Packing Densities: An Update

Jakub Sliačan

Walter Stromquist

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Packing Densities

Let σ be a permutation of size k (the "pattern"),
and let π be any permutation.

The packing density of σ in π is

$\delta(\sigma, \pi)$ = Probability that a random k – subsequence of π is order-isomorphic to σ.

Then:

$$\delta(\sigma, n) = \max_{\pi \in S_n} \delta(\sigma, \pi)$$

and

$$\delta(\sigma) = \lim_{n \to \infty} \delta(\sigma, n).$$

<u>We want to know $\delta(\sigma)$ for various patterns σ.</u>

Packing Densities with Permutons

A *permuton* μ is a probability measure on $[0,1]^2$ with uniform marginals.

The *packing density* of σ in μ is

$\delta(\sigma, \mu)$ = Probability that k points drawn independently from μ are order-isomorphic to the graph of σ.

Then

$$\delta(\sigma) = \max_{\mu} \delta(\sigma, \mu)$$

and a permuton achieving this maximum is an *optimizer* for σ. This definition of $\delta(\sigma)$ agrees with the previous definition.

(Alternatively: If you like, μ is just a permutation of size n, and we only care about the case of $n \to \infty$.)

Packing Densities for Patterns of Size 3, 4

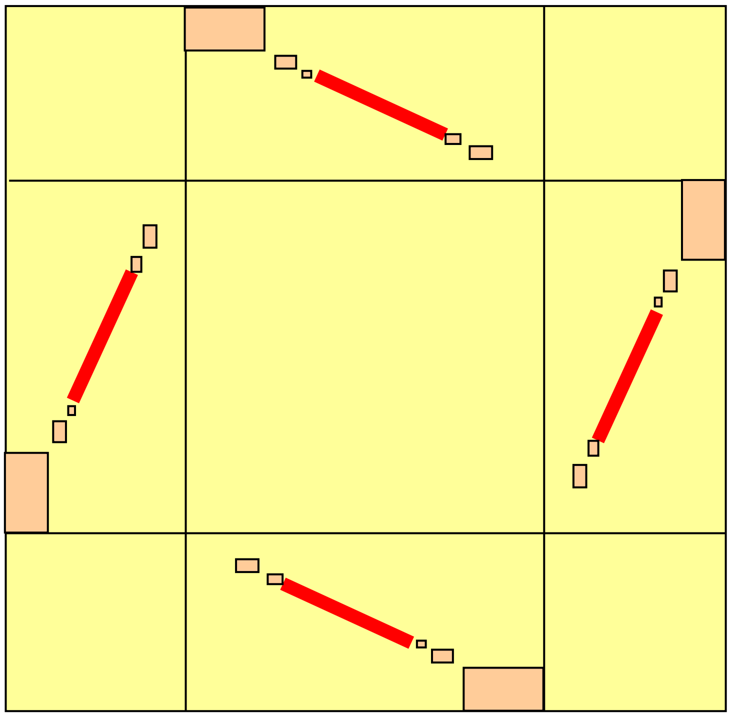
σ	Upper Bound	Lower Bound
123	1	
132	$2\sqrt{3}-3 \approx 0.464$	
1234	1	
1243	3/8	
2143	3/8	
1432	algebraic, ≈ 0.4236	
1324	0.244054523929423295868733…	
2413	≈ 0.104724	≈ 0.104780
1342	≈ 0.198836597	≈ 0.198837287

For 1324, the only update is that we can now compute a hundred digits or more. We still don't know whether the packing density is algebraic.

Packing 2413

Upper Bound ≈ 0.104780 is from Flag Algebra, JS PP 2016.

Lower Bound ≈ 0.104724 is from Presutti and WRS, PP 2007:

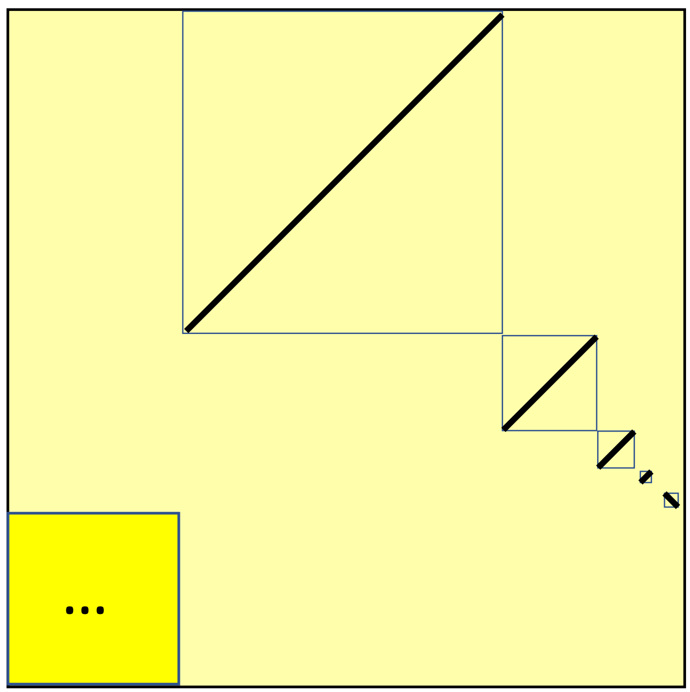


Neither the upper bound nor the lower bound pretends to be exact.

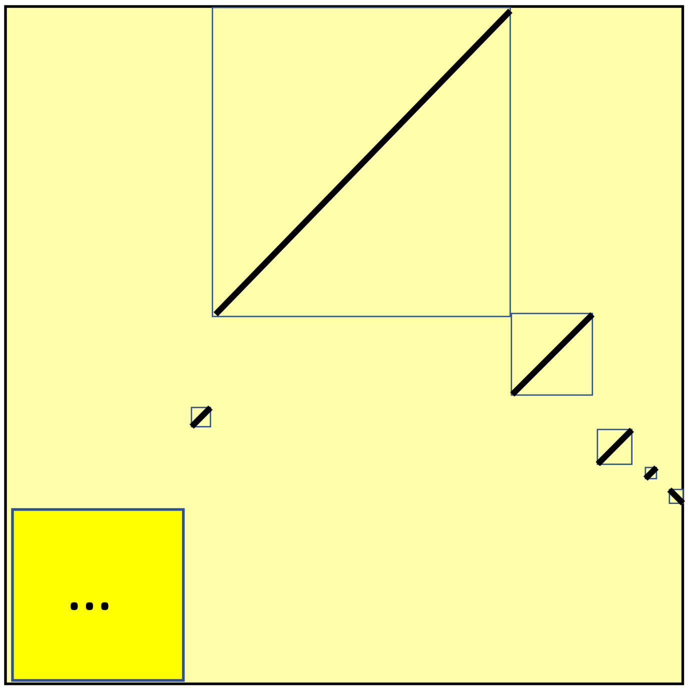
Packing 1342

Our main update this year is for $\delta(1342)$.

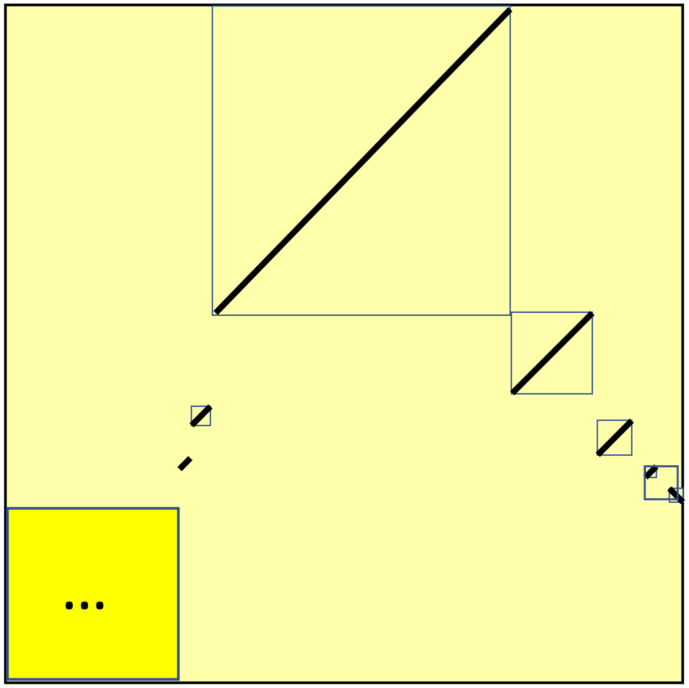
The previous lower bound was ≈ 0.1965796, based on this construction by Batkeyev.



But we can do better by adding a tiny box:



And then, why not another one?



But that's the end of it. Adding more boxes in the middle doesn't help; this is the best we can do with this theme. And since there are only seven boxes, the optimum box sizes (and the packing density) must be algebraic.

Box sizes (left to right)

$a1 = 0.21741277235363473086924448 43$

$a2 = 0.01705980578992427227406205 49$

$a3 = 0.05161014024878922702302309 72$

$a4 = 0.43407228098738649943129530 07$

$a5 = 0.14798956259503904962506118 29$

$a6 = 0.07644572558056569713833513 65$

$a7 = 0.05540971244466052363897874 33$

This construction gives the lower bound ≈ 0.198836597 for $\delta(1342)$.

The current upper bound (Flag Algebra, JS PP2016) is ≈ 0.198837287.

The Density Vector

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For a permuton μ, consider its "density vector" with all of its size-3 pattern densities:

$$X(\mu) = (\delta(123, \mu), \delta(132, \mu), \delta(213, \mu), \delta(231, \mu), \delta(312, \mu), \delta(321, \mu)).$$

I'm tired of typing the μ, so let μ be understood and use this notation:

$$X(\mu) = (\delta_{123}, \ \delta_{132}, \ \delta_{213}, \ \delta_{231}, \ \delta_{312}, \ \delta_{321}) \quad \in \mathbb{R}^6.$$

What can this vector be?

This might be a hard problem. The set of possible vectors is not convex. It has cross sections that aren't even simply connected.

But we do know some things. The six entries add to 1. And…

Linear Constraints on the Density Vector

…the entries are subject to some linear constraints.

δ_{132}	$\leq$	0.464…	(packing density of 132)
$\delta_{123} + \delta_{321}$	$\geq$	1/4	(monotone triples)
$\delta_{213} + \delta_{312}$	$\leq$	1/2	("V-shaped things")

How can we find more of these ?

Specifically, we want formulas of the form

$$a_{123}\,\delta_{123} + a_{132}\,\delta_{132} + a_{213}\,\delta_{213} + a_{231}\,\delta_{231} + a_{312}\,\delta_{312} + a_{321}\,\delta_{321} \;\geq\; 0.$$

For example, suppose we knew (from some magical machine) that

$$3\,\delta_{123} + (-2\sqrt{3})\,\delta_{132} + 3\,\delta_{213} + (1-\sqrt{3})\,\delta_{231} + (1-\sqrt{3})\,\delta_{312} + 3\,\delta_{321} \;\geq\; 0.$$

Then with about two lines of algebra, we can conclude that

$$\boxed{\delta_{132} \;\leq\; (2\sqrt{3}-3) \;-\; (6-3\sqrt{3})\;(\,\delta_{231} + \delta_{312}\,).}$$

That's an upper bound for the packing density! And it's better (because of the extra terms) then the standard version. And---thanks to that magical machine---it is almost free.

Flag Algebra

A *flag* F of size (k, t) is a permutation of size k, together with a designated subsequence of size t.

For example:

5 2 3 1 4

The "outer permutation" is 5 2 3 1 4.
The designated subsequence is 2 4.
That defines the "inner permutation" 1 2. If the inner permutation is τ, then F is called a τ-*flag*. So this is a 12-flag.

There are exactly four flags of size (2,1). They are all "1-flags."

F_1	F_2	F_3	F_4
1 2	1 2	2 1	2 1

The density of a flag F in a permuton μ is

$\delta(F, \mu)$ = Probability that k independent draws from μ,
with t of them randomly designated as a subsequence,
are order-isomorphic to the graph of F.

We also need to know about *joint appearances* of two flags F_1 and F_2, both of which have size (k, t) and the same inner permutation τ.

Suppose a permutation π of size $(2k - t)$ can be divided into two subsequences of length k, which are isomorphic to F_1 and F_2 respectively, and which overlap only on their images of τ. That's a joint appearance.

For example, 2 1 3 is a joint appearance of $F_4 = 2\ 1$ and $F_1 = 1\ 2$.

The same pattern 213 can be interpreted as a joint appearance of F_1 and F_3, or of F_2 and F_2.

Each of the other size-3 patterns can be interpreted as a joint appearance in three ways, also.

Fact:

$$\delta (F1, \mu) \, \delta (F2, \mu) =$$

$$\sum_{\pi} (\delta(\pi, \mu)) \ \text{(Number of ways to represent } \pi \text{ as a joint appearance of } F_1 \text{ and } F_2 \text{)}.$$

The Flag Algebra Machine

Now let's select a collection of flags with the same size and same τ.
Today, we will use the four flags of size (2,1).

And, let's pick ANY four real numbers q_1, q_2, q_3, q_4.

Then:

$$0 \le (q_1 \, \delta(F_1,\mu) + q_2 \, \delta(F_2, \mu) + q_3 \, \delta(F_3, \mu) \; q_4 \, \delta(F_4, \mu))^2$$

$$= \sum_{i,j} q_i q_i \;\; \delta(\mathrm{Fi}, \mu)\delta(\mathrm{Fj}, \mu)$$

$= \sum_{i,j} q_i q_i \;\; \sum_{\pi} \delta(\pi, \mu)$(Number of ways to represent π as a joint appearance of F_1 and F_2)

$$= \sum_{\pi}\left(\sum_{i,j} q_i q_i \; (Number\ of\ ways \ldots)\right) \;\; \delta(\pi, \mu)$$

$= \sum_{\pi} \; (\alpha_{\pi}) \;\; \delta(\pi, \mu).$ THAT'S WHAT WE WANTED.

Very Concrete Results

For ANY choice of q_1, q_2, q_3, q_4, we have:

$$
\begin{aligned}
& (q_1^2 + q_1 q_2 + q_2^2) \quad \delta(123, \mu) \\
+ \quad & (q_1^2 + q_2 q_3 + q_2 q_4) \quad \delta(132, \mu) \\
+ \quad & (q_2^2 + q_1 q_3 + q_1 q_4) \quad \delta(213, \mu) \\
+ \quad & (q_4^2 + q_2 q_3 + q_1 q_3) \quad \delta(231, \mu) \\
+ \quad & (q_3^2 + q_1 q_4 + q_2 q_4) \quad \delta(312, \mu) \\
+ \quad & (q_3^2 + q_3 q_4 + q_4^2) \quad \delta(321, \mu) \quad \geq 0.
\end{aligned}
$$

When $q_1, = 1$, $q_2 = q_3 = 0$, $q_4 = -1$ we get:

$$\delta_{213} + \delta_{312} \leq 1/2 \qquad \text{(“V-shaped things”)}$$

When $q_1 = 0$, $q_2 = \sqrt{3}$, $q_3 = q_4 = -1$ we get:

$$3\,\delta_{123} + (-2\sqrt{3})\,\delta_{132} + 3\,\delta_{213} + (1-\sqrt{3})\,\delta_{231} + (1-\sqrt{3})\,\delta_{312} + 3\,\delta_{321} \geq 0$$

(which gives us the packing density of 132).

Let $\phi = \frac{1+\sqrt{5}}{2}$ = "golden mean" $\approx$ 1.6218.

When $q_1 = -\phi$, $q_2 = \phi$, $q_3 = 1$, $q_4 = -1$ we get

$$(2\,\phi^2)\;(\delta_{123} + \delta_{321}) - (2\,\phi)\,(\delta_{213} + \delta_{312}) \;\geq\; 0$$

which simplifies to

$$(\delta_{213} + \delta_{312}) \,/\, (\delta_{123} + \delta_{321}) \;\leq\; \phi$$

or

$$\boxed{\frac{\text{Number of V-shaped patterns}}{\text{Number of monotone patterns}} \;\leq\; \phi}$$

in any permuton or (in the limit) large permutation. Did we know that?