Math Fundamentals Part 2

- Functions
- Order of Growth

What is a function?

input $\longrightarrow$ output

(domain) (range / codomain)

How do you compare and/or classify functions?

$f(n) = 3n^2 + 4n + 1$ $g(n) = 5n^2$ $\boxed{h(n) = 100n + 100}$ $i(n) = ½\, n^3$ $j(n) = ⅛\, n^4$ $\boxed{k(n) = 0.00000001(2^n)}$

power of n

coefficient

Key Idea: We compare and classify functions by focusing on *the way the function grows.*

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That means looking at the relationship between the input and the output–how the output grows as the input gets larger.

Comparing Functions

$f(n) = n^2 + 1$ vs. $g(n) = \frac{1}{3} n^3$

Comparing Functions: by specific values

$f(n) = n^2 + 1$ vs. $g(n) = \frac{1}{3} n^3$

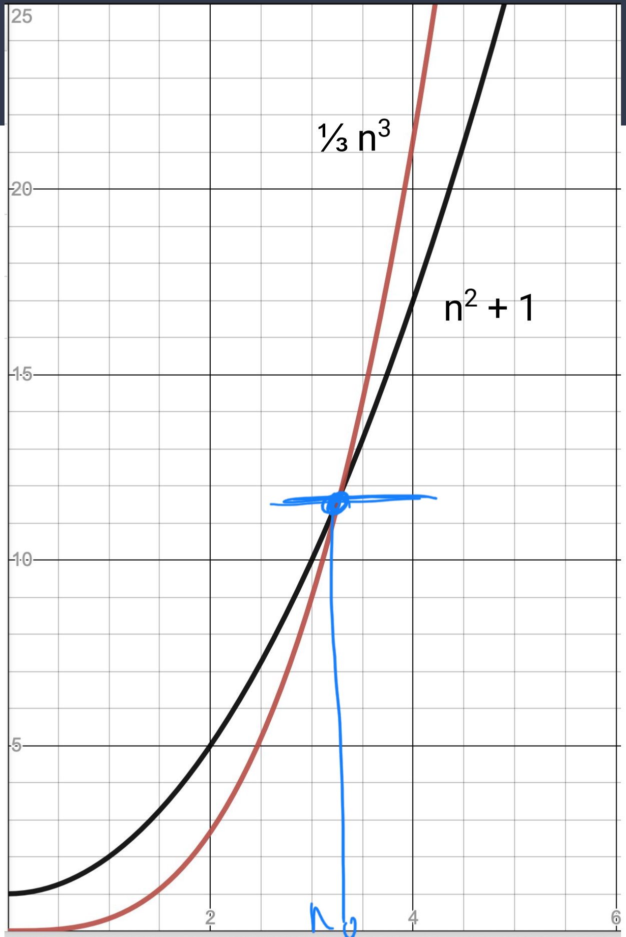
n	f(n)	g(n)
0	1	0
1	2	⅓
2	5	2 ⅔
4	17	21 ⅓
8	65	170 ⅔
16	257	1365 ⅓

Comparing Functions: by graphing

$f(n) = n^2 + 1$ vs. $g(n) = ⅓\, n^3$

$\underline{\underline{g(n) \geq f(n)}}$

for $n \geq n_0$



Comparing Functions: by ratio

$f(n) = n^2 + 1$ vs. $g(n) = ⅓\, n^3$

$$\frac{n^2 + 1}{\frac{1}{3} n^3} = \frac{\cancel{n^2}}{\frac{1}{3} n^{\cancel{3}}} + \frac{1}{\frac{1}{3} n^3}$$

Comparing Functions: by specific values

$f(n) = n^2$ vs. $g(n) = 2n^2$

n	f(n)	g(n)
0	0	0
1	1	2
2	4	8
4	16	32
8	64	128
16	256	512

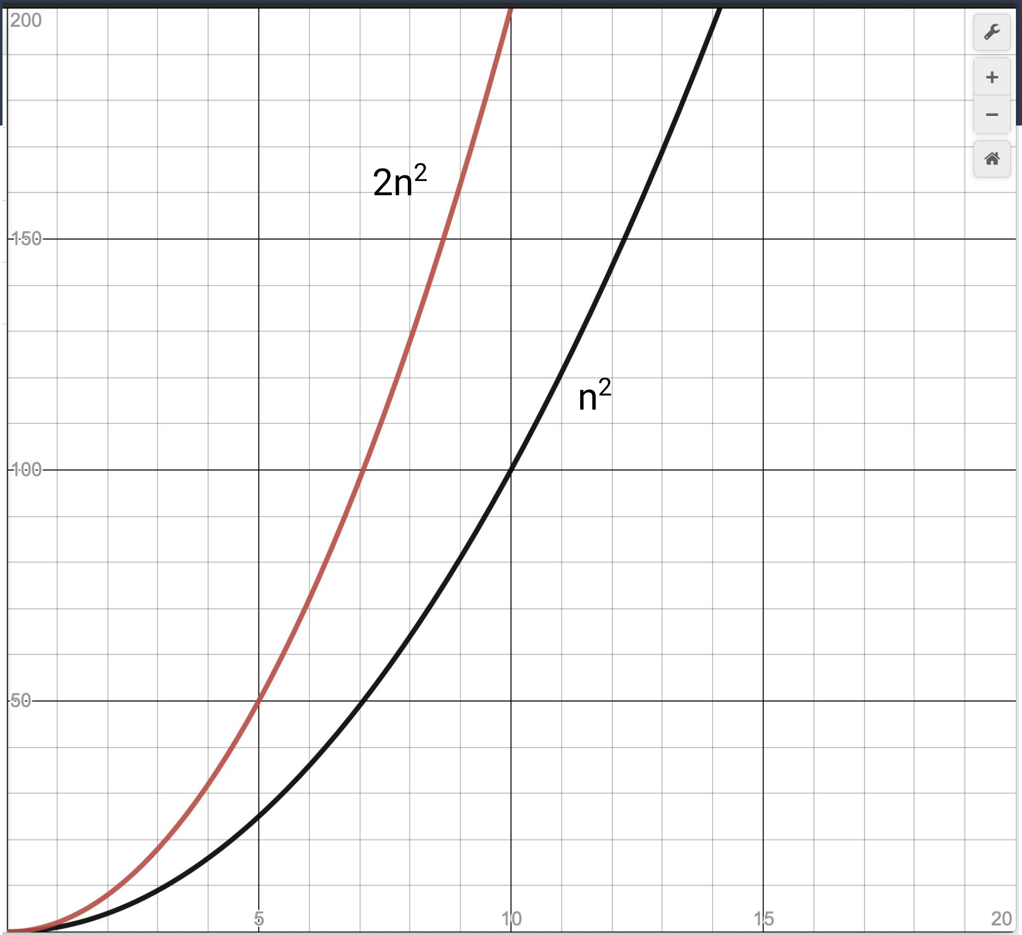
Comparing Functions: by specific values

$f(n) = n^2$ vs. $g(n) = 2n^2$

n	f(n)	f(2n)/f(n)	g(n)	g(2n)/g(n)
0	0	-	0	-
1	1	-	2	-
2	4	4	8	4
4	16	4	32	4
8	64	4	128	4
16	128	4	512	4

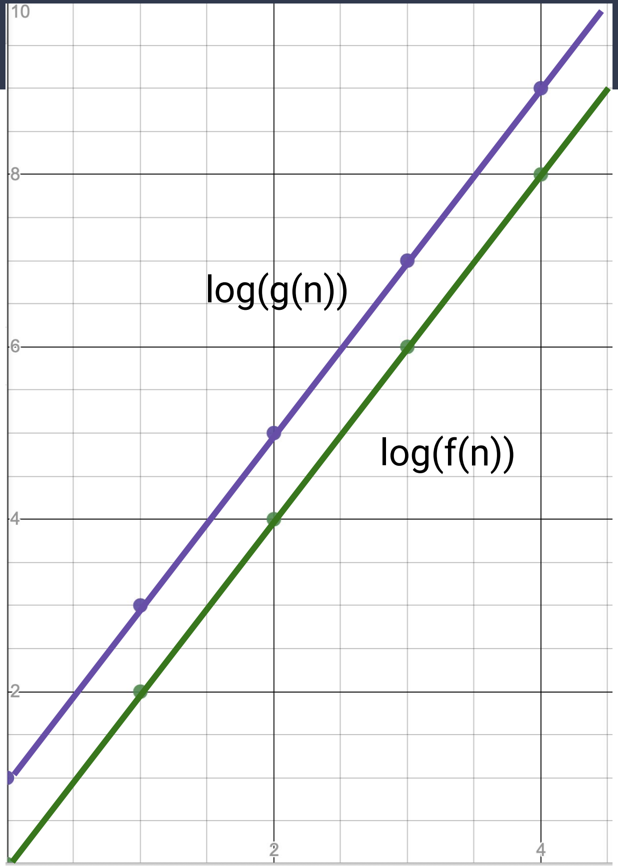
Comparing Functions: by graphing

$f(n) = n^2$ vs. $g(n) = 2n^2$



Comparing Functions: by (log-log) graphing

n	lgn	f(n)	lg(f(n))	g(n)	lg(g(n))
0	-	0	-	0	-
1	0	1	0	2	1
2	1	4	2	8	3
4	2	16	4	32	5
8	3	64	6	128	7
16	4	256	8	512	9



Comparing Functions: by ratio

$f(n) = n^2$ vs. $g(n) = 2n^2$

$$\frac{2\cancel{n^2}}{\cancel{n^2}} = 2$$

Classifying functions

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How do we know if they grow at the same rate?

What does it actually mean to grow at the same rate?

Big-Oh, Big-Omega, and Big-Theta.

- Big-Oh, Big-Omega, and Big-Theta are notations with formal definitions that help us compare and classify functions according to order of growth.
- Big-Oh is used to denote that one function can be an *upper bound* for another function.
- Big-Omega is used to denote that one function can be a *lower bound* for another function.
- Big-Theta is used to denote that one function can be both an *upper bound* and a *lower bound* for another function. (This also means that they grow at the same rate and are therefore in the same order of growth classification.)

Big-Oh, Big-Omega, and Big-Theta.

f(x) is O(g(x)) means

- g(x) is an upper bound for f(x)
- g(x) does not grow more slowly than f(x)
- f(x) does not grow more quickly than g(x)

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f(x) is Ω(g(x)) means

- g(x) is a lower bound for f(x)
- g(x) does not grow more quickly than f(x)
- f(x) does not grow more slowly than g(x)

Big-Oh, Big-Omega, and Big-Theta.

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f(x) is θ(g(x)) means
- g(x) is a tight lower bound and a tight upper bound for f(x)
- f(x) is a tight lower bound and a tight upper bound for g(x)
- f(x) and g(x) grow at the same rate

Examples: Using the proper notation, compare each pair of functions.

1. $f(n) = 4n^3 + n^2$ and $g(n) = 24n^3 + 6n^2$

$f(n)$ is $\Theta(g(n))$
$g(n)$ is $\Theta(f(n))$

$f(n)$ is $O(g(n))$
$f(n)$ is $\Omega(g(n))$

$g(n)$ is $O(f(n))$
$g(n)$ is $\Omega(f(n))$

2. $f(n) = 2n^2$ and $g(n) = n^2 logn$

$f(n)$ is $O(g(n))$
~~$f(n)$ is $\Theta(g(n))$~~

$$\frac{\cancel{n^2}\log n}{2\cancel{n^2}} = \frac{\log n}{2}$$

$g(n)$ is $\Omega(f(n))$

Big-Oh: Formal Definition

f(n) is *O(g(n))* if there exist positive constants *c* and n_0 such that $f(n) \le c \cdot g(n)$ for $n \ge n_0$.

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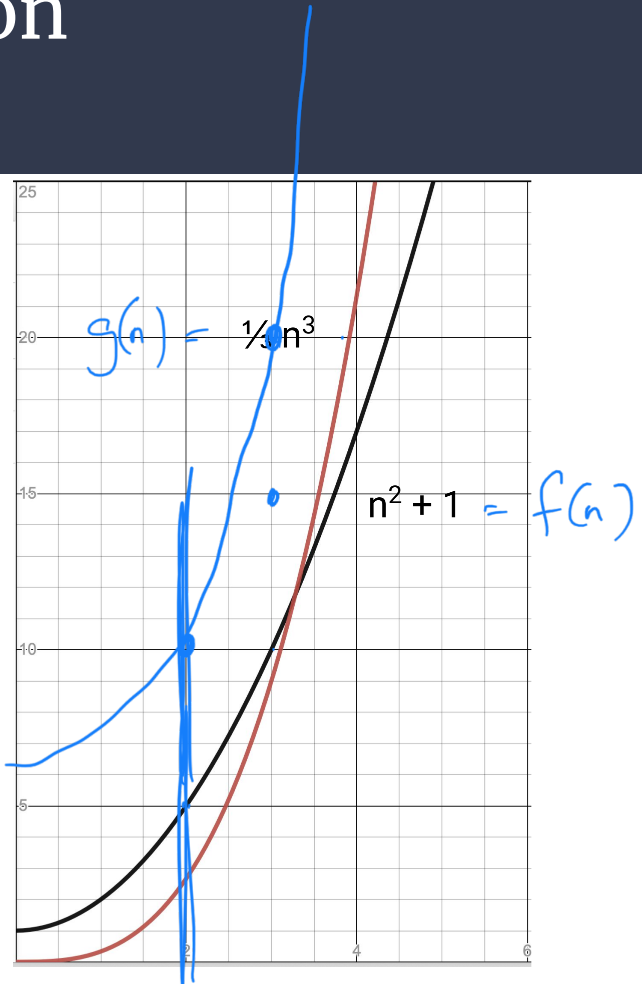
$f(n)$ is $O(g(n))$ if there exist positive constants c and n_0 such that $f(n) \le c \cdot g(n)$ for $n \ge n_0$.

$c = 2$

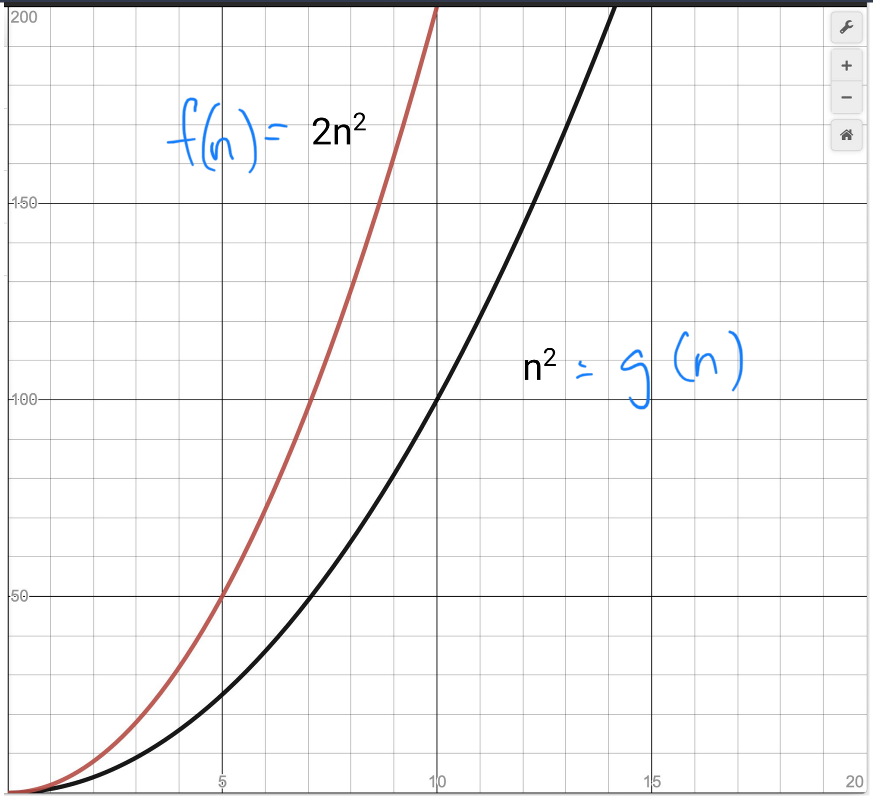
$n_0 = 2$



Big-Oh: Formal Definition

f(n) is *O(g(n))* if there exist positive constants *c* and n_0 such that $f(n) \le c \cdot g(n)$ for $n \ge n_0$.

c = 3

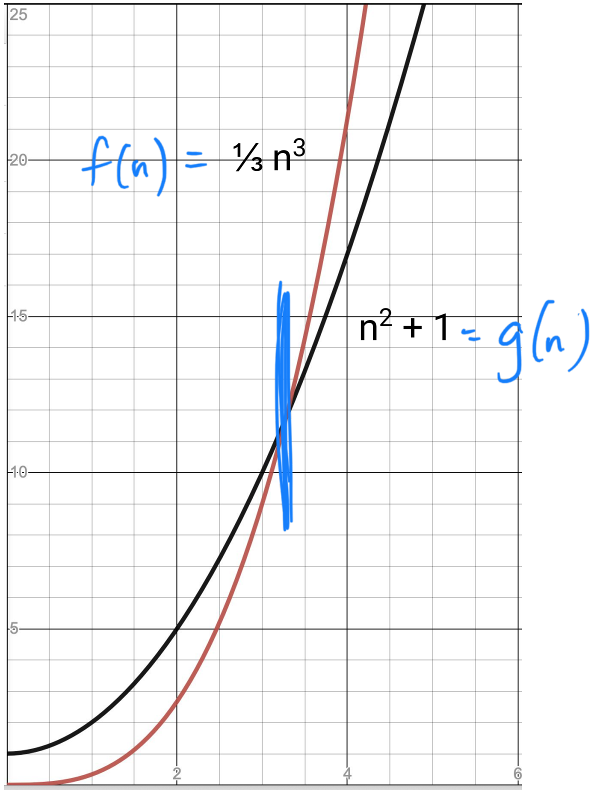


Big-Omega: Formal Definition

$f(n)$ is $\Omega(g(n))$ if there exist positive constants c and n_0 such that $f(n) \geq c \cdot g(n)$ for $n \geq n_0$.

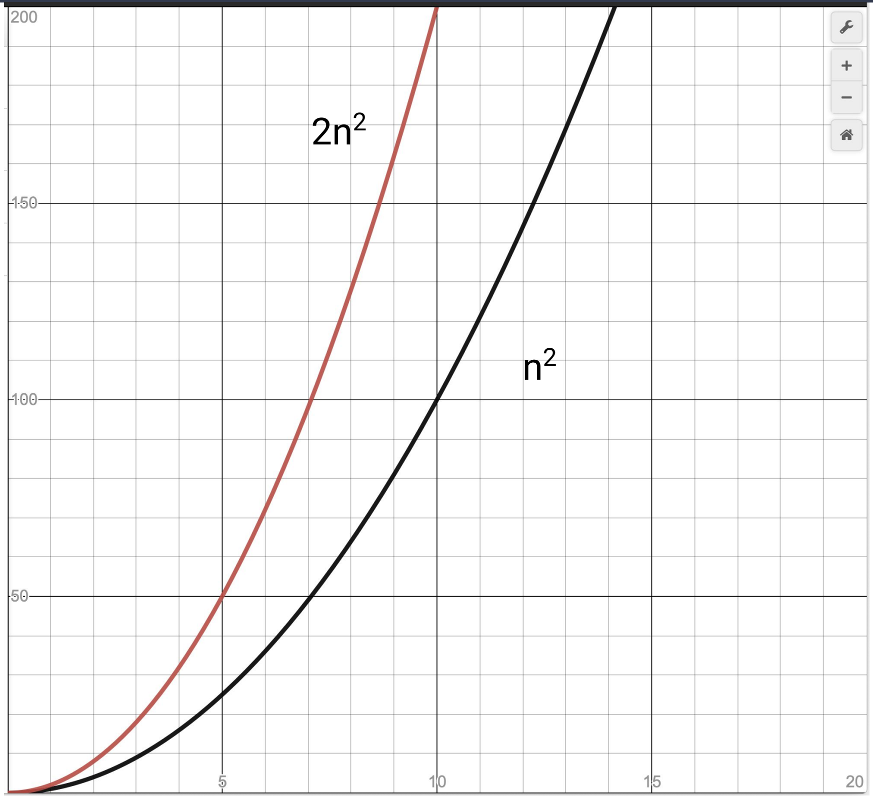
Big-Omega: Formal Definition

f(n) is *Ω(g(n))* if there exist positive constants *c* and n_0 such that $f(n) \geq c \cdot g(n)$ for $n \geq n_0$.



Big-Omega: Formal Definition

f(n) is *Ω(g(n))* if there exist positive constants *c* and n_0 such that $f(n) \geq c \cdot g(n)$ for $n \geq n_0$.



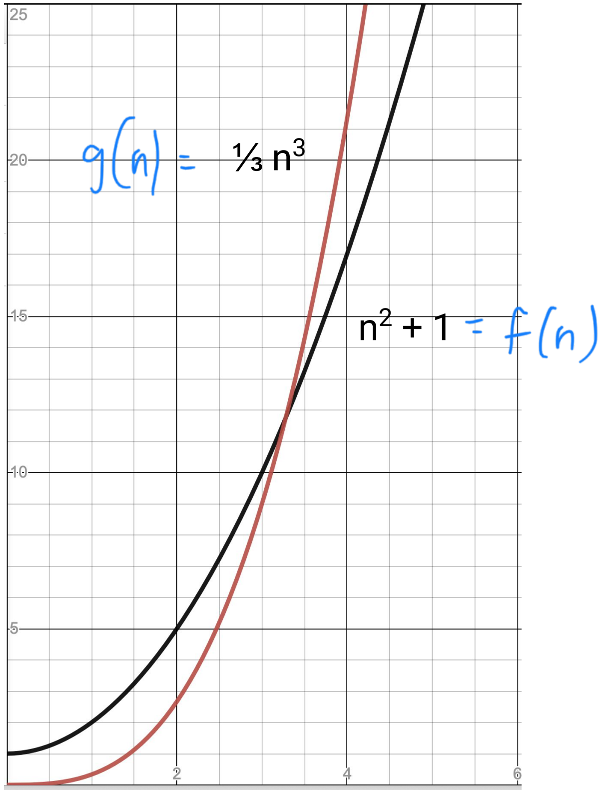
Big-Theta: Formal Definition

$f(n)$ is $\theta(g(n))$ if $f(n)$ is $O(g(n))$ and $f(n)$ is $\Omega(g(n))$.

Big-Theta: Formal Definition

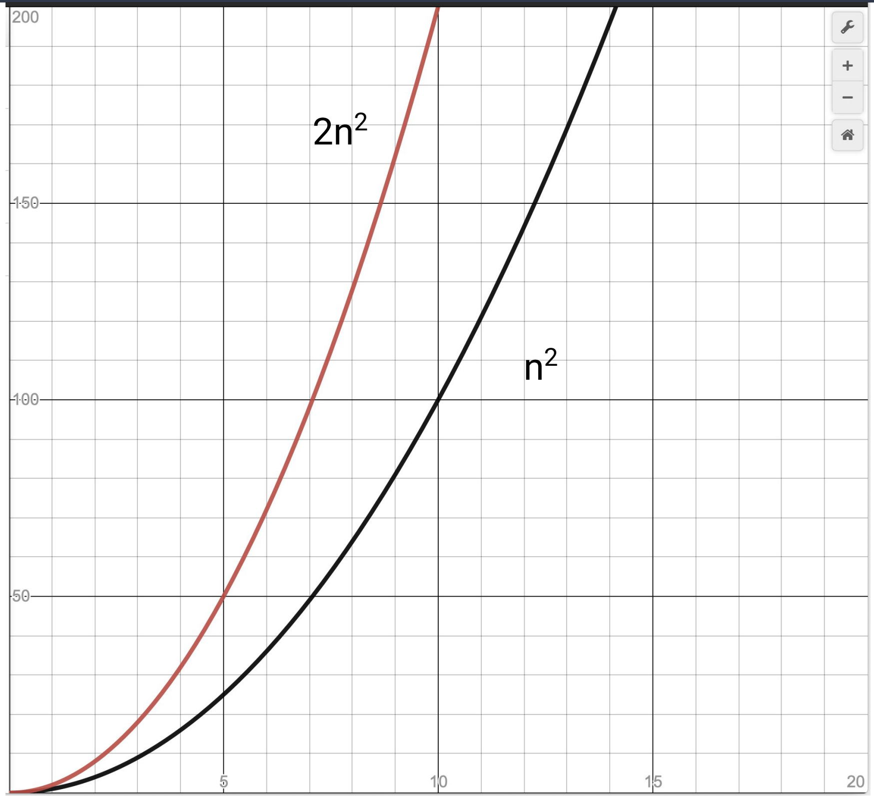
f(n) is *θ(g(n))* if *f(n)* is *O(g(n))* and *f(n)* is *Ω (g(n))*.

~~f(n) is Ω(g(n))~~



Big-Theta: Formal Definition

f(n) is *θ(g(n))* if *f(n)* is *O(g(n))* and *f(n)* is *Ω (g(n))*.



Classifying functions...

- Note that the formal definitions of big-Oh, big-Omega, and bit-Theta only require that a *c* and an n_0 exist.
- This means we can pick any *c* and n_0 that work.
- This also means that when we classify functions in this way we typically
  - ignore lower terms
  - drop the starting coefficient

- Big-Oh is used to describe an upper bound (does not have to be tight).
- Big-Omega is used to describe a lower bound (does not have to be tight).
- Big-Theta is used to classify algorithms according to order of growth. This is because since it establishes both an upper and a lower bound, that bound has to be a tight bound.

Exercise. Classify each function and justify your answer according to the formal definition of big-Theta.

1. $f(n) = 5n^2 + 4n + 1$

Claim: $f(n)$ is $\Theta(n^2)$

$f(n)$ is $O(n^2)$: $5n^2 + 4n + 1 \le 5n^2 + 5n^2 + 5n^2 \le 15n^2$, $c = 15$, $n_0 = 1$

$f(n)$ is $\Omega(n^2)$: $5n^2 + 4n + 1 \ge n^2$ for $n \ge 1$

2. $f(n) = 4n^2 logn^{2} + 5n + 20logn$

$\Theta(n^2 \log n)$

$f(n)$ is $O(n^2 \log n)$: $8n^2 \log n + 5n + 20 \log n \le 20n^2 \log n + 20n^2 \log n + 20n^2 \log n \le 60n^2 \log n$ for $n \ge 1$

$f(n)$ is $\Omega(n^2 \log n)$: $8n^2 \log n + 5n + 20 \log n \ge n^2 \log n$ for $n \ge 1$

3. $f(n) = \dfrac{3^{log_3 n+1} - 1}{2}$

$\Theta(n)$

$3^{\log_3 n + 1} = 3 \cdot 3^{\log_3 n} = 3n$

$\frac{3n-1}{2} = \frac{3n}{2} - \frac{1}{2} \le \frac{3n}{2} + \frac{3n}{2} \le 3n$

$\frac{3n-1}{2} = \frac{3n}{2} - \frac{1}{2} \ge \frac{1}{2}n$ for $n \ge 1$

Common Order of Growth Classifications

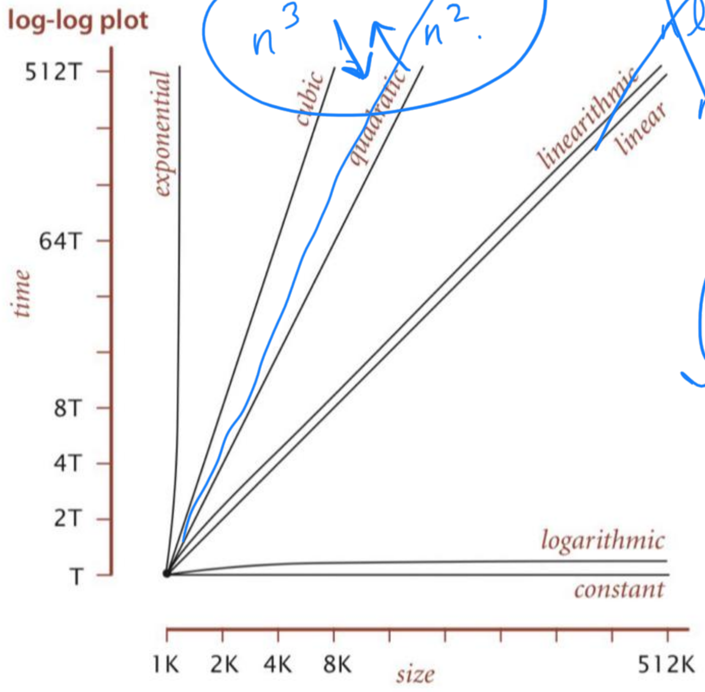
slowest-growing

Name	In Big-Theta	Examples
constant	$\theta(1)$	100, 256, 300000, 1/2
logarithmic	$\theta(logn)$	$log_2 n,\ 4log_3 n,\ logn^6,\ log(n/4)$
linear	$\theta(n)$	$5n + 2,\ 8n + 100,\ \frac{4n-1}{5}$
linearithmic	$\theta(nlogn)$	$4nlogn^3 + 5n + 1,\ \frac{nlog_4 n}{2} + 100$
quadratic	$\theta(n^2)$	$8n^2 + 8n + 1,\ 300nlog(2^n)$
cubic	$\theta(n^3)$	$n^3 + n^2 + n + 1,\ n^2log(16^n)$
exponential	$\theta(2^n)$	$2^{n+1},\ 8(2^n)$

faster-growing

factorial

Common Order of Growth Classifications



Typical orders of growth

$\frac{\log n}{n}$

$$\frac{\cancel{n^2}\log n}{\cancel{n^2}} \quad \text{vs. } n^2$$

$$\frac{n^{\cancel{3}}}{\cancel{n^2}\log n} = \frac{n}{\log n}$$

image from [2]

Exercise. Determine if these statements are true or false and justify your answer.

1. $3n^2 logn^2 + 4n$ is $\Omega(logn)$

T

2. $3n^2 logn + 4n$ is $O(n^2)$

F

3. $6n^2 log2^n + 4n^3$ is $O(n^2 logn)$

F

4. $5n^2 + 5n + 7$ is $\theta(n^2)$

T

5. $4log^2 n + n + 2^n$ is $\theta(n)$

F $O(n)$

6. $15n^3$ is $\Omega(1)$

T

$4(logn)^2$

$logn$

n vs.

$n\,logn$

$logn$ $\quad 1$

Some useful things to keep in mind.

1. If $f(n)$ is $O(g(n))$, then $g(n)$ is $\Omega(f(n))$.

2. If $f(n)$ is $O(g(n))$ and $g(n)$ is $O(h(n))$, then $f(n)$ is $O(h(n))$.

3. If $f(n)$ is $\theta(g(n))$, then $g(n)$ is $\theta(f(n))$.

4. Every order of growth function in this class will always be $\Omega(1)$.

5. If $f(n)$ is $O(g(n))$, then $f(n) + g(n)$ is $\theta(g(n))$.

Given:

$f(x)$ is $O(g(x))$ $\quad$ g(x) is Ω(f(x))

$h(x)$ is $\Omega(g(x))$ $\quad$ g(x) is O(h(x))

$i(x)$ is $\Theta(f(x))$

Exercise. Determine if each statement is definitely true (DT), definitely false (DF), or possibly true/possibly false (PT) and justify your answer.

1. $f(x)$ is $O(h(x))$

DT

i(x) is Θ(f(x))

f(x) is O(g(x)) → g(x) is Ω(~~f(x)~~ i(x))

2. $g(x)$ is $\Omega(i(x))$

DT

h(x)

g(x)

f(x)

3. $f(x) + g(x) + h(x) + i(x)$ is $O(h(x))$

DT

4. $f(x) + g(x) + h(x) + i(x)$ is $\Omega(f(x))$

DT

5. $f(x) + \cancel{g(x)} + \cancel{h(x)} + i(x)$ is $O(i(x))$

PT

References & Resources

1. https://www.desmos.com/calculator
2. Algorithms by Sedgewick & Wayne