Minimum Spanning Tree

- Terminology and Properties
- Prim’s Algorithm
- Kruskal’s Algorithm
- Baruvka’s Algorithm
- Traveling Salesperson Problem

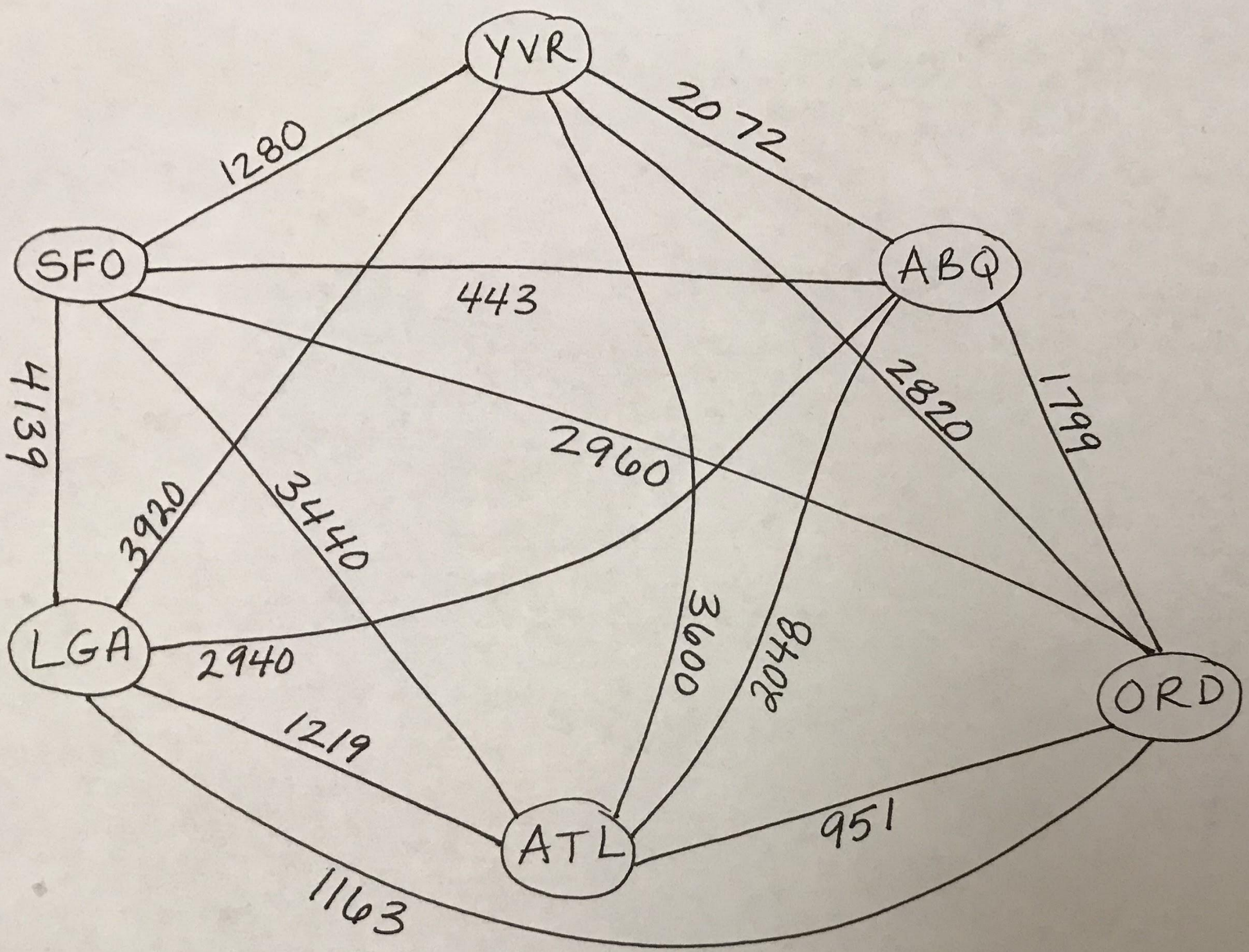
Imagine...

You are employed at a brand new airline, and your first task is to determine which flights the airline should provide according to the following guidelines:

- There needs to be a way to get between each pair of cities/airports in the following list: San Francisco, CA (SFO); Vancouver, BC (YVR); Albuquerque, NM (ABQ); Chicago, IL (ORD); Atlanta, GA (ATL); New York, NY (LGA)
- The route between two cities does not have to be a direct flight.
- The total number of flights should be minimized.
- The total distance covered by all flights should be minimized.
- Assume that if a flight exists, it goes both ways (e.g. SFO→ YVR and YVR→ SFO)

How would you go about solving this problem?

	SFO	YVR	ABQ	ORD	ATL	LGA
SFO	0	1280	443	2960	3440	4139
YVR	1280	0	2072	2820	3600	3920
ABQ	443	2072	0	1799	2048	2940
ORD	2960	2820	1799	0	951	1163
ATL	3440	3600	2048	951	0	1219
LGA	4139	3920	2940	1163	1219	0



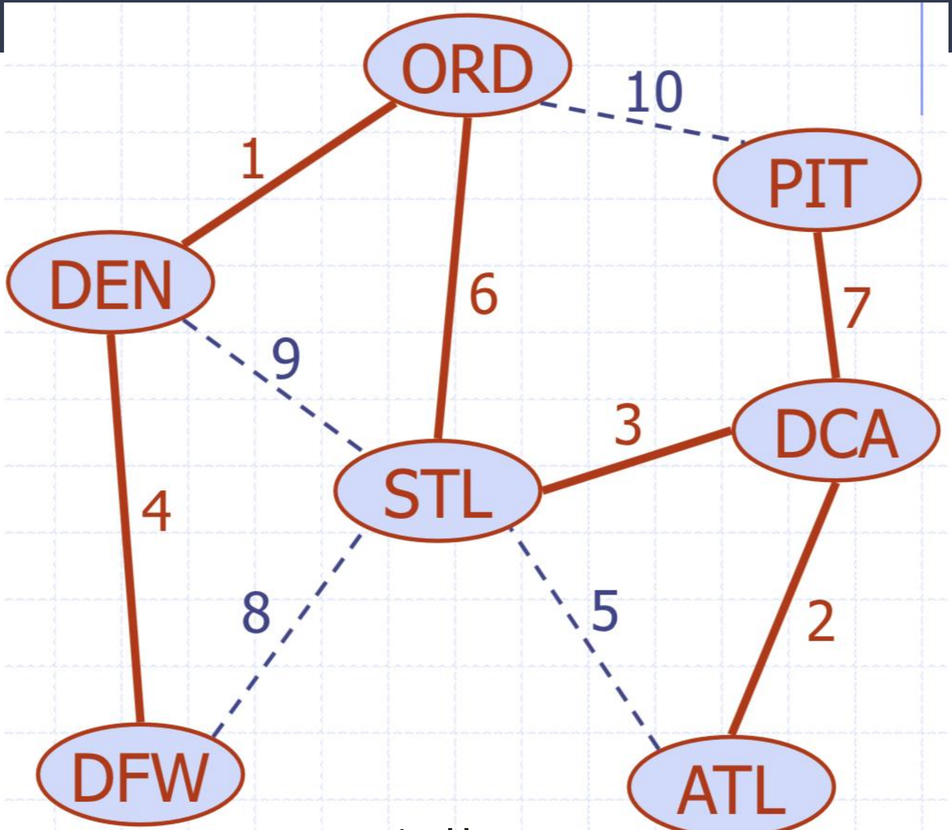
How does this problem differ from a shortest path problem?

In graph terms, how would you describe what we are looking for?

What we really want is a minimum spanning tree (MST)...

A minimum spanning tree of a graph G...

- ...is a spanning subgraph (i.e. it contains all the vertices of G)
- ...is a tree (i.e. it has no cycles)
- ...is minimal weight-wise (i.e. the total weight of all the edges it uses is minimized so that no other spanning tree has a smaller total weight)
- ...does NOT guarantee the shortest path between two vertices!



Picture from [1]

Property 1: The Cycle Property

- Let **T** be a minimum spanning tree of a weighted graph **G**
- Let **e** be an edge of **G** that is not in **T**
- Let **C** be the cycle formed by adding **e** to **T**
- Claim: For every edge **f** of **C**:

  weight(f) <= weight(e)

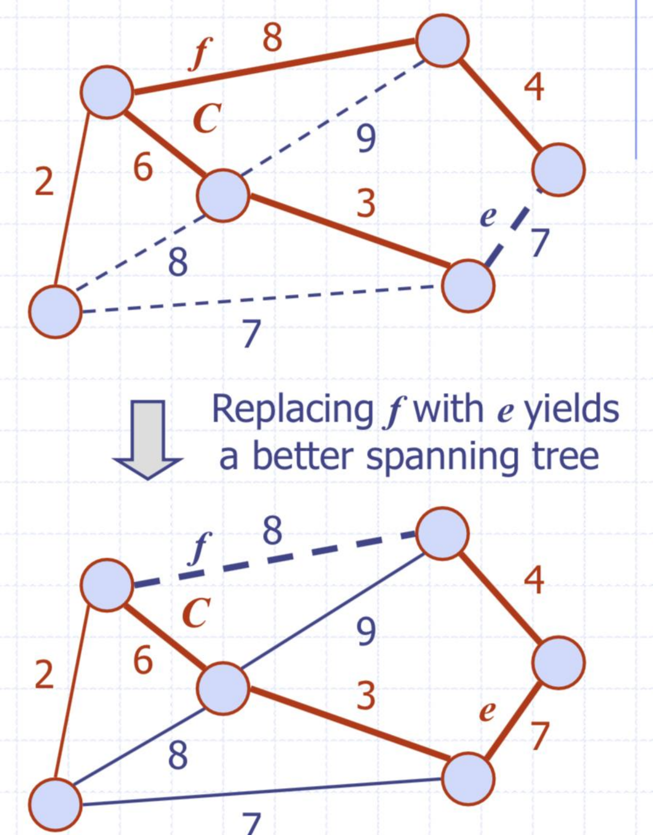
Proof by Contradiction: ???

Property 1: The Cycle Property

- Let **T** be a minimum spanning tree of a weighted graph **G**
- Let **e** be an edge of **G** that is not in **T**
- Let **C** be the cycle formed by adding **e** to **T**
- Claim: For every edge **f** of **C**:

  weight(f) <= weight(e)

Proof by Contradiction: Given all the properties above, assume that for some edge **f** in **C**, ***weight(f) > weight(e).*** Then replacing **f** with **e** will produce a spanning tree **T'** such that the total weight of **T'** is smaller than the total weight of **T**. But that contradicts the definition of **T** as an MST of **G.**



Picture from [1]

Property 2: The Partition Property

- Consider a partition of a graph **G** into subsets **U** and **V**
- Let **e** be an edge of minimum weight across the partition (i.e. **e** has one endpoint in **U** and one endpoint in **V**)
- Claim: There is a minimum spanning tree of **G** that includes edge **e**

Proof: ???

Picture from [1]

Property 2: The Partition Property

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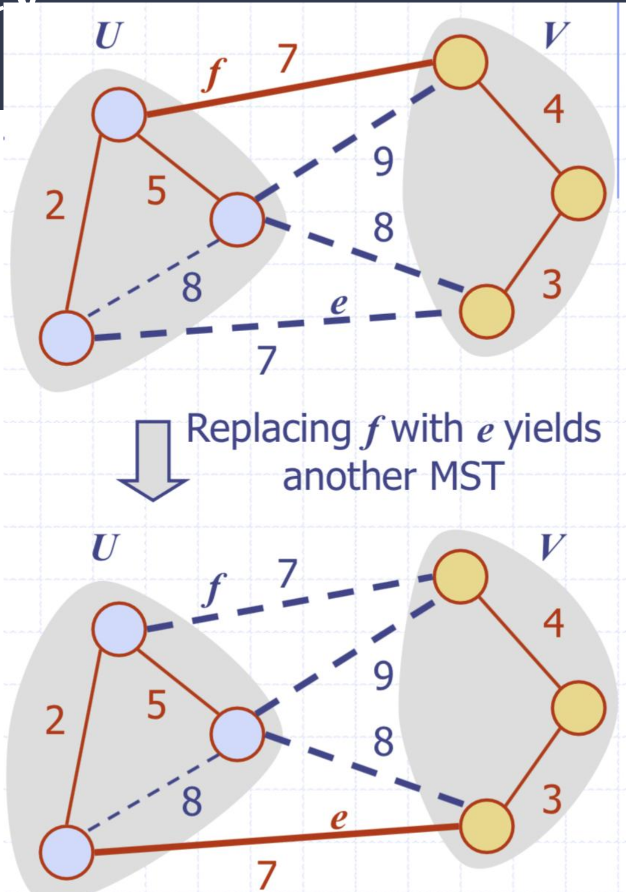
- Let **e** be an edge of minimum weight across the partition (i.e. **e** has one endpoint in **U** and one endpoint in **V**)
- Claim: There is a minimum spanning tree of **G** that includes edge **e**

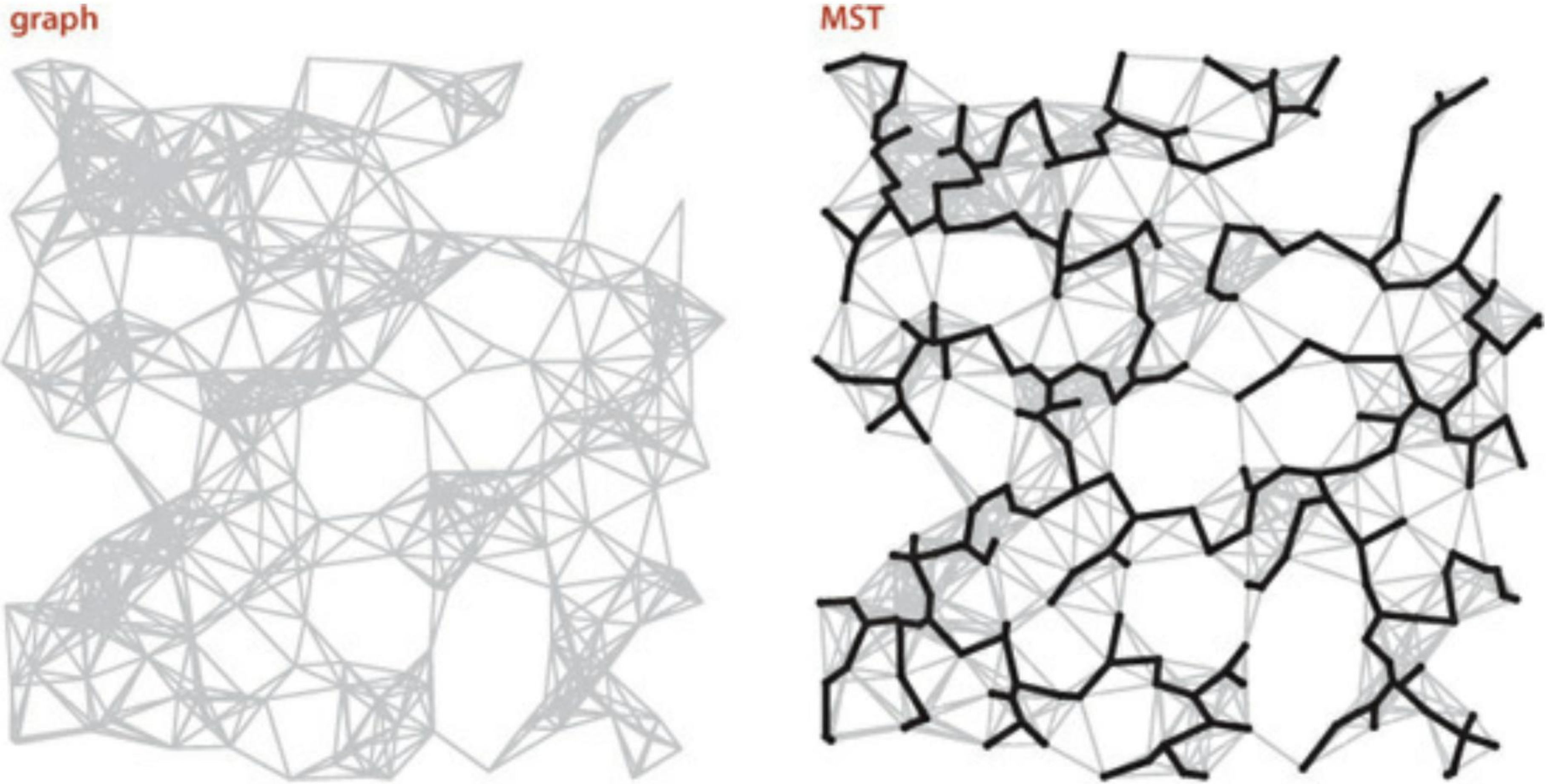
Proof: Let **T** be an MST of **G** and let the definitions above be true**.** If **T** does not contain **e**, consider the cycle **C** formed by **e** with **T** and let **f** be an edge of **C** across the partition. By the cycle property, ***weight(f)<=weight(e)***. Thus, ***weight(f) = weight(e)***, and we obtain another MST by replacing **f** with **e**.



Picture from [1]

Questions to think about...

1. Given an undirected, weighted graph **G**, let **T** be an MST of **G**. Is **T** unique?
2. Does an MST also give you the shortest path between a pair of vertices?
3. Does every weighted undirected graph have an MST?
4. Can you find an MST of a weighted undirected graph if there are negative weight edges?
5. Can you find an MST in a directed graph?
6. How would you find an MST of a weighted undirected graph?

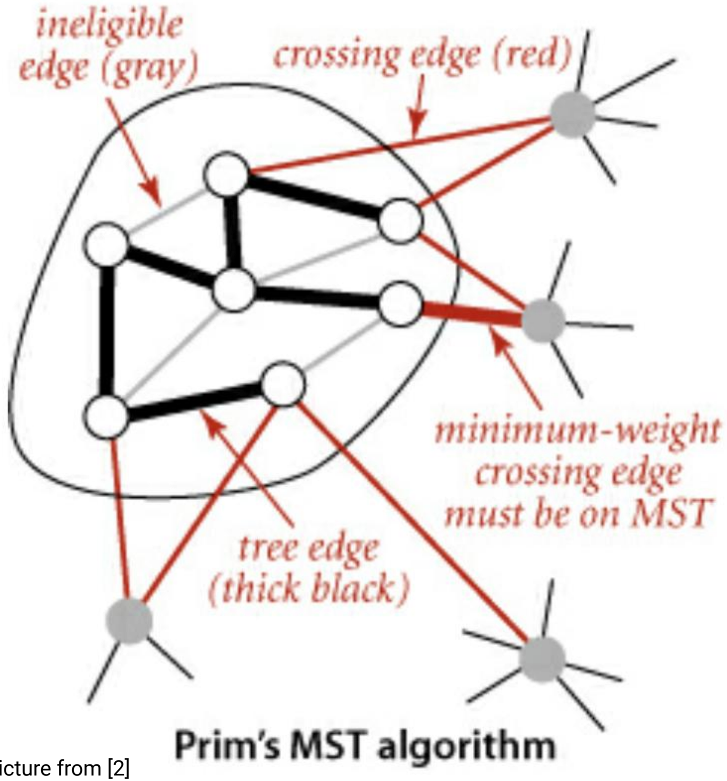


A 250-vertex Euclidean graph (with 1,273 edges) and its MST

Picture from [2]

Prim’s Algorithm (aka Jarnik’s Algorithm)

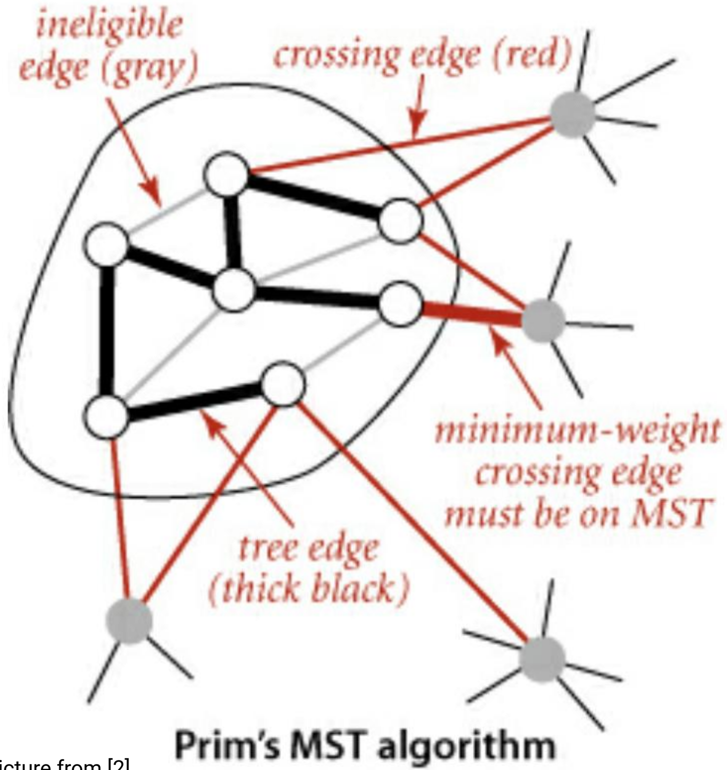
- Grow the tree one edge at a time
- Start with a single vertex (counts as a tree)
- Add |V| - 1 edges to it by adding the minimum-weight edge that connects a new vertex to the tree (crossing the partition that is defined by the current tree vertices--called a *crossing edge*)
- NOTE: When you add an edge to the tree, you are also adding a vertex to the tree.
- Like Dijkstra’s Shortest Path, this is a greedy algorithm.



Picture from [2]

Prim’s Algorithm (aka Jarnik’s Algorithm)

Proof of Correctness: Follows directly from the Partition Property because we are choosing the minimum weight edge across the tree-defined partition.



Picture from [2]

Implementation of Prim's Algorithm

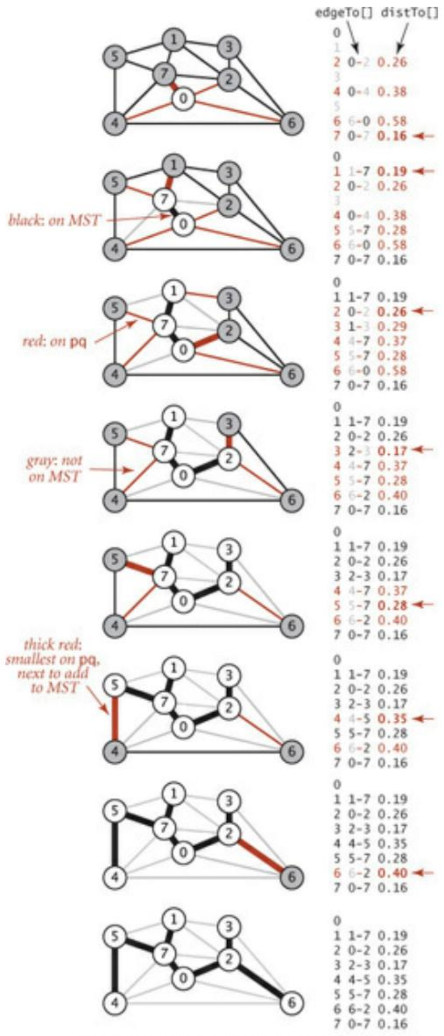
- **marked[]**: an array of booleans to keep track of vertices on the tree
- **edgeTo[]**: an array to keep track of the lightest edge connecting a new vertex to the tree
- **distTo[]**: an array to keep track of the weight of the lightest edge connecting a new vertex to the tree
- **pq**: a minimum priority queue to keep track of eligible crossing edges (key is the weight of the edges)

Algorithm $PrimsMST(G)$
Input: $G = (V, E)$, a weighted, undirected graph
Output: A minimum-weight spanning tree of G

```
edgeTo[] := a |V|-sized array to store the edge
    connecting a vertex to the tree
distTo[] := a |V|-sized array to keep track of
    the distance of the edge connecting a vertex
    to a tree
marked[] := a |V|-sized array to keep track of
    which vertices have been visited
pq := a heap-based min priority queue with
    weights as keys and vertices as values

//initialize structures
for all v ∈ V:
    distTo[v] = ∞
distTo[0] = 0
pq.insert(0, 0)

//main loop
while !pq.isEmpty:
    v = pq.delMin()
    marked[v] = T
    for each edge e = (v, u) adjacent to v:
        if marked[u]
            continue
        if e.weight() < distTo[u]:
            edgeTo[u] = e
            distTo[u] = e.weight()
            if pq.contains(u):
                pq.changeKey(u, distTo[u])
            else:
                pq.insert(u, distTo[u])
```



Trace of Prim's algorithm (eager version)

Picture from [2]

```
//main loop
while !pq.isEmpty:
    v = pq.delMin()
    marked[v] = T
    for each edge e = (v, u) adjacent to v:
        if marked[u]
            continue
        if e.weight() < distTo[u]:
            edgeTo[u] = e
            distTo[u] = e.weight()
            if pq.contains(u):
                pq.changeKey(u, distTo[u])
            else:
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```

Algorithm $PrimsMST(G)$
Input: $G = (V, E)$, a weighted, undirected graph
Output: A minimum-weight spanning tree of G

$edgeTo[] :=$ a $|V|-$sized array to store the edge connecting a vertex to the tree
$distTo[] :=$ a $|V|-$sized array to keep track of the distance of the edge connecting a vertex to a tree
$marked[] :=$ a $|V|-$sized array to keep track of which vertices have been visited
$pq :=$ a heap-based min priority queue with weights as keys and vertices as values

//initialize structures
for all $v \in V$:
    $distTo[v] = \infty$
$distTo[0] = 0$
$pq.insert(0, 0)$

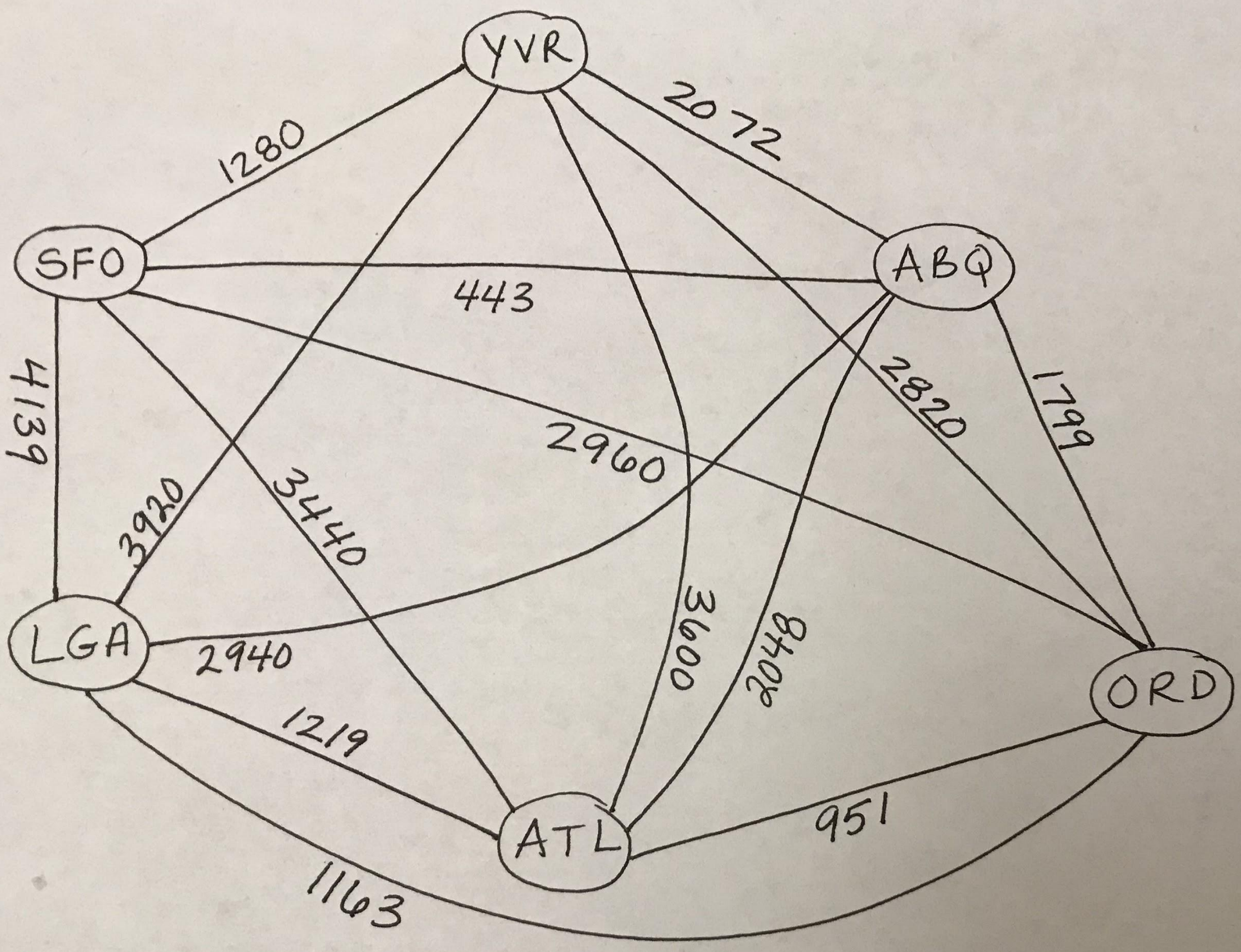
O(|V|)

//main loop
while $!pq.isEmpty$:
    $v = pq.delMin()$
    $marked[v] = T$
    for each edge $e = (v, u)$ adjacent to v:
        if $marked[u]$
            continue
        if $e.weight() < distTo[u]$:
            $edgeTo[u] = e$
            $distTo[u] = e.weight()$
            if $pq.contains(u)$:
                $pq.changeKey(u, distTo[u])$
            else:
                $pq.insert(u, distTo[u])$

```
//main loop
while !pq.isEmpty:
    v = pq.delMin()
    marked[v] = T
    for each edge e = (v, u) adjacent to v:
        if marked[u]
            continue
        if e.weight() < distTo[u]:
            edgeTo[u] = e
            distTo[u] = e.weight()
            if pq.contains(u):
                pq.changeKey(u, distTo[u])
            else:
                pq.insert(u, distTo[u])
```

Assuming adjacency list representation, total time is **O((V + E)logV) = O(ElogV)** for a connected graph

- delMin in a heap-based PQ: O(logV)
- checking if PQ contains a vertex and changing a key is O(logV) as long as there is an auxiliary data structure keeping track of positions in the queue
- key of any vertex *v* is updated at most *deg(v)* times and sum of all degrees is O(E)



YVR
SFO
ABQ
LGA
ATL
ORD
1280
2072
443
4139
2820
1799
2960
3920
3440
3600
2048
2940
1219
951
1163

Imagine...

You are employed at a company that has offices in several different cities. Your first task is to work with a phone company to ensure that there is a connection between every pair of offices according to the following guidelines:

- The connection between two offices does not have to be direct.
- The company wants to minimize the total cost of all the connections.

What kind of a problem is this?

Imagine...

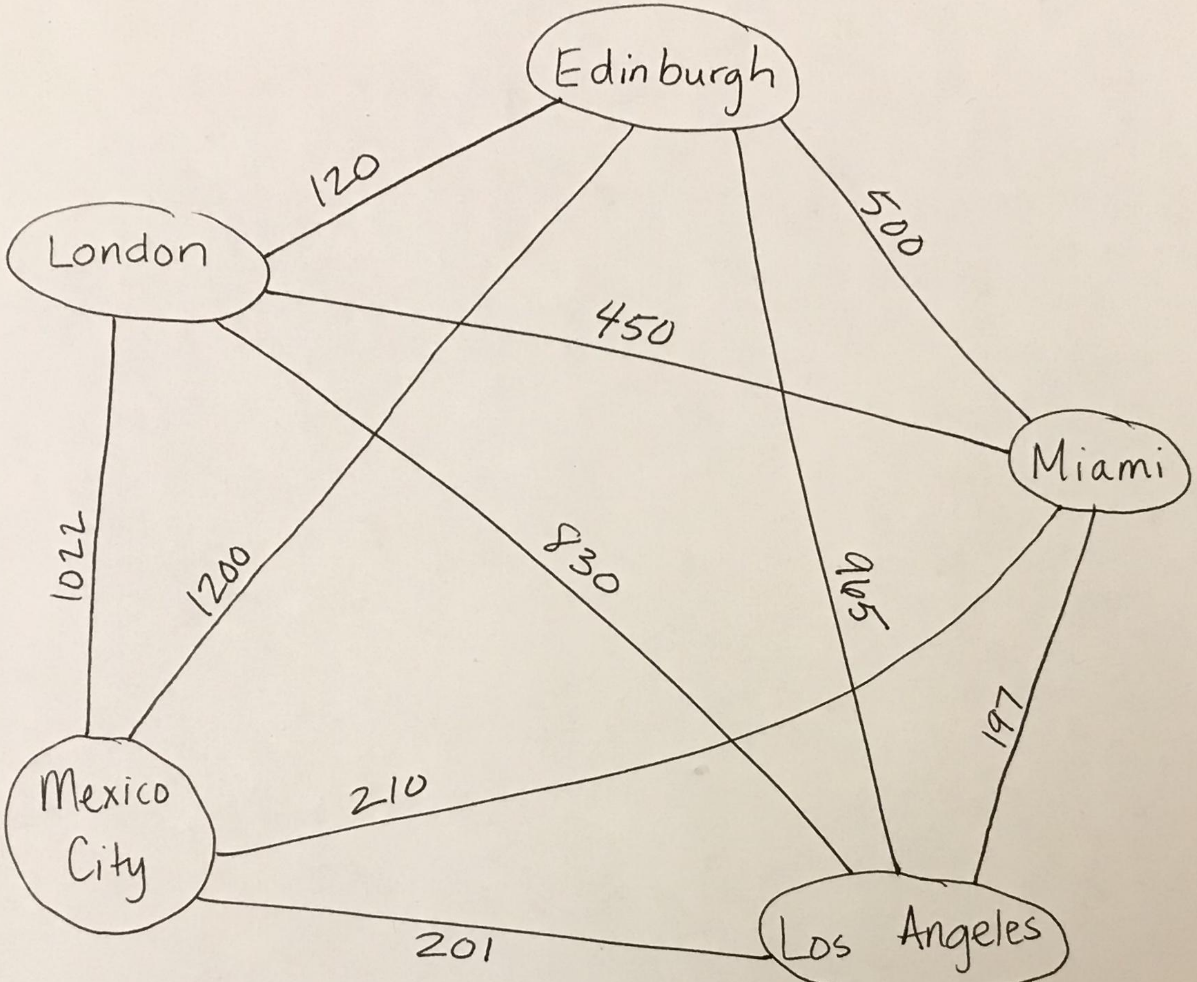
You are employed at a company that has offices in several different cities. Your first task is to work with a phone company to ensure that there is a connection between every pair of offices according to the following specifications:

- The phone company charges different rates to connect pairs of cities, specified on the following slide.
- The connection between two offices does not have to be direct.
- The company wants to minimize the total cost of all the connections.

What kind of a problem is this?

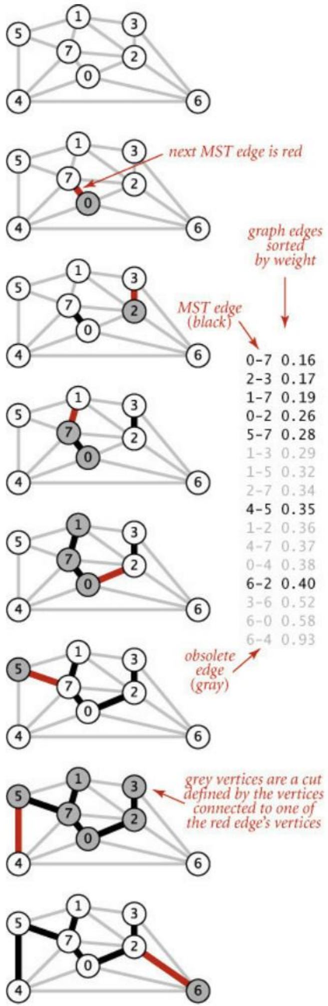
A minimum spanning tree problem!

	London	Edinburgh	Miami	Mexico City	Los Angeles
London	0	$120	$450	$1022	$830
Edinburgh	$120	0	$500	$1200	$965
Miami	$450	$500	0	$210	$197
Mexico City	$1022	$1200	$210	0	$201
Los Angeles	$830	$965	$197	$201	0


Edinburgh
London
Miami
Mexico City
Los Angeles
120
500
450
1022
1200
830
965
197
210
201

Kruskal’s Algorithm

- Process edges, adding edges to the tree smallest-weight first **as long as the new edge does not form a cycle.**
- The result is that the algorithm creates a forest that eventually merges into a single tree.
- What would be some of the challenges in implementing this?
- What data structures would be useful?
- How would you represent the graph?

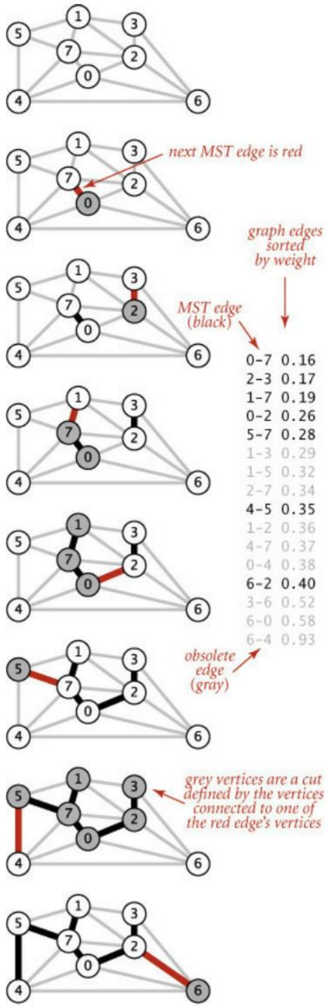


Trace of Kruskal's algorithm

Picture from [2]

Kruskal’s Algorithm

- Can be implemented using disjoint sets for each separate component
- Start with |V| disjoint sets, each containing a single vertex
- Combine sets (union) by adding edges, reducing the number of separate components until there is only one



Trace of Kruskal's algorithm

Picture from [2]

Algorithm $KruskalsMST(G)$
Input: $G = (V, E)$, a connected, weighted, undirected graph
Output: A minimum-weight spanning tree of G

```
pq := a minimum priority queue of edges where keys are weights
for each edge e = (u, v) ∈ E:
    pq.insert(e, e.weight())

A = ∅
for each v ∈ V:
    makeSet(v)
while |A| < |V| − 1
    e = (u, v) = pq.delMin()
    if findSet(u) ≠ findSet(v):
        A = A ∪ (u, v)
        union(u, v)
return A
```

Algorithm $KruskalsMST(G)$
Input: $G = (V, E)$, a connected, weighted, undirected graph
Output: A minimum-weight spanning tree of G

$pq :=$ a minimum priority queue of edges where keys are weights
for each edge $e = (u, v) \in E$:
    $pq.insert(e, e.weight())$

$A = \emptyset$
for each $v \in V$:
    $makeSet(v)$
while $|A| < |V| - 1$
    $e = (u, v) = pq.delMin()$
    if $findSet(u) \neq findSet(v)$:
        $A = A \cup (u, v)$
        $union(u, v)$
return A

- **O(ElogE)** for doing **E** inserts into the **E**-sized PQ
- Alternatively, we could just sort the edges in **O(ElogE)** time

Algorithm $KruskalsMST(G)$
Input: $G = (V, E)$, a connected, weighted, undirected graph
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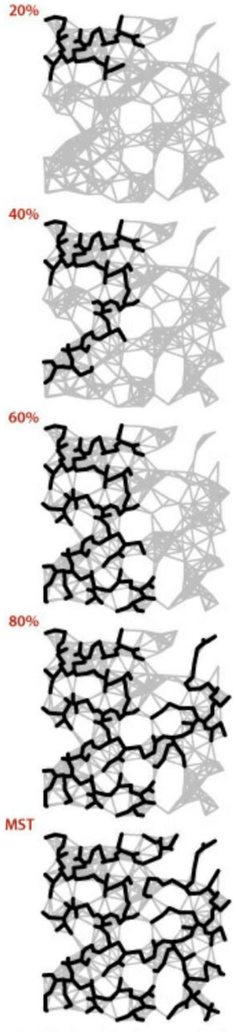
    if $findSet(u) \neq findSet(v)$:
        $A = A \cup (u, v)$
        $union(u, v)$
return A

- *makeSet* is an O(1) operation, so the total time is **O(V)**
- The total time required for **E** *union* and *findSet* operations is **O(ElogV)**

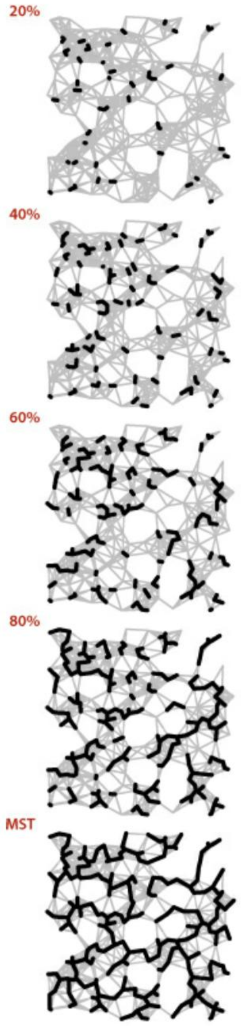
- **Total time: O(ElogE + ElogV)=O(ElogE)**

Prim's vs. Kruskal's

- Space: |**V**| vs. |**E**|
- Time: |**E**|**log**|**V**| vs. |**E**|**log**|**E**|



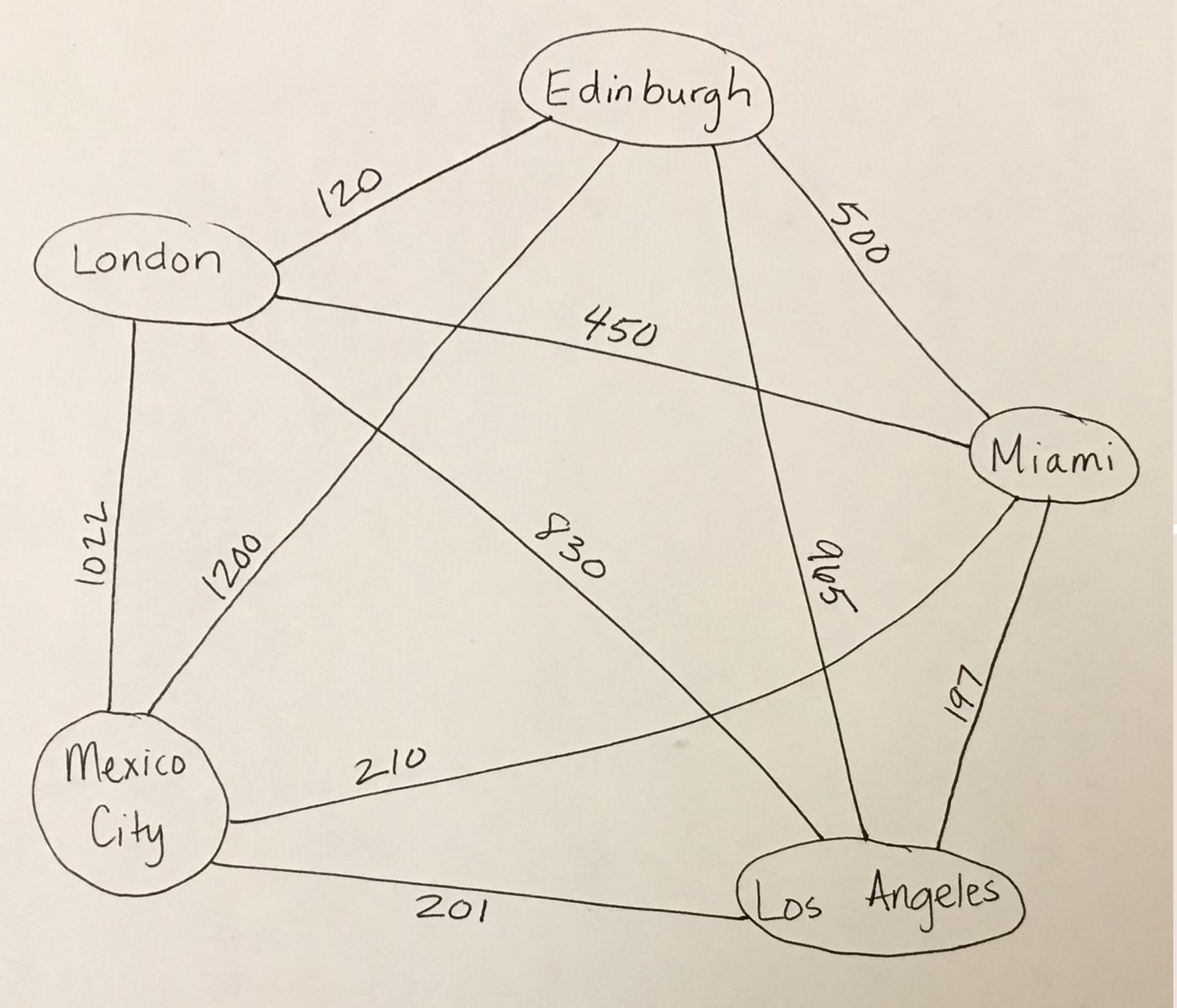
Prim's algorithm (250 vertices)



Kruskal's algorithm (250 vertices)

Pictures from [2]

Goal: Find an MST of the graph using Kruskal's Algorithm...



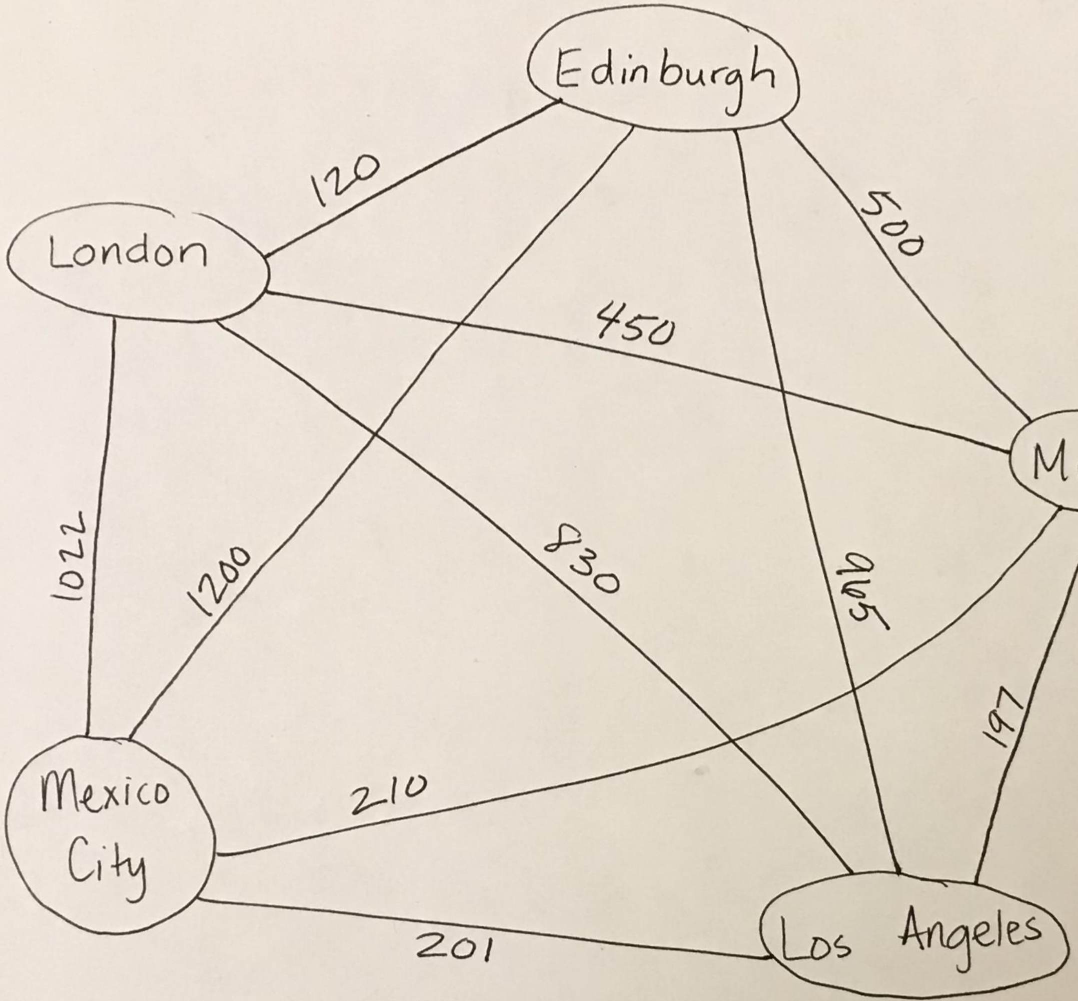
Baruvka's Algorithm

- Like Kruskal's Algorithm, Baruvka's algorithm grows many "clouds" at once.

```
Algorithm BaruvkaMST(G)
    T ← V  {just the vertices of G}
  while T has fewer than n-1 edges do
    for each connected component C in T do
      Let edge e be the smallest-weight edge from C to another component in T.
      if e is not already in T then
        Add edge e to T
  return T
```

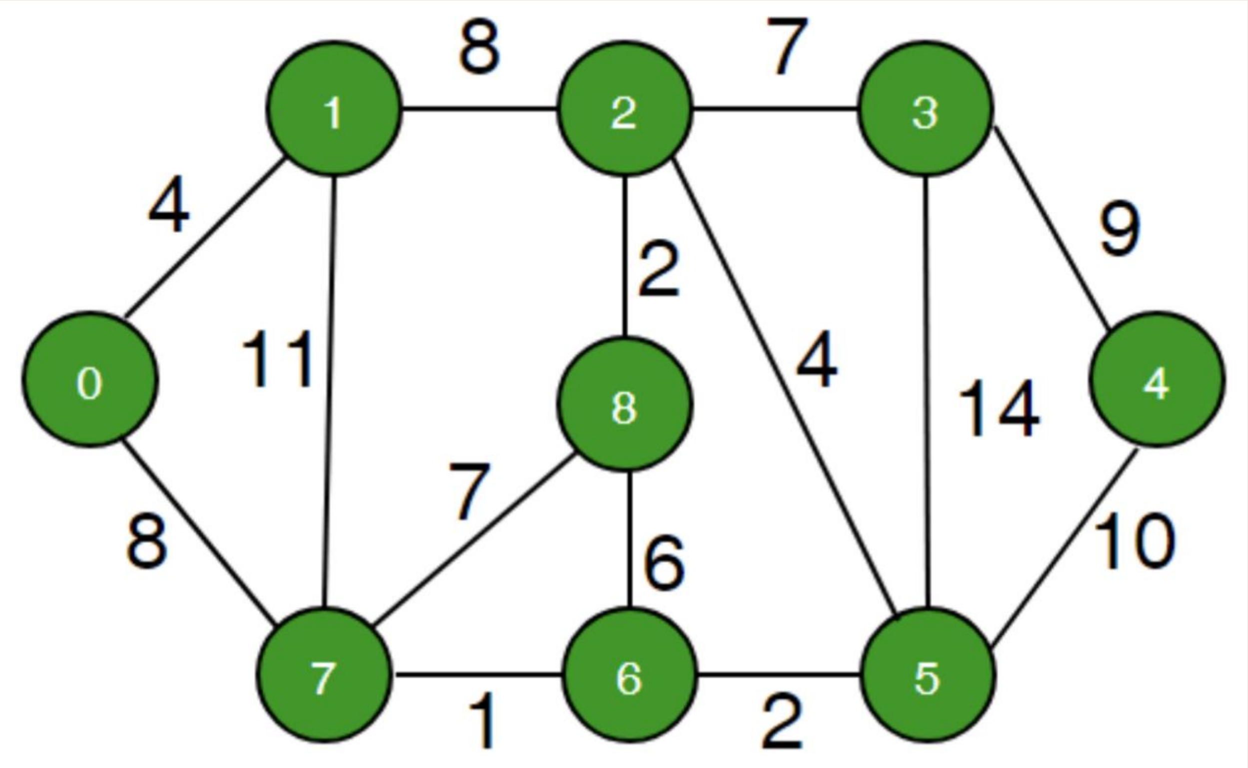
- Each iteration of the while-loop halves the number of connected components in T.
  - The running time is O(m log n).

Picture from [1]



Goal: Find an MST of the graph using Baruvka's Algorithm...

More Examples



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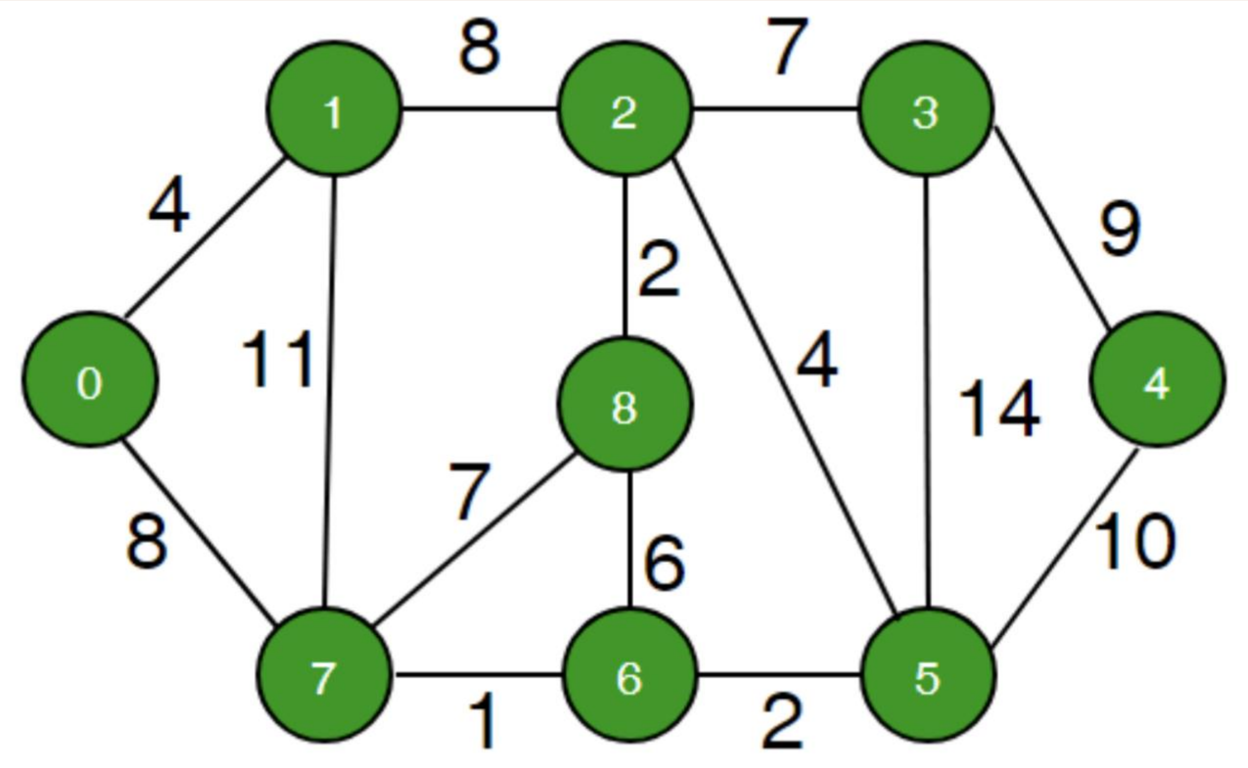

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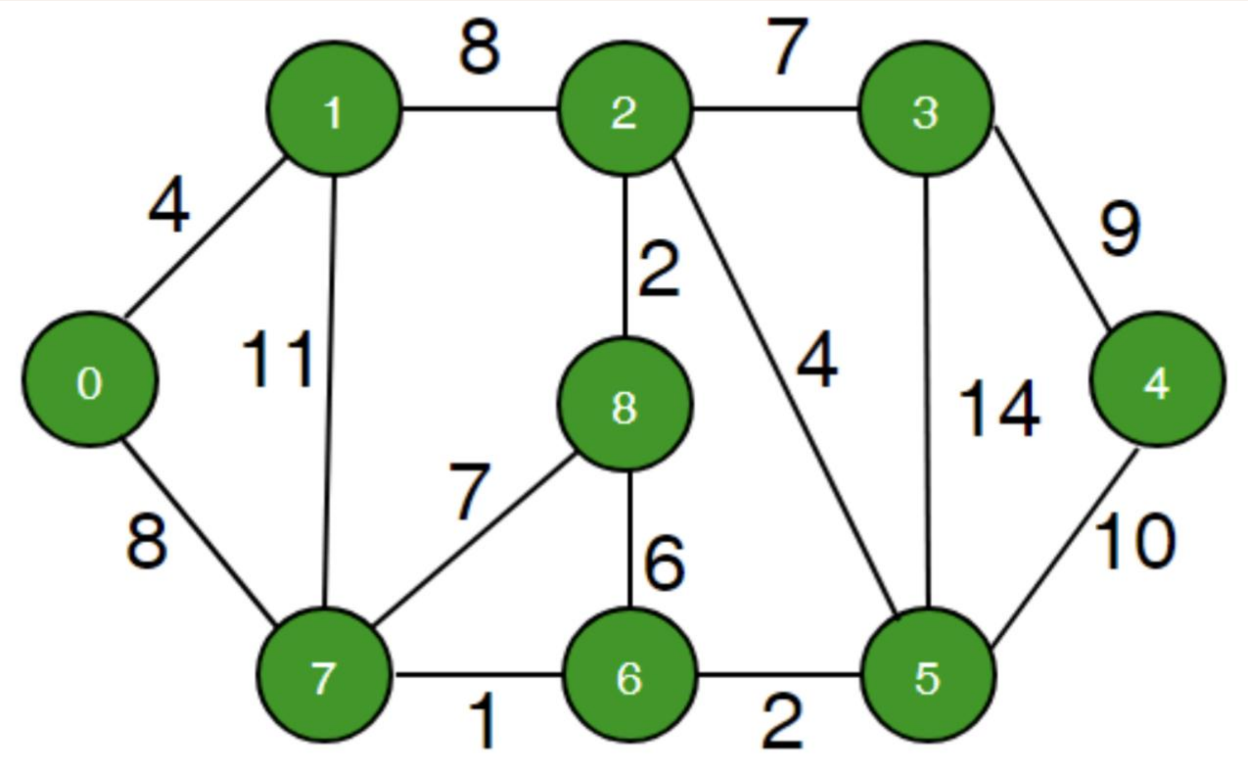
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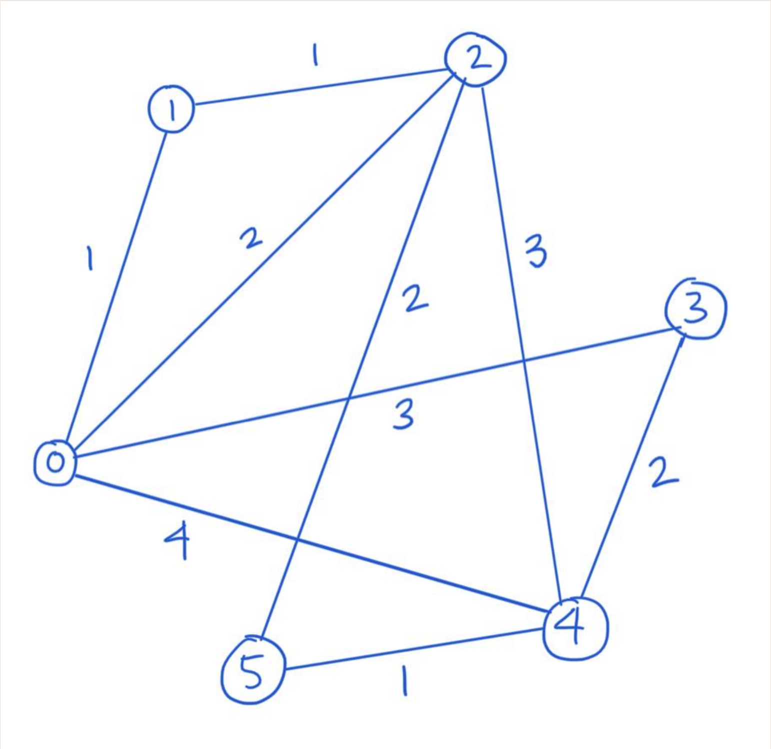
8

How many different MSTs does the graph below have?



A little extra

An NP-Complete Problem

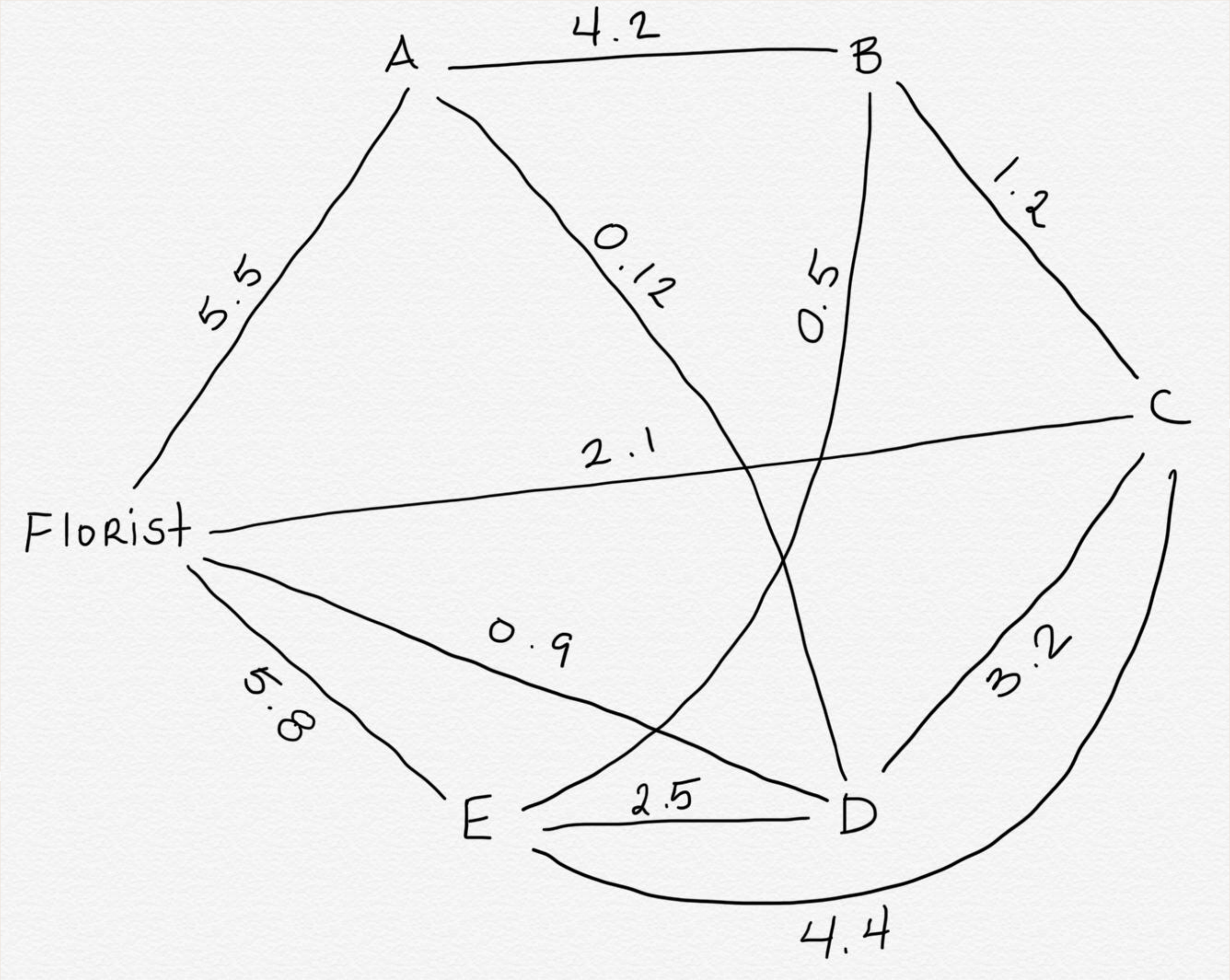
Imagine...

You are employed at a florist shop as a driver during their busy Valentine's Day season. Your first task is to take a list of delivery addresses and plan a route according to the following specifications:

- You want to start and end at the florist shop.
- You want to deliver ALL the flowers
- You want to minimize the distance you have to go.
- You don't want to visit any stop more than once.
- You don't want to drive the same road more than once.

How would you go about solving this problem?

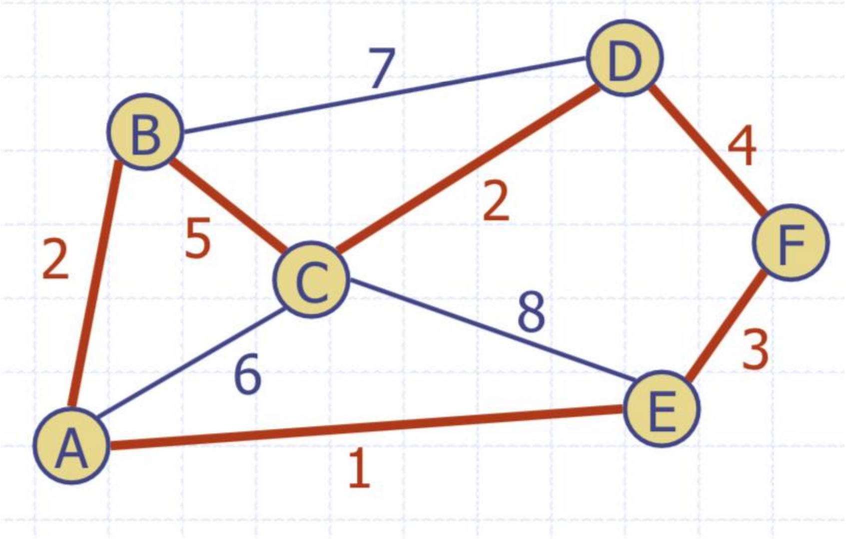
	Florist	A	B	C	D	E
Florist	0	5.5	NONE	2.1	0.9	5.8
A	5.5	0	4.2	NONE	0.12	NONE
B	NONE	4.2	0	1.2	NONE	0.5
C	2.1	NONE	1.2	0	3.2	4.4
D	0.9	0.12	NONE	3.2	0	2.5
E	5.8	NONE	0.5	4.4	2.5	0



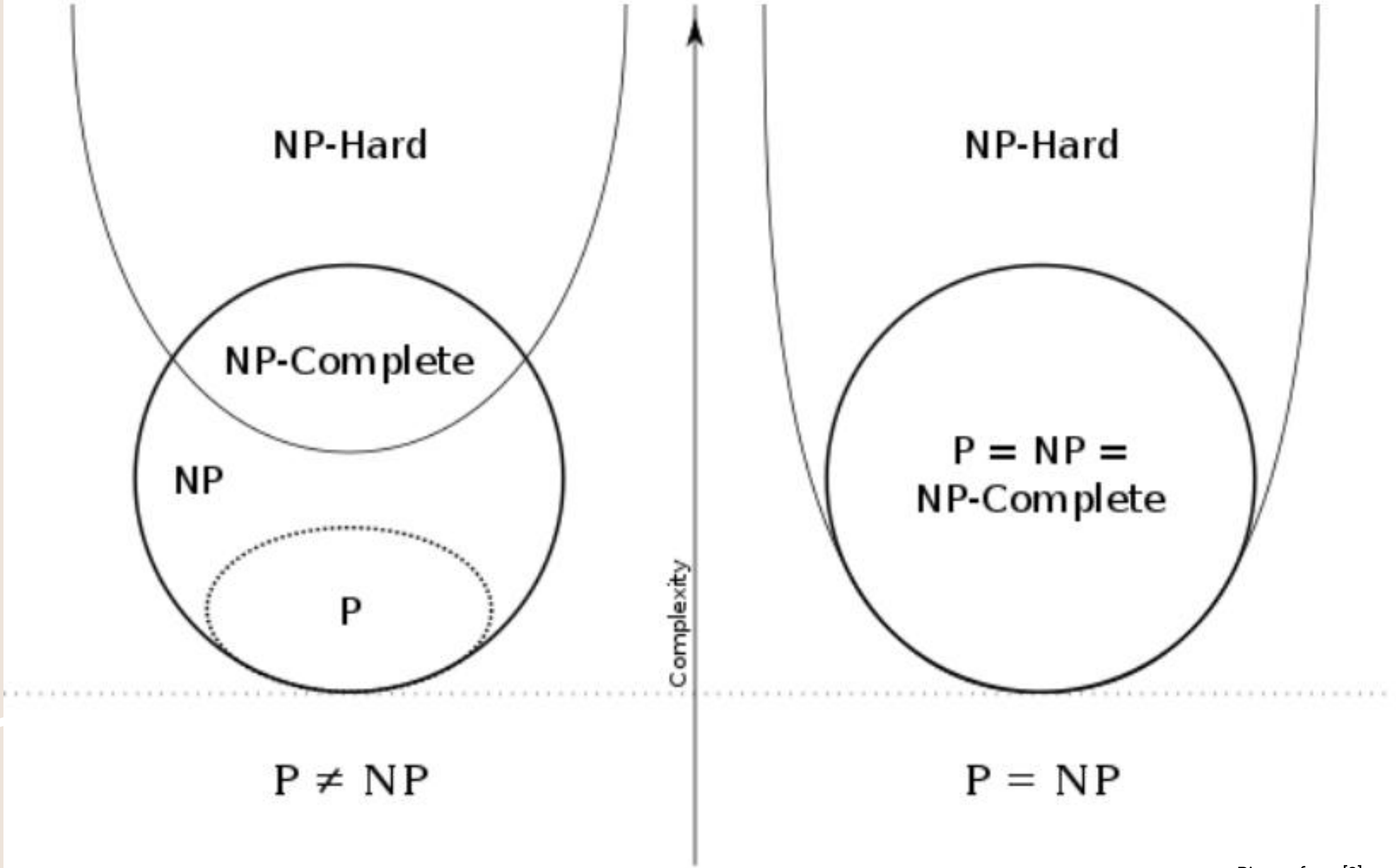
A
B
C
D
E
Florist
4.2
1.2
0.12
0.5
5.5
2.1
0.9
5.8
3.2
2.5
4.4

The Traveling Salesperson Problem

- A **tour** of a graph is a spanning cycle (goes through all the vertices)
- A TSP tour of a weighted graph is a tour that is simple (i.e. repeats no vertices or edges) and has minimum weight
- Determining if a TSP tour shorter than a given length L **exists** in a graph is NP-Complete (i.e. solution can be verified in polynomial time but not discovered in polynomial time—at least not yet)
- Finding a TSP tour is NP-hard (i.e. finding a minimum-weight tour)



Picture from [1]



Picture from [3]

Other NP-Complete Problems

- Boolean Satisfiability Problem (SAT)
- Knapsack Problem
- Hamiltonian Path Problem
- Subgraph Isomorphism Problem
- Subset sum problem
- Clique Problem
- Vertex Cover Problem
- Independent set problem
- Dominating set problem
- Graph Coloring problem

References

[1] Goodrich and Tamassia
[2] Sedgewick and Wayne
[3] en.wikipedia.org