Inversions in randomly labelled trees

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Tony Johansson and Fiona Skerman

Outline

- Inversions in labelled rooted trees;

  definitions, pictures

- Random labelling

  basic properties

  tree parameters, total path length

  results: cumulants, Bernoulli numbers

- Random labelling of random trees

Inversions in a Permutation/Labelled Tree

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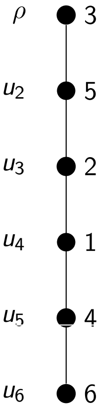

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$\pi \in S_n$

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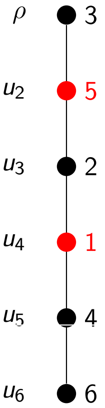

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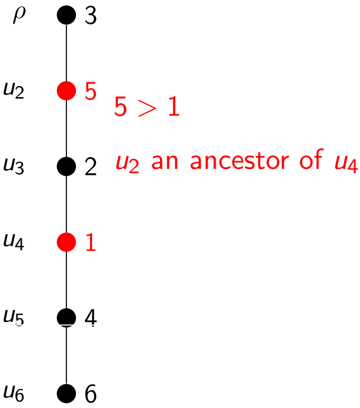

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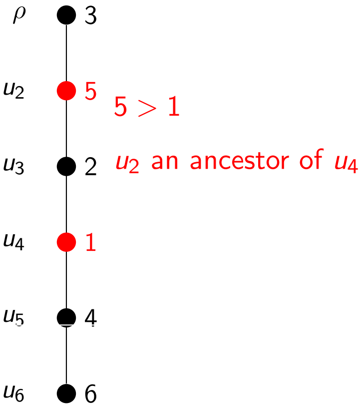

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$\pi : V(T) \to [n]$

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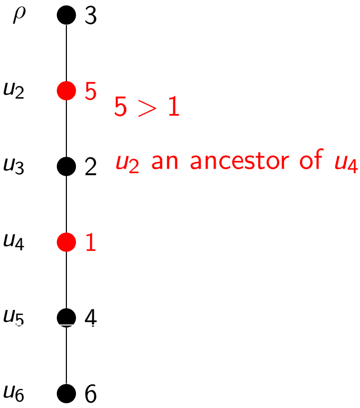

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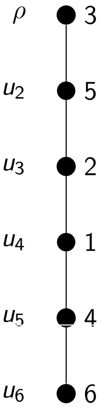

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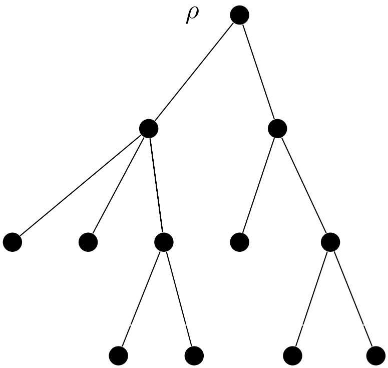

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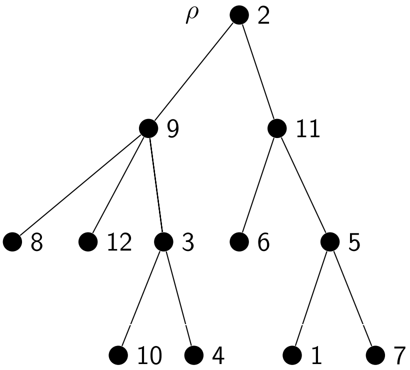

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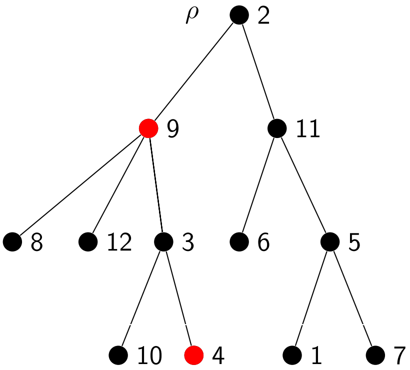

$\#$ inversions $= 8$

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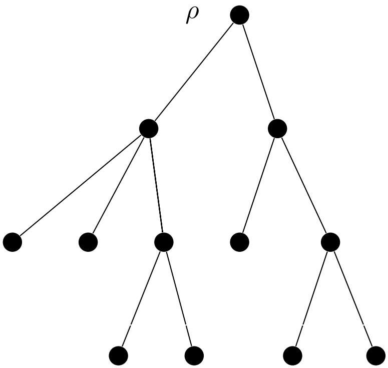

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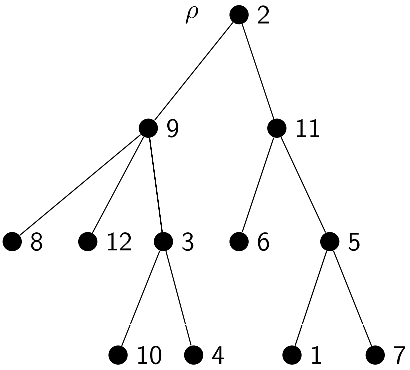

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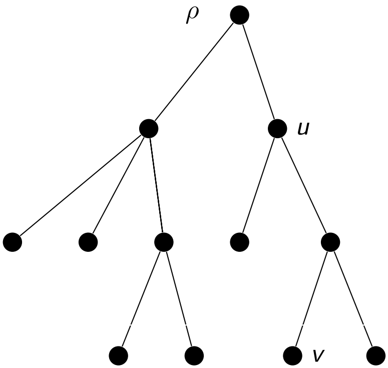

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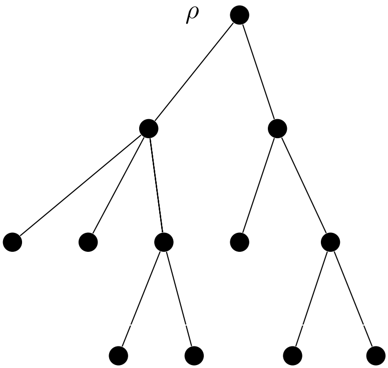

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$$\mathbb{E}[Inv(T)] = \sum_{u<v} \frac{1}{2} = \frac{1}{2}\Upsilon(T)$$

$\Upsilon(T)$ is the **total path length**

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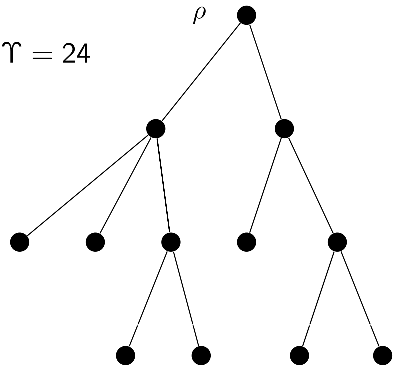

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Results on Fixed Trees

For a path P_n, asymptotic normality.

Theorem Feller '68,

Let π be uniformly random permutation. Moment generating function

$$\mathbb{E}[e^{tInv(\pi)}] = \prod_{j=1}^{n} \frac{e^{jt} - 1}{j(e^t - 1)},$$

and

$$\frac{Inv(\pi) - \mathbb{E}(Inv(\pi))}{\mathbb{V}(Inv(\pi))} \to N(0,1).$$

Cumulant moments of r.v. X, $\kappa_k(X)$: $\ln \mathbb{E}[e^{tX}] = \sum_{k=0}^{\infty} \kappa_k \frac{t^k}{k!}$

For node u, let z_u denote the number of nodes in the tree rooted at u.

B_k denotes the k-th Bernoulli number.

Results on Fixed Trees

Theorem CHJJS '17+

Let T be a fixed tree. Write $X = Inv(T, \pi)$. Let $\kappa_k(X)$ be the k-th cumulant of X. Then

$$\mathbb{E}[X] = \frac{1}{2} \sum_{v \in V} (z_v - 1)$$

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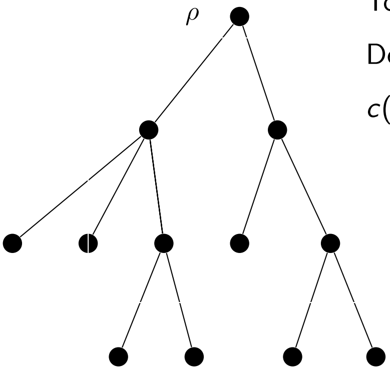
Results: Combinatorial Interpretation



To express the sum $\sum_v z_v^k$.

Denote number of common ancestors

$c(v_1, \dots, v_k) = |\{u : u \leq v_i, \forall i\}|$

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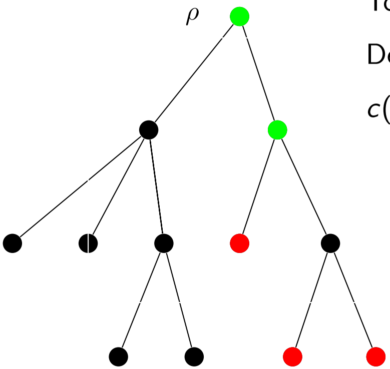

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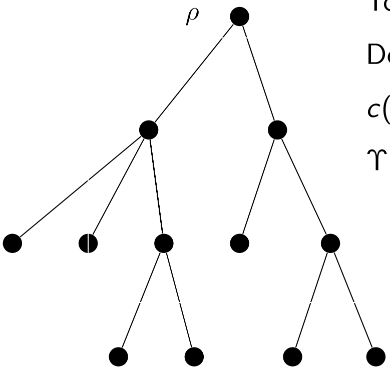

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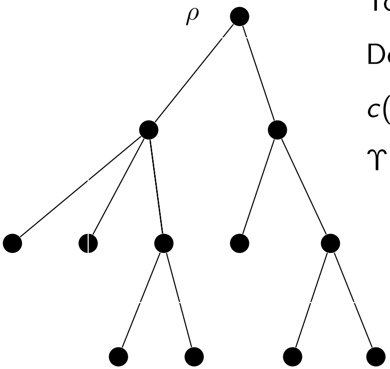

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For one vertex $c(u) = h(u) + 1$,

So $\Upsilon_1(T) = \Upsilon(T) + n$.

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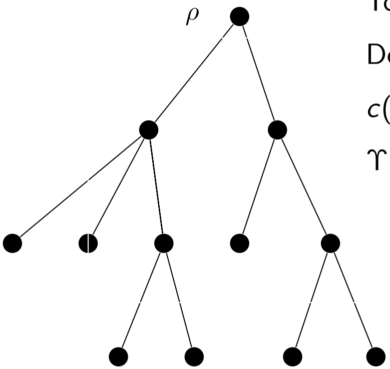

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$$\mathbb{V}[X] = \sum_{v\in V}(z_v^2 - 1) = \frac{1}{12}(\Upsilon_2(T) - n)$$

and more generally,

$$\kappa_k(X) = \frac{B_k}{k}(-1)^k(\Upsilon_k(T) - n).$$

Galton Watson Trees

Begin with one node.

Recursively, each node has a random number of children.

Number of children drawn independently from offspring distribution ξ.

Galton Watson Trees

For a unit Brownian excursion $e(u)$, $\eta = \int_{[0,1]^2} \min_{s \le u \le t} e(u)$.

Theorem CHJJS '17+

Suppose T_n is a conditional Galton-Watson tree with offspring distribution ξ such that $\mathbb{E}[\xi] = 1$ and $\mathbb{V}[\xi] = \sigma^2 \in (0, \infty)$, and define

$$X_n = \frac{Inv(T_n, \pi) - \Upsilon(T_n)/2}{n^{5/4}}.$$

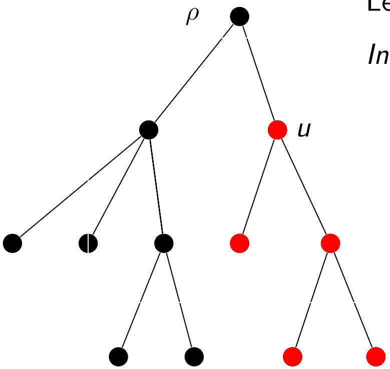
Then we have

$$X_n \to^d X \sim 12\sigma^{1/2}\sqrt{\eta}\, \mathcal{N},$$

where $\mathcal{N}$ is a standard normal random variable, independent from the random variable η.

This strengthen results of Panholtzer and Seitz 2012.

Key Lemma



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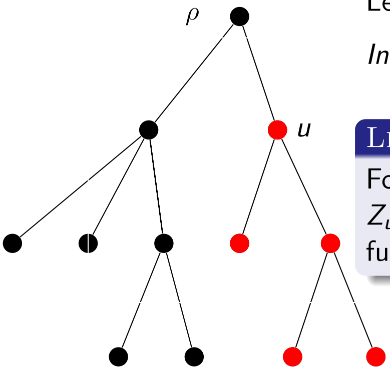

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Lemma

For each node u,
$Z_u \sim Unif\{0, 1, \ldots, z_u - 1\}$ and furthermore the Z_u's are independent.