DAMA 50

Written Assignment I

Submitted by:

Panagiotis Paltsokas

ID: std163861

[DAMA 50] Written Assignment 1

Panagiotis Paltsokas - std163861

Problem 7

Let A be a 8×8 matrix with elements $A_{ij} = j(-1)^i + i(-1)^j + k$ where $i, j = 1, \ldots, 8$, and k is the last digit of your academic ID.

Using `Sagemath`

(a) Compute A.

(b) Find the row reduced echelon form of AA^T.

(c) Find the rank of A.

(d) Compute the expression $\det(A^2 - A + \mathbb{1})$.

Problem 7 Solution

In [1]:

```
%display latex
```

(a)

In [2]:

```
A=matrix(QQ,8,8, lambda i,j: ((j+1)*(-1)^(i+1)+(i+1)*(-1)^(j+1)+1));A
```

Out[2]:

$$\begin{pmatrix} -1 & 0 & -3 & -2 & -5 & -4 & -7 & -6 \\ 0 & 5 & 2 & 7 & 4 & 9 & 6 & 11 \\ -3 & 2 & -5 & 0 & -7 & -2 & -9 & -4 \\ -2 & 7 & 0 & 9 & 2 & 11 & 4 & 13 \\ -5 & 4 & -7 & 2 & -9 & 0 & -11 & -2 \\ -4 & 9 & -2 & 11 & 0 & 13 & 2 & 15 \\ -7 & 6 & -9 & 4 & -11 & 2 & -13 & 0 \\ -6 & 11 & -4 & 13 & -2 & 15 & 0 & 17 \end{pmatrix}$$

(b)

In [3]:

```
Atrans=A.transpose();Atrans
```

Out[3]:

$$\begin{pmatrix} -1 & 0 & -3 & -2 & -5 & -4 & -7 & -6 \\ 0 & 5 & 2 & 7 & 4 & 9 & 6 & 11 \\ -3 & 2 & -5 & 0 & -7 & -2 & -9 & -4 \\ -2 & 7 & 0 & 9 & 2 & 11 & 4 & 13 \\ -5 & 4 & -7 & 2 & -9 & 0 & -11 & -2 \\ -4 & 9 & -2 & 11 & 0 & 13 & 2 & 15 \\ -7 & 6 & -9 & 4 & -11 & 2 & -13 & 0 \\ -6 & 11 & -4 & 13 & -2 & 15 & 0 & 17 \end{pmatrix}$$

In [4]:

```
Aprod=A*Atrans;Aprod
```

Out[4]:

$$\begin{pmatrix} 140 & -184 & 148 & -176 & 156 & -168 & 164 & -160 \\ -184 & 332 & -144 & 372 & -104 & 412 & -64 & 452 \\ 148 & -144 & 188 & -104 & 228 & -64 & 268 & -24 \\ -176 & 372 & -104 & 444 & -32 & 516 & 40 & 588 \\ 156 & -104 & 228 & -32 & 300 & 40 & 372 & 112 \\ -168 & 412 & -64 & 516 & 40 & 620 & 144 & 724 \\ 164 & -64 & 268 & 40 & 372 & 144 & 476 & 248 \\ -160 & 452 & -24 & 588 & 112 & 724 & 248 & 860 \end{pmatrix}$$

In [5]:

```
Aprod.rref()
```

Out[5]:

$$\begin{pmatrix} 1 & 0 & 0 & -1 & -1 & -2 & -2 & -3 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 2 & 2 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(c)

In [6]:

```
print(f'The rank of matrix A is {A.rank()}.')
```

The rank of matrix A is 3.

In [7]:

```
A.rref()
```

Out[7]:

$$\begin{pmatrix} 1 & 0 & 0 & -1 & -1 & -2 & -2 & -3 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 2 & 2 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Indeed the reduced row echelon form has 3 linearly independent vectors.

(d)

In [8]:

```
Identity8x8=identity_matrix(8,8);Identity8x8
```

Out[8]:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

In [15]:

```
B = A^2-A+Identity8x8;exp
```

Out[15]:

$$\begin{pmatrix} 142 & -184 & 151 & -174 & 161 & -164 & 171 & -154 \\ -184 & 328 & -146 & 365 & -108 & 403 & -70 & 441 \\ 151 & -146 & 194 & -104 & 235 & -62 & 277 & -20 \\ -174 & 365 & -104 & 436 & -34 & 505 & 36 & 575 \\ 161 & -108 & 235 & -34 & 310 & 40 & 383 & 114 \\ -164 & 403 & -62 & 505 & 40 & 608 & 142 & 709 \\ 171 & -70 & 277 & 36 & 383 & 142 & 490 & 248 \\ -154 & 441 & -20 & 575 & 114 & 709 & 248 & 844 \end{pmatrix}$$

In [18]:

```
print(f'The determinant of the matrix B is det(A^2-A+I) = {B.det()}')
```

```
The determinant of the matrix B is det(A^2-A+I) = 5051649
```

[DAMA 50] Written Assignment 1

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Problem 8

Assume the system of equations in x, y, z, t

$$\begin{aligned} ax + y + z + t &= 1 \\ x + ay + z + t &= b \\ x + y + az + t &= b^2 \\ x + y + z + at &= b^3 \end{aligned}$$

where a, b are integer parameters.
Using `sagemath`
(a) find a formal solution of the system in terms of a, b;
(b) determine the values of a, b for which the system has infinitely many solutions and the values of a, b for which the system is impossible;
(c) derive the full solution for $a =$ `the first digit of your academic ID`, $b =$ `the last digit of your academic ID`.

Problem 8 Solution

In [1]:

```
restore()
```

In [2]:

```
%display latex
```

In [3]:

```
x,y,z,t,a,b=var('x,y,z,t,a,b')
```

(a)

In [4]:

```
eq=[
a*x+y+z+t==1
, x+a*y+z+t==b
, x+y+a*z+t==b^2
, x+y+z+a*t==b^3]
eq
```

Out[4]:

$$\left[ax + t + y + z = 1, ay + t + x + z = b, az + t + x + y = b^2, at + x + y + z = b^3\right.$$

In [5]:

```
solution=solve(eq,[x,y,z,t]);solution
```

Out[5]:

$$\left[\left[x = -\frac{b^3 + b^2 - a + b - 2}{a^2 + 2\,a - 3}, y = -\frac{b^3 - ab + b^2 - 2\,b + 1}{a^2 + 2\,a - 3}, z = \frac{ab^2 - b^3 + 2\,b^2 -}{a^2 + 2\,a -}\right.\right.$$

(b)

As observed above, for $a = -3$ or $a = 1$, in the system's solution, the denominator (determinant) equals zero. For any value of b that makes the numerator equal to zero, the system has infinite solutions. For any value of b that makes the numerator different than zero, the system has no solutions.

For $a = -3$ the numerators are all equal to $-b^3 - b^2 - b - 1$

In [6]:

```
equations = []
for i in range(4):
    equation = solution[0][i].rhs().numerator().subs(a==-3) == 0
    equations.append(equation)
equations
```

Out[6]:

$$\left[-b^3 - b^2 - b - 1 = 0, -b^3 - b^2 - b - 1 = 0, -b^3 - b^2 - b - 1 = 0, -b^3 - b^2 - b\right.$$

In [7]:

```
solve(equations,b)
```

Out[7]:

$$[[b = i], [b = (-i)], [b = (-1)]]$$

RESULT :

For $a = -3$ and $b = -1$ the system has infinite solutions

For $a = -3$ and $b \neq -1$ the system has no solutions

--

For $a = 1$ the numerators are $-b^3 - b^2 - b + 3$, $-b^3 - b^2 + 3b - 1$, $-b^3 + 3b^2 - b - 1$ and $3b^3 - b^2 - b - 1$

In [8]:

```
equations2 = []
for i in range(4):
    equation = solution[0][i].rhs().numerator().subs(a==1) == 0
    equations2.append(equation)
equations2
```

Out[8]:

$\left[-b^3 - b^2 - b + 3 = 0, -b^3 - b^2 + 3\,b - 1 = 0, -b^3 + 3\,b^2 - b - 1 = 0, 3\,b^3 - b^2 -\right.$

In [9]:

```
solve(equations2, b)
```

Out[9]:

$[[b = 1]]$

RESULT :

For $a = 1$ and $b = 1$ the system has infinite solutions

For $a = 1$ and $b \neq 1$ the system has no solutions

--

(c)

My academic id is 163861 in which case, for $a = b = 1$ the system has infinite solutions. However i will solve it below for two random values of a and b that return a specific, full solution.

For $a = 1$ and $b = 1$

In [10]:

```
eq2=[
x+y+z+t==1
, x+y+z+t==1
, x+y+z+t==1
, x+y+z+t==1]
eq2
```

Out[10]:

$$[t + x + y + z = 1, t + x + y + z = 1, t + x + y + z = 1, t + x + y + z = 1]$$

In [11]:

```
solve(eq2,[x,y,z,t])
```

Out[11]:

$$[[x = -r_1 - r_2 - r_3 + 1, y = r_3, z = r_2, t = r_1]]$$

Infinite solutions in the form of $[x, y, z, t] = [-r_1 - r_2 - r_3 + 1, r_3, r_2, r_1]$

For $a = 2$ and $b = 3$ (two arbitrary values)

In [12]:

```
c_sol2 = []
for i in range(4):
    sol2 = solution[0][i].subs(a==2, b==3)
    c_sol2.append(sol2)
c_sol2
```

Out[12]:

$$[x = (-7), y = (-5), z = 1, t = 19]$$

Another solution for Problem 8 (b)

In [13]:

```
restore()
```

In [14]:

```
%display latex
```

In [15]:

```
x,y,z,t=var('x,y,z,t')
```

In [16]:

```
R.<a,b>=QQ[]
```

In [17]:

```
A=matrix([[a,1,1,1],[1,a,1,1],[1,1,a,1],[1,1,1,a]]);A
```

Out[17]:

$$\begin{pmatrix} a & 1 & 1 & 1 \\ 1 & a & 1 & 1 \\ 1 & 1 & a & 1 \\ 1 & 1 & 1 & a \end{pmatrix}$$

In [18]:

```
v=vector([1,b,b^2,b^3])
```

In [19]:

```
Aa=A.augment(v,subdivide=True);Aa
```

Out[19]:

$$\left(\begin{array}{cccc|c} a & 1 & 1 & 1 & 1 \\ 1 & a & 1 & 1 & b \\ 1 & 1 & a & 1 & b^2 \\ 1 & 1 & 1 & a & b^3 \end{array}\right)$$

In [20]:

```
A1=Aa.subs(a=1).echelon_form();A1
```

Out[20]:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & b-1 \\ 0 & 0 & 0 & 0 & b^2-1 \\ 0 & 0 & 0 & 0 & b^3-1 \end{pmatrix}$$

In [21]:

```
b=var('b')
```

In [22]:

```
eq3=[b-1==0, b**2-1==0, b**3-1==0];eq3
```

Out[22]:

$$\left[b - 1 = 0, b^2 - 1 = 0, b^3 - 1 = 0\right]$$

In [23]:

```
sola1b1=solve(eq3, b);sola1b1
```

Out[23]:

$$[[b = 1]]$$

In [24]:

```
A1.subs(b=1)
```

Out[24]:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

SOLUTION

For $a = 1$ and $b = 1$ the system has infinite solutions

Therefore, for $a = 1$ and $b \neq 1$ the system has no solutions

In [25]:

```
A2=Aa.subs(a=-3).echelon_form();A2
```

Out[25]:

$$\begin{pmatrix} 1 & 0 & 0 & -1 & -\frac{1}{4}b^2 - \frac{1}{4}b - \frac{1}{2} \\ 0 & 1 & 0 & -1 & -\frac{1}{4}b^2 - \frac{1}{2}b - \frac{1}{4} \\ 0 & 0 & 1 & -1 & -\frac{1}{2}b^2 - \frac{1}{4}b - \frac{1}{4} \\ 0 & 0 & 0 & 0 & b^3 + b^2 + b + 1 \end{pmatrix}$$

In [26]:

```
eq4=[b**3+b**2+b+1];eq4
```

Out[26]:

$$\left[b^3 + b^2 + b + 1\right]$$

In [27]:

```
solve(eq4,b)
```

Out[27]:

$$[b = (-i)\,, b = i, b = (-1)]$$

In [28]:

```
A2.subs(b=-1)
```

Out[28]:

$$\begin{pmatrix} 1 & 0 & 0 & -1 & -\frac{1}{2} \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

SOLUTION

For $a = -3$ and $b = -1$ the system has infinite solutions

Therefore, for $a = -3$ and $b \neq -1$ the system has no solutions

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Problem 9

Using by hand calculation

(a) utilizing Gaussian elimination compute the rank of the matrix

$$A = \begin{pmatrix} -1 & 1 & 1 \\ 2 & 1 & 0 \\ 4 & -1 & -2 \end{pmatrix}$$

(b) bring the matrix A in row reduced echelon form.

(a) Using the Gaussian elimination method, we reduce matrix A to row-echelon form. With elementary row operations we get:

$$\begin{pmatrix} -1 & 1 & 1 \\ 2 & 1 & 0 \\ 4 & -1 & -2 \end{pmatrix} \underset{R3=R3+4R1}{\overset{R2=R2+2R1}{\longrightarrow}} \begin{pmatrix} -1 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 3 & 2 \end{pmatrix} \underset{R3=R3-R2}{\overset{R1=R1\cdot(-1)}{\longrightarrow}} \begin{pmatrix} 1 & -1 & -1 \\ 0 & 3 & 2 \\ 0 & 0 & 0 \end{pmatrix} \overset{R2=R2\cdot\frac{1}{3}}{\longrightarrow} \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 2/3 \\ 0 & 0 & 0 \end{pmatrix}$$

The rank of the matrix is the number of nonzero rows, when reduced in row-echelon form. We can see that $\mathrm{rank}(A) = 2$.

(b) To bring the matrix A in reduced row-echelon form, we need every pivot to be the number one and it has to be the sole non-zero element in its column. Continuing from our last matrix:

$$\begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 2/3 \\ 0 & 0 & 0 \end{pmatrix} \overset{R1=R1+R2}{\longrightarrow} \begin{pmatrix} 1 & 0 & -1/3 \\ 0 & 1 & 2/3 \\ 0 & 0 & 0 \end{pmatrix}$$

[DAMA 50] Written Assignment 1

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Problem 10

Ben owns two types of pets: penguins and cats. When we asked him how many of each he has, he tried to answer with the following conundrum: "My pets have 14 legs, 10 eyes and 5 tails."
Using by hand calculation
(a) can we figure out how many penguins and cats he owns ?
(b) Should the answer be "My pets have 5 heads, 10 eyes and 5 tails.", what could we conclude?

(a) Let x be the number of penguins and y be the number of cats, Ben owns. Of course $x, y \in \mathbb{N}$ since we cannot have negative or float numbers when counting animals.

"My pets have 14 legs".

Each penguin has 2 legs and each cat has 4, so in terms of legs: $2x + 4y = 14$

"My pets have 10 eyes".

Each animal has 2 eyes so in terms of eyes: $2x + 2y = 10$
We have enough data to find how many penguins and cats ben has. It is a 2x2 linear system.

$$\begin{cases} 2x+4y=14 \\ x+y=5 \end{cases} \Leftrightarrow \begin{cases} 2\cdot(5-y)+4y=14 \\ x=5-y \end{cases} \Leftrightarrow \begin{cases} 10-2y+4y=14 \\ x=5-y \end{cases} \Leftrightarrow \begin{cases} 2y=4 \\ x=5-y \end{cases} \Leftrightarrow \begin{cases} y=2 \\ x=5-2 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} y=2 \\ x=3 \end{cases}$$

Ben owns **3 penguins and 2 cats**.

(b) If the conundrum was "My pets have 5 heads, 10 eyes and 5 tails" we would get the same equation out of all three clues, since both types of animals have 1 head, 2 eyes and 1 tail.

We would get $\begin{cases} x+y=5 \\ 2x+2y=10 \\ x+y=5 \end{cases} \Leftrightarrow \begin{cases} x+y=5 \\ x+y=5 \\ x+y=5 \end{cases}$

which has more than one solution *(infinite if we would solve for x and y in $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$)*.
Also since he owns two types of pets, that means he owns at least one of each. Hence $x \geq 1$ and $y \geq 1$.
Our possible solutions here would be:
$(x, y) \in \{(4,1),(3,2),(2,3),(1,4)\}$

If we disregard the fact that he has to own at least one of each type of pets, our solutions would be:
$(x, y) \in \{(5,0),(4,1),(3,2),(2,3),(1,4),(0,5)\}$ since he could own just penguins or just cats.