2D Melt Pool Velocity Field

2D Incompressible Flow Equations

Continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Momentum (x-direction)

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

Momentum (y-direction)

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

Staggered Grid

```
┌──v──┐──v──┐
│     │     │
u  p  u  p  u    p = cell center (square middle)
│     │     │    u = vertical faces (left/right edges)
┌──v──┐──v──┐    v = horizontal faces (top/bottom edges)
│     │     │
u  p  u  p  u
│     │     │
┌──v──┐──v──┐
```

Discretizing Momentum (x-direction)

at (i + 1/2, j)

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

We can approximate the following terms as:

$$\frac{\partial u}{\partial t} \approx \frac{u^{n+1}_{i+\frac{1}{2},j} - u^{n}_{i+\frac{1}{2},j}}{\Delta t}$$

$$u\frac{\partial u}{\partial x} \approx u_{i+\frac{1}{2},j} \cdot \frac{u_{i+1+\frac{1}{2},j} - u_{i-1+\frac{1}{2},j}}{2\Delta x}$$

$$v\frac{\partial u}{\partial y} \approx v_{i+\frac{1}{2},j} \cdot \frac{u_{i+\frac{1}{2},j+1} - u_{i+\frac{1}{2},j-1}}{2\Delta y}$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} \approx -\frac{1}{\rho}\frac{p_{i+1,j} - p_{i,j}}{\Delta x}$$

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u_{i+\frac{1}{2}+1,j} - 2u_{i+\frac{1}{2},j} + u_{i+\frac{1}{2}-1,j}}{\Delta x^2}$$

$$\frac{\partial^2 u}{\partial y^2} \approx \frac{u_{i+1/2,j+1} - 2u_{i+1/2,j} + u_{i+1/2,j-1}}{\Delta y^2}$$

$$\boxed{\begin{aligned}&\frac{u^{n+1}_{i+1/2,j} - u^{n}_{i+1/2,j}}{\Delta t} + u_{i+1/2,j} \cdot \frac{u_{i+1/2+1,j} - u_{i+1/2-1,j}}{2\Delta x} + v_{i+1/2,j} \cdot \frac{u_{i+1/2,j+1} - u_{i+1/2,j-1}}{2\Delta y} = -\frac{1}{\rho}\frac{p_{i+1,j} - p_{i,j}}{\Delta x} \\ &+ \nu\left[\frac{u_{i+1/2+1,j} - 2u_{i+1/2,j} + u_{i+1/2-1,j}}{\Delta x^2} + \frac{u_{i+1/2,j+1} - 2u_{i+1/2,j} + u_{i+1/2,j-1}}{\Delta y^2}\right]\end{aligned}}$$

Interpolating ν

Because ν doesn't exist at $i + \frac{1}{2}$ it gets interpolated as:

$$\nu_{i+1/2,j} = \frac{\nu_{i,j} + \nu_{i+1,j}}{2}$$

Takes the average of all the neighbors around it

Discretization with ν

$$\begin{aligned}&\frac{u^{n+1}_{i+1/2,j} - u^{n}_{i+1/2,j}}{\Delta t} + u_{i+1/2,j} \cdot \frac{u_{i+1/2+1,j} - u_{i+1/2-1,j}}{2\Delta x} + v_{i+1/2,j} \cdot \frac{u_{i+1/2,j+1} - u_{i+1/2,j-1}}{2\Delta y} = -\frac{1}{\rho}\frac{p_{i+1,j} - p_{i,j}}{\Delta x} \\ &+ \frac{\nu_{i,j} + \nu_{i+1,j}}{2}\left[\frac{u_{i+1/2+1,j} - 2u_{i+1/2,j} + u_{i+1/2-1,j}}{\Delta x^2} + \frac{u_{i+1/2,j+1} - 2u_{i+1/2,j} + u_{i+1/2,j-1}}{\Delta y^2}\right]\end{aligned}$$

Discretization in the y direction

$$\boxed{\begin{aligned}&\frac{v^{n+1}_{i,j+1/2} - v^{n}_{i,j+1/2}}{\Delta t} + v_{i,j+1/2} \cdot \frac{v_{i,j+1/2+1} - v_{i,j+1/2-1}}{2\Delta y} + u_{i,j+1/2} \cdot \frac{v_{i+1,j+1/2} - v_{i-1,j+1/2}}{2\Delta x} = -\frac{1}{\rho}\frac{p_{i,j+1} - p_{i,j}}{\Delta y} \\ &+ \frac{\nu_{i,j} + \nu_{i,j+1}}{2}\left[\frac{v_{i,j+1/2+1} - 2v_{i,j+1/2} + v_{i,j+1/2-1}}{\Delta y^2} + \frac{v_{i+1,j+1/2} - 2v_{i,j+1/2} + v_{i-1,j+1/2}}{\Delta x^2}\right]\end{aligned}}$$

Discretization of continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial u}{\partial x} = \frac{u_{i+1/2,j} - u_{i-1/2,j}}{\Delta x}$$

$$\frac{\partial v}{\partial y} = \frac{v_{i,j+1/2} - v_{i,j-1/2}}{\Delta y}$$

Discretizes to:

$$\boxed{\frac{u_{i+1/2,j} - u_{i-1/2,j}}{\Delta x} + \frac{v_{i,j+1/2} - v_{i,j-1/2}}{\Delta y} = 0}$$