Characterising inflations of monotone grid classes of permutations

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Abstract

We characterise those permutation classes whose simple permutations are monotone griddable. This characterisation is obtained by identifying a set of nine substructures, at least one of which must occur in any simple permutation containing a long sum of 21s. A copy of the full paper is available on the arXiv:1702.04269.

1 Introduction

A common route to understanding the structure of a permutation class (and hence, e.g. complete its enumeration) is via its simple permutations, as their structure can be considerably easier to characterise than the entire class. Albert, Atkinson, Homberger and Pantone [4] introduced the notion of *deflatability* to study this phenomenon: that is, the property that the simples in a given permutation class $\mathcal{C}$ actually belong to a proper subclass $\mathcal{D} \subsetneq \mathcal{C}$. See also Vatter's recent survey [12].

One general case of deflatability is where the set of simple permutations of a class is finite. Such classes are well-quasi-ordered, finitely based, and have algebraic generating functions [1], and via a Ramsey-type result for simple permutations [8], it is decidable when a permutation class has this property [9].

Here, we look beyond classes with finitely many simples to those whose simples are 'monotone griddable', and prove the following characterisation. For definitions common to the wider study of

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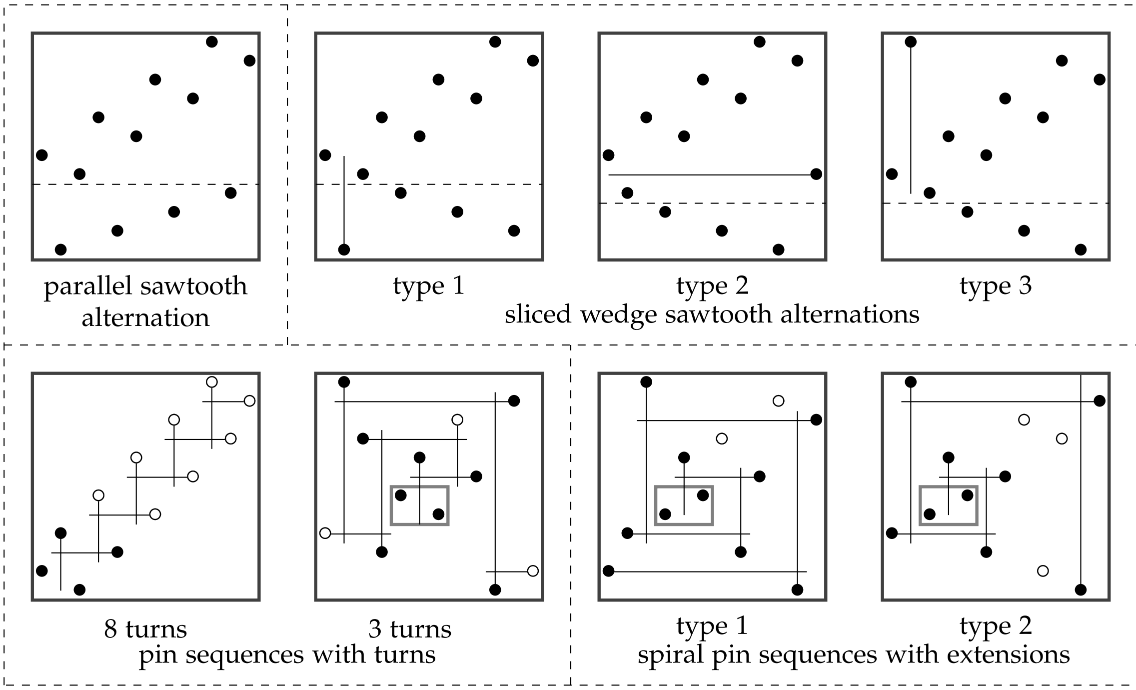

Figure 1: Examples of the permutations characterising the griddability of simples in Theorem 1.1.

permutation patterns, we refer the reader to Bevan's introduction [7], and, in lieu of definitions, the reader is invited to look at Figure 1 for examples of the structures mentioned.

Theorem 1.1. *The simple permutations in a class $\mathcal{C}$ are monotone griddable if and only if $\mathcal{C}$ does not contain the following structures, or their symmetries:*

- *arbitrarily long parallel sawtooth alternations,*
- *arbitrarily long sliced wedge sawtooth alternations,*
- *proper pin sequences with arbitrarily many turns, and*
- *spiral proper pin sequences with arbitrarily many extensions.*

In general, classes whose simple permutations are monotone griddable do not immediately possess the range of properties that classes with only finitely many simples do. Indeed, few general properties are known even for classes that are themselves wholly monotone griddable, but this has not diminished the efficacy of the following characterisation for the structural understanding and enumeration of many classes (see, for example [3]).

A *sum of k copies of 21* is the permutation $21\,43\,65 \cdots (2k)(2k-1)$, written in one line notation, and we will abbreviate this to $\oplus_k 21$. Similarly, a *skew sum of k copies of 12* is the permutation $\ominus_k 12 = (2k-1)(2k)\cdots 34\,12$.

Theorem 1.2 (Huczynska and Vatter [10])**.** *A class $\mathcal{C}$ is monotone griddable if and only if it does not admit arbitrarily long sums of 21 or skew sums of 12. That is, for some k neither $\oplus_k 21$ nor $\ominus_k 12$ belong to $\mathcal{C}$.*

We will take the above characterisation of "monotone griddability" as our definition. However, for completeness, we observe that the standard definition is as follows: A class $\mathcal{C}$ is *monotone griddable* if there exist integers h and v such that for every permutation $\pi \in \mathcal{C}$, we may divide the plot of π into cells using at most h horizontal and v vertical lines, in such a way as the points in each cell form a monotone increasing or decreasing sequence (or the cell is empty).

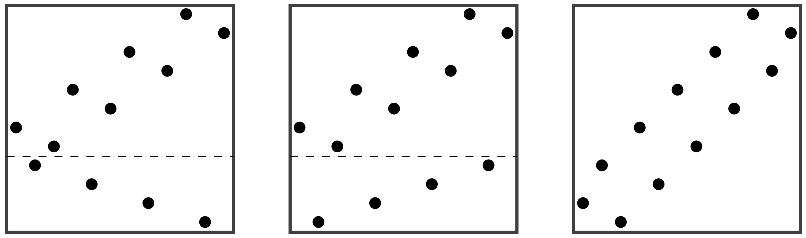

Figure 2: Examples of the two types of sawtooth alternation and an increasing oscillation in the statement of Theorem 1.3.

Aside from the structural information that monotone griddability provides in its own right, the reason that Theorem 1.2 has proved so useful is that the classes to which it has been applied typically in fact possess the stronger property of being *geometrically griddable*. Such classes are well-quasi-ordered, finitely based and have rational generating functions [2]. As we have no direct characterisation for a class to be geometrically griddable, the above theorem (which certainly provides a necessary condition) has often provided enough of a 'hook' to solve the task at hand.

It is our hope that Theorem 1.1 can provide a similar 'hook' to ease the study of classes whose simple permutations are geometrically griddable. Any such class is known to be well-quasi-ordered, finitely based, and strongly algebraic (meaning that it and every subclass have algebraic generating functions), see Albert, Ruškuc and Vatter [6]. Furthermore, every class with growth rate less than $\kappa \approx 2.20557$, is of this form [6]. For instances of practical enumeration tasks that have exploited the geometric griddability of the simple permutations, see Albert, Atkinson and Vatter [5], and Pantone [11].

Our characterisation in Theorem 1.1 relies on the following auxiliary result, which guarantees the existence of certain types of structure in simple permutations that contain a long sum of 21s.

Theorem 1.3. *There exists a function $f(n)$ such that every simple permutation that contains a sum of $f(n)$ copies of 21 must contain a parallel or wedge sawtooth alternation of length $3n$ or an increasing oscillation of length n.*

See Figure 2 for examples of the three types of unavoidable structure. Note that wedge sawtooth alternations are not necessarily simple, so the existence of wedge sawtooth alternations in a permutation class do not guarantee that the simple permutations are not monotone griddable, but Theorem 1.3 does nevertheless provide a sufficient condition.

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