each new entry on the left, insert some entries at a particular location in the middle, called the f-insertion point.

If σ contains a pattern 123, determine the f-insertion point of σ as follows:

- Let $A(\sigma)$ be the smallest “1” in an instance of 123 in σ.
- Let $B(\sigma)$ be the rightmost “2” in an instance of 123 in σ that starts with $A(\sigma)$.
- Then the f-insertion point of σ is immediately to the left of $B(\sigma)$. Denote this position by $\text{ins}_f(\sigma)$.

Example

Let $\sigma = 245316$, then $A(\sigma) = 2$ and $B(\sigma) = 3$, so the f-insertion point is immediately to the left of 3, and $\text{ins}_f(\sigma) = 4$.

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- Let $B(\sigma)$ be the rightmost "2" in an instance of 123 in σ that starts with $A(\sigma)$.
- Then the f-insertion point of σ is immediately to the left of $B(\sigma)$. Denote this position by $\text{ins}_f(\sigma)$.

Example

Let $\sigma = 245316$, then $A(\sigma) = 2$ and $B(\sigma) = 3$, so the f-insertion point is immediately to the left of 3, and $\text{ins}_f(\sigma) = 4$.

f-insertion point

Idea for f_2 and f_3: instead of inserting each new entry on the left, insert some entries at a particular location in the middle, called the f-insertion point.

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The bijection f_2

Let $\pi^{(h(e))} = g_{123}(\text{head}(e))$. Then for each $i = h(e)+1, \ldots, n$, let $\pi' = \pi^{(i-1)}(1, e_i+1)$, then $\pi^{(i)} = \pi^{(i-1)}(\text{ins}_f(\pi') - 1, e_i + 1)$. In other words,

- tentatively insert $e_i + 1$ to the left of π to obtain π',
- find the f-insertion point of π',
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A step in the f_2 algorithm

Inserting an entry into a permutation avoiding 123

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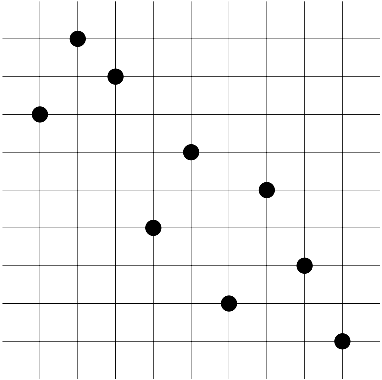

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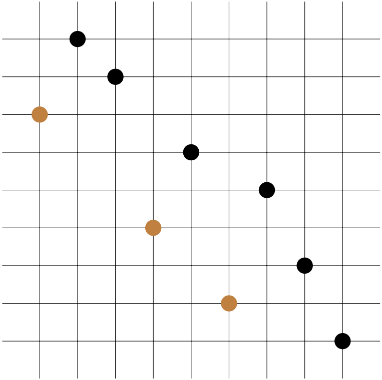

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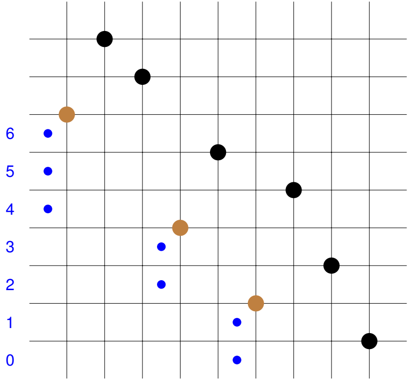

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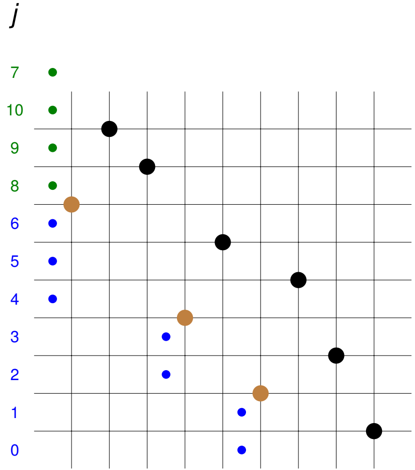

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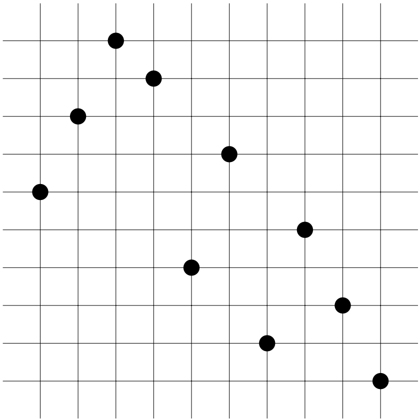

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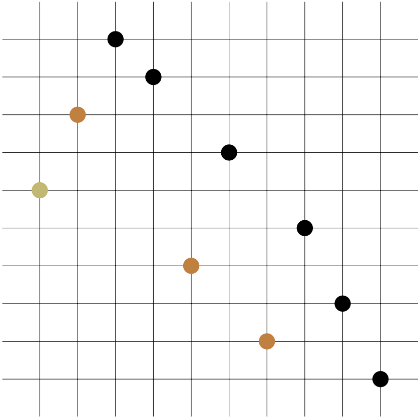

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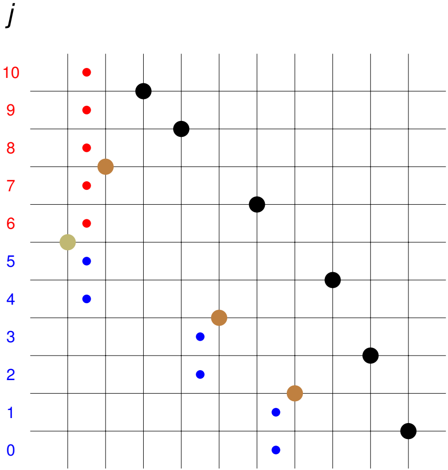

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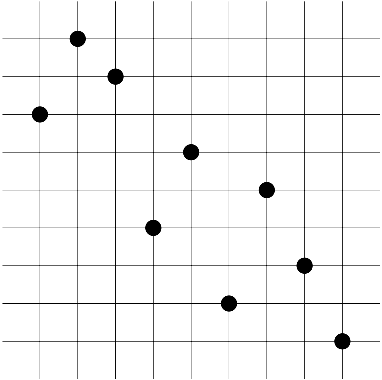

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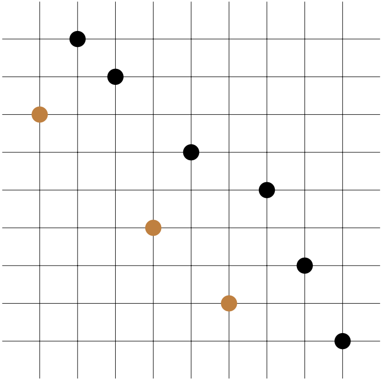

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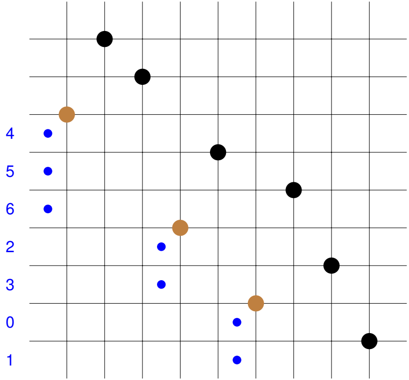

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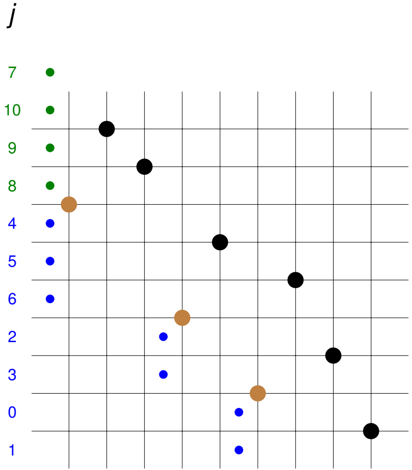

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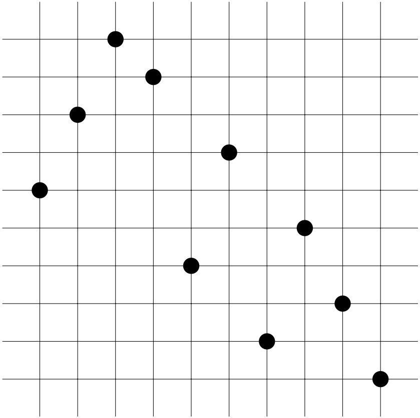

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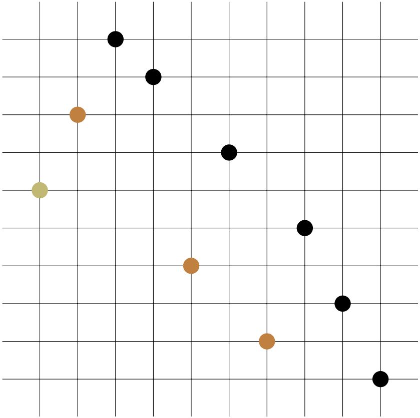

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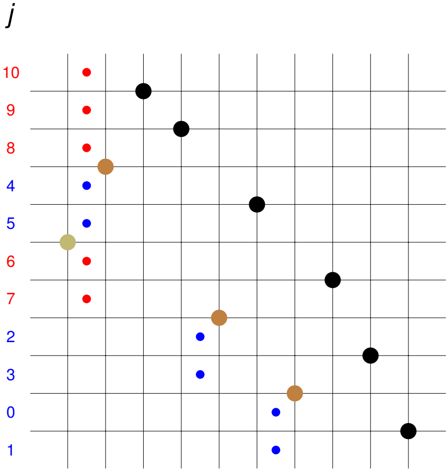

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Inflations of T_i's

Let σ and τ be shape-Wilf-equivalent on top-right-justified Ferrers boards. Recall: in this case, $\sigma \overset{s}{\sim} \tau \implies 1 \oplus \sigma \overset{s}{\sim} 1 \oplus \tau$.

Theorem

The following patterns are Wilf-equivalent:

$T_1(\sigma) =$ $\sim$ $T_2(\sigma) =$ $\sim$ $T_3(\sigma) =$

$\wr \qquad \wr \qquad \wr$

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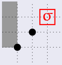
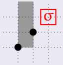
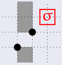
$T_1(\sigma) =$ 

$\sim$ $T_2(\sigma) =$ 

$\sim$ $T_3(\sigma) =$ 

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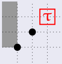
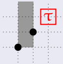
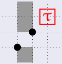
$T_1(\tau) =$ 

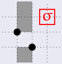
$\sim$ $T_2(\tau) =$ 

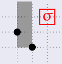
$\sim$ $T_3(\tau) =$ 

Inflations of T_i's

Conjecture

The following patterns are Wilf-equivalent: $T_1(\sigma)$, $T_2(\sigma)$, $T_3(\sigma)$, *and*

$\cdots \sim T_4(\sigma) =$ 

$\sim$ $T_5(\sigma) =$ 

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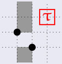
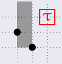
$\cdots \sim T_4(\tau) =$ 

$\sim$ $T_5(\tau) =$ 

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Inflations of T_i's

Conjecture

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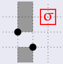
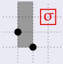
$\cdots \sim T_4(\sigma) =$ 

$\sim$ $T_5(\sigma) =$ 

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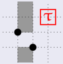
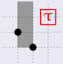
$\cdots \sim T_4(\tau) =$ 

$\sim$ $T_5(\tau) =$ 

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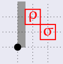
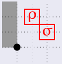
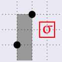
$T_7(\rho, \sigma) =$ 

$\sim$ $T_8(\rho, \sigma) =$ 

Inflations of T_i's

Conjecture

All patterns $T_i(\sigma)$ $(i = 1, \dots, 5)$, $T_6(\sigma) =$  *, as well as* $T_7(1, \sigma)$, $T_8(1, \sigma)$ *are Wilf-equivalent for* $\sigma = r(\mathrm{id})$.

Questions

- Are there other Wilf-equivalent inflations?
- Are all these Wilf-equivalences instances of more general theorem(s) about mesh patterns?
- f_i^{-1}'s encode permutations as subexcedant sequences. Sum of their entries is a Mahonian statistic fstat_i. For a given $i = 1, \ldots, 9$, what does it count?
  - E.g. $f_8^{-1} = \mathrm{invcode}$ and $\mathrm{fstat}_8 = \mathrm{inv}$ (obvious).
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Thank you!

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