Gray codes for restricted ascent sequences

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Abstract

Sabri and Vajnovszki gave a Gray code for $\mathcal{A}(n)$ [4]. In this presentation, we focus on pattern avoiding ascent sequences. From ad-hoc approach, we obtain that for $A_{010}(n)$, $A_{001}(n)$ and $A_{011}(n)$. From general approach, we obtain Gray code listings for many cases.

1. Introduction and Preliminaries

An *ascent sequence* is a sequence $x_1x_2\cdots x_n$ of nonnegative integers such that $x_1 = 0$ and

$$x_i \le \mathbf{asc}(x_1x_2\cdots x_{i-1}) + 1 \tag{1}$$

for $2 \le i \le n$, where $\mathbf{asc}(x_1x_2\cdots x_k)$ is the number of *ascents* in the sequence $x_1x_2\cdots x_k$, that is, the number of $1 \le j$ such that $x_j < x_{j+1}$. For example, 01201014216 is an ascent sequence of length 11. There are many enumerative results for pattern avoiding acent sequence [1] [3].

$$\begin{aligned}\mathcal{A}(n) &: \text{the set of ascent sequences of length} n\\ \mathcal{A}_p(n) &: \text{the set of ascent sequences of length } n \text{avoiding a pattern } p\end{aligned} \tag{2}$$

- $\mathcal{A}(4) = \{0000, 0001, 0010, 0011, 0012, 0100, 0101, 0102, 0110, 0111, 0112, 0120, 0121, 0122, 0123\}$
- $\mathcal{A}_{001}(4) = \{0000, 0100, 0110, 0111, 0120, 0121, 0122, 0123\}$

Combinatorial Gray code is a method to generate objects in a combinatorial class so that the successive objects differ in some pre-specified, usually small way, see [5].

Sabri and Vajnovszki gave a Gray code for $\mathcal{A}(n)$ [4]. We focus on pattern avoiding ascent sequences.

We define two distances.

$$\begin{aligned}d_{Ham}(x,y) &:= |\{i | x_i \ne y_i, 1 \le i \le n\}|\\ d_{Str}(x,y) &:= \sum_{1\le i\le n} |x_i - y_i|.\end{aligned} \tag{3}$$

2. Two graphs on binary sequences

Let $X(n)$ and $Y(n)$ be graphs on binary sequences of size n.

In $X(n)$, two vertices are connected by an edge iff only the last entries are different and the remaining entries are all same or they differ by an adjacent transposition.

For example, $001011\mathbf{1}$ and $001011\mathbf{0}$ are adjacent in $X(n)$ and also $0\mathbf{01}0111$ and $0\mathbf{10}0111$ are so.

In $Y(n)$, two vertices are connected by an edge iff they differ by an adjacent transposition or they differ only one position, i-th position, and j-th entries with $j \ge i+1$ are all 0.

For example, $0\mathbf{01}011000$ and $0\mathbf{10}011000$ are adjacent in $Y(n)$ and also $001011\mathbf{0}00$ and $001011\mathbf{1}00$ are so.

Remark that $001011\mathbf{0}00$ and $001011\mathbf{1}00$ are adjacent in $Y(n)$ but they are not adjacent in $X(n)$.

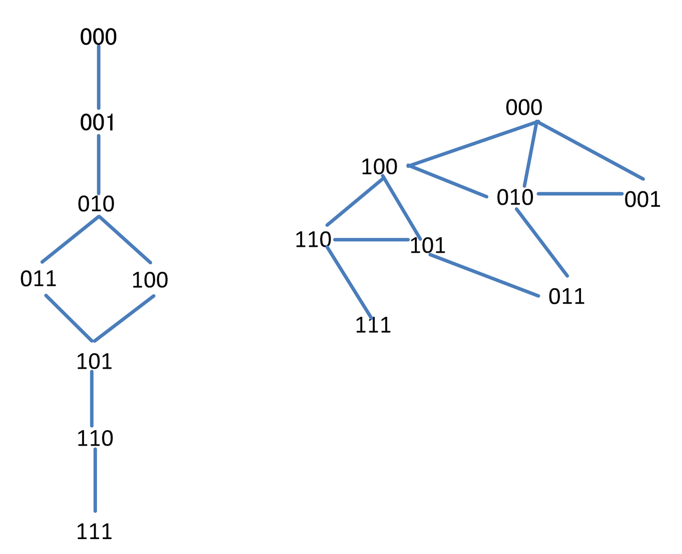

Figure 1: *Left-hand side is $X(3)$ and right-hand side is $Y(3)$*

Theorem 1 *[6] There is a listing of the binary sequences of length n which starts from $00\cdots0$ and ends at $100\cdots0$ such that every distance of two successive sequences is at most two in $X(n)$. Furthermore, distance two jumps do not consecutive.*

Proposition 1 *There are two Hamiltonian paths in $Y(n)$. One starts from $00\cdots0$ and ends at $11\cdots1$. The other starts from $011\cdots11$ and ends at $11\cdots1$.*

3. The cases of $\mathcal{A}_{010}(n)$

The 010 avoiding ascent sequences is the set of weakly increasing ascent sequence and hence the number of $\mathcal{A}_{010}(n)$ is 2^{n-1} [3]. For example, 00122345566 is 010 avoiding.

There is a bijection from $\mathcal{A}_{010}(n)$ to the vertices of $X(n-1)$ such that

the strong distance of two ascent sequences is one iff the distance of the corresponding binary sequence in $X(n-1)$ equals one.

Theorem 2 *$\mathcal{A}_{010}(n)$ has a listing which starts from $00\cdots0$ and ends at $00\cdots01$ such that every strong distance of successive sequences is at most two. Moreover, distance two jumps do not appear consecutively.*

Here is a Gray code for $\mathcal{A}_5(010)$

$$(00000, 00011, 00012, 00122, 00123, 00112, 01112, 01222, 01223, 01234, 01233, 01123, 01122, 01111, 00111, 00001) \tag{4}$$

4. The case of $\mathcal{A}_{001}(n)$

The 001 avoiding ascent sequences of length n is written

$$012\cdots(k-1)a_ka_{k+1}\cdots a_n, \tag{5}$$

where $k-1 \ge a_k \ge a_{k+1} \ge \cdots \ge a_n \ge 0$ and the number of $\mathcal{A}_{001}(n)$ is 2^{n-1} [3]. For example, 12345677644211 is 001 avoiding.

There is a bijection from $\mathcal{A}_{001}(n)$ to the vertices of $X(n-1)$ such that

the strong distance of two ascent sequences is one iff the distance of the corresponding binary sequence in $X(n-1)$ equals one.

Theorem 3 *$\mathcal{A}_{001}(n)$ has a listing which starts from $00\cdots0$ and ends at $010\cdots00$ such that every strong distance of successive sequences is at most two. Moreover, distance two jumps do not appear consecutively.*

Here is a Gray code for $\mathcal{A}_5(001)$.

$$(00000, 01100, 01200, 01220, 01230, 01210, 01211, 01222, 01232, 01234, 01233, 01231, 01221, 01111, 01110, 01000) \tag{6}$$

5. The case of $\mathcal{A}_{011}(n)$

A 001 avoiding ascent sequences is the shuffle of the strictly increasing sequence and 0s and hence the number of 011 ascent sequences of length n is 2^n.

There is a bijection from $\mathcal{A}_{011}(n)$ to the vertices of $Y(n-1)$ such that

two ascent sequence differ by one position or one adjacent transposition iff the distance of the corresponding binary sequence in $Y(n-1)$ equals one.

Theorem 4 *$\mathcal{A}_{011}(n)$ has two listing such that one starts from $00\cdots0$ and ends at $012\cdots(n-2)(n-1)$ and the other starts from $0012\cdots(n-2)$ and ends at $012\cdots(n-2)(n-1)$. Each of listints, successive sequences differ by an adjacent transposition or one position.*

Here are two Gray codes for $\mathcal{A}_5(011)$

$$(00000, 00001, 00010, 00012, 00102, 00100, 00120, 00123, 01023, 01020, 01002, 01000, 01200, 01203, 01230, 01234) \tag{7}$$

$$(00123, 00120, 00100, 00102, 00012, 00010, 00001, 00000, 01000, 01002, 01020, 01023, 01203, 01200, 01230, 01234) \tag{8}$$

6. A General approach

By using a generating tree method [2], we can prove general results.

Theorem 5 *For a reduced sequence $p_1p_2\cdots p_k$ with $\max\{p_1, p_2, \cdots p_k\} \ge 2$, $\mathcal{A}_{p_1p_2\cdots p_k}(n)$ has a Gray code of Hamming distance one.*

From Theorem 5, the remaining case is to give Gray codes of given Hamming distance for $A_{p_1p_2\cdots p_k}(n)$ with $p_1, p_2, \cdots, p_k \in \{0,1\}$.

More restrictive cases, we have Gray codes with strong distance.

Theorem 6 *For a reduced sequence $p_1p_2\cdots p_k$ with $p_k \ge \max\{p_1, p_2, \cdots p_{k-1}\}+1$ and $p_k \ge 2$, $\mathcal{A}_{p_1p_2\cdots p_k}(n)$ has a Gray code of strong distance at most two.*

References

[1] A. Baxter, L. Pudwell, Ascent sequences avoiding pairs of patterns, Electronic Journal of Combinatorics 22.1, P1.58, 2015

[2] A. Bernini, E. Grazzini, E. Pergola, and R. Pinzani. A general exhaustive generation algorithm for Gray structures. Acta Informatica, 44(5): 361-376, 2007.

[3] P. Duncan and E. Steingrimsson. Pattern avoidance in ascent sequences. Electron. J.Combin. 18(1): P226, 2011.

[4] A. Sabri, V. Vajnovszki, Reflected Gray code based orders on some restricted growth sequences, The Computer Journal 58(5), 1099-1111, 2015.

[5] C. Savage. A Survey of Combinatorial Gray Codes, SIAM Review, v.39 n.4, p.605-629, Dec, 1997.

[6] M. Tomie, Two listing of binary words and Gray codes for 132-312 avoiding permutations, arXiv:1612.03620.

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