Patterns and Cycles in Dynamical Systems

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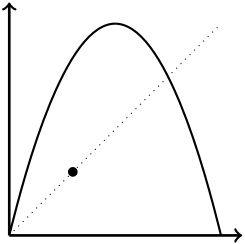
Function Iteration

Example:

Let $f(x) = 4x(1-x)$. Then

$(x, f(x), f^2(x), f^3(x)) = (.30, -, -, -)$



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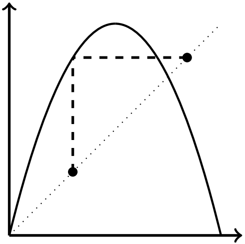
$$(x, f(x), f^2(x), f^3(x)) = (.30, .84, -, -)$$



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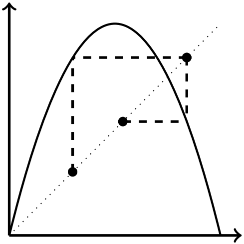
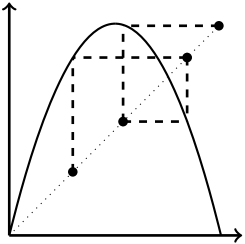


Function Iteration

Example:

Let $f(x) = 4x(1-x)$. Then

$(x, f(x), f^2(x), f^3(x)) = (.30, .84, .53, .99)$



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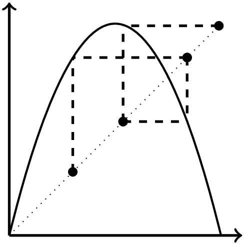
and so

$$\text{Pat}(.3, f, 4) = \text{st}(.30, .84, .53, .99) = 1324$$



Patterns and Dynamical Systems

Theorem (Bandt-Keller-Pompe): Every piecewise-monotone map $f : [0, 1] \to [0, 1]$ has forbidden patterns, i.e. patterns that never arise as iterates.

$\#$ Allowed patterns $\longleftrightarrow$ complexity (i.e. topological entropy)

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Example: Let $f(x) = 4x(1-x)$.

$$321 \notin \mathrm{Allow}(f) \to \underbrace{4321, 1432, 54213,}_{\text{contain consecutive 321}} \ldots \notin \mathrm{Allow}(f)$$

Sarkovskii's Theorem

An *n−periodic point* of a map is a point such that

$$f^n(x) = x \text{ and } f^i(x) \neq x \text{ for all } 1 \leq i < n.$$

Theorem (Sarkovskii):
If a continuous map f of the unit interval has an m-periodic point and $\ell \lhd m$ in the Sarkovskii ordering

$$1 \lhd 2 \lhd 2^2 \lhd \cdots \lhd 2^n \lhd \cdots \lhd 5{\cdot}2^n \lhd 3{\cdot}2^n \lhd \cdots \lhd 7{\cdot}2 \lhd 5{\cdot}2 \lhd 3{\cdot}2 \lhd \cdots \lhd 7 \lhd 5 \lhd 3$$

then f must also have an ℓ-periodic point.

Question: Is there a similar order for the permutation structure of periodic points?

Cycle Type

Let x be a periodic point of order n and $\mathrm{Pat}(x, f, n) = \pi$.

The *cycle type* of x is $\hat{\pi} \in \mathcal{C}_n$ where

$$\pi = \pi_1\pi_2 \dots \pi_n \to \hat{\pi} = (\pi_1, \pi_2, \dots, \pi_n).$$

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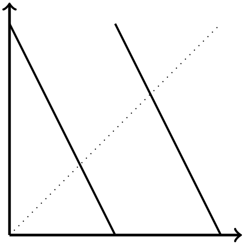
Example: Consider $G_2(x) = \{-2x\}$.



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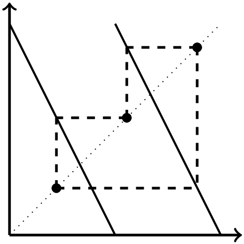
Example: Consider $G_2(x) = \{-2x\}$.
A 3-periodic orbit of G_2 is:

$$(x, G_2(x), G_2^2(x)) = \left(\frac{8}{9}, \frac{2}{9}, \frac{5}{9}\right)$$

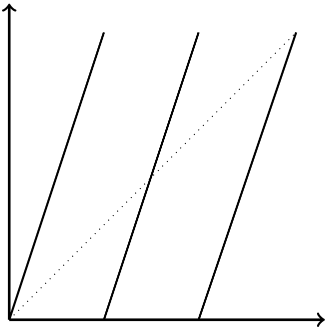
Giving $\mathrm{Pat}(\frac{8}{9}, G_2, 3) = 312$ and

$$\hat{\pi} = (3, 1, 2) = 231$$



The Shape of Cycles

The representative of a 6-periodic orbit of $F_3(x) = \{3x\}$ is $x = \frac{13}{14}$.

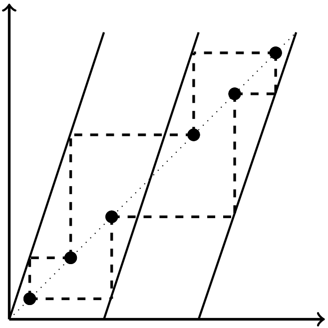


The Shape of Cycles

The representative of a 6-periodic orbit of $F_3(x) = \{3x\}$ is $x = \frac{13}{14}$.

$$\mathrm{Pat}(x, F_3, 6) = \mathrm{st}\left(\frac{13}{14}, \frac{11}{14}, \frac{5}{14}, \frac{1}{14}, \frac{3}{14}, \frac{9}{14}\right) = 653124$$

The *cycle type* of the orbit is $\hat{\pi} = (6, 5, 3, 1, 2, 4) = 241635$.



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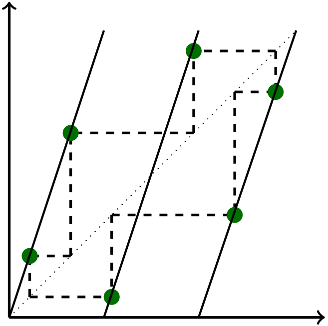

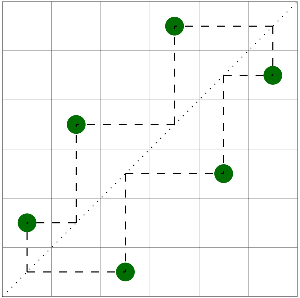

$\hat{\pi} = 241635$

Forcing Order

For a family of interval maps $\mathcal{F}$, a cycle $\hat{\pi}$ *forces* a cycle $\hat{\tau}$ if, for any $f \in \mathcal{F}$, whenever $\hat{\pi} \in \mathrm{AlCyc}(f)$ then $\hat{\tau} \in \mathrm{AlCyc}(f)$ as well.

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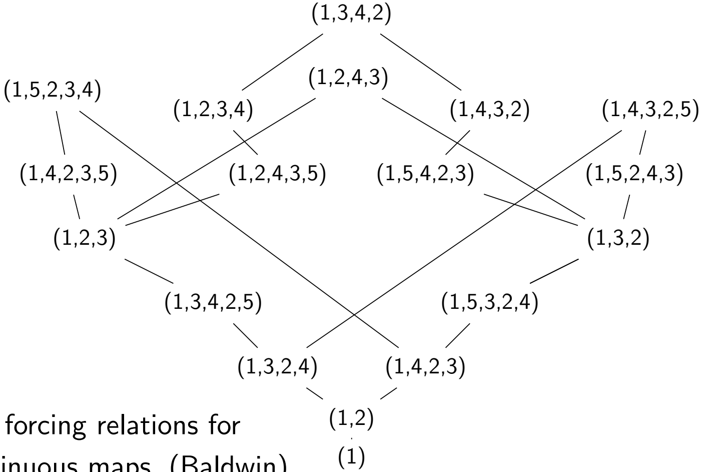
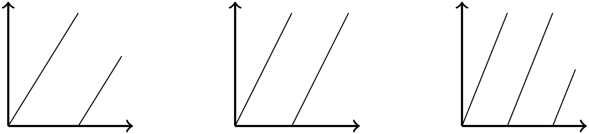


The forcing relations for continuous maps, (Baldwin).

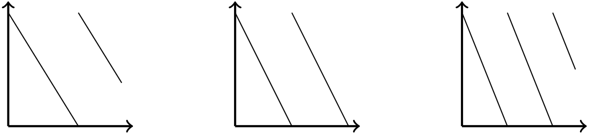
Beta Shifts and Negative Beta Shifts

For $\beta > 1$, consider the classes of functions,

$$F_\beta(x) = \{\beta x\} \text{ and } G_\beta(x) = 1 - \{\beta x\} \text{``} = \text{''} \{-\beta x\}.$$



The graphs of $F_\beta(x) = \{\beta x\}$ for
(a) $\beta = \frac{1+\sqrt{5}}{2}$, (b) $\beta = 2$, and (c) $\beta = 2.5$.



The graphs of $G_\beta(x) = 1 - \{\beta x\} \text{``} = \text{''} \{-\beta x\}$ for
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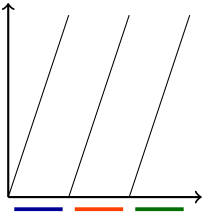
Itineraries

Example: $F_3(x) = \{3x\}$

Name the monotonic intervals: **0**, **1**, **2**



$$
\begin{array}{ccc}
(x, F_3(x), F_3^2(x), \ldots) & = & (.13, .39, .17, .51, .53, .59, .77, \ldots) \\
 & \to & \mathbf{0} \quad \mathbf{1} \quad \mathbf{0} \quad \mathbf{1} \quad \mathbf{1} \quad \mathbf{1} \quad \mathbf{2} \ldots
\end{array}
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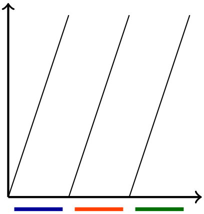

$$\begin{array}{rcl} (x, F_3(x), F_3^2(x), \ldots) & = & (.13, .39, .17, .51, .53, .59, .77, \ldots) \\ & \to & \mathbf{0} \quad \mathbf{1} \quad \mathbf{0} \quad \mathbf{1} \quad \mathbf{1} \quad \mathbf{1} \quad \mathbf{2} \ldots \end{array}$$

Why? Applying F_3 is now a *shift* of the word.

$$\Sigma(w_1 w_2 w_3 \ldots) = w_2 w_3 \ldots.$$

$$\begin{array}{rcl} \mathrm{Pat}(x, F_3, 4) & = & \mathrm{Pat}(010112\ldots, \Sigma_3, 4) \\ & = & \mathrm{st}(010112\ldots, 10112\ldots, 0112\ldots, 112\ldots) = 1324 \end{array}$$

Beta and Negative Beta Expansions

For $F_\beta(x) = \{\beta x\}$, itineraries correspond to β-expansions:

$$x = \frac{w_1}{\beta} + \frac{w_2}{\beta^2} + \frac{w_3}{\beta^3} + \dots.$$

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For $G_\beta(x) = \{-\beta x\}$, itineraries correspond to $(-\beta)$-expansions:

$$x = \frac{w_1 + 1}{(-\beta)} + \frac{w_2 + 1}{(-\beta)^2} + \frac{w_3 + 1}{(-\beta)^3} + \dots.$$

Alternating Order: In odd positions, 0 is low and $\lfloor \beta \rfloor$ is high, in even positions, $\lfloor \beta \rfloor$ is low and 0 is high.

$$0101\ldots <_{alt} 0000\ldots <_{alt} 1111\ldots <_{alt} 1010\ldots$$

Segmentations

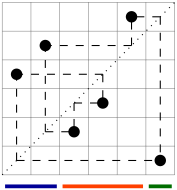
Ask: Is $\hat{\pi}$ a cycle of $F_N(x) = \{Nx\}$?

An N-segmentation of $\hat{\pi}$ is a sequence $0 = e_0 \leq e_1 \leq \cdots \leq e_N = n$ such that each segment $\hat{\pi}_{e_t+1}\hat{\pi}_{e_t+2}\ldots\hat{\pi}_{e_{t+1}}$ is *increasing*.

A 3-segmetation of

$$\hat{\pi} = (6, 1, 4, 3, 2, 5) = 452361$$

$$\mathbf{4\ 5} \mid \mathbf{2\ 3\ 6} \mid \mathbf{1}$$



From this, define a word ω by

$$\begin{array}{rcccccc} \pi = & 6 & 1 & 4 & 3 & 2 & 5 \\ \omega = & \_\_\_ & \_\_\_ & \_\_\_ & \_\_\_ & \_\_\_ & \_\_\_ \end{array}$$

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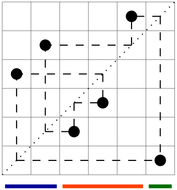
4 5 | **2 3 6** | **1**



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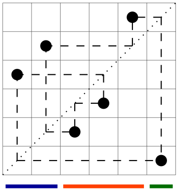

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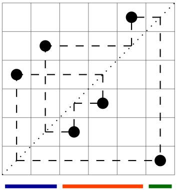


From this, define a word ω by

$$\begin{array}{cccccccc} \pi = & 6 & 1 & 4 & 3 & 2 & 5 \\ \omega = & \mathbf{2} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} \end{array}$$

Theorem (Archer-Elizalde): $\quad \mathrm{Pat}(\omega^\infty, \Sigma_N, n) = \pi$

$$\min\{N : \hat{\pi} \in \mathrm{AlCyc}(F_N)\} = 1 + \mathrm{des}(\hat{\pi})$$

Negative Segmentations

Ask: Is $\hat{\pi}$ a cycle of $G_N(x) = \{-Nx\}$?

A $-N$-segmentation of $\hat{\pi}$ is a sequence $0 = e_0 \leq e_1 \leq \cdots \leq e_k = n$ such that each segment $\hat{\pi}_{e_t+1}\hat{\pi}_{e_t+2}\ldots\hat{\pi}_{e_{t+1}}$ is *decreasing*.

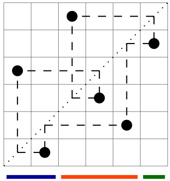
A -3-segmetation of

$\hat{\pi} = (6, 5, 2, 1, 4, 3) = 416325$

4 1 | **6 3 2** | **5**

$\pi =$ 6 5 2 1 4 3

$\omega =$ **2 1 0 0 1 1**



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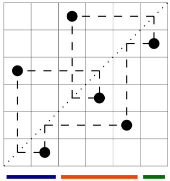

Theorem (Archer-Elizalde): If ω is primitive, then

$$\mathrm{Pat}(\omega^\infty, \Sigma_{-N}, n) = \pi$$

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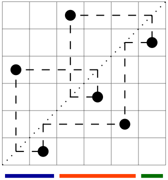
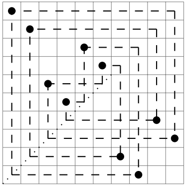


Collapsed cycles:



$\omega = 2011020110$

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β-shifts and $-\beta$-shifts

Theorem: Let $B_p(\hat{\pi}) = \inf\{\beta : \hat{\pi} \in \mathrm{AlCyc}(F_\beta)\}$. Then $B_p(\hat{\pi})$ is equal to the largest real root of

$$p_\omega(x) = x^n - 1 - \sum_{j=1}^{n} w_j x^{n-j},$$

where $\omega = w_1 w_2 \dots w_n$ is the word defined by the unique $(1 + \mathrm{des}(\hat{\pi}))$-segmentation of $\hat{\pi}$.

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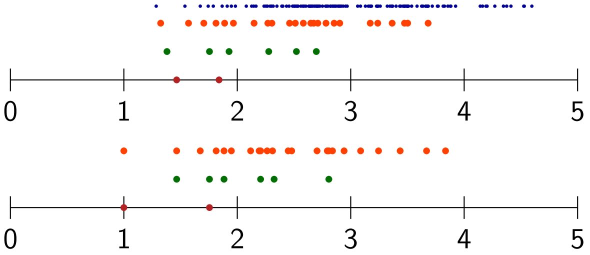
Theorem: Let $\overline{B}_p(\hat{\pi}) = \inf\{\beta : \hat{\pi} \in \mathrm{AlCyc}(G_\beta)\}$. Then $\overline{B}_p(\hat{\pi})$ is equal to the largest real root of

$$\bar{p}_\omega(x) = (-x)^n - 1 + \sum_{j=1}^{n} (w_j + 1)(-x)^{n-j},$$

where $\omega = w_1 w_2 \dots w_n$ is the word defined by a (usually unique) $(1 + \mathrm{asc}(\hat{\pi}) + \epsilon(\hat{\pi}))$-segmentation of $\hat{\pi}$.

(If $\epsilon(\hat{\pi}) = 1$, take ω to be the smallest with respect to $<_{alt}$.)

Distributions

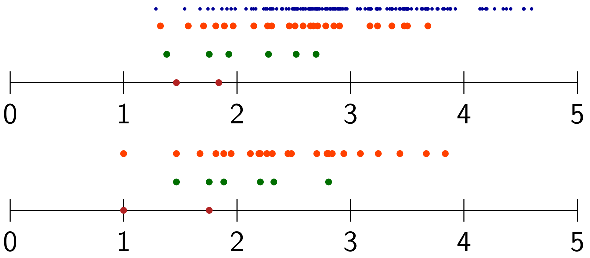


Plots of $B_p(\hat{\pi})$ (top) and $\overline{B}_p(\hat{\pi})$ (bottom) for $\pi \in \mathcal{C}_n$ and $n = 3, 4, 5, 6$.

Distributions

Theorem: The distribution of $\lceil B_p(\hat{\pi}) \rceil = 1 + \mathrm{des}(\hat{\pi})$ (resp. $\lceil \overline{B}_p(\hat{\pi}) \rceil = 1 + \mathrm{asc}(\hat{\pi}) + \epsilon(\hat{\pi})$) is asymptotically normal with mean $\mu = \frac{n+1}{2}$ and variance $\sigma^2 = \frac{n-1}{12}$.

Why? Descents in cycles are normal, (Fulman).



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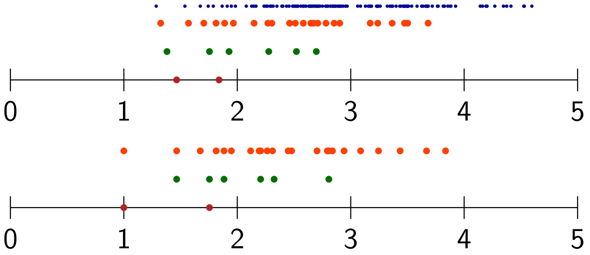
Why? Descents in cycles are normal, (Fulman).

Conjecture: The distribution of $B_p(\hat{\pi})$ (resp. $\overline{B}_p(\hat{\pi})$) is asymptotically normal with mean $\mu = \frac{n}{2}$ and variance $\sigma^2 = \frac{n-1}{12}$.



Plots of $B_p(\hat{\pi})$ (top) and $\overline{B}_p(\hat{\pi})$ (bottom) for $\pi \in \mathcal{C}_n$ and $n = 3, 4, 5, 6$.

References

Remember that we started with patterns realized by *any* point in the interval?

Come talk to me here or at FPSAC about it:

"Patterns of Negative Shifts and Signed Shifts."

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[1] K. Archer and S. Elizalde, Cyclic permutations realized by signed shifts, *Journal Combinatorics* 5 (2014), 1–30.

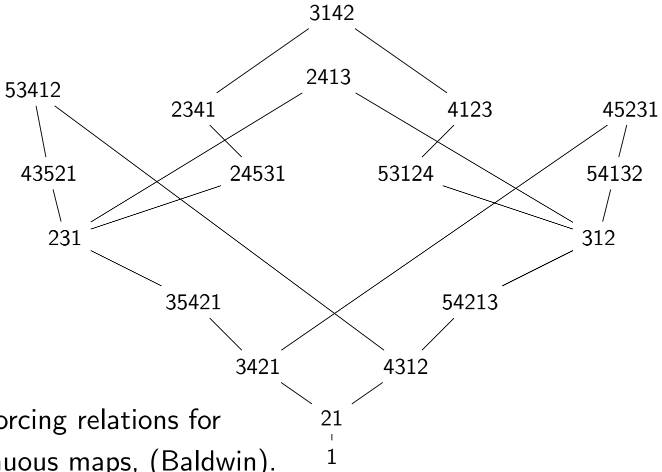
[2] C. Bandt, G. Keller and B. Pompe, Entropy of interval maps via permutations, *Nonlinearity* 15 (2002), 1595–1602.

[3] S. Baldwin, Generalizations of a theorem of Sarkovskii on orbits of continuous real-valued functions *Discrete Mathematics* 76 (1987), 111–127.

[4] J. Fulman, The Distribution of Descents in Fixed Conjugacy Classes of the Symmetric Groups, *Journal of Combinatorial Theory* 84 (1998), 171–180.

Forcing Order

For a family of interval maps $\mathcal{F}$, a cycle $\hat{\pi}$ *forces* a cycle $\hat{\tau}$ if, for any $f \in \mathcal{F}$, if $\hat{\pi} \in \mathrm{AlCyc}(f)$ then $\hat{\tau} \in \mathrm{AlCyc}(f)$ as well.



The forcing relations for continuous maps, (Baldwin).

Smallest of the $B_p(\hat{\pi})$

The $\hat{\pi} \in \mathcal{C}_n$ minimizing $B_p(\hat{\pi})$ is of the form:

$$\hat{\pi} = (n, 1, 2, 3, \ldots, n{-}1) = 234\ldots(n{-}1)n1,$$

giving

$$\omega = 10^{n-1}.$$

And so $B_p(\hat{\pi})$ is the largest real root of

$$x^n + x^{n-1} - 1.$$

n	2	3	4	5	6	$\cdots$	100
$B(\pi)$	$\frac{1+\sqrt{5}}{2}$	1.466	1.383	1.325	1.285	$\cdots$	1.034

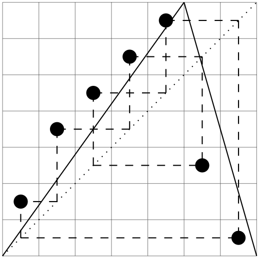
This $\hat{\pi}$ maximizes $\lceil \overline{B}_p(\hat{\pi}) \rceil = 1 + \mathrm{asc}(\hat{\pi}) = n - 1$.

Enumerations

Theorem (Archer-Elizalde):

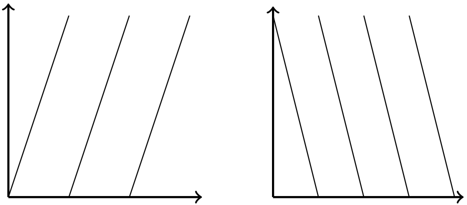
$$|\mathcal{C}_n(213, 312)| = \frac{1}{2n} \sum_{\substack{d|n \\ d \text{ odd}}} \mu(d) 2^{n/d},$$

these are n-cycles whose one-line permutation can be drawn on $\sigma = +-$.

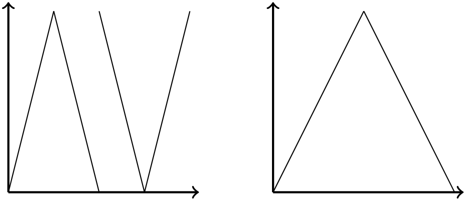


Also formulas for the number of cycles according to the number of descents, ascents.

Signed Shifts



The graphs of M_σ for $\sigma = +^3$, $\sigma = -^4$.



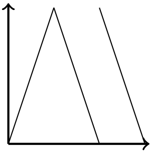
The graphs of M_σ for $\sigma = + - - +$, and $\sigma = +-$, respectively.

Signed-Shift Characterization

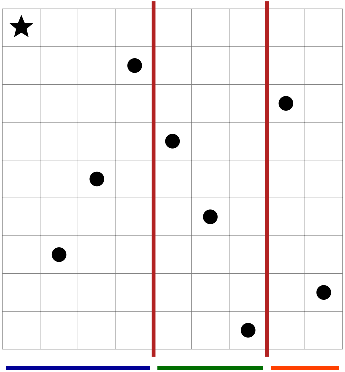
Theorem (Archer-Elizalde-M.): A permutation π is realized by the signed-shift M_σ if and only if $\hat{\pi}$ is the same shape as M_σ.

Example:

Consider $\sigma = + - -$



The permutation $\pi = 923564871$ is realized by M_σ because $\hat{\pi} = \star 35864172$ is the same shape as M_σ.



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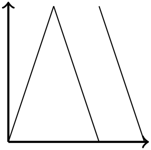

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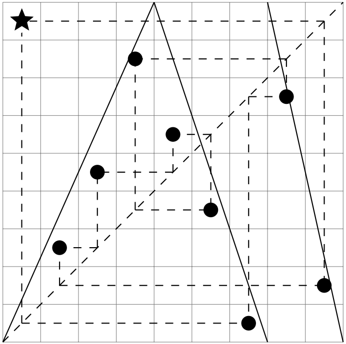

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Under the Rug:

Collapsed permutations and $\pi_{n-1}\pi_n = (n-1)n$ when $\sigma_k = +$, etc.

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The Reason Why:

The cycle diagram of $\hat{\pi}$ is the cobweb diagram for M_σ starting at a point that induces π.