
GAMES IN EXTENSIVE FORM AND REPEATED GAMES

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*To be written.

1 Introduction

Many strategic interactions take place repeatedly over time or involve players making sequential moves. For example, Coca-Cola’s “duel” with Pepsi goes on week after week. An incumbent firm, such as Microsoft, has to decide how to respond to entry into its markets (*e.g.*, how to respond to IBM pushing Linux). To understand games with a time dimension, we need to go beyond the analysis in “Introduction to Game Theory & the Bertrand Trap” to consider games in *extensive form*. The insights we derive from doing so will, then, allow us to consider *repeated games*.

In the earlier reading, we considered only games in *normal form*; that is, we listed the (pure) strategies of the players in a table or matrix form (recall, *e.g.*, the Prisoners’ Dilemma shown in Figure 1). The structure and timing of moves was ignored.

NORMAL FORM

		Player 2	
		Aggressive	Passive
Player 1	Aggressive	3, 3	2, 6
	Passive	6, 2	4, 4

Figure 1: Prisoners’ Dilemma

In this reading, we will explicitly consider the structure and timing of moves. When we do this, we’re said to be considering the game in *extensive form*.

EXTENSIVE FORM

2 Games in Extensive Form

Figure 2 shows a game in extensive form. Note the game resembles a decision tree. It is read in a similar fashion, except, now, there are two decision makers (*i.e.*, players). The game begins on the left, with a player, “the entrant,” deciding whether to *enter* or *stay out* of a market. If the entrant decides to stay out, then the entrant gets \$0 and the other player, “the incumbent,” gets \$10 million. Note the convention of listing the payoffs in the order in which the players move (*i.e.*, in this case, the entrant’s payoffs are listed first). If, however, the entrant decides to enter, then the incumbent must choose a response. The incumbent can be *tough* or *easy* (accommodating). If the incumbent elects to be tough, then the payoffs are −\$5 million for the entrant and \$2 million for the incumbent. If the incumbent is easy, then the payoffs are \$5 million for the entrant and \$5 million for the incumbent.

We can also represent the game of Figure 2 in *normal form*. See Figure 3. Observe that, in normal form, we lose any notion of timing. As we will see, this has important implications when we seek to solve the game.

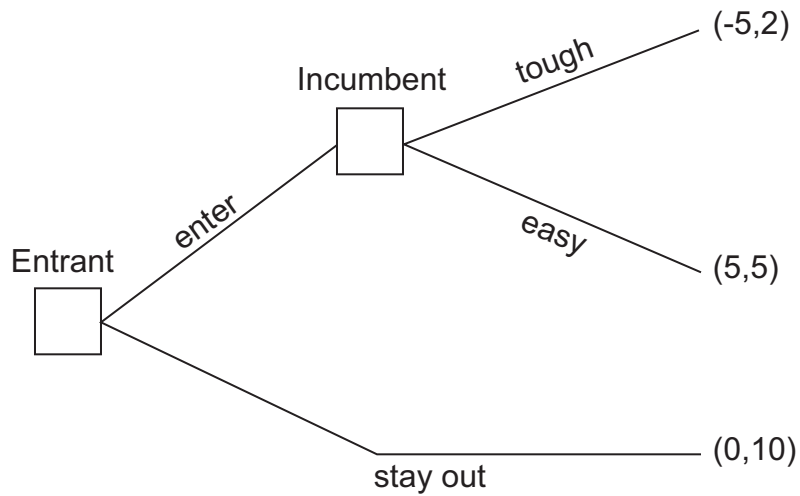


Figure 2: An entry deterrence game. Payoffs in millions of dollars. First number in each pair is the payoff to the entrant. Second number in each pair is the payoff to the incumbent.

		Incumbent	
		Tough	Easy
Entrant	Enter	-5 <u>2</u>	<u>5</u> <u>5</u>
	Stay out	<u>0</u> <u>10</u>	0 <u>10</u>

Figure 3: Entry game of Figure 2 in normal form. Payoffs in millions of dollars.

3 Solving Games in Extensive Form

As with any game, we wish to solve the game in Figure 2; that is, make predictions about how the players would play the game.

One seemingly plausible method for doing so would be to look at the game in normal form (see Figure 3) and find the Nash equilibrium (or equilibria). Recall that a Nash equilibrium occurs when *each* player is playing a best response to the strategies being played by her opponents (see §2.3 of “Introduction to Game Theory & the Bertrand Trap”). From Figure 3, the entrant’s best response to *easy* by the incumbent is *enter* (a payoff of 5 beats a payoff of 0). Hence, the lower 5 in the top-right cell of the matrix is underlined. It is also the case that *easy* is the incumbent’s best response to *enter* (a payoff of 5 beats a payoff of 2). Hence, the upper 5 in the top-right cell of the matrix is underlined. We’ve shown that *easy* and *enter* are *mutual* best responses, which means that a Nash equilibrium of the game is for the entrant to *enter* and for the incumbent to be *easy* on the entrant.

There is, however, another Nash equilibrium. Observe that *stay out* is a best response to *tough* and *vice versa*. Hence, these two strategies are also mutual best responses, which means that the other Nash equilibrium is the entrant *stays out* because the incumbent threatens to be *tough* should the entrant enter.

But looking at the game in extensive form (*i.e.*, Figure 2), we see there’s a problem with this second Nash equilibrium. If the entrant entered, then why would the incumbent be *tough*? Once the entrant has entered, the incumbent faces the choice of getting \$2 million by being *tough* or \$5 million by being *easy*. Five million dollars is better than two million dollars, so the threat to be *tough* is not credible — if given the move, the incumbent would be *easy*. In the parlance of game theory, the incumbent’s threat to be tough is *incredible*. We should not think it reasonable to suppose the entrant would be fooled by an incredible threat, so we shouldn’t accept this second Nash equilibrium as being a reasonable prediction.

INCREDIBLE
THREAT

We need, therefore, a means of solving games in extensive form that doesn’t predict outcomes that rely on incredible threats. Fortunately, there is one — it’s called *backward induction* or, equivalently, *subgame perfection*.¹ Backward induction is simply the reasonable solution concept that we solve the game “backwards.” Start at the last decision node. What would a rational player do if he or she had the move at this node? That is the action that we will suppose is

BACKWARD
INDUCTION

SUBGAME
PERFECTION

¹A *subgame* is the part of a game following a decision node (square box). The game in Figure 2 has two subgames. One is the somewhat trivial game that begins when the incumbent decides between *tough* and *easy*. The other is the whole game (*i.e.*, the game starting from the entrant’s decision). Subgame perfection is the requirement that the solution to a game be a Nash equilibrium in each subgame. The second Nash equilibrium fails this test because *tough* is not a best response in the subgame that begins with the incumbent’s decision. The first Nash equilibrium does, however, pass this test because *easy* is a best response in the subgame that begins with the incumbent’s decision and, as shown in Figure 3, *enter* and *easy* is a Nash equilibrium for the whole game; that is, the subgame that begins with the entrant’s decision. As this last observation makes clear, a subgame-perfect solution is also a Nash equilibrium. But, as just seen, not all Nash equilibria need be subgame perfect.

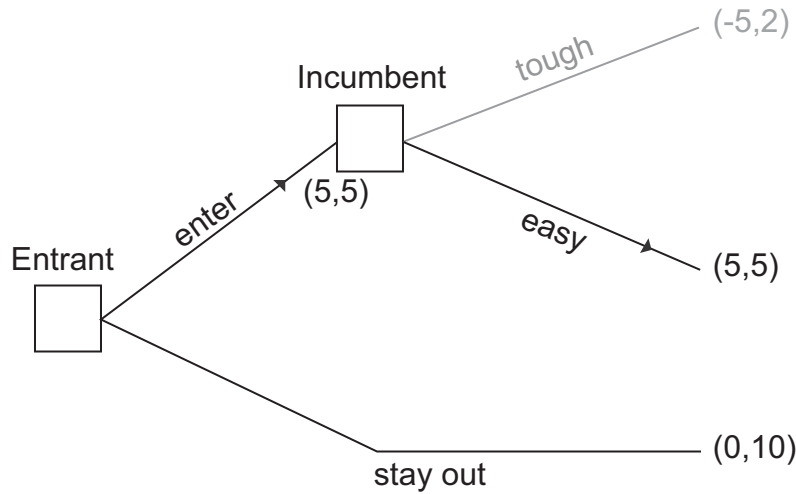


Figure 4: The entry game of Figure 2 showing backward induction (payoffs in millions of dollars). Because $5 > 2$, the incumbent will choose *easy* rather than *tough*. Hence, the entrant's decision is between getting 5 by entering and getting 0 by staying out. $5 > 0$, so the entrant enters.

made at that decision node. Now move back to the penultimate decision node. Given the action we concluded would be taken at the last decision node, what would a rational player do at this penultimate node if he or she had to move? That is the action we will suppose is made at that decision node. Now keep moving back, node by node, and repeat the process, taking as given the actions determined for the later nodes. To put backward induction to work in the entry game, consider the last decision node. As noted, the rational decision is to play *easy* (\$5 million $>$ \$2 million). Hence, the entrant presumes that if she enters, then the incumbent will play *easy*. Hence, if she enters, she will get \$5 million, the consequence of *enter* being followed by *easy*. If she stays out, she gets \$0. Five million dollars beats zero dollars, so she will *enter*. Figure 4 illustrates this reasoning: The *tough* branch has been grayed because the incumbent won't play it and a little arrow has been put on the *easy* branch. Moreover, the payoffs from the *easy* branch have been "moved up," to indicate the decision that the entrant really sees herself making.

As a further illustration of backward induction, consider the game in Figure 5. Player 1 plays first, followed by Player 2. Player 1 chooses to play *up* or *down* and, then, Player 2 gets a choice of *high* or *low*. To solve the game by backward induction, consider Player's 2 decision at his top decision node (the one labeled *a*). If he plays *high*, he gets 3. If he plays *low*, he gets 5. Five is greater than three, so he would play *low* if given the move at node *a*. To help us "remember" this, the *low* branch has a little arrow and the payoff is "moved

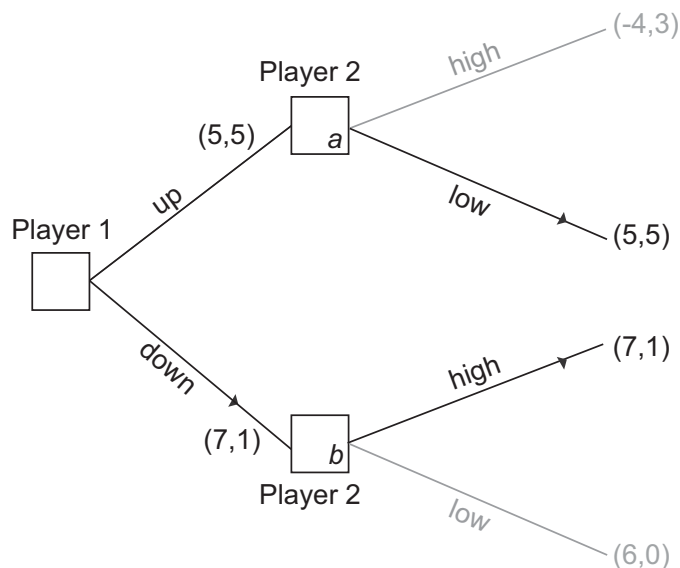


Figure 5: Another game in extensive form. The first number in each payoff pair is the payoff to Player 1 and the second number is the payoff to Player 2.

up” to node a . Now consider his decision at the bottom decision node (the one labeled b). If Player 2 plays *high* he gets 1. If he plays *low*, he gets 0. One is greater zero, so he would play *high* if given the move at node b . Again, we’ve noted this with a little arrow on the appropriate branch and moved forward the relevant payoff. Now consider Player 1: If she plays *up*, she knows, given our analysis, that she will ultimately get 5. If she plays *down*, she knows, given our analysis, that she will ultimately get 7. Seven is greater than five, so she will play *down*. The solution of the game is, thus, Player 1 plays *down* and Player 2 plays the strategy choose *low* if Player 1 plays *up*, but choose *high* if Player 1 plays *down*.

3.1 A Note on Strategies

We have so far been somewhat loose in our use of the word “strategy.” In game theory, the word strategy has a precise meaning:

Formal Definition of Strategy: A strategy is a complete contingent plan of action for a player.

STRATEGY

One way to think about a strategy is that it is a very detailed set of instructions for how to play a given game. So detailed, in fact, that, if you left the instructions with a trusted friend, he or she would be able to play the game on your behalf and realize the same outcome that you would have if you had

		Player 2			
		(high,high)	(high,low)	(low,high)	(low,low)
Player 1	up	-4 <u>3</u>	-4 <u>3</u>	5 <u>5</u>	5 <u>5</u>
	down	<u>7</u> <u>1</u>	<u>6</u> 0	<u>7</u> <u>1</u>	<u>6</u> 0

Figure 6: The game of Figure 5 in normal form. Note the convention for the strategies of Player 2: (A,B) means play action A if node a is reached and play action B if node b is reached.

played the game yourself. Note this requires that the instructions — your strategy — be sufficiently detailed that your friend always knows what move he or she should make whenever he or she might be called upon to move. For instance, consider the game shown in Figure 5. A complete contingent plan for Player 2 — that is, a *strategy* for Player 2 — must say what he should do at all decision nodes at which he has the choice of moves. That is, it must say what he will do at the node labeled a and it must say what he will do at the node labeled b . One strategy, the one Player 2 plays in the subgame-perfect equilibrium (the solution derived by backward induction), is *low* if node a is reached and *high* if node b is reached. There are, however, three other strategies that Player 2 could have chosen:

- play *high* if node a is reached and play *high* if node b is reached;
- play *high* if node a is reached and play *low* if node b is reached; and
- play *low* if node a is reached and play *low* if node b is reached.

Because Player 2 has four possible strategies, if we put the game from Figure 5 into normal form, it would be a 2×4 matrix, as shown in Figure 6.

4 Repeated Games

In many situations, a strategic situation can be modeled as a repeated game. A *repeated game* is a game in which a given game, the *stage game*, is played repeatedly. For example, consider two manufacturers who can change their prices once a week each Monday morning. One can view this as the manufacturers' repeating the Bertrand game on a weekly basis. In this example, the Bertrand game is the stage game.

In the analysis of repeated games, a distinction is made between *finitely* repeated games and *infinitely* repeated games. As their names suggest, a finitely repeated game is one in which the stage game is repeated a finite number of periods, while an infinitely repeated game is potentially repeated an infinite number of periods. To be precise, in a finitely repeated game, all the players

REPEATED GAME

STAGE GAME

FINITELY REPEATED

INFINITELY REPEATED

know the game will be repeated exactly T times. In an infinitely repeated game, the players either (i) know the game will be repeated infinitely; or (ii) they don't know when the game will end and, furthermore, believe that, at the end of each period, there is a positive probability that the game will be repeated next period.

4.1 An Insight

Consider a repeated game that consists of a stage game, denoted by \mathcal{G} . Consider a subgame-perfect Nash equilibrium of the stage game \mathcal{G} . Suppose each period, the players repeated the strategies that they are to play in this Nash equilibrium. Would that be a subgame-perfect equilibrium of the *repeated game*? The answer is *yes*. To see why, observe first that starting at any period t , the play of \mathcal{G} in t and subsequent periods represents a subgame of the whole repeated game. Recall, too, that an outcome is a subgame-perfect equilibrium if it is a Nash equilibrium in each subgame (see footnote 1). Hence, we need to establish that, starting at any arbitrary period, each player playing his or her part of the Nash equilibrium of \mathcal{G} in that and subsequent periods is a Nash equilibrium of the repeated game from that period on.

In a Nash equilibrium, remember, each player is playing a best response to the strategies that he or she believes the other players will play. If you suppose the other players are playing, each period, the strategies they would play in a given Nash equilibrium of \mathcal{G} , then you can do no better, by definition, in any given period than to play your best response to these strategies; that is, to play your part of the Nash equilibrium of \mathcal{G} . This is true in any period, thus true in all periods. This insight, therefore, establishes that repeating your equilibrium strategy for the Nash equilibrium of \mathcal{G} in each repetition of \mathcal{G} is a best response for the *repeated game*. Moreover, this is clearly true of all players, so each player's playing, period by period, his or her equilibrium strategy for \mathcal{G} , is a Nash equilibrium of the *repeated game*. To summarize:

Conclusion. *Let $\mathcal{N}_{\mathcal{G}}$ be a subgame-perfect Nash equilibrium of the stage game \mathcal{G} . Then a subgame-perfect equilibrium of the repeated game in which \mathcal{G} is the stage game is for each player to play, in each period, the same strategy he or she would play as part of $\mathcal{N}_{\mathcal{G}}$.*

Of course, if all there were to repeated games was repetition of a Nash equilibrium of the stage game, repeated games wouldn't be particularly interesting. As we will see, there are contexts in which there is, indeed, more to repeated games than simple repetition of a Nash equilibrium of the stage game. But, as we will also see, the fact that repetition of a Nash equilibrium of the stage game is a subgame-perfect equilibrium of the repeated game will prove useful in what follows.

4.2 Finitely Repeated Games

Consider repeating a stage game, \mathcal{G} , T times. Let t index the periods; that is, t ranges from 1 (the first period) to T (the last period). To begin, let's suppose that \mathcal{G} has only one Nash equilibrium. This Nash equilibrium is also a subgame-perfect equilibrium (recall footnote 1). What is the subgame-perfect equilibrium of this *repeated* game?

Relying on our knowledge of backward induction, the obvious way to proceed is to begin at the end and work our way backward. At the end (*i.e.*, period T), what happens? Well, the players are going to play \mathcal{G} one last time. Whatever happened in the past (*i.e.*, previous periods) is now sunk and, thus, irrelevant to what happens now. That is, at period T , it is as if the players are simply playing \mathcal{G} once. We know what happens if the players play \mathcal{G} once — they play the Nash equilibrium (recall, we've assumed that \mathcal{G} has only one). So, in period T , the prediction is that the players will play the Nash equilibrium of \mathcal{G} . Now go back to period $T - 1$. What happens here? Again periods $t = 1, \dots, T - 2$ are in the past, thus, sunk, and hence irrelevant. The players all know that in period T they will play the Nash equilibrium of \mathcal{G} regardless of what they do today in period $T - 1$. Hence, again, it is as if they are simply playing \mathcal{G} once. From which we conclude they will play the Nash equilibrium of \mathcal{G} in period $T - 1$. If we go back to period $T - 2$, the same analysis applies yet again: Periods $t = 1, \dots, T - 3$ are sunk and what happens in the future, periods $t = T - 1$ and T , is also independent of what's done in the current period. Hence, again, the players play the Nash equilibrium of \mathcal{G} in period $T - 2$. And so forth, all the way back to period 1. We've shown:

Conclusion (Unraveling). *Consider the finite repetition of a stage game. Assume that (i) all players know how many times it will be repeated and (ii) the stage game has a single Nash equilibrium. Then the subgame-perfect equilibrium of the finitely repeated game is repetition of the Nash equilibrium of the stage game every period.*

UNRAVELING

This argument and its conclusion are known in game theory as the *unraveling* result.

Observe that the argument about the last period, T , holds more generally. Because previous periods are past (*i.e.*, sunk), *the players must play a Nash equilibrium of the stage game in the last period.*

LAST PERIOD IS
NASH

The unraveling result relied on three assumptions: The game has finite length; everyone knows when it will end; and the stage game has a single Nash equilibrium. The first two assumptions are, as we will see, somewhat the same — a finitely repeated game with an unknown ending is a lot like an infinitely repeated game. But more on this in Section 4.4. Let's, here, consider the consequences of the third assumption.

Figure 7 shows a possible stage game. Contrary to the premise of the unraveling result, this game has three Nash equilibria. Two are pure-strategy equilibria: In one, Player 1 plays *up* and Player 2 plays *left*; while, in the other, Player 1 plays *down* and Player 2 plays *right*. The third equilibrium is in mixed

		Player 2									
		<i>left</i>	<i>right</i>								
Player 1	<i>up</i>	<table><tr><td><u>6</u></td><td><u>0</u></td></tr><tr><td>5</td><td><u>0</u></td></tr></table>	<u>6</u>	<u>0</u>	5	<u>0</u>	<table><tr><td>-1</td><td><u>6</u></td></tr><tr><td>5</td><td><u>0</u></td></tr></table>	-1	<u>6</u>	5	<u>0</u>
	<u>6</u>	<u>0</u>									
5	<u>0</u>										
-1	<u>6</u>										
5	<u>0</u>										
	<i>down</i>										

Figure 7: Stage Game with Three Nash Equilibria

strategies. To find this third Nash equilibrium, let p be the probability that Player 2 plays *left*; hence $1 - p$ is the probability that he plays *right*. If Player 1 is willing to mix between *up* and *down*, then she must like her expected payoff from the two strategies equally well:

$$\frac{\text{Expected payoff from up}}{p \times 6 + (1 - p) \times (-1)} = \frac{\text{Expected payoff from down}}{p \times 5 + (1 - p) \times 0}$$

The solution to this equation is $p = \frac{1}{2}$; that is, in the mixed-strategy Nash equilibrium, Player 2 must play *left* and *right* with equal probability. Because the game is symmetric, it is readily seen that Player 2 is willing to mix in equilibrium only if Player 1 plays *up* and *down* with equal probability. In summary, the mixed-strategy equilibrium is both players mix between their pure strategies with equal probability. Note, as a consequence, the probability of any given cell in Figure 7 being realized is $\frac{1}{4}$. Hence, each player's expected payoff in the mixed-strategy equilibrium is

$$\frac{1}{4} \times (-1) + \frac{1}{4} \times 0 + \frac{1}{4} \times 6 + \frac{1}{4} \times 5 = \frac{10}{4} = \frac{5}{2}.$$

Observe that the *total* payoffs that the players enjoy in either of the two pure-strategy Nash equilibria is 6 ($= 6 + 0$). In the mixed-strategy equilibrium the sum of the expected payoffs is 5. Note, though, that if the players could somehow get themselves into the bottom-left cell of Figure 7, they would enjoy total payoffs of 10. That is, the players would do best if they played *down* and *left* respectively. But *down* and *left* is not a Nash equilibrium, so it would never be the outcome if the game were played once.

But suppose we repeat the game in Figure 7, could the players somehow do better? The answer is yes (at least if the players care enough about the future). To see how, consider the following strategies:

1. In periods $t = 1$ through $t = T - 1$ (*i.e.*, all periods *but* the last), Player 1 plays *down* and Player 2 plays *left*.
2. If, in period $t \leq T - 1$, Player 1 alone deviates — plays *up* while Player 2 plays *left* — then in *all* subsequent periods Player 1 is to play *down* and Player 2 is to play *right*; that is, following a deviation by Player 1, the players are to play the pure-strategy Nash equilibrium in which Player 1 gets 0 and Player 2 gets 6.

3. If, in period $t \leq T - 1$, Player 2 alone deviates — plays *right* while Player 1 plays *down* — then in *all* subsequent periods Player 2 is to play *left* and Player 1 is to play *up*; that is, following a deviation by Player 2, the players are to play the pure-strategy Nash equilibrium in which Player 1 gets 6 and Player 2 gets 0.
4. If, in period $t \leq T - 1$, both players deviate (*i.e.*, Player 1 plays *up* and Player 2 plays *right*), then in *all* subsequent periods the players mix over their pure strategies with equal probability; that is, play the mixed-strategy Nash equilibrium.
5. Finally, if, in periods $t = 1$ through $t = T - 1$, both players played cooperatively — that is, played as instructed in Step 1 — then, in the last period ($t = T$), the players mix over their pure strategies with equal probability; that is, play the mixed-strategy Nash equilibrium.

In other words, the idea is that the players play cooperatively (*down-left*) every period but the last. It is, unfortunately, impossible to get them to play cooperatively in the last period because, as we saw above, they must play a Nash equilibrium of the stage game in the last period. If one player deviates from being cooperative, then that player is punished for the remaining periods by having the worst Nash equilibrium for him or her played. If both players deviate simultaneously from being cooperative, then the players play the mixed-strategy Nash equilibrium in the remaining periods.

Will these strategies work to achieve cooperation in all but the last period? That is, do these strategies constitute a subgame-perfect equilibrium? Consider, first, what happens following any deviation. Because the players play a Nash equilibrium in the following periods, that is subgame perfect — recall that repetition of a Nash equilibrium of the stage game is always subgame perfect. Hence, what we need to check is that no player wishes to deviate. Suppose the players reach the last period without a deviation (*i.e.*, they have been playing according to Step 1 in each of the previous periods). Now they are supposed to play the mixed-strategy equilibrium. Because it's an equilibrium, neither player will wish to deviate from playing it. Step back to period $T - 1$. If there have been no deviations prior to period $T - 1$, how should the players play? Each player knows that if he or she cooperates in this period, then he or she will get, in expectation, $5/2$ next period (period T). So if he or she expects the other player to cooperate this period, what is his or her best response? If he or she also cooperates, then he or she gets 5 this period and $5/2$ next period, or a total of $15/2$. If he or she deviates, then he or she gets 6 this period, but, because next period the equilibrium that is worst for him or her will be played, he or she gets 0 in the next period. So the total payoff from deviating in period $T - 1$ is $6 (= 6 + 0)$. Six is less than $15/2$, so clearly deviating is not a best response to the other player cooperating: The best response to a player who will cooperate in period $T - 1$ is to cooperate also. So, for the subgame consisting of the last two periods, cooperating in period $T - 1$ and playing the mixed strategy in period T is a Nash equilibrium. Now consider period $T - 2$. Again suppose no deviations

prior to $T - 2$. As just shown, if both players cooperate in period $T - 2$, they expect to receive $15/2$ over the last two periods (periods $T - 1$ and T). If a player deviates unilaterally in period $T - 2$, he or she can expect to receive 0 over the last two periods because the worst Nash equilibrium for him or her will be played over these last two periods. So the total payoff to deviating in period $T - 2$ is just 6, whereas the total payoff to cooperating in period $T - 2$ is $5 + 15/2 = 25/2 > 6$. Again, it's better to cooperate. Clearly, this argument can be repeated all the way back to period 1; hence, we may conclude that the strategies enumerated above do constitute a subgame-perfect equilibrium of the repeated game.

Conclusion. *Consider the finite repetition of a stage game. Assume that all players know how many times it will be repeated, but assume the game has multiple Nash equilibria. Then it can be possible to sustain cooperative play in all but the last period by threatening to revert to the worst Nash equilibrium of the stage game for a player if that player deviates unilaterally from cooperation.*

You might object to this conclusion on the grounds that, when we considered the players' decisions at $T - 1$, we had them count the value of playing the mixed-strategy equilibrium in period T at the full $5/2$. Unless the amount of time between periods is exceedingly small, shouldn't the players *discount* future payoffs? That is, if r is the interest rate, shouldn't each player value, at time $T - 1$, his or her payoff at time T as

$$\frac{1}{1+r} \times \frac{5}{2}?$$

The answer, of course, is yes. Letting $\delta = 1/(1+r)$ denote the *discount factor*, the analysis at period $T - 1$ should have been whether 6 at $t = T - 1$ and nothing in the future was worth more or less than $5 + \delta \frac{5}{2}$. That is, it's a best response to play cooperatively today if

DISCOUNT FACTOR

$$5 + \delta \frac{5}{2} \geq 6.$$

That expression holds provided $\delta \geq 2/5$; that is, provided $r \leq 3/2$. Unless interest rates are particularly usurious or the length of time between periods is particularly great, it is likely that this condition is met. Going back to period $T - 2$, the comparison is between 6 and $5 + \delta 5 + \delta^2 \frac{5}{2}$. Tedious algebra reveals that 6 is smaller provided

$$\delta \geq \frac{\sqrt{35} - 5}{5},$$

where the right-hand side is, note, smaller than $1/5$. And so forth. In summary, then, our previous conclusion is still valid even when the players discount future payoffs provided they don't discount them by too much.

4.3 Infinitely Repeated Games

We now turn to infinitely repeated games. Now there is no last period. Again, let \mathcal{G} denote the stage game and t denote a period. Let δ , $0 < \delta < 1$, denote the discount factor.

To begin our analysis, recall the Prisoners' Dilemma (see Figure 1 on page 1). Suppose that were our stage game. We know that if it is played once, then the sole Nash equilibrium is for both players to be *aggressive*, which means each player receives a payoff of 3. In contrast, if they would both be *passive*, they would each receive a payoff of 4. But both playing *passive* is not a Nash equilibrium of the Prisoners' Dilemma.

From the unraveling result, it is also the case that the only subgame-perfect equilibrium of a game in which the Prisoners' Dilemma is repeated a *finite* number of periods is for the players to play *aggressive* every period. That is, their individual per-period payoffs will be 3.

But what happens if the game is repeatedly *infinitely*? Consider a period t and the following strategies:

1. In this period, play *passive* unless a player was *aggressive* in any previous period.
2. If either player ever played *aggressive* in a previous period, then both players are to play *aggressive* in this period.

In other words, each player is to cooperate — play *passive* — unless someone deviated in the past by being *aggressive*, in which case the players are to be *aggressive* forever after. Observe that the penalty phase, Step 2, in which everyone is forever *aggressive* if someone was previously *aggressive*, is a credible threat (*i.e.*, subgame perfect) because we know that repetition of a Nash equilibrium of the stage game is a subgame-perfect equilibrium of any subgame of a repeated game. Hence, the only thing to check is whether the players would wish to cooperate (*i.e.*, abide by Step 1). Consider one player. Suppose she speculates that her rival will cooperate as long as she cooperates. What is her best response to such cooperation from her rival? If she too cooperates, then her payoff this period is 4 and it's 4 in every subsequent period. The present value of cooperating is, thus,

$$\begin{aligned}
 PV_{\text{cooperate}} &= 4 + 4\delta + 4\delta^2 + \cdots \\
 &= \sum_{\tau=0}^{\infty} 4\delta^{\tau} \\
 &= 4 \times \sum_{\tau=0}^{\infty} \delta^{\tau}, \tag{1}
 \end{aligned}$$

where τ is an index of periods.² If, instead, this player deviates, then she gets 6 this period, but thereafter she gets only 3 per period. The present value of

²Recall that $x^0 = 1$ for any real number x .

such a deviation is, thus,

$$\begin{aligned}
 PV_{\text{deviate}} &= 6 + 3\delta + 3\delta^2 + \dots \\
 &= 6 + \sum_{\tau=1}^{\infty} 3\delta^{\tau} \\
 &= 6 + 3 \times \sum_{\tau=1}^{\infty} \delta^{\tau}.
 \end{aligned} \tag{2}$$

Before determining which present value is greater, it is useful to recall a formula for such infinite sums. Let S be the value of such a sum; that is,

$$S = \sum_{\tau=t}^{\infty} \gamma^{\tau} = \gamma^t + \gamma^{t+1} + \gamma^{t+2} + \dots \tag{3}$$

(where $0 < \gamma < 1$). If we multiply through by γ , we get

$$\gamma S = \sum_{\tau=t+1}^{\infty} \gamma^{\tau} = \gamma^{t+1} + \gamma^{t+2} + \gamma^{t+3} + \dots \tag{4}$$

Observe that we can subtract expression (4) from expression (3) to obtain:

$$S - \gamma S = (1 - \gamma)S = \gamma^t.$$

This follows because each γ^{τ} term in (4) subtracts away a corresponding term in (3); except there is no γ^t term in (4), so γ^t is “left standing” after all this subtraction. Dividing both sides of this last expression by $1 - \gamma$, we have

INFINITE SUM
FORMULA

$$S = \sum_{\tau=t}^{\infty} \gamma^{\tau} = \frac{\gamma^t}{1 - \gamma}. \tag{5}$$

Using the infinite sum formula, expression (5), we see that $PV_{\text{cooperate}}$ is greater than or equal to PV_{deviate} if

$$4 \frac{1}{1 - \delta} \geq 6 + 3 \frac{\delta}{1 - \delta};$$

or, multiplying both sides by $1 - \delta$, if

$$4 \geq 6(1 - \delta) + 3\delta.$$

This will be true if $\delta \geq \frac{2}{3}$. In other words, if the players are sufficiently patient — that is, value future payoffs sufficiently highly — then it is a best response to cooperate with your rival if you think your rival will cooperate with you. Because we’ve shown that cooperating represents mutual best responses, cooperating is a Nash equilibrium for each subgame and, thus, cooperation is a subgame-perfect equilibrium.

This argument is quite general and so we can, in fact, conclude:

Conclusion (Folk Theorem). *Consider an infinitely repeated game among N players in which the stage game, \mathcal{G} , has a single Nash equilibrium. Let $\bar{\pi}_1, \dots, \bar{\pi}_N$ be the payoffs to the N players in this Nash equilibrium of \mathcal{G} . Then any outcome of \mathcal{G} in which the players' per-period payoffs are π_1, \dots, π_N and $\pi_n > \bar{\pi}_n$ for each player n can be sustained as part of a cooperative subgame-perfect equilibrium provided the players are sufficiently patient.*

This conclusion, known in game theory as the *folk theorem*,³ is quite powerful because it says that if players are sufficiently patient, then they can do a lot better in an infinitely repeated game than a one-shot analysis of the stage game might suggest.

FOLK THEOREM

4.4 Infinitely Repeated Games as Games with Unknown Ends

Nobody lives forever. Firms go out of business. Even the sun and the universe have finite lives according to the laws of physics. How, then, can any game be truly infinite? At first glance, this seems an important concern: From the unraveling result, we know that finite repetition of the Prisoners' Dilemma has a unique equilibrium in which play is non-cooperative (*e.g.*, the players are aggressive every period). Moreover, according to the unraveling result, it doesn't matter how many periods the game lasts as long as it has finite length (*e.g.*, $T = 10$ or 100 or 1000 or $10,000$, etc.). **It's only when the game is infinite in length that we invoke the folk theorem and hope to see an equilibrium in which play is cooperative** (*e.g.*, the players are passive every period).

Fortunately, this initial impression is somewhat misleading. Remember that what is essential for the unraveling result is that the players know when the game will end; that is, which is the last period.⁴ Suppose, instead, that rather than knowing when the game ends, each player merely knows that, if they are playing the stage game in the current period, then the probability they will play the stage game in the next period is β , $0 < \beta < 1$. Take this to be true of any period; hence, if the players are at some arbitrary period, t , they assess the probability of playing next period (*i.e.*, at $t + 1$) as β . If the game "survives" to $t + 1$, then the probability of a game at $t + 2$ is β ; hence, back at t , the probability of a game at $t + 2$ is $\beta \times \beta = \beta^2$. More generally, the probability of playing a game τ periods in the future is β^τ . Observe that, as τ gets really big, β^τ gets really small. Put slightly differently, there is absolutely no chance of

³To be precise, the folk theorem is really a collection of similar results. I've given just the one for when the stage game has a single Nash equilibrium. There are variations of this result to cover situations in which the stage game has multiple Nash equilibria.

⁴In fact, to be precise, we need not only that each player know when the game will end, but that each player understand that every player knows when the game will end. That is, in the parlance of game theory, the date the game ends must be *common knowledge*: Every player must know when the game ends; every player must know that every other player knows when the game will end; every player must know that every other player knows that every player knows when the game will end; and so on.

the game truly lasting forever — with probability one it will end in finite time. *But no one knows when.*

Consider, then, the consequence of this for the Prisoners' Dilemma game (Figure 1). Ignore, for the moment, financial discounting (*i.e.*, set $\delta = 1$). Consider the cooperative equilibrium developed in the previous section. What's the best response to cooperating now? If you too cooperate this period, you get 4. If the game survives to next period and you continue to cooperate, you get 4; but, from the perspective of this period, that 4 has an *expected* value of 4β (with probability β you play again and get 4 and with probability $1 - \beta$ the game ends and you get 0; so the expected value is $4 \times \beta + 0 \times (1 - \beta) = 4\beta$). If the game were to survive two periods more and you cooperate, then you will get 4 in two periods time; but because the probability of surviving two more periods is just β^2 , the expected value is $4\beta^2$. Similarly, the plan to cooperate τ periods in the future has an expected return of $4\beta^\tau$. The expected value of playing cooperatively given you anticipate the other player will too is, therefore,

$$\begin{aligned} EV_{\text{cooperate}} &= 4 + 4\beta + 4\beta^2 + \dots \\ &= \sum_{\tau=0}^{\infty} 4\beta^\tau \\ &= 4 \frac{1}{1 - \beta}, \end{aligned}$$

where the last equality follows from the infinite sum formula (expression (5)). If you deviate (*i.e.*, play *aggressive*), then you get 6 in this period, but are doomed to get 3 in each future period. Of course, those future threes are not for sure — there is a chance the game will end before those periods are reached — hence we need to take expected values. The expected value of playing non-cooperatively (deviating) given you anticipate the other player will cooperate in this period is, thus,

$$\begin{aligned} EV_{\text{deviate}} &= 6 + 3\beta + 3\beta^2 + \dots \\ &= 6 + \sum_{\tau=1}^{\infty} 3\beta^\tau \\ &= 6 + 3 \frac{\beta}{1 - \beta}, \end{aligned}$$

where, again, the infinite sum formula has been used to simplify the last expression. From these two expressions, we see that $EV_{\text{cooperate}}$ is greater than or equal to EV_{deviate} if

$$4 \frac{1}{1 - \beta} \geq 6 + 3 \frac{\beta}{1 - \beta};$$

or, multiplying both sides by $1 - \beta$, if

$$4 \geq 6(1 - \beta) + 3\beta.$$

This will be true if $\beta \geq \frac{2}{3}$. In other words, if the players are sufficiently confident the game will continue — that is, the expected values of future payoffs are sufficiently great — then it is a best response to cooperate with your rival if you think your rival will cooperate with you. Because we’ve shown that cooperating represents mutual best responses, cooperating is a Nash equilibrium for each subgame and, thus, cooperation is a subgame-perfect equilibrium.

This argument is, in fact, general. We can, therefore, conclude

Conclusion (Folk Theorem — 2nd version). *Consider a repeated game among N players in which the stage game, \mathcal{G} , has a single Nash equilibrium. Suppose, in any given period, the game repeats with probability β and ends with probability $1 - \beta$, where $0 < \beta < 1$. Let $\bar{\pi}_1, \dots, \bar{\pi}_N$ be the payoffs to the N players in this Nash equilibrium of \mathcal{G} . Then any outcome of \mathcal{G} in which the players’ per-period payoffs are π_1, \dots, π_N and $\pi_n > \bar{\pi}_n$ for each player n can be sustained as part of a cooperative subgame-perfect equilibrium provided the players are sufficiently confident that the game will continue (i.e., provided β is sufficiently great).*

You might wonder about the fact that we’ve done this analysis without financial discounting. But putting back in financial discounting (i.e., $\delta < 1$) is not a problem. We need only compare the expected present value of cooperating, which is

$$\begin{aligned} EPV_{\text{cooperate}} &= 4 + 4\delta\beta + 4\delta^2\beta^2 + \dots \\ &= \sum_{\tau=0}^{\infty} 4(\delta\beta)^{\tau} \\ &= 4 \frac{1}{1 - \delta\beta}, \end{aligned}$$

to the expected present value of deviating, which is

$$\begin{aligned} EPV_{\text{deviate}} &= 6 + 3\delta\beta + 3\delta^2\beta^2 + \dots \\ &= 6 + \sum_{\tau=1}^{\infty} 3(\delta\beta)^{\tau} \\ &= 6 + 3 \frac{\delta\beta}{1 - \delta\beta}. \end{aligned}$$

Provided $EPV_{\text{cooperate}} \geq EPV_{\text{deviate}}$, which is true if $\delta\beta \geq \frac{2}{3}$, then it’s a subgame-perfect equilibrium to have cooperation period by period.

Conclusion (Folk Theorem — 3rd version). *Consider a repeated game among N players in which the stage game, \mathcal{G} , has a single Nash equilibrium. Suppose, in any given period, the game repeats with probability β and ends with probability $1 - \beta$, where $0 < \beta \leq 1$. Let δ , $0 < \delta < 1$, be the common discount factor. Let $\bar{\pi}_1, \dots, \bar{\pi}_N$ be the payoffs to the N players in this Nash equilibrium of \mathcal{G} . Then any outcome of \mathcal{G} in which the players’ per-period payoffs are*

π_1, \dots, π_N and $\pi_n > \bar{\pi}_n$ for each player n can be sustained as part of a co-operative subgame-perfect equilibrium provided the players are both sufficiently patient and sufficiently confident that the game will continue (i.e., provided $\delta \times \beta$ is sufficiently great).

5 Insights from Repeated Games for Competitive Strategy

In this section, we apply our insights from the analysis of repeated games to competitive strategy.

5.1 Tacit Collusion

Consider two firms, Firm A and Firm B, which are Bertrand competitors. Specifically, suppose that, in each period, the firms set their prices, p_A and p_B . Assume each firm has constant marginal cost c for producing the good in question and that each firm has the capacity to serve all the demand it faces. The demand faced by firm i is given by the following (where p_i is its price and p_j is its rival's price):

$$d_i(p_i, p_j) = \begin{cases} D, & \text{if } p_i < p_j \text{ \& } p_i \leq V \\ \frac{1}{2}D, & \text{if } p_i = p_j \leq V \\ 0, & \text{otherwise} \end{cases} . \quad (6)$$

This demand structure can be interpreted as there exist D consumers, each of whom wishes to purchase at most one unit, and each of whom is willing to pay at most V for that unit. Assume $V > c$. Consumers go to the firm with the lower price; hence, provided Firm i 's price doesn't exceed V , it gets all D customers if it is the firm with the lower price. If the two firms charge the same price and that price doesn't exceed V , then the firms split demand equally. Finally, if a firm charges more than its rival or charges a price above what customers are willing to pay, then it has zero sales.

From "Introduction to Game Theory & the Bertrand Trap", we know that were this game played once, the equilibrium (outcome) would be the Bertrand Trap: Both firms would set price equal to marginal cost. Each firm's profit would be $\frac{1}{2}D \times (p - c) = 0$. We also know that repetition of the Bertrand Trap is an equilibrium of the repeated game. The question is whether any another equilibrium can be sustained in an *infinitely* repeated game.⁵

If the firms could sustain another outcome, what outcome would they like? Well one possibility is $p_A = p_B = V$; that is, both firms charge the maximum price at which the consumers are willing to buy. Observe that $p = V$ is the price that a monopolist would charge. A firm's per-period profit at $p = V$ is

⁵Because the Bertrand game has a single equilibrium, the Bertrand Trap, the unraveling result tells us that the only subgame-perfect equilibrium of a *finitely* repeated game with known end is simply period-by-period repetition of the Bertrand Trap.

$\frac{1}{2}D \times (V - c) > 0$. We will see whether this outcome can be sustained as a subgame-perfect equilibrium of an *infinitely* repeated game.

Let δ , $0 < \delta < 1$, be the common discount factor.⁶ Consider the following strategy:

1. Charge $p = V$ in the current period unless either firm charged a lower price in any previous period.
2. If either firm ever charged a lower price in a previous period, then both firms are to charge $p = c$ in this period.

In other words, as long as your rival has cooperated by charging the monopoly price ($p = V$), you should cooperate by charging $p = V$; but if you or your rival ever played uncooperatively in the past (charged a lower price), then play the Bertrand Trap forever after.

Does this constitute a subgame-perfect equilibrium? We know that the *punishment* rule (Step 2) is subgame perfect because it is simply repetition of the Nash equilibrium of the stage game. Hence, the only thing to check is whether cooperating is a best response to your rival cooperating. Cooperating yields $\frac{1}{2}D \times (V - c)$ this period and in each future period. Hence,

PUNISHMENT

$$\begin{aligned} PV_{\text{cooperate}} &= \frac{1}{2}D \times (V - c) + \frac{1}{2}D \times (V - c)\delta + \frac{1}{2}D \times (V - c)\delta^2 + \dots \\ &= \frac{1}{2}D \times (V - c) \frac{1}{1 - \delta}. \end{aligned} \quad (7)$$

Not cooperating means charging a price below V . The best such deviation is to charge a price just below V : That ensures you get all the customers and do so at nearly the monopoly price. Let ε denote the little bit by which you undercut $p = V$. So the profit in the period you deviate is $D \times (V - \varepsilon - c)$. Of course, having deviated in the current period, your future profits are zero because a deviation is punished by reversion to the Bertrand Trap. Hence,

$$PV_{\text{deviate}} = D \times (V - \varepsilon - c) + 0\delta + 0\delta^2 + \dots = D \times (V - \varepsilon - c).$$

Cooperating will be a best response — and, thus, sustainable as part of a subgame-perfect equilibrium — if $PV_{\text{cooperate}} \geq PV_{\text{deviate}}$; that is, if

$$\frac{1}{2}D \times (V - c) \frac{1}{1 - \delta} \geq D \times (V - \varepsilon - c). \quad (8)$$

Because the $\varepsilon > 0$ in (8) is arbitrary, (8) needs to hold for as small an ε as possible, which means we can consider $\varepsilon = 0$ in (8). This, in turn, means that (8) holds if $\delta \geq \frac{1}{2}$.

This outcome is known as *tacit collusion*. The firms are *not* getting together

TACIT COLLUSION

⁶If we wish, we can consider the risk of the game ending as being embedded into δ ; that is, δ accounts for both financial discounting and the discounting for the probability that game might not continue.

to agree on prices (*i.e.*, overtly colluding) — which would be illegal under antitrust laws. But, even without an explicit agreement, they understand they have an incentive not to undercut the monopoly price due to the threat of profit-eroding price competition were they to undercut; that is, the start of a price war if they undercut.

PRICE WAR

5.1.1 More than two competitors

What if we changed the analysis so that, instead of two firms, there were N firms. It is readily shown that

$$PV_{\text{cooperate}} = \frac{1}{N} D \times (V - c),$$

because, now, the market is split N ways if all N firms tacitly collude at the monopoly price. The present value of deviating, however, isn't changed because deviating means capturing the entire market (at the cost of Bertrand competition in all subsequent periods). So the condition for tacit collusion to hold, expression (8), becomes

$$\frac{1}{N} D \times (V - c) \frac{1}{1 - \delta} \geq D \times (V - \varepsilon - c). \quad (9)$$

Again this must hold for all ε , so it must hold if $\varepsilon = 0$. Hence, the condition for tacit collusion to be sustainable as a subgame-perfect equilibrium with N firms depends on whether

$$\delta \geq 1 - \frac{1}{N}. \quad (10)$$

If it is, then tacit collusion can be sustained as a subgame-perfect equilibrium. If it isn't, then tacit collusion cannot be sustained and the only equilibrium is infinite repetition of the Bertrand Trap.

Notice that (10) becomes a harder and harder condition to satisfy as N gets larger. For $N = 2$, we just need for $\delta \geq \frac{1}{2}$, but for $N = 3$, we need $\delta \geq \frac{2}{3}$. Another way to put this is as follows:

Conclusion. *Tacit collusion becomes harder to sustain the more firms there are in an industry.*

5.1.2 Interpreting δ

Recall that δ is a measure of how much weight players put on future periods. It is, in part, determined by financial discounting — a dollar tomorrow is worth less than a dollar today. It is — or at least can be — a reflection of the likelihood that the game will continue (recall Section 4.4). This last interpretation sheds light on the phenomenon of fiercer price competition in declining industries. For instance, consider an industry that is being eliminated due to environmental regulation (*e.g.*, the industry that produced lead additives for gasoline). Because the future is not particularly valuable (the probability of playing in the future

is relatively low), tacit collusion becomes harder to sustain. In the limit, if the “end date” becomes known, then the unraveling result tells us that fierce price competition should break out immediately. This is one reason, for instance, that reports of over-stocked stores prior to Christmas can cause big pre-Christmas sales — the end date, December 24, is known. Furthermore, everyone realizes that everyone has too much capacity; that is, the conditions of the Bertrand Trap hold. The unraveling result then applies and we have lots of big sales well before Christmas.

Conclusion. *Tacit collusion requires that firms put sufficient weight on the future. In particular, the probability of continued play must be sufficiently high.*

5.1.3 Price wars

Tacit collusion is supported by the credible threat of reverting to the Bertrand Trap if any firm undercuts the collusive (monopoly) price. The threat is credible because repetition of the Bertrand Trap is an subgame-perfect equilibrium of the repeated game. This reversion to Bertrand competition can be seen as a price war. Of course you might object that few price wars last forever. That is, *infinite* reversion to the Bertrand Trap seems unrealistic (as well as unduly harsh).

Fortunately, in some contexts, we don’t need to use *infinite* reversion to the Bertrand Trap as the threat that supports tacit collusion. If the price war is severe enough for long enough, then it will serve as a suitable threat. To see this, return to an industry with two firms and suppose that, instead of infinite reversion, the firms play the Bertrand Trap for 5 periods if either firm undercuts. Because, in any given period, a best response to your rival pricing at marginal cost is for you to also price at marginal cost, this 5-period punishment phase or price war, is credible. Will this support tacit collusion? The present value of cooperating stays the same, but the present value of deviating is, now,

$$\begin{aligned}
 PV_{\text{deviate}} &= D \times (V - c) + \underbrace{0\delta + \cdots + 0\delta^5}_{\text{5 periods of war}} + \underbrace{\frac{D}{2}(V - c)\delta^6 + \frac{D}{2}(V - c)\delta^7 + \cdots}_{\text{return to tacit collusion}} \\
 &= D \times (V - c) + \sum_{\tau=6}^{\infty} \frac{D}{2}(V - c)\delta^{\tau} \\
 &= D \times (V - c) + \frac{D}{2}(V - c) \frac{\delta^6}{1 - \delta}
 \end{aligned}$$

(where, for convenience, the ε term has been set to zero). Comparing this expression with $PV_{\text{cooperate}}$ (see expression (7)), tacit collusion will be sustainable — even with just a 5-period price war as a punishment for undercutting — if

$$2\delta - \delta^6 \geq 1.$$

Tedious algebra reveals that this holds if $\delta \geq .50866$. In other words, the threat of a 5-period price war works almost as well as the threat of an infinite price

war. We would expect, therefore, that this would be a sufficient deterrent in many contexts.

Conclusion. *Tacit collusion be supported by the threat of a price war of finite duration in the event a firm deviates if the firms put sufficient weight on the future.*

Another, more subtle, complaint might arise: In the equilibria considered, either tacit collusion is sustainable or it's not. If it's not, then the firms are in the Bertrand Trap every period starting with the first. If it is, then the firms tacitly collude every period starting with the first. For these second equilibria, price wars are just a threat, but a threat that's never realized. In reality, however, we do see periods in which firms in a given industry seem engaged in tacit collusion and periods in which these same firms seem to be in a price war. How do we explain that?

The answer is as follows. In the analysis to this point, we have assumed that the firms all know if some firm has deviated by undercutting the (tacit) collusive price. In many situations, this is a reasonable assumption because prices are posted publicly. But in other situations, prices are not posted publicly (*e.g.*, prices are negotiated) or the posted prices only partially reflect a consumer's true cost (*e.g.*, there may be deals on shipping or extras can be thrown in, etc.). If the prices being charged by the various firms aren't common knowledge, then the firms have to rely on other evidence to decide whether their rivals are tacitly colluding with them or are undercutting them. One such piece of evidence is own demand — if your own demand suddenly drops, then this could be evidence that your rivals have undercut you. In the models considered above, a drop in demand would, in fact, be conclusive evidence: You expect to get a positive fraction of total demand, D , but if you are undercut you get no demand. In real life, however, demand fluctuates for other reasons (*e.g.*, because household incomes fluctuate due to random events such as layoffs or new jobs). Hence, in a more realistic model, you wouldn't know if a drop in own demand is due to undercutting by a rival or some such random event. In this world, with *imperfect monitoring* of rivals' prices, tacit collusion can only be sustained by so-called *trigger strategies*: You play the collusive price (*e.g.*, $p = V$) each period unless, in the previous period, your own demand fell below a certain threshold — the *trigger* — in which case you engage in Bertrand pricing (*i.e.*, $p = c$) for that period and some number of future periods. That is, you play collusively unless you have sufficient evidence to suspect undercutting, in which case you engage in a price war for some period of time. After the price war, you resume tacitly colluding. In this world, even if all firms try to collude tacitly, it can happen that random events lead to sufficiently low demand that a price war still gets triggered — we can see periods of tacit collusion punctuated by periods of price war.

IMPERFECT
MONITORING

TRIGGER
STRATEGIES

Although a realistic examination of trigger strategies is beyond these notes, we can gain insight by considering a simple model. Suppose, with probability g times are good, in which case total demand is $D > 0$ at prices $p \leq V$. With probability $1 - g$ times are bad, in which case there is no demand regardless of

price (this is, to be sure, an unrealistic assumption; but it will suffice to explore the intuition behind trigger strategies). The probability of good or bad times is independent across periods (*i.e.*, there is no correlation across time; just as the outcome of a given coin flip doesn't depend on the outcome of previous coin flips). Assume there are two firms. Some new notation will be useful in what follows:

- Let $\Pi^M = D \times (V - c)$ denote the profit a monopolist would realize in a good-time period. Note this is (modulo a trivially small ε) what a firm that unilaterally deviates (undercuts) would get in a good-time period. Observe, then, that $\Pi^M/2$ is the profit that a tacitly colluding firm earns in a good-time period.
- Let PV^+ be the expected present discounted value of a firm's profit from the current period on assuming that the firms start the current period in a collusive phase (*i.e.*, both firms expect their rivals to charge $p = V$ in this period).
- Let PV^- be the expected present discounted value of a firm's profit from the current period on assuming that the firms start the current period in the *first period* of a price-war phase (*i.e.*, both firms expect their rivals to charge $p = c$ in this period).
- Let ℓ denote the length of a price war (number of periods of a price war).

Because the probabilities are independent and the game is repeated infinitely, the game is *stationary*; that is, if the game is in the same phase in period t as it is in period t' , then the present values going forward from those times are the same.

STATIONARY

What are PV^+ and PV^- ? The second one is easy:

$$PV^- = \underbrace{0 + 0\delta^1 + \dots + 0\delta^{\ell-1}}_{\ell \text{ periods of price war}} + \delta^\ell PV^+, \quad (11)$$

where the last term in expression (11) arises because ℓ periods in the future the firms return to the collusive phase, which has a present value starting, then, of PV^+ . What about PV^+ ? Suppose you're in the collusive phase in the given period. Observe that with probability g it will be a good time, so you earn $\Pi^M/2$ this period and you start the *next* period in the collusive phase. The present value next period of being in the collusive phase is PV^+ , by definition, and the value *today* of being in the collusive phase *next* period is, thus, δPV^+ . So the total present value if it is the good time this period is $\Pi^M/2 + \delta PV^+$. If, instead, it's a bad time this period, you get 0 this period. Moreover, this will trigger a price war, so *next* period begins a price war. The present value at the start of a price war is, by definition, PV^- . The price war begins *next* period, so its present value *today* is δPV^- . So the total present value if it is the bad time today is $0 + \delta PV^- = \delta PV^-$. We can thus conclude:

$$PV^+ = g \times \left(\frac{\Pi^M}{2} + \delta PV^+ \right) + (1 - g) \times \delta PV^-. \quad (12)$$

We can use equations (11) and (12) to solve for PV^+ and PV^- in terms of the primitives of the problem:

$$PV^+ = \frac{g \frac{\Pi^M}{2}}{1 - g\delta - (1 - g)\delta^{\ell+1}} \quad (13)$$

$$PV^- = \frac{\delta^\ell g \frac{\Pi^M}{2}}{1 - g\delta - (1 - g)\delta^{\ell+1}} \quad (14)$$

Lastly, for tacit collusion to be sustained, neither firm can wish to start a price war; that is, neither firm should wish to undercut if it thought its rival would play cooperatively. Hence, tacit collusion can be sustained only if

$$PV^+ \geq g\Pi^M + \delta PV^-, \quad (15)$$

where $g\Pi^M = g\Pi^M + (1 - g) \times 0$ is the *expected* profit if you undercut your rival. Substituting for PV^+ and PV^- using expressions (13) and (14), canceling like terms, and rearranging, it can be shown that (15) is equivalent to

$$2g\delta + (1 - 2g)\delta^{\ell+1} \geq 1. \quad (16)$$

Observe, first, that this expression *cannot* be satisfied if $\ell = 0$, because $\delta \not\geq 1$. Hence, there must be at least some period of price war following bad times. This makes perfect sense — without any threat of punishment, neither firm has any incentive to cooperate.

We would like the length of the price war to be as small as possible because, from expression (13), the present value of tacit collusion is greater, the smaller is ℓ .⁷ So we seek the smallest ℓ that satisfies (16).⁸ If, for example, $\delta = 9/10$ and $g = 3/4$, then that ℓ would be 3; a price war of three periods must follow bad times if the firms are to sustain tacit collusion.

Observe that expression (16) *cannot* be satisfied for any values of $\delta < 1$ and ℓ if $g \leq \frac{1}{2}$. That is, if the probability of a good times is less than $1/2$, then it is impossible for the firms to sustain tacit collusion. The intuition of this result is the following: A firm has an incentive to cooperate today because of the threat of punishment tomorrow. What is the cost of this punishment? It is the lost profits from not colluding in the future (*i.e.*, from being in a price war). But if the likelihood of profits in the future from colluding is sufficiently small — bad times are very likely — then the risk of losing these profits is not particularly threatening, which means the punishment is not much of a deterrent.

Note that because $g > 1/2$ if tacit collusion is possible, we know that $2g\delta > 2g\delta + (1 - 2g)\delta^{\ell+1}$. Hence, a *necessary* condition for tacit collusion

⁷Because $\delta < 1$, $\delta^{\ell+1}$ gets smaller as ℓ gets bigger. Hence, $(1 - g)\delta^{\ell+1}$ gets smaller as ℓ gets bigger. But $(1 - g)\delta^{\ell+1}$ enters the denominator negatively, so increasing ℓ makes the denominator get bigger, which makes the fraction (*i.e.*, PV^+) get smaller.

⁸That smallest ℓ is the smallest integer not smaller than

$$\frac{\log\left(\frac{1-2g\delta}{1-2g}\right)}{\log(\delta)} - 1.$$

to be sustainable is that $g\delta \geq 1/2$. From this, it follows that the lower the probability of good times, the more the firms must weight the future if tacit collusion is to be feasible; or, conversely, the higher the probability of good times, the less the firms need weight the future for tacit collusion to be feasible.

Finally, as should be apparent, imperfect monitoring — that is, a need to employ trigger strategies — makes sustaining tacit collusion harder than it is when all firms know all firms' prices. Moreover, not only does imperfect monitoring make it harder to sustain tacit collusion, but even if the firms can sustain it, they enjoy less profits because of the, now, unavoidable price wars that punctuate periods of tacit collusion.

To summarize:

Conclusion. *When firms can imperfectly monitor the prices of their rivals and must, therefore, rely on demand to make inferences about undercutting, then*

- *firms must employ trigger strategies if they wish to sustain tacit collusion;*
- *even tacitly colluding firms will suffer periods of price wars;*
- *the ability to sustain tacit collusion is made harder if the likelihood of high-demand states (i.e., good times) is relatively low; in fact, if the likelihood of high-demand states is sufficiently low, then tacit collusion is impossible; and*
- *firms do worse with imperfect monitoring than they would with perfect monitoring; hence, firms have an incentive to improve their monitoring of their rivals' prices.*

5.2 Reputation I: Product Quality

Rolex watches are sold in stores for many thousands of dollars. Yet, if a stranger on the street offered to sell you a what he claimed was a genuine Rolex for \$90, you'd probably decline his offer. Why? The answer is straightforward, while a *genuine* Rolex may be worth thousands of dollars, an imitation Rolex may only be worth a few dollars. Unless you're extremely knowledgeable about watches, it would likely be difficult for you to distinguish between a genuine Rolex and a knock-off. The risk is that a street vendor is selling you a knock-off that wouldn't even be worth the \$90 he asks. Indeed, if it were truly a genuine Rolex, why would he be selling it for so little?⁹ But there's a more subtle question: Why do you trust the jewelry store that offers a genuine Rolex? You still can't determine whether the one in the jewelry store is genuine (or even that it's not stolen).

The answer is repeated games. You are playing a repeated game with the jewelry store (or, more precisely, the jewelry store is playing a repeated game

⁹There is, of course, the possibility that the watch is genuine, but stolen. But, morals aside, it's still true that a genuine but stolen Rolex is a less valuable good than a genuine and legitimate Rolex — the watch being offered by the street vendor is unlikely to be of the same quality as the "same" watch at a jeweler's.

with customers). You aren't likely to be playing a repeated game with the street vendor, as you are unlikely to ever see him again.

To formalize this intuition, consider a store that sells an *experience* good; that is, a good whose quality is learned by the customer only over time (*i.e.*, once she's taken it home). To get away from the potential liability a store would incur from selling fake or stolen watches, let's focus on a slightly different problem: The store can sell a high-quality product, which costs it c_H per unit; or it can sell a low-quality product, which costs it c_L . Not surprisingly, assume $c_L < c_H$. A consumer values the high-quality product at v_H and the low-quality product at v_L , where $v_L < v_H$. Because the product is an experience good, the customer does not know whether she's buying a high or low-quality product at the time of purchase. It is only over time, once the purchase has been made, that she learns its quality.

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The most the customer will pay for a product she believes to be high quality is v_H . The most she will pay for a product she believes to be low quality is v_L . Her net gain is $v - p$, where v denotes the actual quality and p is the price she pays. The store's profit is $p - c$, where c is determined by the quality of the product actually sold. Assume the store has market power, so it sets price. Clearly, if the customer believed the the quality is high, the store could charge $p = v_H$. But if the customer believes the quality is low, then the most the store can charge is $p = v_L$. Hence, there are only two prices that are relevant to the problem: $p = v_H$ and $p = v_L$.

Suppose the game were played once. Because $c_L < c_H$, the store has the incentive to *provide* the low-quality product, although it may claim it is the high-quality product. Recognizing this, the customer would know the product was low quality and pay no more than v_L . The store's profit would, thus, be $v_L - c_L$. But if $v_H - c_H > v_L - c_L$, then the store would be happier if it could somehow convince the customer it would provide high quality. But, in a one-shot game, that's impossible — the sole Nash equilibrium of this game played just once is for the store to provide low quality and the customer to buy provided the price doesn't exceed v_L .

Now consider an infinitely repeated version of this game. Assume the customer learns the quality before she next visits the store. Consider the following strategies:

1. Each period, the store charges $p = v_H$, provides high quality, and the customer believes the store will provide high quality unless the store has ever sold a low-quality product in any previous period.
2. If the store has ever sold a low-quality product in a previous period, then, in the current period, the store charges $p = v_L$, provides low quality, and the customer believes the store will provide low quality.

Step 2 is clearly the punishment phase. Because the punishment phase is simply repetition of the Nash equilibrium of the stage game, it is subgame perfect. The only question, therefore, is whether the store wishes to cooperate as set forth in

Step 1. If the store cooperates, the present value of its profit is

$$PV_{\text{honest}} = \sum_{\tau=0}^{\infty} (v_H - c_H) \delta^\tau = (v_H - c_H) \frac{1}{1 - \delta}. \quad (17)$$

If, however, it cheats the customer — provides low quality at the high-quality price — then the present value of its profit is

$$PV_{\text{cheat}} = v_H - c_L + \sum_{\tau=1}^{\infty} (v_L - c_L) \delta^\tau = v_H - c_L + (v_L - c_L) \frac{\delta}{1 - \delta}. \quad (18)$$

The present value of honesty exceeds that of dishonesty (cheating) if

$$(v_H - c_H) \frac{1}{1 - \delta} \geq v_H - c_L + (v_L - c_L) \frac{\delta}{1 - \delta}$$

or, equivalently, if

$$\delta \times (v_H - v_L) \geq c_H - c_L.$$

Because $v_H - c_H > v_L - c_L$, this will be true for δ large enough.

Conclusion. *If the store weights the future sufficiently highly, then it is possible for the store to develop a reputation for providing high-quality products (for which it can charge a high price).*

In other words, repetition allows the store to develop a *reputation* for providing high quality. It does because the if it loses its reputation — becomes seen as the provider of low-quality products — then its profit per sale goes down. This is the reason you trust the jewelry store, it will be there tomorrow, but not the street vendor, who won't.

REPUTATION

The above model is admittedly simple and you can no doubt think of how real life varies from the assumptions. Nonetheless, for the most part, it should be apparent that the complications of real life don't fundamentally alter this conclusion. However, one real life concern might yet make you worried: While the jewelry store plays repeatedly, customers frequently do not. That is, few people buy expensive watches with any frequency and, moreover, given mobility, lack of store loyalty, and so forth, even when people do buy another watch, it may well be from another store. How, then, do repeated games work?

The answer is that only the store needs to be a long-lived player provided that each period's consumers know the store's reputation. That is, one could imagine that each period a new customer arrives who knows the store's past history (*i.e.*, whether it ever sold low quality instead of high quality). If it's sold low quality at any time in the past, this period's customer expects low quality and, thus, won't pay more than v_L . If, however, it's never sold low quality in the past, then this period's customer expects high quality and, thus, will pay up to v_H . It is clear that under this variant of the model, expressions (17) and (18) still define the relevant payoffs for the firm if it is honest (*i.e.*, provides high quality) or cheats (*i.e.*, provides low quality). In other words, the same conclusion applies even though the store is the only player playing repeatedly.

5.3 Reputation II: Internal Relations*

To be written.

5.4 Reputation III: Entry Deterrence

Recall from the analysis of Figure 2 (see page 2) that deterring entry can be difficult. In the game shown in Figure 2, the subgame-perfect equilibrium is one in which entry occurs: Any threat by the incumbent to be *tough* is incredible; hence, recognizing this, the entrant enters.

If, however, we consider entry deterrence over time, then incumbents can develop reputations for playing tough that can serve to deter entry. In this section, we investigate how.

First, let's consider the following modification of the game. Suppose that the game is potentially repeated as follows: Each period, the entrant can decide to enter or stay out. If it stays out, then it can decide to enter at some future date. If it enters, then the incumbent decides between being easy or tough. If it is tough, the -5 for the entrant means it goes bankrupt and disappears forever. This leaves the incumbent to earn 10 in all *subsequent* periods. If the incumbent is easy, then the two firms co-exist that period and forever after. Now the present value of payoffs to the incumbent from being easy are

$$\frac{5}{1 - \delta},$$

where δ is the discount factor. The payoffs to the incumbent from being tough are

$$2 + \frac{10\delta}{1 - \delta}.$$

Provided $\delta \geq 3/8$, it is better to be tough than easy.

Now you might object that all I've done is transform the payoffs of the game so the game looks like the game in Figure 8. Provided $\delta \geq 3/8$, the subgame-perfect equilibrium of the game in Figure 8 is for the incumbent to play *tough* if the entrant *enters* and for the entrant to *stay out*. This may be a reasonable transformation, but it doesn't have much to do with repeated games.

Consider, therefore, an alternative version of the game. Suppose that the incumbent lives forever, but that the entrant — regardless of the play in the period — lives just one period. (This is, to be sure, somewhat unrealistic, but it can be seen as a model of “hit-and-run” entry, of the sort that airlines engaged in on certain routes in the early days that followed airline deregulation.) Assume, however, that the period- t entrant knows the history of the game up to period t . Now, even if the entrant “crushes” a given entrant, a new potential entrant will pop up next period. This might seem to put the incumbent back into the one-shot of Figure 2, but in fact the situation is more analogous to that in Figure 8. To see why, consider the following strategy for the incumbent:

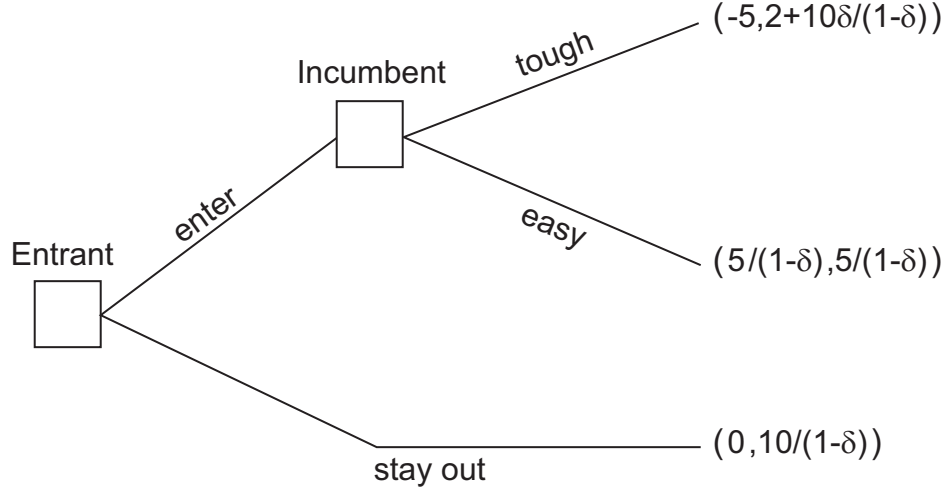


Figure 8: The entry deterrence game of Figure 2 transformed.

- If it has never been easy on an entrant in any previous period or if no entrant has ever entered in a previous period, then play tough this period if the entrant enters; but
- if it has ever been easy on an entrant, then be easy on all subsequent entrants.

If the entrant in a given period thought the incumbent would play according to this strategy and if the incumbent had either always been tough against past entrants or had never faced entry, then the best response of that given period's entrant would be *stay out* because $0 > -5$. The question is, thus, whether the incumbent's threat to be tough is credible in this repeated game. The answer is *yes* provided $\delta \geq 3/8$. If, in a given period, the incumbent is easy following entry, then it gets 5 today. Moreover, it will be expected to be easy in all future periods and, thus, will be entered against in all future periods. Its best response if it will *always* be entered against is to be easy, so it will get 5 in all future periods; that is, the present value of being easy today is

$$\frac{5}{1-\delta}. \quad (19)$$

If, however, it is tough on entrant today, it gets 2 today. But it maintains its reputation for toughness, so no future entrants should enter, which means it gets 10 in every future period. The present discounted value of being tough today is, therefore,

$$2 + \frac{10\delta}{1-\delta}.$$

As we saw above, this exceeds the value in expression (19), which means it *is* credible to be tough against any entrant. Note that this outcome is very much dependent on the game being repeated.