DAMA 50

Written Assignment II

Submitted by:

Panagiotis Paltsokas

ID: std163861

[DAMA 50] Written Assignment 2

Panagiotis Paltsokas - std163861

Problem 7

Consider the Euclidean vector space $\mathbb{R}^5$ with the Euclidean inner product. A subspace $U \subset \mathbb{R}^5$ is defined by

$$U = \operatorname{span}\left\{ \begin{bmatrix} 0 \\ -1 \\ 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 5 \\ 0 \\ 7 \end{bmatrix} \right\}.$$

Using `Sagemath`

(a) check if the vectors that span U are linearly independent;

(b) find the projection matrix that maps $\mathbb{R}^5$ onto U;

(c) find the projection of the vector $x = \begin{bmatrix} -1 & 9 & -1 & 4 & 1 \end{bmatrix}^T$ onto U;

(d) compute the projection error in question (c) both analytically and numerically.

(a)

Define the vectors u_1, u_2, u_3 and u_4

In [1]:
```
%display latex
```

In [2]:
```
u1=vector(QQ, [0,-1,2,0,2])
u2=vector(QQ, [1,-3,1,-1,2])
u3=vector(QQ, [-3,4,1,2,1])
u4=vector(QQ, [-1,-3,5,0,7])
show(u1,u2,u3,u4)
```

$(0,\ -1,\ 2,\ 0,\ 2)\ (1,\ -3,\ 1,\ -1,\ 2)\ (-3,\ 4,\ 1,\ 2,\ 1)\ (-1,\ -3,\ 5,\ 0,\ 7)$

Compute the matrix M

In [3]:
```
M=column_matrix([u1,u2,u3,u4]);M
```

Out[3]:

$$\begin{pmatrix} 0 & 1 & -3 & -1 \\ -1 & -3 & 4 & -3 \\ 2 & 1 & 1 & 5 \\ 0 & -1 & 2 & 0 \\ 2 & 2 & 1 & 7 \end{pmatrix}$$

Check for linear dependence with Row Echelon Form

In [4]:
```
M.echelon_form()
```

Out[4]:

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The vectors that span U are not linearly dependent. Only u_1, u_2 and u_3 are linearly independent because they are the only pivot columns.

(b)

Compute the matrix A of the basis vectors of U

In [5]:
```
A=column_matrix([u1,u2,u3]);A
```

Out[5]:

$$\begin{pmatrix} 0 & 1 & -3 \\ -1 & -3 & 4 \\ 2 & 1 & 1 \\ 0 & -1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$$

In [6]:
```
At=transpose(A);At
```

Out[6]:

$$\begin{pmatrix} 0 & -1 & 2 & 0 & 2 \\ 1 & -3 & 1 & -1 & 2 \\ -3 & 4 & 1 & 2 & 1 \end{pmatrix}$$

Let $\vec{x}$ be an arbitrary vector of $\mathbb{R}_5$. Then the projection of $\vec{x}$ onto U is $Proj_U\vec{x} = A(A^TA)^{-1}A^T\vec{x}$ and the projection matrix is $P_\pi = A(A^TA)^{-1}A^T$

In [7]:
```
Pπ=A*((At*A).inverse())*At;Pπ
```
Out[7]:
$$\begin{pmatrix} \frac{10}{21} & -\frac{2}{7} & \frac{4}{21} & -\frac{1}{7} & -\frac{1}{3} \\ -\frac{2}{7} & \frac{43}{63} & \frac{4}{63} & \frac{2}{7} & -\frac{2}{9} \\ \frac{4}{21} & \frac{4}{63} & \frac{58}{63} & \frac{1}{7} & \frac{1}{9} \\ -\frac{1}{7} & \frac{2}{7} & \frac{1}{7} & \frac{1}{7} & 0 \\ -\frac{1}{3} & -\frac{2}{9} & \frac{1}{9} & 0 & \frac{7}{9} \end{pmatrix}$$

(c)

The simple and easy way

We can just multiply the projection matrix P_π with the vector $\vec{x}$ to find $\pi_U(x)$

In [8]:
```
x=vector(QQ,[-1,9,-1,4,1]);x
```
Out[8]:
$$(-1,\ 9,\ -1,\ 4,\ 1)$$

In [9]:
```
px=Pπ*x;px
```
Out[9]:
$$\left(-\frac{29}{7},\ \frac{51}{7},\ \frac{1}{7},\ \frac{22}{7},\ -1\right)$$

The more analytical approach as shown in Example 3.11 of "Mathematics for Machine Learning"

We compute the matrix A^TA and the vector A^Tx

In [10]:
```
x=matrix([-1,9,-1,4,1]).transpose();x
```
Out[10]:
$$\begin{pmatrix} -1 \\ 9 \\ -1 \\ 4 \\ 1 \end{pmatrix}$$

In [11]:
```
AtA=At*A;AtA
```
Out[11]:
$$\begin{pmatrix} 9 & 9 & 0 \\ 9 & 16 & -14 \\ 0 & -14 & 31 \end{pmatrix}$$

In [12]:
```
Atx=At*x;Atx
```
Out[12]:
$$\begin{pmatrix} -9 \\ -31 \\ 47 \end{pmatrix}$$

We now solve the equation $A^T A\lambda = A^T x$ to find λ:

In [13]:
```
var('λ1,λ2,λ3,λ')
```
Out[13]: $(\lambda_1, \lambda_2, \lambda_3, \lambda)$

In [14]:
```
λ=matrix([λ1,λ2,λ3]).transpose();λ
```
Out[14]:
$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix}$$

In [15]:
```
B=AtA*λ;B
```
Out[15]:
$$\begin{pmatrix} 9\,\lambda_1 + 9\,\lambda_2 \\ 9\,\lambda_1 + 16\,\lambda_2 - 14\,\lambda_3 \\ -14\,\lambda_2 + 31\,\lambda_3 \end{pmatrix}$$

In [16]:
```
equations = [B[0][0]==Atx[0][0], B[1][0]==Atx[1][0], B[2][0]==Atx[2][0]];eq
uations
```
Out[16]: $[9\,\lambda_1 + 9\,\lambda_2 = (-9)\,, 9\,\lambda_1 + 16\,\lambda_2 - 14\,\lambda_3 = (-31)\,, -14\,\lambda_2 + 31\,\lambda_3 = 47]$

In [17]:
```
solutions = solve(equations, [λ1,λ2,λ3]);solutions
```
Out[17]:
$$\left[\left[\lambda_1 = \left(\frac{1}{7}\right), \lambda_2 = \left(-\frac{8}{7}\right), \lambda_3 = 1\right]\right]$$

In [18]:
```
λt=matrix([solutions[0][0].rhs(),solutions[0][1].rhs(), solutions[0][2].rhs
()]).transpose();λt
```
Out[18]:
$$\begin{pmatrix} \frac{1}{7} \\ -\frac{8}{7} \\ 1 \end{pmatrix}$$

The projection of $\pi_U(x)$ of x onto U can be computed via $\pi_U(x) = A\lambda$

In [19]:
```
ΠuX = A*λt; ΠuX
```
Out[19]:
$$\begin{pmatrix} -\frac{29}{7} \\ \frac{51}{7} \\ \frac{1}{7} \\ \frac{22}{7} \\ -1 \end{pmatrix}$$

(d)

The projection error in question (c) is the norm of the difference vector between the original vector and its projection onto U. That is $\|x - \pi_U(x)\|$

In [20]:
```
x=vector(QQ,[-1,9,-1,4,1]);x
```
Out[20]:
$$(-1, 9, -1, 4, 1)$$

In [21]:
```
px=Pπ*x;px
```
Out[21]:
$$\left(-\frac{29}{7}, \frac{51}{7}, \frac{1}{7}, \frac{22}{7}, -1\right)$$

In [22]:
```
analytical = norm(x-px);analytical
```
Out[22]:
$$2\sqrt{\frac{33}{7}}$$

In [23]:
```
numerical= n(analytical);numerical
```
Out[23]:
$$4.34248118673448$$

[DAMA 50] Written Assignment 2

Panagiotis Paltsokas - std163861

Problem 8

Given a non-zero vector $v \in \mathbb{R}^m$ the Householder matrix P is defined as

$$P = \mathbb{1} - \beta v v^T \ , \ \beta = \frac{2}{v^T v}$$

where $\mathbb{1}$ is the m-by-m identity matrix. If a vector $x \in \mathbb{R}^m$ is multiplied by P, then it is reflected in the hyperplane $\{\text{span}(v)\}^{\perp}$. Given $v = (1, -1, 1)^T$, use `Sagemath` to
(a) compute the associated Householder matrix P as well as P^2;
(b) compute the vector $x' = Px$, where $x = (1, 1, 3)^T$;
(c) compute and the lengths of both x and x' and comment on the result;
(d) plot the vectors v, x, x' in a common 3-dimensional plot.

In [3]:
```
restore()
```

In [4]:
```
%display latex
```

(a)

Define vector $v = (1, -1, 1)^T$

In [5]:
```
v=vector([1,-1,1])
```

In [6]:
```
id3=identity_matrix(3)
```

Compute the Householder matrix P which is defined as $P = 1 - \beta v v^T$

In [7]:
```
P=id3-2/(v.row()*v.column())[0][0]*v.column()*v.row();P
```

Out[7]:

$$\left(\begin{array}{rrr} \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{array}\right)$$

Verify that P is symmetrical

In [8]:
```
P==P.transpose()
```

Out[8]: True

Compute P^2

In [9]:
```
P**2
```

Out[9]:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We notice that P is a square, symmetrical matrix which implies that P and P^T are equal. We also see that $P^2 = I_{3x3} \Rightarrow P \cdot P = I_{3x3} \Rightarrow P \cdot P^T = I_{3x3}$ which implies that P is orthogonal.

(b)

First we define vector $x = (1, 1, 3)^T$

In [10]:
```
x=vector([1,1,3]);x
```

Out[10]: $(1, 1, 3)$

Compute vector $x' = Px$

In [11]:
```
x2=P*x.column();vector(x2.list())
```

Out[11]: $(-1, 3, 1)$

(c)

Compute the length of x

In [12]:
```
norm(x)
```

Out[12]: $\sqrt{11}$

Compute the length of x'

In [13]:
```
norm(vector(x2.list()))
```

Out[13]: $\sqrt{11}$

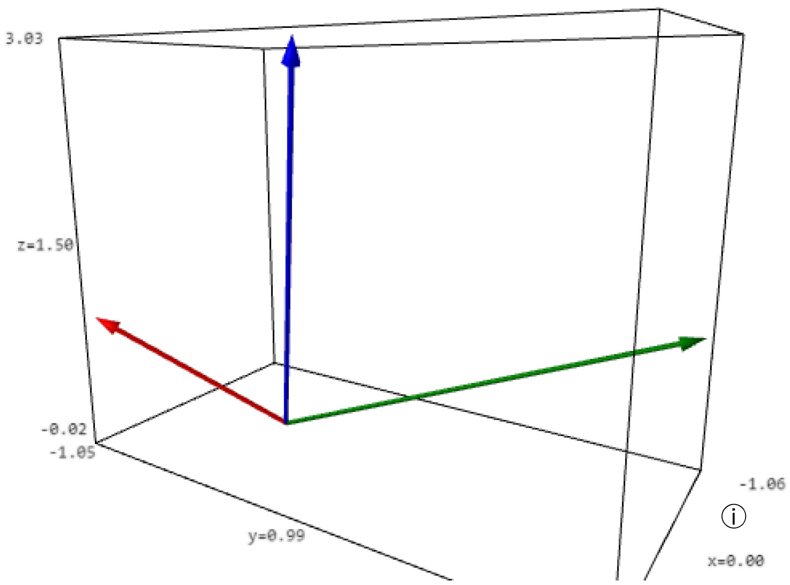
The lengths of x and x' are equal, which is to be expected since the length of a vector x is not changed when transforming it using an orthogonal matrix such as P.

(d)

In [14]:

```
arrows= arrow((0,0,0),(1,-1,1), color = "red")+\
arrow((0,0,0),(1,1,3), color = "blue")+\
arrow((0,0,0),(-1,3,1), color = "green");arrows
```

Out[14]:



[DAMA 50] Written Assignment 2

Panagiotis Paltsokas - std163861

Problem 9

Using by-hand calculation,
(a) find the 3 by 3 real symmetric matrix S given that

$$x^T S\, x = 4(x_1 - x_2 + 2x_3)^2$$

given $x = (x_1, x_2, x_3)^T$.
(b) Is S positive definite ? Is it positive semi-definite?

(a) Let S be a 3 by 3 real symmetric matrix. Then for $a,b,c,d,e,f \in \mathbb{R}$ we are given that

$$x^T S\, x = 4\left(x_1 - x_2 + 2x_3\right)^2 \Leftrightarrow \left(x_1\ x_2\ x_3\right)\cdot\begin{pmatrix} a & d & e \\ d & b & f \\ e & f & c \end{pmatrix}\cdot\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 4\left(x_1 - x_2 + 2x_3\right)^2 \Leftrightarrow$$

$$\Leftrightarrow \left(ax_1 + dx_2 + ex_3 \quad dx_1 + bx_2 + fx_3 \quad ex_1 + fx_2 + cx_3\right)\cdot\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} =$$

$$= 4\left(x_1^{\,2} + x_2^{\,2} + 4x_3^{\,2} - 2x_1x_2 - 4x_2x_3 + 4x_1x_3\right) \Leftrightarrow$$

$$\Leftrightarrow ax_1^{\,2} + dx_1x_2 + ex_1x_3 + dx_1x_2 + bx_2^{\,2} + fx_2x_3 + ex_1x_3 + fx_2x_3 + cx_3^{\,2} =$$
$$= 4x_1^{\,2} + 4x_2^{\,2} + 16x_3^{\,2} - 8x_1x_2 - 16x_2x_3 + 16x_1x_3 \Leftrightarrow$$

$$\Leftrightarrow ax_1^{\,2} + bx_2^{\,2} + cx_3^{\,2} + 2dx_1x_2 + 2ex_1x_3 + 2fx_2x_3 =$$
$$= 4x_1^{\,2} + 4x_2^{\,2} + 16x_3^{\,2} - 8x_1x_2 - 16x_2x_3 + 16x_1x_3$$

The two polynomials are equal if and only if they have the same degree and corresponding terms have equal coefficients. Therefore

$$\begin{cases} a = 4 \\ b = 4 \\ c = 16 \\ 2d = -8 \iff d = -4 \\ 2e = 16 \iff e = 8 \\ 2f = -16 \iff f = -8 \end{cases}$$

So the 3 by 3 real symmetric matrix S is:

$$S = \begin{pmatrix} 4 & -4 & 8 \\ -4 & 4 & -8 \\ 8 & -8 & 16 \end{pmatrix}$$

(b)

If matrix S was positive definite then $\forall x \in \mathbb{R}^3 \setminus \{0\}: \ \ x^T S x > 0$.

We can see that for $x = (0,2,1)^T$ we get $x^T S x = 0$.

Therefore, matrix S is **not** positive definite.

However, since $x^T S x = 4(x_1 - x_2 + 2x_3)^2 \geq 0, \ \ \forall x_1, x_2, x_3 \in \mathbb{R}$,

Matrix S is positive semi-definite.

[DAMA 50] Written Assignment 2

Panagiotis Paltsokas - std163861

Problem 10

Consider the vector space $\mathbb{R}^4$ with a Euclidean inner product.
You are given three vectors $(1,0,1,0)^T$, $(2,2,0,-3)^T$, $(0,3,6,5)^T$. Using by-hand calculation
(a) verify that these vectors are linearly independent;
(b) apply the Gram–Schmidt procedure to transform the above basis vectors to obtain an orthonormal basis.

(a) We compute the column matrix A of the given vectors and we put A in reduced row echelon form. With elementary row operations we get:

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 3 \\ 1 & 0 & 6 \\ 0 & -3 & 5 \end{pmatrix} \xrightarrow{R3=R3-R1} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 3 \\ 0 & -2 & 6 \\ 0 & -3 & 5 \end{pmatrix} \xrightarrow[R4=R4+\frac{3}{2}R2]{R3=R3+R2} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & 9 \\ 0 & 0 & \frac{19}{2} \end{pmatrix} \xrightarrow[R4=\frac{2}{19}\cdot R4]{R2=\frac{1}{2}\cdot R2,\ R3=\frac{1}{9}\cdot R3} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow[R4=R4-R3]{R2=R2-\frac{2}{3}\cdot R3} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R1=R1-2R2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Every column is a pivot column which implies that the given vectors are linearly independent. So a basis for the column space of A is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 6 \\ 5 \end{bmatrix} \right\}$

(b)

$$v_1 = (1, 0, 1, 0), \quad v_2 = (2, 2, 0, -3), \quad v_3 = (0, 3, 6, 5)$$

$$\boldsymbol{w_1} = v_1 = (\mathbf{1},\ \mathbf{0},\ \mathbf{1},\ \mathbf{0}) \quad \text{with } \|w_1\| = \sqrt{1^2+0+1^2+0} = \sqrt{2}$$

$$\boldsymbol{w_2} = v_2 - \frac{v_2 \cdot w_1}{\|w_1\|^2}\cdot w_1 = (2,\ 2,\ 0,\ -3) - \frac{(2,\ 2,\ 0,\ -3)\cdot(1,\ 0,\ 1,\ 0)}{1^2+0+1^2+0}\cdot(1,\ 0,\ 1,\ 0) =$$

$$= (2,\ 2,\ 0,\ -3) - \frac{2+0+0+0}{2}\cdot(1,\ 0,\ 1,\ 0) = (2,\ 2,\ 0,\ -3) - (1,\ 0,\ 1,\ 0) =$$

$$= (\mathbf{1},\ \mathbf{2},\ -\mathbf{1},\ -\mathbf{3})$$

where $\|w_2\| = \sqrt{1^2+2^2+(-1)^2+(-3)^2} = \sqrt{1+4+1+9} = \sqrt{15}$

$$\boldsymbol{w_3} = v_3 - \frac{v_3 \cdot w_1}{\|w_1\|^2}\cdot w_1 - \frac{v_3 \cdot w_2}{\|w_2\|^2}\cdot w_2 = (0,\ 3,\ 6,\ 5) - \frac{(0,\ 3,\ 6,\ 5)\cdot(1,\ 0,\ 1,\ 0)}{2}\cdot(1,\ 0,\ 1,\ 0) -$$

$$- \frac{(0,\ 3,\ 6,\ 5)\cdot(1,\ 2,\ -1,\ -3)}{1+4+1+9}\cdot(1,\ 2,\ -1,\ -3) =$$

$$= (0,\ 3,\ 6,\ 5) - \frac{0+0+6+0}{2}\cdot(1,\ 0,\ 1,\ 0) - \frac{0+6-6-15}{15}\cdot(1,\ 2,\ -1,\ -3) =$$

$$= (0,\ 3,\ 6,\ 5) - (3,\ 0,\ 3,\ 0) + (1,\ 2,\ -1,\ -3) = (-3,\ 3,\ 3,\ 5) + (1,\ 2,\ -1,\ -3) =$$

$$= (-\mathbf{2},\ \mathbf{5},\ \mathbf{2},\ \mathbf{2})$$

where $\|w_3\| = \sqrt{(-2)^2+5^2+2^2+2^2} = \sqrt{4+25+4+4} = \sqrt{37}$

Verifying the orthogonality:

$$w_1 \cdot w_2 = (1,\ 0,\ 1,\ 0)\cdot(1,\ 2,\ -1,\ -3) = 1+0-1+0 = 0$$

which implies that $w_1 \perp w_2$

$$w_2 \cdot w_3 = (1,\ 2,\ -1,\ -3)\cdot(-2,\ 5,\ 2,\ 2) = -2+10-2-6 = -10+10 = 0$$

which implies that $w_2 \perp w_3$

$$w_1 \cdot w_3 = (1, 0, 1, 0) \cdot (-2, 5, 2, 2) = -2 + 0 + 2 + 0 = 0$$

which implies that $w_1 \perp w_3$.

Indeed $w_1,\ w_2,\ w_3$ is an orthogonal basis.

Now

$$u_1 = \frac{1}{\sqrt{2}}(1, 0, 1, 0) = \left(\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}, 0\right)$$

$$u_2 = \frac{1}{\sqrt{15}}(1, 2, -1, -3) = \left(\frac{\sqrt{15}}{15}, \frac{2\sqrt{15}}{15}, -\frac{\sqrt{15}}{15}, -\frac{\sqrt{15}}{5}\right)$$

$$u_3 = \frac{1}{\sqrt{37}}(-2, 5, 2, 2) = \left(-\frac{2\sqrt{37}}{37}, \frac{5\sqrt{37}}{37}, \frac{2\sqrt{37}}{37}, \frac{2\sqrt{37}}{37}\right)$$

is an orthonormal basis since $u_1 \perp u_2$, $u_2 \perp u_3$, $u_1 \perp u_3$ and $\|u_1\| = \|u_2\| = \|u_3\| = 1$.