Math Fundamentals Part 1

- Exponents & Logarithms
- Sequences & Summations

Exponents & Logarithms

$a^3 = a \cdot a \cdot a$

$ab^2 = a \cdot b \cdot b$

$(ab)^2 = (ab)(ab) = a^2 b^2$

$b^x = a \iff \log_b a = x$

If the log base is not specified, assume it's 2.

Notation:

$\log^2 a = (\log a)^2$

$\log a^2 = \log(a^2)$

Rules of Exponents & Logarithms

1. $b^x \cdot b^y = b^{x+y}$ $\qquad \log_b(xy) = \log_b x + \log_b y$

2. $\dfrac{b^x}{b^y} = b^{x-y}$ $\qquad \log_b(x/y) = \log_b x - \log_b y$

3. $(b^x)^y = b^{xy}$ $\qquad \log_b(x^y) = y \log_b x$

4. $\log_a b = \dfrac{\log_c b}{\log_c a}$

5. $\log_b(b^x) = x$ and $b^{\log_b x} = x$

Simplify each of the following without using a calculator.

(a) $4 \times (4^2)^6 =$ $4^1 \cdot 4^{12} = 4^{13}$

(b) $5^{(2^3)} =$ 5^8

(c) $log_2 16^2 =$ $2 \log_2 16 = 2 \cdot 4 = 8$

(d) $log_3 \left(\frac{1}{9}\right) =$ $\log_3\left(\frac{1}{3^2}\right) = \log_3\left(3^{-2}\right) = -2$

(e) $log_{16} 32 =$ $\log_{16}(16 \cdot 2) = \log_{16} 16 + \log_{16} 2$

$= 1 + \frac{1}{4} = \frac{5}{4}$

$\frac{\log_2 32}{\log_2 16} = \frac{5}{4}$

Determine what's wrong with the following "proof."

Conjecture: 8 = 16
Direct Proof.

Let a and k be any positive integers.

a^k

$= a^{k+1}/a$

$\neq a^{k+2}/a^2$

$\neq a^k \cdot a^k/a^2$

$= a^{2k}/a^2$

$= (a^2)^k/(a^2)^1$

$= (a^2)^{k-1}$

Let a = 2 and k = 3.

Then $2^3 = 8 = (2^2)^2 = 4^2 = 16$.

$$\frac{a^{k+1}\cdot a}{a^1 \cdot a} = \frac{a^{k+2}}{a^2}$$

$$a^{k+2} = a^k \cdot a^2$$

$$\frac{b^k}{b^1}$$

Summations

$$\sum_{k=1}^{N} k = \frac{N(N+1)}{2}$$ Arithmetic Series

$$\sum_{k=1}^{N} k^2 = \frac{N(N+1)(2N+1)}{6}$$

$b \neq 1$, $$\sum_{k=0}^{N} b^k = \frac{b^{N+1}-1}{b-1}$$ geometric series

$1+4+9+16+\ldots+N$

$1^2+2^2+3^2+4^2+\ldots+(\sqrt{N})^2$

$$\sum_{k=1}^{\sqrt{N}} k^2$$

$$\sum_{k=1}^{N} 1 = N$$

$$\sum_{k=0}^{N} 1 = N+1$$

$\sum c f(k)$

$= c \sum f(k)$

$$\sum_{k=1}^{N} (f(k)+g(k)) = \sum_{k=1}^{N} f(k) + \sum_{k=1}^{N} g(k)$$

Simplify each of the following.

$\log a + \log b = \log(a \cdot b)$

(*a*) $1 + 2 + 3 + \ldots. + N - 1 + N =$

$$\sum_{k=1}^{N} k = \frac{N(N+1)}{2}$$

(*b*) $\boxed{logN + log(N-1) + \ldots + log2 + log1} =$

$$\sum_{k=1}^{N} \log k = \log\left(\underline{N \cdot (N-1) \cdot (N-2) \cdot \ldots . (1)}\right) = \log(N!)$$

$$3^{\log_3 N + 1} = 3^1 \cdot \boxed{3^{\log_3 N}} = 3N$$

$$\log_3 \underline{N} = \cancel{\log_3} 3^k$$

$$k = \log_3 N$$

(*c*) ~~$1 + 3 + 9 + 27 + \ldots + N$~~ $=$

$$\sum_{k=0}^{\log_3 N} 3^k = \frac{3^{\log_3 N + 1} - 1}{3 - 1} = \frac{3N - 1}{2}$$

(*d*) $\sum_{k=1}^{N} 1 =$ N

(*e*) $\sum_{i=1}^{logN} 1 =$ $\log N$

More Examples

1. Write the following as a summation:

$$\underline{5} + \underline{15} + \underline{25} + \ldots + N \quad \frac{N+5}{10}$$

$5(2\cdot\boxed{1}-1)$ $5(2\cdot\boxed{2}-1)$ $5(2\cdot\boxed{3}-1)$

$$\sum_{k=1}^{\frac{N+5}{10}}(10k-5)$$

quadratic

2. Simplify the following:

$$\sum_{k=0}^{N}\sum_{j=1}^{N} j = \sum_{k=0}^{N}\frac{N(N+1)}{2} = \frac{N(N+1)}{2}\sum_{k=0}^{N} 1 = \frac{N(N+1)^2}{2}$$

3. Simplify the following:

$$\sum_{k=1}^{N}\sum_{j=1}^{N} kj = \sum_{k=1}^{N} k \sum_{j=1}^{N} j = \sum_{k=1}^{N} k\frac{N(N+1)}{2} = \frac{N(N+1)}{2}\sum_{k=1}^{N} k = \frac{N^2(N+1)^2}{4}$$

$$k(k+1) = k^2 + k$$

4. Simplify the following:

$$\sum_{k=1}^{N}\sum_{j=1}^{k} j = \sum_{k=1}^{N}\frac{k(k+1)}{2} = \frac{1}{2}\sum_{k=1}^{N}(k^2+k) = \frac{1}{2}\left[\sum_{k=1}^{N} k^2 + \sum_{k=1}^{N} k\right] =$$

cubic

$$\frac{1}{2}\left[\frac{N(N+1)(2N+1)}{6} + \frac{N(N+1)}{2}\right]$$