DAMA 50

Written Assignment III

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[DAMA 50] Written Assignment 3

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Problem 7

Consider the matrix

$$A = \begin{pmatrix} -2 & 6 & -3 \\ 6 & 3 & 2 \\ -3 & 2 & a \end{pmatrix} \tag{1.1}$$

where a is a real parameter and $x = (1, 1, 1)^T$.
Consider the case $a = -18$. Using `Sagemath`
(a) compute the eigenvalues and the eigenvectors of A;
(b) normalize the eigenvectors of A;
(c) compute (numerically with 3 significant digits) the components of x in the orthonormal basis of question (b).

Consider the case $a = 6$. Using `Sagemath`
(d) compute the eigenvalues and the eigenvectors of A;
(e) what is the algebraic multiplicity of each eigenvalue?
(f) are the eigenvectors orthogonal ? If not find an orthonormal basis.
(g) compute (numerically with 3 significant digits) the components of x in the orthonormal basis of question (f).

In [1]:
```
restore()
```

In [2]:
```
%display latex
```

For $a = -18$

In [3]:
```
A=matrix([[-2,6,-3],[6,3,2],[-3,2,-18]]);A
```

Out[3]:

$$\begin{pmatrix} -2 & 6 & -3 \\ 6 & 3 & 2 \\ -3 & 2 & -18 \end{pmatrix}$$

In [4]:
```
x=vector([1,1,1]);x.column()
```
Out[4]:
$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

(a)

Compute the diagonal matrix of eigenvalues eA which bears all the eigenvalues in its diagonal and the right eigenmatrix vA with all the eigenvectors as its columns

In [5]:
```
eA,vA=A.eigenmatrix_right();
```
In [6]:
```
eA
```
Out[6]:
$$\begin{pmatrix} 7 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -19 \end{pmatrix}$$

In [7]:
```
vA
```
Out[7]:
$$\begin{pmatrix} 1 & 1 & 1 \\ \frac{3}{2} & -\frac{2}{3} & -\frac{2}{3} \\ 0 & -\frac{1}{3} & \frac{13}{3} \end{pmatrix}$$

Extract the eigenvalues

In [8]:
```
Aeigenvalues=A.eigenvalues();Aeigenvalues
```
Out[8]: $[7, -5, -19]$

Matrix A eigenvalues are : $\lambda_1 = 7, \lambda_2 = -5, \lambda_3 = -19$

Extract the eigenvectors

In [9]:
```
A.eigenvectors_right()
```
Out[9]:
$$\left[\left(7, \left[\left(1, \frac{3}{2}, 0\right)\right], 1\right), \left(-5, \left[\left(1, -\frac{2}{3}, -\frac{1}{3}\right)\right], 1\right), \left(-19, \left[\left(1, -\frac{2}{3}, \frac{13}{3}\right)\right], 1\right)\right]$$

In [10]:
```
Ae1=vector(vA.column(0))
Ae2=vector(vA.column(1))
Ae3=vector(vA.column(2));Ae1,Ae2,Ae3
```
Out[10]:
$$\left(\left(1, \frac{3}{2}, 0\right), \left(1, -\frac{2}{3}, -\frac{1}{3}\right), \left(1, -\frac{2}{3}, \frac{13}{3}\right)\right)$$

Matrix A eigenvectors are : $\vec{v_1} = [1, \frac{3}{2}, 0]^T \quad \vec{v_2} = [1, -\frac{2}{3}, -\frac{1}{3}]^T \quad \vec{v_3} = [1, -\frac{2}{3}, \frac{13}{3}]^T$

(b)

Normalize the eigenvectors

In [11]:
```
An1=Ae1.normalized()
An2=Ae2.normalized()
An3=Ae3.normalized();An1,An2,An3
```
Out[11]:
$$\left(\left(\frac{2}{13}\sqrt{13},\ \frac{3}{13}\sqrt{13},\ 0\right),\left(\frac{3}{14}\sqrt{14},\ -\frac{1}{7}\sqrt{14},\ -\frac{1}{14}\sqrt{14}\right),\left(\frac{3}{182}\sqrt{182},\ -\frac{1}{91}\sqrt{182},\ \frac{1}{14}\sqrt{182}\right)\right)$$

The vectors An1, An2, An3 form an orthonormal basis

(c)

To compute the components of x in the ONB mentioned above, we take its dot product with each vector of the ONB

In [12]:
```
x_comp = [x.dot_product(An1), x.dot_product(An2), x.dot_product(An3)];x_comp
```
Out[12]:
$$\left[\frac{5}{13}\sqrt{13}, 0, \frac{1}{13}\sqrt{182}\right]$$

In [13]:
```
x_comp_num = [x_comp[0].n(digits=3), x_comp[1].n(digits=3), x_comp[2].n(digits=3)];x_comp_num
```
Out[13]: $[1.39, 0.000, 1.04]$

(d)

For $a = 6$

In [14]:
```
A2=matrix([[-2,6,-3],[6,3,2],[-3,2,6]]);A2
```
Out[14]:
$$\begin{pmatrix} -2 & 6 & -3 \\ 6 & 3 & 2 \\ -3 & 2 & 6 \end{pmatrix}$$

Compute the diagonal matrix of eigenvalues $eA2$ which bears all the eigenvalues in its diagonal and the right eigenmatrix $vA2$ with all the eigenvectors as its columns

In [15]:
```
eA2,vA2=A2.eigenmatrix_right()
```
In [16]:
```
eA2
```
Out[16]:
$$\begin{pmatrix} -7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{pmatrix}$$

In [17]:
```
vA2
```
Out[17]:
$$\begin{pmatrix} 1 & 1 & 0 \\ -\frac{2}{3} & 0 & 1 \\ \frac{1}{3} & -3 & 2 \end{pmatrix}$$

Extract the eigenvalues

In [18]:
```
A2eigenvalues=A2.eigenvalues();A2eigenvalues
```
Out[18]: $[-7, 7, 7]$

Sort the eigenvalues in descending order

In [19]:
```
A2eigenvalues_ord = [A2eigenvalues[1],A2eigenvalues[2],A2eigenvalues[0]];A2eigenvalues_ord
```

Out[19]: $[7, 7, -7]$

Matrix A2 eigenvalues are : $\lambda_1 = 7$ with algebraic multiplicity 2 and $\lambda_2 = -7$

Extract the eigenvectors

In [20]:
```
A2eigen = A2.eigenvectors_right();A2eigen
```

Out[20]:
$$\left[\left(-7, \left[\left(1, -\frac{2}{3}, \frac{1}{3}\right)\right], 1\right), (7, [(1, 0, -3), (0, 1, 2)], 2)\right]$$

Store the eigenvectors in descending eigenvalue order

In [21]:
```
A2e1=vector(vA2.column(1))
A2e2=vector(vA2.column(2))
A2e3=vector(vA2.column(0));A2e1,A2e2,A2e3
```

Out[21]:
$$\left((1, 0, -3), (0, 1, 2), \left(1, -\frac{2}{3}, \frac{1}{3}\right)\right)$$

Matrix A eigenvectors are : $\overrightarrow{v_1} = [1, 0, -3]^T \quad \overrightarrow{v_2} = [0, 1, 2]^T \quad \overrightarrow{v_3} = [1, -\frac{2}{3}, \frac{1}{3}]^T$

(e)

In [22]:
```
print(f"Eigenvalue {A2eigen[0][0]} has an algebraic multiplicity of {A2eigen[0][2]}")
print(f"Eigenvalue {A2eigen[1][0]} has an algebraic multiplicity of {A2eigen[1][2]}")
```

```
Eigenvalue -7 has an algebraic multiplicity of 1
Eigenvalue 7 has an algebraic multiplicity of 2
```

(f)

Checking if all dot products among the eigenvectors are equal to 0

In [23]:
```
print(f"The dot produtcs are {A2e1*A2e2},{A2e1*A2e3},{A2e2*A2e3}")
```

```
The dot produtcs are -6,0,0
```

We notice that the eigenvectors $\overrightarrow{v_1} = [1, 0, -3]^T$ and $\overrightarrow{v_2} = [0, 1, 2]^T$ have a dot product of -6 which means that they are not orthogonal

Compute an orthogonal basis using the Gram-Schmidt orthogonalization process

In [24]:
```
u1 = A2e1
u2 = A2e2 - ((A2e2.dot_product(u1)) / (u1.dot_product(u1))) * u1
u3 = A2e3 - ((A2e3.dot_product(u1)) / (u1.dot_product(u1))) * u1 - ((A2e3.dot_product(u2)) / (u2.d
ot_product(u2))) * u2;u1,u2,u3
```

Out[24]:
$$\left((1, 0, -3), \left(\frac{3}{5}, 1, \frac{1}{5}\right), \left(1, -\frac{2}{3}, \frac{1}{3}\right)\right)$$

Normalize the vectors of the orthogonal basis to get an orthonormal basis

In [25]:
```
A2_orthonormal_basis=[u1.normalized(),u2.normalized(),u3.normalized()];A2_orthonormal_basis
```

Out[25]:

$$\left[\left(\frac{1}{10}\sqrt{10},\, 0,\, -\frac{3}{10}\sqrt{10}\right), \left(\frac{3}{7}\sqrt{\frac{7}{5}},\, \frac{5}{7}\sqrt{\frac{7}{5}},\, \frac{1}{7}\sqrt{\frac{7}{5}}\right), \left(\frac{3}{14}\sqrt{14},\, -\frac{1}{7}\sqrt{14},\, \frac{1}{14}\sqrt{14}\right)\right]$$

(g)

To compute the components of x in the ONB mentioned above, we take its dot product with each vector of the ONB

In [26]:
```
x_comp2 = [x.dot_product(u1.normalized()), x.dot_product(u2.normalized()), x.dot_product(u3.normal
ized())];x_comp2
```

Out[26]:

$$\left[-\frac{1}{5}\sqrt{10},\, \frac{9}{7}\sqrt{\frac{7}{5}},\, \frac{1}{7}\sqrt{14}\right]$$

In [27]:
```
x_comp2_num = [x_comp2[0].n(digits=3), x_comp2[1].n(digits=3), x_comp2[2].n(digits=3)];x_comp2_num
```

Out[27]: $[-0.632, 1.52, 0.534]$

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Problem 8

Consider the 6×2 matrix A with elements

$$a_{ij} = i^2 + 2ij,\ \ i = 1, \ldots, 6,\ \ j = 1, 2$$

Using `sagemath`
(a) Compute A;
(b) Compute the SVD of A and print the left and right singular vector matrices and the singular value matrix using 3 significant digits;
(c) Compute the rank-1 approximation $A1$ of A and print using 3 significant digits;
(d) Compute the Frobenius norm of the difference $A - A1$ (e) Compute the rank-2 approximation $A2$ of A and print using 3 significant digits. Compute the Frobenius norm of the difference $A - A2$. Comment on these results.

Solution

In [28]:
```
restore()
```

In [29]:
```
%display latex
```

(a)

Compute matrix A

In [30]:
```
A=matrix(QQ,6,2, lambda i,j: ((i+1)^2+2*(i+1)*(j+1)));A
```

Out[30]:

$$\begin{pmatrix} 3 & 5 \\ 8 & 12 \\ 15 & 21 \\ 24 & 32 \\ 35 & 45 \\ 48 & 60 \end{pmatrix}$$

(b)

Compute the left singular vector matrix U, the singular value matrix S and the right singular vector matrix V for the SVD of A

In [31]:
```
U,S,V = (A.change_ring(RDF)).SVD()
```

In [32]:
```
U.n(digits=3)
```

Out[32]:

$$\begin{pmatrix} -0.0536 & -0.340 & -0.293 & -0.398 & -0.506 & -0.618 \\ -0.133 & -0.511 & -0.480 & -0.286 & 0.0854 & 0.634 \\ -0.239 & -0.514 & 0.802 & -0.159 & -0.0659 & 0.0796 \\ -0.370 & -0.349 & -0.174 & 0.823 & -0.152 & -0.0992 \\ -0.528 & -0.0152 & -0.104 & -0.177 & 0.731 & -0.381 \\ -0.712 & 0.487 & 0.0107 & -0.160 & -0.419 & 0.235 \end{pmatrix}$$

In [33]:
```
S.n(digits=3)
```

Out[33]:

$$\begin{pmatrix} 108. & 0.000 \\ 0.000 & 2.07 \\ 0.000 & 0.000 \\ 0.000 & 0.000 \\ 0.000 & 0.000 \\ 0.000 & 0.000 \end{pmatrix}$$

In [34]:
```
V.n(digits=3)
```

Out[34]:

$$\begin{pmatrix} -0.614 & 0.789 \\ -0.789 & -0.614 \end{pmatrix}$$

(c)

We construct the matrix $A1$ as $A_1 = u_1 v_1^T$ which is formed by the outer product of the first orthogonal column vector of U and V

In [35]:
```
u1=U[:,0];u1.n(digits=3)
```
Out[35]:
$$\begin{pmatrix} -0.0536 \\ -0.133 \\ -0.239 \\ -0.370 \\ -0.528 \\ -0.712 \end{pmatrix}$$

In [36]:
```
v1T=V[:,0].transpose();v1T.n(digits=3)
```
Out[36]:
$$\begin{pmatrix} -0.614 & -0.789 \end{pmatrix}$$

In [37]:
```
A1=u1*v1T;A1.n(digits=3)
```
Out[37]:
$$\begin{pmatrix} 0.0329 & 0.0423 \\ 0.0818 & 0.105 \\ 0.147 & 0.188 \\ 0.228 & 0.292 \\ 0.324 & 0.417 \\ 0.437 & 0.561 \end{pmatrix}$$

We compute the rank-1 approximation $\hat{A}(1) := \sigma_1 A_1$ where σ_1 is the first singular value

In [38]:
```
σ1=S[0][0];σ1.n(digits=3)
```
Out[38]: $108.$

In [39]:
```
A1aprx=σ1*A1;A1aprx.n(digits=3)
```
Out[39]:
$$\begin{pmatrix} 3.56 & 4.57 \\ 8.84 & 11.3 \\ 15.8 & 20.3 \\ 24.6 & 31.6 \\ 35.0 & 45.0 \\ 47.2 & 60.6 \end{pmatrix}$$

(d)

Compute the difference $A - A_1$ and store it in a matrix $B1$

In [40]:
```
B1=A-A1aprx;B1.n(digits=3)
```
Out[40]:
$$\begin{pmatrix} -0.556 & 0.433 \\ -0.837 & 0.651 \\ -0.842 & 0.655 \\ -0.571 & 0.445 \\ -0.0249 & 0.0194 \\ 0.797 & -0.620 \end{pmatrix}$$

Compute the Frobenius norm of $B1$

In [41]:
```
frobnorm1 = B1.norm('frob');frobnorm1
```
Out[41]: 2.0746347947807724

In [42]:
```
frobnorm1.n(digits=3)
```
Out[42]: 2.07

(e)

For the rank-2 approximation we will need to compute $\hat{A}(2) := \sigma_1 A_1 + \sigma_2 A_2$

In [43]:
```
u2=U[:,1];u2.n(digits=3)
```
Out[43]:
$$\begin{pmatrix} -0.340 \\ -0.511 \\ -0.514 \\ -0.349 \\ -0.0152 \\ 0.487 \end{pmatrix}$$

In [44]:
```
v2T=V[:,1].transpose();v2T.n(digits=3)
```
Out[44]: $\begin{pmatrix} 0.789 & -0.614 \end{pmatrix}$

In [45]:
```
A2=u2*v2T;A2.n(digits=3)
```
Out[45]:
$$\begin{pmatrix} -0.268 & 0.209 \\ -0.403 & 0.314 \\ -0.406 & 0.316 \\ -0.275 & 0.214 \\ -0.0120 & 0.00936 \\ 0.384 & -0.299 \end{pmatrix}$$

In [46]:
```
σ2=S[1][1];σ2.n(digits=3)
```
Out[46]: 2.07

$\hat{A}(2) := \sigma_1 A_1 + \sigma_2 A_2$

In [47]:
```
A2aprx = σ1*A1+σ2*A2;A2aprx.n(digits=3)
```
Out[47]:
$$\begin{pmatrix} 3.00 & 5.00 \\ 8.00 & 12.0 \\ 15.0 & 21.0 \\ 24.0 & 32.0 \\ 35.0 & 45.0 \\ 48.0 & 60.0 \end{pmatrix}$$

Compute the difference $A - A_2$ and store it in a matrix $B2$

In [48]:
```
B2=A-A2aprx;B2.n(digits=3)
```
Out[48]:
$$\begin{pmatrix} 2.66 \times 10^{-15} & 4.44 \times 10^{-15} \\ 2.66 \times 10^{-15} & 3.55 \times 10^{-15} \\ 3.55 \times 10^{-15} & 3.55 \times 10^{-15} \\ 3.55 \times 10^{-15} & 3.55 \times 10^{-15} \\ 7.11 \times 10^{-15} & 7.11 \times 10^{-15} \\ 7.11 \times 10^{-15} & 7.11 \times 10^{-15} \end{pmatrix}$$

Compute the Frobenius norm of $B2$

In [49]:
```
frobnorm2 = B2.norm('frob');frobnorm2
```
Out[49]: $1.7290988493396307 \times 10^{-14}$

In [50]:
```
frobnorm2.n(digits=3)
```
Out[50]: 1.73×10^{-14}

Comment

The matrices $B_1 = A - A_1$ and $B_2 = A - A_2$ capture the information that cannot be represented by the rank-1 and rank-2 approximation respectively (the error). The Frobenius norm of a matrix A, which is defined by $\|A\|_F = \left(\sum_{i=1}^{n} \sum_{j=1}^{n} a_{i,j}^2\right)^{\frac{1}{2}}$ can give us an idea of the magnitude of the errors in the matrices. We observe tha the error that corresponds to the rank-2 approximation is significantly smaller than the error that coresponds to the rank-1 approximation, which means that the rank-2 captures the essential features of the original matrix better than the rank-1 approximation.

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Problem 9

Given the eigenvalues $\lambda_1 = 9, \lambda_2 = 0, \lambda_3 = -9$ and the associated eigenvectors $e_1 = \frac{1}{3}(-2, 2, 1)$, $e_2 = \frac{1}{3}(2, 1, 2)$, $e_3 = \frac{1}{3}(-1, -2, 2)$ of a real symmetric matrix A, compute A using by-hand calculation.

Theorem 4.20 (Eigendecomposition). *A square matrix $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ can be factored into*

$$\boldsymbol{A} = \boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{-1}, \tag{4.55}$$

where $\boldsymbol{P} \in \mathbb{R}^{n \times n}$ and $\boldsymbol{D}$ is a diagonal matrix whose diagonal entries are the eigenvalues of $\boldsymbol{A}$, if and only if the eigenvectors of $\boldsymbol{A}$ form a basis of $\mathbb{R}^n$.

Theorem 4.21. *A symmetric matrix $\boldsymbol{S} \in \mathbb{R}^{n \times n}$ can always be diagonalized.*

Since we have a real symmetric matrix $\boldsymbol{A}$, it will be a square matrix and given its eigenvalues and eigenvectors we can compute the matrices $\boldsymbol{P}$ and $\boldsymbol{D}$ mentioned in Theorem (4.20) such that:

$$\mathrm{P} = \begin{pmatrix} -2/3 & 2/3 & -1/3 \\ 2/3 & 1/3 & -2/3 \\ 1/3 & 2/3 & 2/3 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} -2 & 2 & -1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{pmatrix} \quad \text{and } \mathrm{D} = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -9 \end{pmatrix} = 9\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Therefore, we can compute A which will be:

$$\mathrm{A} = \mathrm{P} \cdot \mathrm{D} \cdot \mathrm{P}^{\mathrm{T}} = \frac{1}{3} \cdot \begin{pmatrix} -2 & 2 & -1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{pmatrix} \cdot 9 \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \cdot \frac{1}{3} \cdot \begin{pmatrix} -2 & 2 & 1 \\ 2 & 1 & 2 \\ -1 & -2 & 2 \end{pmatrix} =$$

$$= \frac{9}{9} \cdot \begin{pmatrix} -2 & 2 & -1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} -2 & 2 & 1 \\ 2 & 1 & 2 \\ -1 & -2 & 2 \end{pmatrix} = \begin{pmatrix} -2 & 0 & 1 \\ 2 & 0 & 2 \\ 1 & 0 & -2 \end{pmatrix} \cdot \begin{pmatrix} -2 & 2 & 1 \\ 2 & 1 & 2 \\ -1 & -2 & 2 \end{pmatrix} =$$

$$= \begin{pmatrix} 3 & -6 & 0 \\ -6 & 0 & 6 \\ 0 & 6 & -3 \end{pmatrix}$$

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Problem 10

Given the matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, answer the following questions, by providing all the necessary by-hand calculations:
(i) Compute the characteristic polynomial and the eigenvalues of the matrix A.
(ii) Compute the eigenvectors of the matrix A.
(iii) Find a general formula for the matrix A^n with $n \in \mathbb{Z}^+$, utilizing the diagonalizable nature of the original matrix A.

(i) We are given matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

The characteristic polynomial will be $p_A(\lambda) := \det(A - \lambda I)$ where

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 1 = \lambda^2 - 4\lambda + 4 - 1 = \lambda^2 - 4\lambda + 3$$

$$p_A(\lambda) = \lambda^2 - 4\lambda + 3$$

The eigenvalues of $\boldsymbol{A}$ are the roots of the characteristic polynomial.

$$p_A(\lambda) = 0 \Leftrightarrow \lambda^2 - 4\lambda + 3 = 0 \Leftrightarrow (\lambda - 3)(\lambda - 1) = 0 \Leftrightarrow \lambda = 3 \text{ or } \lambda = 1$$

(ii) We find the eigenvectors that correspond to these eigenvalues by looking at vectors $\boldsymbol{x}$ such that:

$$\begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix} x = 0$$

<u>For **λ=3**</u> we will solve the homogenous system:

$$\begin{pmatrix} 2-3 & 1 \\ 1 & 2-3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \Leftrightarrow \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \Leftrightarrow \begin{cases} -x_1 + x_2 = 0 \\ x_1 - x_2 = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x_1 = x_2 \\ x_1 = x_2 \end{cases} \Leftrightarrow \begin{cases} x_1 = x_2 = t \\ x_2 = t \end{cases} \Leftrightarrow \vec{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} t$$

so we obtain a solution space $E_3 = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$

<u>For **λ=1**</u> we will solve the homogenous system:

$$\begin{pmatrix} 2-1 & 1 \\ 1 & 2-1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \Leftrightarrow \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \Leftrightarrow \begin{cases} x_1 + x_2 = 0 \\ x_1 + x_2 = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x_1 = -x_2 \\ x_1 = -x_2 \end{cases} \Leftrightarrow \begin{cases} x_1 = -x_2 = -t \\ \quad x_2 = t \end{cases} \Leftrightarrow \vec{x} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} t$$

so we obtain a solution space $E_1 = \text{span}\left\{\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right\}$

Therefore, the eigenvalues of ***A*** are $\lambda_1 = 3$ and $\lambda_2 = 1$ (the roots of the characteristic polynomial), and the associated eigenvectors are $\vec{e}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\vec{e}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

(iii)

Theorem 4.20 (Eigendecomposition)**.** *A square matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ can be factored into*

$$\boldsymbol{A} = \boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{-1}\,, \tag{4.55}$$

where $\boldsymbol{P} \in \mathbb{R}^{n\times n}$ and $\boldsymbol{D}$ is a diagonal matrix whose diagonal entries are the eigenvalues of $\boldsymbol{A}$, if and only if the eigenvectors of $\boldsymbol{A}$ form a basis of $\mathbb{R}^n$.

$\det(\vec{e}_1, \vec{e}_2) = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 1 + 1 = 2 \neq 0$. So $\vec{e}_1$ and $\vec{e}_2$ are linearly independent so they form a basis of $\mathbb{R}^2$. Therefore, according to theorem (4.55), the matrix ***A*** can be factored into $A = PDP^{-1}$

where $P = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$ and $P^{-1} = \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ since

$$\left(\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right) \overset{R2 \leftarrow R2 - R1}{=} \left(\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 1 \end{array}\right) \overset{R2 \leftarrow \frac{R2}{2}}{=} \left(\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 0 & 1 & -1/2 & 1/2 \end{array}\right) =$$

$$\overset{R1 \leftarrow R1 + R2}{=} \left(\begin{array}{cc|cc} 1 & 0 & 1/2 & 1/2 \\ 0 & 1 & -1/2 & 1/2 \end{array}\right)$$

Also according to the property (4.62) we can compute $\boldsymbol{A}^n$

- Diagonal matrices $\boldsymbol{D}$ can efficiently be raised to a power. Therefore, we can find a matrix power for a matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ via the eigenvalue decomposition (if it exists) so that

$$\boldsymbol{A}^k = (\boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{-1})^k = \boldsymbol{P}\boldsymbol{D}^k\boldsymbol{P}^{-1}\,. \tag{4.62}$$

Computing $\boldsymbol{D}^k$ is efficient because we apply this operation individually to any diagonal element.

$$A^n = P \cdot D^n \cdot P^{-1} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 3^n & 0 \\ 0 & 1^n \end{pmatrix} \cdot \frac{1}{2} \cdot \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2} \cdot \begin{pmatrix} 3^n & -1 \\ 3^n & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2} \cdot \begin{pmatrix} 3^n+1 & 3^n-1 \\ 3^n-1 & 3^n+1 \end{pmatrix}$$