# AUTOMATIC ENUMERATION OF RESTRICTED PERMUTATIONS

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The ATRAP algorithm builds upon the work and the conjectured covers found by the Struct algorithm by Bean et al. [2017]. Rather than search for conjectured covers as Struct we instead search for a cover in a way that ensures it is valid. The structures used are similar to generalized grid classes, however, the cells of the grids can also include the single point, i.e. the set containing just the permutation 1. We will define these here as our algorithm also relies heavily upon these. We will use the notation given by Vatter [2011].

**Tilings.** Given a permutation $\pi$ of length $n$, and two subsets $X, Y \subseteq [n]$, then $\pi(X \times Y)$ is the permutation that is order isomorphic to the subword with indices from $X$ and values in $Y$. For example $35216748([3,7] \times [2,6]) = 132$, from the subword 264.

Suppose $M$ is a $t \times u$ matrix (indexed from bottom to top and left to right) whose entries are permutation classes or the set $\{1\}$ which we call the point and denote $\bullet$. An *$M$-gridding* of a permutation $\pi$ of length $n$ is a pair of sequences $1 = c_1 \leqslant \cdots \leqslant c_{t+1} = n+1$ and $1 = r_1 \leqslant \cdots \leqslant r_{u+1} = n+1$ such that $\pi([c_k, c_k+1) \times [r_l, r_{l+1}))$ is in $M_{k,l}$ for all $k$ in $[t]$ and $\ell$ in $[u]$. The set $\mathrm{Grid}(M)$ consists of all permutations with an $M$-gridding.

A tiling $T$ is a matrix as above which satisfies the property that all permutations can have at most one $T$-gridding. In an abuse of notation, we refer to both $T$ and $\mathrm{Grid}(T)$ as the tiling $T$. Let $T_n$ be the set of length $n$ permutations in $T$.

For a permutation class $\mathcal{C}$, a *cover* is a finite set of disjoint tilings such that their union is $\mathcal{C}$. Struct searches for a conjectured cover.

The ATRAP algorithm also allows tilings to have cells that are permutation classes without the empty permutation. It essentially searches for a combinatorial specification (see Flajolet and Sedgewick [2009]) using tilings where we take some liberties with the cartesian product constructor. There are five types of strategies used in trying to find a structure for $\mathcal{C} = \mathrm{Av}(B)$.

**Verification.** Verification strategies are used to say when a tiling is understood. After applying many strategies a tiling may become a subset of the class. In this case, we consider it to be understood. It is possible to check for this using the following lemma, where $|B|_\infty$ denotes the length of the longest pattern in $B$.

**Lemma 1.** *For a permutation class $\mathcal{C} = \mathrm{Av}(B)$ and a tiling $T$ with $k$ points, if $T_{k+|B|_\infty} \subseteq \mathcal{C}$ then $T \subseteq \mathcal{C}$.*

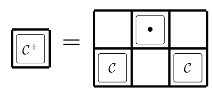
*Proof.* Assume that $T_{k+|B|_\infty} \subseteq \mathcal{C}$. Let $\pi$ be a permutation in $T$ with length greater than $k + |B|_\infty$ that is not in $\mathcal{C}$. There is an occurence in $\pi$ of some pattern $p$ in $B$. Define $\pi'$ to be the permutation formed by removing all the points from $\pi$ that are not in the occurence of $p$, or those in which the $T$-gridding used a point from $T$. Then $\pi'$ is a permutation in $T$ that is not in $\mathcal{C}$ and of length at most $k$ plus the size of the pattern $p$, so at most $k + |B|_\infty$. Therefore $T_{k+|B|_\infty} \nsubseteq \mathcal{C}$, a contradiction. $\square$

**Batch.** Given a tiling $T$, a batch strategy is a disjoint set of tilings $T_1, T_2, \ldots, T_k$, such that when each $T_i$ is intersected with $\mathcal{C}$ their disjoint union is equal to the intersection of $T$ and $\mathcal{C}$ i.e.

$$(T \cap \mathcal{C}) = (T_1 \cap \mathcal{C}) \sqcup (T_2 \cap \mathcal{C}) \sqcup \ldots \sqcup (T_k \cap \mathcal{C}).$$

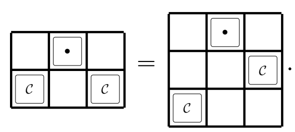
For example, the following gives a batch strategy. For a tiling $T$ with a cell that is a permutation class (which is therefore possibly empty) is the disjoint union of the tiling where the cell is empty, and the tiling where the cell contains at least a point.

**Equivalence.** Given a tiling $T$, an equivalent strategy gives a tiling $T'$ such that $T$ and $T'$ are isomorphic. One possible equivalence strategy would be for a tiling with a cell that contains at least a point, isolate an extreme point, i.e

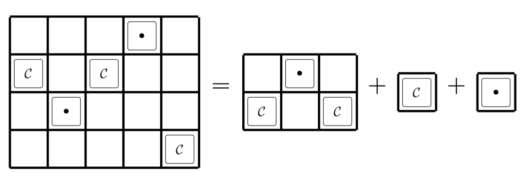


where we use $\mathcal{C}^+$ to denote the permutation class without the empty permuation.

**Inferral.** Given a tiling $T$, an inferral strategy gives a tiling $T'$ such that $T' \subset T$, but moreover $T \cap \mathcal{C} = T' \cap \mathcal{C}$. For example, for a tiling $T$ with two cells in the same row, if you can't place a 21 across the cells, then separate them. For the avoiders of 231 we get



**Decomposition.** For a tiling $T$ a recursive strategy is a set of tilings $T_1, T_2, \ldots, T_k$ such that there is a way to "glue" $T_1, T_2$ and $T_k$ together to give $T$. For example, for a tiling $T$ consider creating the graph where each cell corresponds to a vertex. Two vertices are connected if there exists a gridding of a pattern in $B$ that uses both corresponding cells.



Also, fitting in this strategy is generalization of reversibly deletable point as defined in Vatter [2008].

**Proof Trees.** For a permutation class $\mathcal{C}$, consider the tree where the root node is the $1 \times 1$ tiling with $\mathcal{C}$. The children of a node are the tilings obtained using some strategy on it. If all leaves are subsets or satisfy the property that their tiling exists on an ancestor node within the tree, we call this a *proof tree*. The ATRAP algorithm searches for a proof tree using any set of strategies that fit into one of the five strategy types discussed.

## PermPAL

The proof trees found by ATRAP, alongside the enumerations when possible, can be found on the Permutation Pattern Avoidance Library (or PermPAL for short), Arnarson et al. [2017]. This includes proof trees for all bases with at most length four patterns that contain at least one pattern of length less than four. There are also proof trees for non-regular-insertion-encodable classes whose basis contains length four patterns, including all whose basis has length greater than four.

## References

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