Math Fundamentals Part 3

- Direct Proof
- Proof by Contradiction
- Proof by Contraposition
- Inductive Proof

Direct Proof: to prove something of the form $p \rightarrow q$...

- Let *p* be true…
- ...and show that *q* must also be true.

Proof by Contraposition: to prove something of the form p → q...

$$\neg q \rightarrow \neg p$$

- Let $\neg q$ be true...
- ...and show that $\neg p$ must also be true.

Proof by Contradiction: to prove something of the form *p*...

$P \to F$

F

- Assume that *p* is false...
- ...and show that it leads to a contradiction.

Proof by Contradiction: to prove something of the form p → q...

$\neg(p \to q)$

$\neg(\neg p \vee q)$

$p \wedge \neg q$

$\neg$: not

$\to$ implies

$\wedge$ and

$\vee$ or

- Assume that *p* is true and *q* is false...
- ...and show that it leads to a contradiction.

Proof by Induction: to prove that something is true for all natural numbers (or a subset of natural numbers)...

- Goal: Prove that P(*n*) is true for all $n \geq n_0$.
- Basis Step: Show that P(*n*) is true for n_0 (and possible more if necessary).
- Inductive Step:
  - Prove $P(k) \rightarrow P(k+1)$ for weak induction
  - Prove $P(j \leq k) \rightarrow P(k+1)$ for strong induction

Give a direct proof that if n is an odd integer, then n^2 is also an odd integer.

p

q

Let $n = 2k+1$. (Show $(2k+1)^2$ is odd.)

$n = 2k+1$

$n^2 = (2k+1)^2 = 4k^2 + 4k + 1$

$= 2(\underbrace{2k^2 + 2k}) + 1$, which is odd.

$2(\text{integer}) + 1 = 2j + 1$

Prove that for all integers n, n is even if and only if n^2 is even.

$\underbrace{n \text{ is even}}_{p}$ $\underbrace{n^2 \text{ is even}}_{q}$

$p \to q$: Let $n = 2k$. Then $(2k)^2 = 4k^2 = 2(2k^2)$, which is even.

$q \to p$: Proof by Contraposition (see previous slide.)

$\neg p \to \neg q$

Prove by contradiction that if $3n+2$ is an even number, then n is an even number.

p

q

Assume

$\neg(p \to q) \equiv p \land \neg q$

Let $3n+2 = 2j$. Let $n = 2k+1$.

Then $3n+2 = 3(2k+1) + 2$

$= 6k+3+2$

$= 6k+5$

$= 6k+4+1$

$= 2(3k+2)+1$, which is odd

contradiction

$T \to F$

$2 = 3$

What is wrong with this "proof" that 1 = 2?

$a^2 - b^2 = (a-b)(a+b)$

Step	Reason
1. $a = b$	Premise
2. $a^2 = a \times b$	Multiply both sides of (1) by a
3. $a^2 - b^2 = a \times b - b^2$	Subtract b^2 from both sides of (2)
4. $(a-b)(a+b) = b(a-b)$	Algebra on (3)
5. $a + b = b$ Step 5	~~Divide both sides by $a - b$~~
6. $2b = b$	Replace a by b in (5) because $a = b$
7. $2 = 1$	Divide both sides of (6) by b

Parts of an Inductive Proof.

→ Basis Step

→ Inductive Step

- State the inductive hypothesis (IH)
- Put P(k) in terms of smaller k-values.
- Apply the IH (and state it.)
- Math

Prove by induction that the sum of the first n positive odd integers is n^2.

Conjecture: $\sum_{k=1}^{n}(2k-1) = n^2$

$\boxed{\boxed{P(j-1)} \rightarrow \boxed{P(j)}}$

Basis Step.

$\sum_{k=1}^{1}(2k-1) = 2(1)-1 = 1 = 1^2$ ✓

$P(j)$: $\sum_{k=1}^{j}(2k-1) = j^2$

Inductive Step.

Assume as the IH that $\sum_{k=1}^{j-1}(2k-1) = (j-1)^2$

$\sum_{k=1}^{j}(2k-1) = \boxed{\sum_{k=1}^{j-1}(2k-1)} + 2j-1 = (j-1)^2 + 2j - 1$ by the IH

$= j^2 - \cancel{2j} + \cancel{1} + \cancel{2j} - \cancel{1} = j^2$

Use induction to prove that $2^n < n!$ for all positive integers $n \geq 4$.

Basis Step

$2^4 = 16 < 4! = 24$

Inductive Step.

Assume as the IH that $2^{k-1} < (k-1)!$ for some $k \geq 5$.

$$
\begin{aligned}
2^k &= 2 \cdot 2^{k-1} \\
&< 2 \cdot (k-1)! \quad \text{by the IH} \\
&< k\,(k-1)! \\
&= k!
\end{aligned}
$$

Therefore $2^k < k!$.

Use induction to prove that $2 \mid n^2 + n$ for all positive integers.

Basis Step.

$2 \mid 1^2 + 1 = 2 \mid 2$ ✓

Show:

$2 \mid (k+1)^2 + (k+1)$

Inductive Step.

Assume as the IH that $2 \mid \underline{\underline{k^2 + k}}$ for some $k \geq 1$.

$(k+1)^2 + (k+1) = k^2 + 2k + 1 + k + 1$

$= \boxed{k^2 + k} + \boxed{2k + 2}$ → it's all div. by 2

div. by 2 by the IH

div. by 2 b/c $2k+2 = 2(k+1)$

Use induction to prove that if *n* is an integer greater than 1, then *n* can be written as the product of primes.

Basis Step

2 is prime

$P(2 \le j \le k-1) \rightarrow P(k)$

Inductive Step.

Assume as the IH that j can be written as the product of primes for $2 \le j \le k-1$.

If k is prime, done. If k is composite, then $k = m \cdot n$ where $2 \le m \le n \le k-1$. By the IH, m and n can be written as the product of primes. So k is a product of primes.

Let T(N) = T(floor(N/2)) + N and T(1) = 1. Prove that T(N) ≤ 2N for all N ≥ 1.

Basis Step.

$T(1) = 1 \leq 2(1)$ ✓

Inductive Step.

Assume as the IH $T(j) \leq 2j$ for $\overbrace{1 \leq j \leq k-1}^{\lfloor k/2 \rfloor}$.

$$T(k) = \boxed{T(\lfloor k/2 \rfloor)} + k$$

$$\leq \boxed{2\lfloor k/2 \rfloor} + k \quad \text{by the IH}$$

$$\leq 2(k/2) + k$$

$$= 2k$$

$P(\lfloor k/2 \rfloor) \to P(k)$

$1 \to 2, 3$

$\to 4, 5$

$\to 6, 7$

$\longrightarrow$ T(N) is O(N)

Let T(N) = T(floor(N/3)) + N and T(1) = T(2) = 1. Prove that T(N) ≤ 3N for all N ≥ 1.

Basis Step.

$T(1) = 1 \leq 3(1)$ ✓

$T(2) = 1 \leq 3(2)$ ✓

Inductive Step.

Assume as the IH that $T(j) \leq 3j$ for $2 \leq j \leq k-1$.

$$
\begin{aligned}
T(k) &= T(\lfloor k/3 \rfloor) + k \\
&\leq 3\lfloor k/3 \rfloor + k \\
&\leq \frac{3k}{3} + k \\
&= k + k \leq 3k
\end{aligned}
$$

$\longrightarrow$ $T(N)$ is $O(N)$

Determine if the following proof is valid or not. If it is not, explain what is wrong with it and correct it if possible.

Given. T(N) = T(floor(N/3)) + 1 and T(1) = T(2) = 1
Conjecture. T(N) is O(logN)

Proof.
We will show that T(N) ≤ 2logN for all ~~N ≥ 1~~. $N \geq 2$.

Basis Step.
T(1) = 1 ≤ 2(log1) = 2

Inductive Step.
Assume as the IH that T(k-1) ≤ 2log(k-1) for some k ≥ 1.
We will show that T(k) ≤ 2logk.

T(k) = T(floor(k/3)) + 1 ✓
≤ 2log(floor(k/3)) + 1 by the IH
≤ 2log(k/3) + 1
= 2logk - 2log3 + 1
≤ 2logk

$T(1) = 1 \not\leq 2\log 1 = 0$

3 base cases

$T(2), T(3), T(4), T(5)$

$T(j) \leq 2\log j$ for $4 \leq j \leq k-1$

change to strong induction

$k \geq 4$

Let T(N) = T(floor(N/4)) + 1 and T(1) = T(2) = T(3) = 1. Prove that T(N) is O(logN).

Conjecture: $T(N) \le \log N + 1$ for $N \ge 1$

Basis Step.

$T(1) = 1 \le \log 1 + 1 = 1$

$T(2) = 1 \le \log 2 + 1 = 2$

$T(3) = 1 \le \log 3 + 1 =$ ✓

Inductive Step.

Assume as the IH that $T(j) \le \log j + 1$ for $3 \le j \le k-1$.

$$T(k) = T(\lfloor k/4 \rfloor) + 1$$

$$\le \log \lfloor k/4 \rfloor + 1 + 1 \text{ by the IH}$$

$$\le \log (k/4) + 1 + 1$$

$$= \log k - \log 4 + 1 + 1 = \log k \cancel{- 2 + 1 + 1} \le \log k + 1$$