## Introduction

The purpose of this document is to show how characteristic vectors (also known as eigenvectors, latent vectors or proper vectors) and characteristic roots (also known as eigenvalues, latent roots or proper values) and Cholesky decomposition can be used to decompose covariance matrices and do some "clever" matrix operations. A MATLAB implementation is included and certain peculiarities of MATLAB are discussed concerning these calculations.

## Example

Bivariate normal random variables may be generated by the following procedures. Let Σ be the covariance matrix, C be the corresponding matrix of characteristic vectors (also known as eigenvectors, latent vectors or proper vectors) and D be the diagonal matrix of characteristic roots (also known as eigenvalues, latent roots or proper values). Then $C*\Sigma*C^T = D$ and $D^{-1/2}*C*\Sigma*C^T*D^{-1/2} = D^{-1/2}*D*D^{-1/2} = I$ and $\Sigma = C^T*D^{1/2}*D^{1/2}*C$. Where I is the identity matrix, the superscript "T" denotes the transpose

As an example, assume that the covariance matrix is

$$\Sigma = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

With the corresponding matrix of characteristic vectors

$$C = \begin{bmatrix} 0.5257 & 0.8506 \\ 0.8506 & -0.5257 \end{bmatrix}$$

And the diagonal matrix of characteristic roots 2.6180 and 0.3820. Let

$$C = \begin{bmatrix} \sqrt{2.6180} & 0 \\ 0 & \sqrt{0.3820} \end{bmatrix} = \begin{bmatrix} 1.6180 & 0 \\ 0 & 0.6180 \end{bmatrix}$$

Thus

$$C^T*D^{1/2} = \begin{bmatrix} 0.8506 & 0.5257 \\ 1.3763 & -0.3249 \end{bmatrix}$$

Given the foregoing matrix, each pair of generated independent standard normal random variates, e1 and e2 are used with $C^T*D^{1/2}$ to obtain bivariate normal variates $e_1^*$ and $e_2^*$.

$$\begin{bmatrix} 0.8506 & 0.5257 \\ 1.3763 & -0.3249 \end{bmatrix} * \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} e_1^* \\ e_2^* \end{bmatrix}$$

The bivariate normal random variates $e_1^*$ and $e_2^*$ have mean

$$E\begin{bmatrix} e_1^* \\ e_2^* \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

And covariance matrix

$$E\begin{bmatrix} e_1^* \\ e_2^* \end{bmatrix} * \begin{bmatrix} e_1^* & e_2^* \end{bmatrix} = C^T * D^{1/2} * \left( E\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} * \begin{bmatrix} e_1 & e_2 \end{bmatrix} \right) * D^{1/2} * C$$

$$= \begin{bmatrix} 0.8506 & 0.5257 \\ 1.3763 & -0.3249 \end{bmatrix} * \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} * \begin{bmatrix} 0.8506 & 1.3763 \\ 0.5257 & -0.3249 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \Sigma$$

Alternatively, if a computer package is available, the Cholesky matrix decomposition method may be used to obtain a nonsingular triangular matrix $P$ such that $P * P^T = \Sigma$ or

$$P\left( E\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} * \begin{bmatrix} e_1 & e_2 \end{bmatrix} \right) * P^T = \Sigma$$

Consequently,

$$P\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} e_1^* \\ e_2^* \end{bmatrix}$$

For example, if

$$\Sigma = \begin{bmatrix} 3 & 2 \\ 2 & 8 \end{bmatrix}$$

This yield a Cholesky matrix

$$P = \begin{bmatrix} 1.732051 & 0 \\ 1.154701 & 2.581989 \end{bmatrix}$$

And

$$P\left(E\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} * \begin{bmatrix} e_1 & e_2 \end{bmatrix}\right) * P^T = \begin{bmatrix} 1.732051 & 0 \\ 1.154701 & 2.581989 \end{bmatrix} * \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} * \begin{bmatrix} 1.732051 & 1.154701 \\ 0 & 2.581989 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 \\ 2 & 8 \end{bmatrix} = \Sigma$$

In a Monte Carlo sampling study, using the transformation matrix, these procedures are repeated for each of T pairs of random variables for each of the N samples.

## A MATLAB Implementation

```
close all;clear;clc
format bank; % two decimal points

% Bivariate normal random variables may be generated by the following
procedures
% Let Sigma be the covariance matrix, C be the corresponding matrix of
characteristic vectors
% ,also known as eigenvectors, latent vectors or proper vectors, and D be the
diagonal matrix of characteristic
% ,roots, also known as eigenvalues, latent roots or proper values. Then
C*Sigma*C' = D and
% D^(-1/2)*C*Sigma*C'*D^(-1/2) = D^(-1/2)*D*D^(-1/2) = I, where I is the
identity matrix and
% Sigma = C'*D^(1/2)*D^(1/2)*C.

% As an example, assume that the covariance matrix is

mSigma = [ 1 1; 1 2];
display(mSigma)

[C, d] = eig(mSigma);% In matlab with two outputs we get first the
eigenvectors and then the eigenvalues
d=diag(d);% only want the diagonal, since we want only a column vector
containing the eigenvalues
% Note that MATLAB orders the characteristic roots and the corresponding
characteristic
% vectors from smallest to largest. To conform to the text order the roots
% from largest to smallest and reverse the order of the columns of the matrix
of
% characteristic vectors to match.
d = flipud(d);
C = flipud(C');
% Create the diagonal matrix D of characteristic roots
D = zeros( length(d), length(d));
D(logical(eye(size(D)))) = d;
% Due to the normalization issue of characteristic vectors in MATLAB the sign
of the last characteristic vector
% must be changed for us to exactly replicate the results.
C(end,:) = -C(end,:);
```

```
% with the corresponding matrix of characeteristic vectors
display(C)

% and the diagonal matrix of characteristic roots 2.6180 and 0.3820.
display(D)

% Let
SqrtD = sqrt(D);
display(SqrtD);

% Thus
CTransposeSqrtD = C'*SqrtD;
display(CTransposeSqrtD);

% Define some random variables
e1 = randn(1000,1);
e2 = randn(1000,1);
e = [e1 e2];

% Given the foregoing matrix, each pair of generated independent standard
normal random variates e1 and e2 are used with
% C'sqrt(D) to obtain bivariate normal variates estar1 and estar2
estar = e * CTransposeSqrtD';
estar1 = estar(:,1);
estar2 = estar(:,2);

% The bivariate normal random variates estar1 and estar2 have a theoretical
mean 0 (empirically the will be close to zero)
disp(mean(estar1))
disp(mean(estar2))

% and covariance matrix Sigma
SqrtDC = sqrt(D)*C;
mSigma1 = CTransposeSqrtD*eye(2)*SqrtDC;
display(mSigma1)

%%
% Alternatively, if a computer package such is available, the Cholesky matrix
decomposition
% method may be used to obtain a nonsingular triangular matrix P such that
P*PTranspose = Sigma

% For example if
mSigma2 = [3 2; 2 8];
display(mSigma2)

% This yields a Cholesky matrix
P = chol(mSigma2);
P = P';
display(P)

% and
mSigma3 = P*eye(2)*P';
display(mSigma3)
```

```
% In a monte Carlo sampling study, using the transformation matrix, these procedures are repeated
% for each of T pairs of random variables for each of the N samples
```

## Conclusions

Calculating characteristic roots, vectors and Cholesky decomposition in MATLAB is straightforward. However some peculiarities do exist compared to what one get with "standard calculations”. This documentation highlights these peculiarities for which the user should be aware of. In particular

- MATLAB orders the characteristic roots and the corresponding characteristic vectors from smallest to largest.
- Due to normalization issues (which normally do not make a difference), the sign of the last characteristic vector has to be changed (once we have ordered them from largest to smallest)

## References


1) Judge et al, Introduction to the Theory and Practice of Econometrics 2ed, Wiley 1988, Pages 494-496